CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-5-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. All questions are compulsory.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Question Number **1** to **18** are Multiple Choice Questions (MCQ) type and Question Number **19** & **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions of **2** marks each.

(v) In **Section C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Question Number **32** to **35** are Long Answer (LA) type questions carrying **5** marks each.

(vii) In **Section E** – Question Number **36** to **38** are case study based questions carrying **4** marks each where **2** VSA type questions are of **1** mark each and **1** SA type question is of **2** marks. Internal choice is provided in **2** marks question in each case-study.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

(Multiple Choice Questions)

Each question carries 1 mark.

Select the correct option out of the four given options :

1. $\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$ **is equal to:** [1 mark]

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{1}{4}$

Correct Answer: (A) 1

Solution:

Concept:

- The principal value of $\sin^{-1} u$ lies in $[-\pi/2, \pi/2]$.
- For a standard angle, $\sin(\pi/6) = 1/2$ and $\sin(\pi/2) = 1$.

Step 1: Evaluate the inverse sine Since $\pi/6$ lies in the principal range and its sine is $1/2$, we have $\sin^{-1}(1/2) = \pi/6$. The inverse-sine notation means an angle, not the reciprocal of sine.

Step 2: Add the angles and evaluate

$$\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right] = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1.$$

Options (B), (C) and (D) give fractions smaller than this value and therefore fail; the angle sum is a right angle.

Step 3: Conclusion The required value is 1. Hence, the correct option is (A).

Quick Tip to Solve: First replace the inverse sine by its principal standard angle; then add angles before taking sine.

2. Let $A = \{3, 5\}$. Then the number of reflexive relations on A is: [1 mark]

(A) 2

(B) 4

(C) 0

(D) 8

Correct Answer: (B) 4

Solution:

Concept:

- A relation on A is a subset of $A \times A$. Reflexivity requires every pair (a, a), for $a \in A$, to be included.
- Each remaining ordered pair can be included or excluded independently.

Step 1: Identify compulsory pairs Here $A \times A = \{(3,3), (3,5), (5,3), (5,5)\}$. Every reflexive relation must contain both $(3, 3)$ and $(5, 5)$. These compulsory pairs create no choice.

Step 2: Count the independent choices The two off-diagonal pairs $(3, 5)$ and $(5, 3)$ are optional. Thus the number of relations is $2^2 = 4$: include neither, either one separately, or both. Option (A) undercounts these four possibilities, (C) ignores existing reflexive relations, and (D) counts too many choices.

Step 3: Conclusion There are 4 reflexive relations. Hence, the correct option is (B).

Quick Tip to Solve: For a set of size n, reserve the n diagonal pairs and count the remaining n squared minus n binary choices.

---

3. If $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}$ and $A = B^2$, then x equals: [1 mark]

(A) ± 1

(B) -1

(C) 1

(D) 2

Correct Answer: (C) 1

Solution:

Concept:

- Matrix multiplication uses row-by-column products; B^2 means BB.
- Two matrices are equal only when all corresponding entries are equal.

Step 1: Compute the square

$$B^2 = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ x+1 & 1 \end{pmatrix}.$$

In particular, the lower-left entry is $1 \cdot x + 1 \cdot 1 = x + 1$.

Quick Tip to Solve: Use the off-diagonal entry as well as the squared diagonal entry to remove the spurious negative root.

Step 2: Use both entry conditions Comparing with A gives $x^2 = 1$ and $x+1 = 2$. The second equation forces $x = 1$, which also satisfies the first. Option (A) includes the inadmissible value -1; (B) gives lower-left entry zero; and (D) gives diagonal entry four, so all three fail.

Step 3: Conclusion $x = 1$. Hence, the correct option is (C).

Quick Tip to Solve: Use the off-diagonal entry as well as the squared diagonal entry to remove the spurious negative root.

---

4. If $A = [a_{ij}]$ is a square matrix of order 2 such that $a_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}$, then A^2 is: [1 mark]

(A) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

Concept:

- The entry rule specifies diagonal entries as zero and off-diagonal entries as one.
- The (i, j) entry of A^2 is the dot product of row i and column j of A.

Step 1: Form the matrix Because the order is two, the given rule gives $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. There are exactly two diagonal positions and two off-diagonal positions.

Step 2: Multiply row by column

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Options (A), (B) and (C) each have an incorrect off-diagonal or diagonal entry; none equals the product.

Step 3: Conclusion $A^2 = I_2$. Hence, the correct option is (D).

Quick Tip to Solve: This matrix swaps the two coordinates; applying the swap twice returns each coordinate to its original position.

---

5. The value of the determinant $\begin{vmatrix} 6 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{vmatrix}$ **is:** [1 mark]

(A) 10

(B) 8

(C) 7

(D) -7

Correct Answer: (D) -7

Solution:

Concept:

- Expand a determinant along a row using alternating cofactor signs $+, -, +$.
- A two-by-two determinant equals $ad - bc$ for entries a, b, c, d in row order.

Step 1: Expand along the first row The zero in the first row eliminates the middle term. With the positive cofactor sign at position $(1, 3)$, the entry -1 retains its negative sign:

$$D = 6\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}.$$

Step 2: Evaluate the minors The minors are $1 \cdot 3 - 4 \cdot 1 = -1$ and $2 \cdot 1 - 1 \cdot 1 = 1$. Hence $D = 6(-1) + (-1)(1) = -6 - 1 = -7$. The positive choices $10, 8, 7$ in (A), (B), (C) do not equal this signed cofactor sum; dropping the last negative sign would produce an incorrect result.

Step 3: Conclusion The determinant is -7. Hence, the correct option is (D).

Quick Tip to Solve: Expand along the row containing zero and keep the entry sign separate from its cofactor sign.

6. The function $f(x) = [x]$**, where** $[x]$ **denotes the greatest integer less than or equal to** x**, is continuous at:** [1 mark]

(A) $x = 1$

(B) $x = 1.5$

(C) $x = -2$

(D) $x = 4$

Correct Answer: (B) $x = 1.5$

Solution:

Concept:

- The greatest integer function is constant on each open interval $(n, n + 1)$, with n an integer.
- At an integer n, its left limit is $n - 1$, while its right limit and value are n.

Step 1: Check the noninteger point A sufficiently small neighbourhood of 1.5 lies inside $(1, 2)$.

Quick Tip to Solve: For the greatest integer function, select a noninteger point; each integer is a jump discontinuity.

Throughout that neighbourhood, $[x] = 1$, so the left and right limits both equal $f(1.5) = 1$. Thus the function is continuous at 1.5.

Step 2: Check the integer alternatives At 1, the one-sided limits are 0 and 1; at -2, they are -3 and -2; at 4, they are 3 and 4. Each pair is unequal, so options (A), (C) and (D) are discontinuity points. This also shows why the sign of an integer does not remove the jump.

Step 3: Conclusion The function is continuous at $x = 1.5$. Hence, the correct option is (B).

Quick Tip to Solve: For the greatest integer function, select a noninteger point; each integer is a jump discontinuity.

7. The derivative of x^{2x} w.r.t. x is: [1 mark]

(A) x^{2x-1}

(B) $2x^{2x} \log x$

(C) $2x^{2x}(1 + \log x)$

(D) $2x^{2x}(1 - \log x)$

Correct Answer: (C) $2x^{2x}(1 + \log x)$

Solution:

Concept:

- For $x > 0$, logarithmic differentiation handles a variable base raised to a variable power.
- If $y > 0$, then $d(\log y)/dx = y'/y$, and the product rule differentiates $2x \log x$.

Step 1: Take logarithms Let $y = x^{2x}$, with $x > 0$. Then $\log y = 2x \log x$. Differentiating both sides gives

$$\frac{y'}{y} = 2 \log x + 2x\frac{1}{x} = 2(1 + \log x).$$

The extra constant term comes from differentiating $\log x$.

Step 2: Restore the original function Multiplying by $y = x^{2x}$ gives $y' = 2x^{2x}(1 + \log x)$. Option (A), x^{2x-1}, omits the variable-exponent contribution; (B) omits the term from differentiating $\log x$; and (D) gives that term an incorrect relative sign. Only (C) includes both product-rule terms.

Step 3: Conclusion $\frac{d}{dx}x^{2x} = 2x^{2x}(1 + \log x)$, for $x > 0$. Hence, the correct option is (C).

Quick Tip to Solve: When both base and exponent contain x, take logarithms before differentiating; the ordinary power rule is insufficient.

8. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is: [1 mark]

(A) $(-1, \infty)$

(B) $(-2, -1)$

(C) $(-\infty, -2)$

(D) $[-1, 1]$

Correct Answer: (B) $(-2, -1)$

Solution:

Concept:

- A differentiable function is decreasing on an interval where its derivative is negative.
- The product $(x+1)(x+2)$ is negative between its distinct roots -2 and -1.

Step 1: Differentiate and factor

$$f\prime(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2) = 6(x+1)(x+2).$$

The derivative vanishes at $x = -2$ and $x = -1$; these points divide the real line into three intervals.

Step 2: Determine the derivative sign For $x < -2$, both factors are negative, so $f\prime(x) > 0$. For $-2 < x < -1$, the factors have opposite signs, so $f\prime(x) < 0$. For $x > -1$, both are positive, so $f\prime(x) > 0$. Thus (A) and (C) are increasing intervals, and (D) lies in an increasing region apart from its endpoint. Only (B) has the required negative derivative throughout its interior.

Step 3: Conclusion The decreasing interval is $(-2, -1)$. Hence, the correct option is (B).

Quick Tip to Solve: Factor the derivative and use a sign chart; a quadratic with positive leading coefficient is negative between its roots.

9. The function $f(x) = x|x|,\ x \in \mathbb{R}$ is differentiable: [1 mark]

(A) only at $x = 0$

(B) only at $x = 1$

(C) in $\mathbb{R}$

(D) in $\mathbb{R} - \{0\}$

Correct Answer: (C) in $\mathbb{R}$

Solution:

Concept:

- A function is differentiable at a point when its difference quotient has a finite, common limit from both sides.
- For $x \geq 0$, $x|x| = x^2$; for $x < 0$, $x|x| = -x^2$.

Step 1: Differentiate away from zero On $x > 0$, $f\prime(x) = 2x$, while on $x < 0$, $f\prime(x) = -2x$. Thus the function is differentiable at every nonzero real number.

Quick Tip to Solve: Check the difference quotient at zero; multiplying $|x|$ by x removes its derivative jump.

Step 2: Check the joining point Since $f(0) = 0$, the derivative at the origin is

$$f\prime(0) = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0.$$

Both one-sided limits agree. Options (A) and (B) wrongly restrict differentiability to one point, while (D) wrongly excludes the origin.

Step 3: Conclusion The function is differentiable throughout $\mathbb{R}$. Hence, the correct option is (C).

Quick Tip to Solve: Check the difference quotient at zero; multiplying $|x|$ by x removes its derivative jump.

---

10. $\int \frac{\sec x}{\sec x - \tan x}\, dx$ **equals:** [1 mark]

(A) $\sec x - \tan x + c$

(B) $\sec x + \tan x + c$

(C) $\tan x - \sec x + c$

(D) $-(\sec x + \tan x) + c$

Correct Answer: (B) $\sec x + \tan x + c$

Solution:

Concept:

- Use $\sec^2 x - \tan^2 x = 1$ to rationalise the denominator.
- The standard derivatives are $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Step 1: Simplify the integrand On any interval where the original integrand is defined,

$$\frac{\sec x}{\sec x - \tan x} = \frac{\sec x(\sec x + \tan x)}{\sec^2 x - \tan^2 x} = \sec^2 x + \sec x \tan x.$$

Step 2: Integrate and test the options Therefore

$$\int \frac{\sec x}{\sec x - \tan x}\, dx = \tan x + \sec x + c.$$

Option (A) differentiates to $\sec x \tan x - \sec^2 x$; option (C) gives $\sec^2 x - \sec x \tan x$; option (D) gives the negative of the required integrand. Only (B) has both required positive terms.

Step 3: Conclusion The integral is $\sec x + \tan x + c$. Hence, the correct option is (B).

Quick Tip to Solve: Multiply by $\sec x + \tan x$; the denominator becomes exactly one.

---

11. The value of $\int_0^{\pi/4} \sin 2x\, dx$ is: [1 mark]

(A) 0

(B) 1

(C) $\frac{1}{2}$

(D) $-\frac{1}{2}$

Correct Answer: (C) $\frac{1}{2}$

Solution:

Concept:

- For a constant $a \neq 0$, $\int \sin(ax)\, dx = -\cos(ax)/a + c$.
- A definite integral equals the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Find an antiderivative The factor 2 in the angle requires a factor $1/2$ in the antiderivative:

$$\int \sin 2x\, dx = -\frac{1}{2}\cos 2x + c.$$

Differentiation confirms this, because the chain-rule factor 2 cancels $1/2$.

Step 2: Substitute both limits

$$\int_0^{\pi/4} \sin 2x\, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4} = -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0 = 0 + \frac{1}{2}.$$

The integrand is positive inside the interval, excluding (A) and (D). Option (B) misses the chain-rule factor in integration.

Step 3: Conclusion The value is $\frac{1}{2}$. Hence, the correct option is (C).

Quick Tip to Solve: Double the limits inside the cosine, and retain the outside factor $1/2$.

---

12. The sum of the order and the degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right]$ is: [1 mark]

(A) 2

(B) 3

(C) 5

(D) 0

Correct Answer: (B) 3

Solution:

Concept:

- Order is the order of the highest derivative present after carrying out the indicated differentiation.

Quick Tip to Solve: Expand the outer derivative first; degree follows the highest derivative, not the largest visible power.

- Degree is the power of the highest-order derivative when the differential expression is polynomial in derivatives.

Step 1: Expand the indicated derivative Apply the chain rule to the cube of dy/dx:

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right] = 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2}.$$

The highest derivative appearing is the second derivative, so the order is 2.

Step 2: Identify its degree The expanded expression is polynomial in derivatives. The highest-order derivative d^2y/dx^2 occurs to the first power, so the degree is 1. The square on the first derivative does not determine degree. Thus the required sum is $2 + 1 = 3$. Option (A) gives the order alone, (C) incorrectly counts the original cube, and (D) ignores the second derivative.

Step 3: Conclusion The sum of order and degree is 3. Hence, the correct option is (B).

Quick Tip to Solve: Expand the outer derivative first; degree follows the highest derivative, not the largest visible power.

13. Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if: [1 mark]

(A) $a_1b_1 + a_2b_2 + a_3b_3 = 0$
(B) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$
(C) $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$
(D) $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$
Correct Answer: (B) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Solution:

Concept:

- Two nonzero vectors are collinear when one is a scalar multiple of the other: $\vec{a} = \lambda\vec{b}$.
- Comparing components gives $a_1 = \lambda b_1$, $a_2 = \lambda b_2$, and $a_3 = \lambda b_3$.

Step 1: Express the common multiplier Where the displayed denominators are nonzero, the component equations give

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \lambda.$$

This is the standard proportionality criterion represented by option (B). If a component of $\vec{b}$ is zero, use the scalar-multiple equations directly instead of dividing by zero.

Step 2: Compare the other conditions Option (A) states a zero dot product, which describes perpendicular nonzero vectors. Option (C) describes equal vectors, a special case of collinearity, but excludes unequal scalar multiples. Option (D) equates component sums; for example, $(1, 0, 0)$ and $(0, 1, 0)$ have equal sums but are not collinear.

Step 3: Conclusion The general proportional-component condition is option (B).

Quick Tip to Solve: Look for a common multiplier across all components; equality of vectors is only one special case.

14. A unit vector $\hat{a}$ makes equal but acute angles on the coordinate axes. The projection of the vector $\hat{a}$ on the vector $\vec{b} = 5\hat{i} + 7\hat{j} - \hat{k}$ is: [1 mark]

(A) $\frac{11}{15}$

(B) $\frac{11}{5\sqrt{3}}$

(C) $\frac{4}{5}$

(D) $\frac{3}{5\sqrt{3}}$

Correct Answer: (A) $\frac{11}{15}$

Solution:

Concept:

- The direction cosines l, m, n of a unit vector satisfy $l^2 + m^2 + n^2 = 1$.
- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $(\vec{a} \cdot \vec{b})/|\vec{b}|$.

Step 1: Find the unit vector Equal acute angles imply equal positive direction cosines. Setting each equal to l gives $3l^2 = 1$, so

$$\hat{a} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}).$$

The positive sign is essential because all three angles are acute.

Step 2: Compute the projection on the given vector Here

$$\hat{a} \cdot \vec{b} = \frac{5 + 7 - 1}{\sqrt{3}} = \frac{11}{\sqrt{3}}, \qquad |\vec{b}| = \sqrt{25 + 49 + 1} = 5\sqrt{3}.$$

Hence the scalar projection is

$$\frac{\hat{a} \cdot \vec{b}}{|\vec{b}|} = \frac{11/\sqrt{3}}{5\sqrt{3}} = \frac{11}{15}.$$

Option (B) omits one factor of $\sqrt{3}$; (C) and (D) do not match the dot product and magnitude.

Step 3: Conclusion The projection is $\frac{11}{15}$. Hence, the correct option is (A).

Quick Tip to Solve: For equal acute axis angles use $(1, 1, 1)/\sqrt{3}$, then divide the dot product by the target magnitude.

15. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is: [1 mark]

(A) 0°

(B) 30°

(C) 45°

(D) 90°

Correct Answer: (D) 90°

Solution:

Concept:

- A line written with a common parameter has direction ratios equal to the coefficients of that parameter in x, y, z.
- If the dot product of direction vectors is zero, the lines are perpendicular.

Step 1: Obtain the direction ratios Write $2x = 3y = -z = t$. Then $x = t/2$, $y = t/3$, and $z = -t$, giving ratios $(3, 2, -6)$ after scaling by 6. For $6x = -y = -4z = s$, we get $x = s/6$, $y = -s$, $z = -s/4$, giving ratios $(2, -12, -3)$ after scaling by 12.

Step 2: Use the dot product For these two direction vectors,

$$3(2) + 2(-12) + (-6)(-3) = 6 - 24 + 18 = 0.$$

Both direction vectors are nonzero, so the cosine of the angle between them is zero. Therefore the angle is a right angle; the cosines for options (A), (B), and (C) are all nonzero.

Step 3: Conclusion The angle is 90°. Hence, the correct option is (D).

Quick Tip to Solve: Take reciprocals of the coefficients in the chained equations before calculating the dot product.

16. If a line makes angles of 90°, 135° and 45° with the x, y and z axes respectively, then its direction cosines are: [1 mark]

(A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

(B) $-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}$

(D) $0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Correct Answer: (A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Solution:

Concept:

- Direction cosines are the cosines of the angles with the positive x, y, z axes, in that order.
- They satisfy $l^2 + m^2 + n^2 = 1$, with a negative cosine for an obtuse axis angle.

Quick Tip to Solve: Keep the axis order fixed: x, y, z; the obtuse angle supplies the negative cosine.

Step 1: Evaluate the three cosines The specified axis angles give

$$l = \cos 90^\circ = 0, \qquad m = \cos 135^\circ = -\frac{1}{\sqrt{2}}, \qquad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

The zero belongs to the x-component, and the negative value belongs to the y-component.

Step 2: Check the ordering and signs The values obey

$$l^2 + m^2 + n^2 = 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Options (B) and (C) incorrectly place the zero in the y-component. Option (D) gives a positive y-component and would correspond to 45°, whereas the given angle with the y-axis is obtuse.

Step 3: Conclusion The direction cosines are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. Hence, the correct option is (A).

Quick Tip to Solve: Keep the axis order fixed: x, y, z; the obtuse angle supplies the negative cosine.

---

17. If for any two events A and B, $P(A) = \frac{4}{5}$ and $P(A \cap B) = \frac{7}{10}$, then $P(B/A)$ is equal to: [1 mark]

(A) $\frac{1}{10}$

(B) $\frac{1}{8}$

(C) $\frac{7}{8}$

(D) $\frac{17}{20}$

Correct Answer: (C) $\frac{7}{8}$

Solution:

Concept:

- For events A, B with $P(A) > 0$, $P(B \mid A) = P(A \cap B)/P(A)$.
- Conditioning on A restricts the sample space to A; the favourable outcomes are those in $A \cap B$.

Step 1: Identify the conditioning event The notation $P(B/A)$ means the probability of B given A, so the denominator is $P(A) = 4/5$, not $P(B)$. The numerator is the given intersection probability $7/10$.

Step 2: Substitute and compare the options

$$P(B \mid A) = \frac{7/10}{4/5} = \frac{7}{10} \times \frac{5}{4} = \frac{7}{8}.$$

Options (A), (B) and (D) have values $1/10, 1/8, 17/20$, respectively, and none equals this ratio. In particular, $1/8$ is its complement, not the required probability.

Step 3: Conclusion $P(B/A) = \frac{7}{8}$. Hence, the correct option is (C).

Quick Tip to Solve: In $P(B/A)$, use the event after the slash as the denominator.

18. If A and B are two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B'/A)$ is:

[1 mark]

(A) $\frac{1}{4}$
(B) $\frac{1}{8}$
(C) $\frac{3}{4}$
(D) 1

Correct Answer: (C) $\frac{3}{4}$

Solution:

Concept:

- Independent events satisfy $P(A \cap B) = P(A)P(B)$.
- The complement B' has probability $1 - P(B)$, and remains independent of A.

Step 1: Find the required intersection Since A splits into the disjoint events $A \cap B$ and $A \cap B'$,

$$P(A \cap B') = \frac{1}{3} - \frac{1}{3} \times \frac{1}{4} = \frac{1}{4}.$$

Step 2: Condition on A and check the choices

$$P(B' \mid A) = \frac{P(A \cap B')}{P(A)} = \frac{1/4}{1/3} = \frac{3}{4}.$$

Option (A), $1/4$, is $P(B)$, not its complement. Option (B), $1/8$, does not equal the conditional ratio, and option (D), 1, would mean B' is certain given A, which it is not.

Step 3: Conclusion $P(B'/A) = \frac{3}{4}$. Hence, the correct option is (C).

Quick Tip to Solve: For independent events, conditioning leaves the probability unchanged; take the complement of $P(B)$.

19. Assertion (A): $\displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx = 3.$

Reason (R): $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$ [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- For an integrable function, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.
- Adding an integral to its reflected form is useful when the two integrands sum to a constant.

Step 1: Judge the assertion Let $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$. Reflecting x about the midpoint 5 gives

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\,dx.$$

Adding the two expressions gives $2I = \int_2^8 1\,dx = 6$, so $I = 3$. Assertion (A) is true.

Step 2: Judge the reason In $\int_a^b f(x)\,dx$, put $u = a+b-x$. Then $dx = -du$, and the limits a, b become b, a. Reversing them removes the minus sign and proves the identity in (R). Thus Reason (R) is true.

Step 3: Decide whether the reason explains the assertion Here $a+b = 2+8 = 10$. Applying precisely the identity in (R) interchanges the two square roots and supplies the reflected integral used to obtain $2I = 6$. Hence (R) directly explains (A); the other options incorrectly deny either the truth or this explanatory connection.

Step 4: Conclusion Both (A) and (R) are true, and (R) is the correct explanation of (A). The correct option is (A).

Quick Tip to Solve: When the radicals contain x and $a+b-x$, add the original and reflected integrands.

20. Assertion (A): Two coins are tossed simultaneously. The probability of getting two heads, if it is known that at least one head comes up, is $\frac{1}{3}$.

Reason (R): Let E and F be two events with a random experiment, then $P(F/E) = \frac{P(E \cap F)}{P(E)}$.

[1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- For events E, F with $P(E) > 0$, $P(F \mid E) = P(E \cap F)/P(E)$.
- Two fair coins have four equally likely ordered outcomes: HH, HT, TH, TT.

Step 1: Judge the assertion Let E denote at least one head and F denote two heads. Then $E = \{HH, HT, TH\}$ and $F = \{HH\}$. Within the three equally likely outcomes of E, exactly one gives two heads. The conditional probability is therefore $1/3$, so Assertion (A) is true.

Step 2: Judge the reason Reason (R) states the conditional-probability formula, which is valid when the conditioning event has nonzero probability. Here $P(E) = 3/4 > 0$, so it applies. Reason (R) is true; its conditioning event here is the event of at least one head.

Step 3: Check the explanatory connection Since $F \subseteq E$, $E \cap F = F$ and $P(E \cap F) = 1/4$. Thus

$$P(F \mid E) = \frac{1/4}{3/4} = \frac{1}{3}.$$

This is exactly the value in (A), derived using (R). Option (B) incorrectly denies the explanation, while (C) and (D) incorrectly declare one statement false.

Step 4: Conclusion Both (A) and (R) are true, and (R) correctly explains (A). The correct option is (A).

Quick Tip to Solve: For "at least one head", remove TT first and count only the three remaining outcomes.

Section B (10 marks)

This section comprises of Very Short Answer (VSA) type questions of 2 marks each.

21. Draw the graph of the principal branch of the function $f(x) = \cos^{-1} x$. [2 marks]

Solution:

Concept:

- The principal branch $y = \cos^{-1} x$ is the inverse of $x = \cos y$ restricted to $0 \le y \le \pi$.
- Its domain is $[-1, 1]$ and its range is $[0, \pi]$; all angles are in radians.

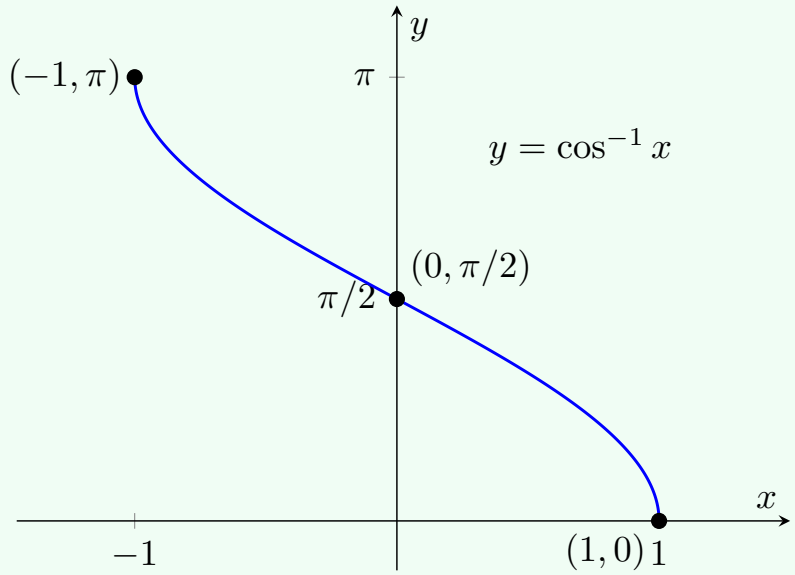
Quick Tip to Solve: Mark $(-1, \pi)$, $(0, \pi/2)$ and $(1, 0)$ before drawing the inverse-cosine curve.

Step 1: Locate the key points Using $\cos 0 = 1$, $\cos(\pi/2) = 0$ and $\cos \pi = -1$, the graph passes through

$$(1, 0), \qquad (0, \pi/2), \qquad (-1, \pi).$$

Both endpoints are included because the domain is closed. These points fix the intercepts and the full height of the principal branch.

Step 2: Draw the decreasing principal branch Cosine is strictly decreasing on $[0, \pi]$, so its inverse is also strictly decreasing. Draw a smooth curve from $(-1, \pi)$ through $(0, \pi/2)$ to $(1, 0)$, staying within $-1 \le x \le 1$ and $0 \le y \le \pi$. For $-1 < x < 1$, $dy/dx = -1/\sqrt{1 - x^2} < 0$; the curve becomes steep near each endpoint. Do not extend the inverse beyond its real domain.



Principal branch of inverse cosine

Step 3: Conclusion The required principal-branch graph is the decreasing curve shown, with domain $[-1, 1]$ and range $[0, \pi]$.

Quick Tip to Solve: Mark $(-1, \pi)$, $(0, \pi/2)$ and $(1, 0)$ before drawing the inverse-cosine curve.

22(a). If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \dfrac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. [2 marks]

Solution:

Concept:

- For nonzero vectors $\vec{a}, \vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$, where $0 \le \theta \le \pi$ is their angle.
- A unit vector has magnitude 1. The sine of an angle in $[0, \pi]$ need not determine a unique angle.

Step 1: Use the magnitude of the cross product The cross product itself is a unit vector, so $|\vec{a} \times \vec{b}| = 1$.

Quick Tip to Solve: After using a cross-product magnitude, retain both supplementary angles unless an acute angle is specified.

Substituting the given magnitudes,

$$1 = 3 \times \frac{2}{3} \sin\theta = 2\sin\theta, \qquad \sin\theta = \frac{1}{2}.$$

The magnitude formula contains sine rather than cosine; a dot-product formula would not use the given information correctly.

Step 2: Find all admissible angles The angle between two nonzero vectors lies in $[0, \pi]$. In this interval, $\sin\theta = 1/2$ at both $\theta = \pi/6$ and $\theta = 5\pi/6$. Both give $2\sin\theta = 1$, as required. The question gives no dot product or other information to distinguish an acute angle from its supplementary obtuse angle, so both must be retained.

Step 3: Conclusion The angle between the vectors is $\boxed{30^\circ \text{ or } 150^\circ}$, equivalently $\pi/6$ or $5\pi/6$.

Quick Tip to Solve: After using a cross-product magnitude, retain both supplementary angles unless an acute angle is specified.

OR

22(b). Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$. [2 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$.
- The cross product is perpendicular to both sides, and its magnitude is $|\vec{a}||\vec{b}|\sin\theta$, the product of base and perpendicular height.

Step 1: Calculate the cross product Using the components of the given adjacent sides,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$
$$= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2)$$
$$= 20\hat{i} + 5\hat{j} - 5\hat{k}.$$

The negative sign before the middle cofactor is essential.

Step 2: Find the magnitude The required area is the length of this cross-product vector, so

$$A = \sqrt{20^2 + 5^2 + (-5)^2}$$
$$= \sqrt{400 + 25 + 25} = \sqrt{450} = 15\sqrt{2}.$$

This is the parallelogram area; taking half of it would instead give the area of the triangle formed by the same two adjacent vectors.

Quick Tip to Solve: Expand the cross product with the middle cofactor negated; use its full magnitude for a parallelogram.

Step 3: Conclusion The area of the parallelogram is $\boxed{15\sqrt{2} \text{ square units}}$.

Quick Tip to Solve: Expand the cross product with the middle cofactor negated; use its full magnitude for a parallelogram.

23(a). **If** $f(x) = \begin{cases} x^2, & x \geq 1 \\ x, & x < 1 \end{cases}$**, then show that** f **is not differentiable at** $x = 1$**.** [2 marks]

Solution:

Concept:

- Differentiability at $x = a$ requires equal finite left-hand and right-hand derivatives at that point.
- Use the difference quotient $\dfrac{f(1+h) - f(1)}{h}$, choosing the branch according to the sign of h.

Step 1: Check the function value and continuity Since the branch x^2 applies at $x = 1$, we have $f(1) = 1$. Also, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1$. Thus the function is continuous at the joining point, but we must still compare its derivatives.

Step 2: Calculate the left-hand derivative For $h < 0$, $1 + h < 1$, so $f(1+h) = 1 + h$. Hence

$$f'_-(1) = \lim_{h \to 0^-} \frac{(1+h) - 1}{h} = \lim_{h \to 0^-} 1 = 1.$$

Step 3: Calculate the right-hand derivative For $h > 0$, $1 + h > 1$, so $f(1+h) = (1+h)^2$. Therefore

$$f'_+(1) = \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} (2 + h) = 2.$$

The two finite one-sided derivatives have different values.

Step 4: Conclusion Since $f'_-(1) = 1 \neq 2 = f'_+(1)$, $\boxed{f \text{ is not differentiable at } x = 1}$.

Quick Tip to Solve: At a piecewise join, calculate both one-sided slopes even when the two function values match.

OR

23(b). **Find the value(s) of** λ**, if the function** $f(x) = \begin{cases} \dfrac{\sin^2 \lambda x}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **is continuous at** $x = 0$**.** [2 marks]

Solution:

Concept:

- Continuity at zero requires $\lim_{x \to 0} f(x) = f(0)$. Here the prescribed value is $f(0) = 1$.

Quick Tip to Solve: Factor out the square of the parameter before using the sine limit, and check a zero parameter separately.

- The standard trigonometric limit is $\lim_{u\to 0}\dfrac{\sin u}{u}=1$, with angles measured in radians.

Step 1: Evaluate the limit for a nonzero parameter For a fixed $\lambda \neq 0$, rewrite the expression so that the standard limit applies:

$$\lim_{x\to 0}\frac{\sin^2(\lambda x)}{x^2}=\lambda^2\lim_{x\to 0}\left(\frac{\sin(\lambda x)}{\lambda x}\right)^2$$
$$=\lambda^2.$$

The substitution $u=\lambda x$ tends to zero from both sides as $x\to 0$, so this is a two-sided limit.

Step 2: Impose continuity and check zero separately Equating this limit to $f(0)=1$ gives $\lambda^2=1$, hence $\lambda=1$ or $\lambda=-1$. If $\lambda=0$, the expression for every $x\neq 0$ is zero, so its limit is zero and cannot equal $f(0)=1$. For each of $\lambda=1,-1$, the computed limit is indeed one, verifying continuity.

Step 3: Conclusion The required values are $\boxed{\lambda=\pm 1}$.

Quick Tip to Solve: Factor out the square of the parameter before using the sine limit, and check a zero parameter separately.

24. Sketch the region bounded by the lines $2x+y=8$, $y=2$, $y=4$ and the y-axis. Hence, obtain its area using integration. [2 marks]

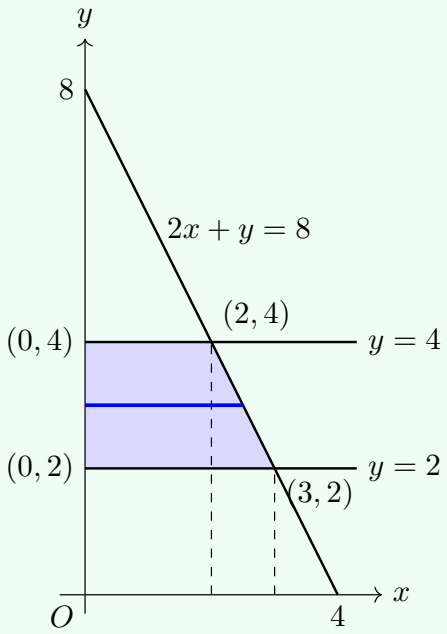
Solution:

Concept:

- For horizontal strips, the area between $x=x_{\text{left}}(y)$ and $x=x_{\text{right}}(y)$ is $\int[x_{\text{right}}(y)-x_{\text{left}}(y)]\,dy$.
- The y-axis gives $x=0$, and $2x+y=8$ gives $x=(8-y)/2$. The horizontal boundaries give $2\le y\le 4$.

Step 1: Identify and sketch the bounded region The line $2x+y=8$ meets $y=2$ at $(3,2)$ and $y=4$ at $(2,4)$. The other two vertices on the y-axis are $(0,2)$ and $(0,4)$. Thus the required region is the shaded trapezium with these four vertices. For every y from 2 to 4, its horizontal strip runs from the y-axis to the sloping line.

Quick Tip to Solve: Use horizontal strips because both horizontal boundaries are fixed y-values; this avoids splitting the integral.



Region bounded by the four given lines

Step 2: Integrate the horizontal width The width of each strip is $(8-y)/2-0$. Therefore

$$\begin{aligned} A &= \int_2^4 \frac{8-y}{2}\,dy \\ &= \left[4y - \frac{y^2}{4}\right]_2^4 \\ &= (16-4)-(8-1) = 12-7 = 5. \end{aligned}$$

The integrand is positive throughout the interval, as the sloping line lies to the right of the y-axis there.

Step 3: Conclusion The area of the bounded region is $\boxed{5 \text{ square units}}$.

Quick Tip to Solve: Use horizontal strips because both horizontal boundaries are fixed y-values; this avoids splitting the integral.

25. Find the angle between the following two lines:

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$ [2 marks]

Solution:

Concept:

- The coefficient of the parameter in a vector line is its direction vector; the fixed position vector locates a point on the line.

Quick Tip to Solve: Use only the parameter coefficients for the angle; the constant position vectors are irrelevant.

- For direction vectors $\vec{b}_1, \vec{b}_2$, the acute angle θ between the lines satisfies $\cos\theta = \dfrac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.

Step 1: Identify the direction vectors From the coefficients of λ and μ, respectively, we get

$$\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}, \qquad \vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}.$$

The position vectors $2\hat{i} - 5\hat{j} + \hat{k}$ and $7\hat{i} - 6\hat{k}$ do not enter the angle formula.

Step 2: Evaluate the dot product and magnitudes

$$\vec{b}_1 \cdot \vec{b}_2 = 3(1) + 2(2) + 6(2) = 19,$$
$$|\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7,$$
$$|\vec{b}_2| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

Hence

$$\cos\theta = \frac{|19|}{7 \times 3} = \frac{19}{21}.$$

Since this ratio lies between zero and one, it gives the smaller angle between the lines.

Step 3: Conclusion The angle between the two lines is $\boxed{\theta = \cos^{-1}(19/21)}$ (approximately $25.21°$).

Quick Tip to Solve: Use only the parameter coefficients for the angle; the constant position vectors are irrelevant.

Section C (18 marks)

This section comprises of Short Answer (SA) type questions of 3 marks each.

26. Using determinants, find the area of $\triangle PQR$ with vertices $P(3,1)$, $Q(9,3)$ and $R(5,7)$. Also, find the equation of line PQ using determinants. [3 marks]

Solution:

Concept:

- The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$.
- Three points are collinear exactly when their coordinate determinant is zero. Applying this to a variable point (x, y) and the two given points gives their line equation.

Step 1: Set up and evaluate the area determinant For $P(3,1), Q(9,3), R(5,7)$, put

$$D = \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ 5 & 7 & 1 \end{vmatrix}.$$

Quick Tip to Solve: Use the same coordinate determinant twice: take half its absolute value for area and set it to zero for a line.

Expanding along the first row gives

$$\begin{aligned} D &= 3(3\times 1 - 7\times 1) - 1(9\times 1 - 5\times 1) + 1(9\times 7 - 3\times 5) \\ &= 3(-4) - 4 + (63 - 15) \\ &= -12 - 4 + 48 = 32. \end{aligned}$$

Therefore the area is $\frac{1}{2}|32| = 16$ square units. The absolute value ensures that reversing the order of the vertices does not change the area.

Step 2: Find the equation of PQ using collinearity Let $T(x, y)$ be any point on PQ. Since T, P, Q are collinear,

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0.$$

Expanding along the first row,

$$\begin{aligned} x(1-3) - y(3-9) + (3\times 3 - 1\times 9) &= 0, \\ -2x + 6y &= 0. \end{aligned}$$

Dividing by -2 gives $x - 3y = 0$. Both P and Q satisfy this equation, since $3 - 3(1) = 0$ and $9 - 3(3) = 0$.

Step 3: Conclusion $\boxed{\text{Area of } \triangle PQR = 16 \text{ square units}}$, and $\boxed{PQ:\ x - 3y = 0}$.

Quick Tip to Solve: Use the same coordinate determinant twice: take half its absolute value for area and set it to zero for a line.

27(a). **Differentiate** $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ **w.r.t.** $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$**.** [3 marks]

Solution:

Concept:

- If $u = u(x)$ and $v = v(x)$, then $\frac{du}{dv} = \frac{du/dx}{dv/dx}$ where the denominator is nonzero.
- Principal inverse trigonometric values must be respected: $\sqrt{x^2} = |x|$, and $\sqrt{(1-2x^2)^2} = |1-2x^2|$.

Step 1: Differentiate the first inverse trigonometric expression Write

$$u = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), \qquad v = \sin^{-1}\left(2x\sqrt{1-x^2}\right).$$

For real finite u, take $-1 < x < 1$. Put $z = (1-x^2)^{-1/2}$, so $z \geq 1$ and $z' = x(1-x^2)^{-3/2}$. For $x \neq 0$,

$$\frac{du}{dx} = \frac{z'}{z\sqrt{z^2-1}} = \frac{x}{|x|\sqrt{1-x^2}},$$

using $\sqrt{z^2-1} = |x|/\sqrt{1-x^2}$.

Quick Tip to Solve: With x = sin t, check the inverse-function ranges before replacing the inverse expressions by t and 2t.

Step 2: Differentiate the second expression and take the ratio For $w = 2x\sqrt{1-x^2}$, the product rule gives

$$w' = 2\sqrt{1-x^2} - \frac{2x^2}{\sqrt{1-x^2}} = \frac{2(1-2x^2)}{\sqrt{1-x^2}}.$$

Also $1 - w^2 = 1 - 4x^2(1-x^2) = (1-2x^2)^2$. Thus, for $x \neq \pm 1/\sqrt{2}$,

$$\frac{dv}{dx} = \frac{2(1-2x^2)}{\sqrt{1-x^2}\,|1-2x^2|}.$$

Dividing gives

$$\frac{du}{dv} = \frac{x\,|1-2x^2|}{2|x|(1-2x^2)}.$$

On the usual branch $0 < x < 1/\sqrt{2}$, put $x = \sin t$, $0 < t < \pi/4$; then $u = t$, $v = 2t$, confirming the value $1/2$.

Step 3: State the branch dependence Using principal values, the ratio is $1/2$ on $(-1, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$, and $-1/2$ on $(-1/\sqrt{2}, 0)$ and $(1/\sqrt{2}, 1)$. At $x = 0$, $u = \sin^{-1}|x|$ has a cusp; at $x = \pm 1/\sqrt{2}$, v has a cusp. The ordinary two-sided derivative with respect to v is therefore not defined there.

Step 4: Conclusion On the standard branch $0 < x < 1/\sqrt{2}$, $\boxed{du/dv = 1/2}$. In full principal-value form, $\boxed{du/dv = \dfrac{x|1-2x^2|}{2|x|(1-2x^2)}}$, for $-1 < x < 1$, excluding $0, \pm 1/\sqrt{2}$.

Quick Tip to Solve: With x = sin t, check the inverse-function ranges before replacing the inverse expressions by t and 2t.

OR

27(b). If $y = \tan x + \sec x$, then prove that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$. [3 marks]

Solution:

Concept:

- Use $\dfrac{d}{dx}(\tan x) = \sec^2 x$ and $\dfrac{d}{dx}(\sec x) = \sec x \tan x$, where $\cos x \neq 0$.
- The quotient rule is $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$. The identity $\cos^2 x = 1 - \sin^2 x$ will simplify the first derivative.

Step 1: Differentiate y and simplify the first derivative Given $y = \tan x + \sec x$, differentiating term by term gives

$$\begin{aligned}\frac{dy}{dx} &= \sec^2 x + \sec x \tan x \\ &= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \\ &= \frac{1+\sin x}{\cos^2 x}.\end{aligned}$$

Quick Tip to Solve: Simplify the first derivative to 1/(1 - sin x) before differentiating again; it avoids a long product expansion.

Since $\cos^2 x = (1-\sin x)(1+\sin x)$, we obtain

$$\frac{dy}{dx} = \frac{1}{1-\sin x}.$$

The cancellation is valid on the original domain $\cos x \neq 0$: there $\sin x$ cannot be 1 or -1, so neither factor is zero.

Step 2: Differentiate once more Now differentiate $(1-\sin x)^{-1}$ by the chain rule. The derivative of its inner expression is $-\cos x$, so

$$\begin{aligned}\frac{d^2y}{dx^2} &= -1(1-\sin x)^{-2}(-\cos x)\\ &= \frac{\cos x}{(1-\sin x)^2}.\end{aligned}$$

The two negative signs cancel. Equivalently, applying the quotient rule to $1/(1-\sin x)$ gives numerator $0(1-\sin x) - 1(-\cos x) = \cos x$, which verifies the sign. All steps hold wherever the given y is defined.

Step 3: Conclusion Hence $\boxed{\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}}$, as required, for $\cos x \neq 0$.

Quick Tip to Solve: Simplify the first derivative to 1/(1 - sin x) before differentiating again; it avoids a long product expansion.

28(a). Evaluate: $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\cos 2x}{1+\cos 2x}\,dx$. [3 marks]

Solution:

Concept:

- Use the double-angle identity $1+\cos 2x = 2\cos^2 x$. On the interval $[-\pi/4, \pi/4]$, $\cos x \neq 0$, so division by $\cos^2 x$ is valid.
- An antiderivative of $\sec^2 x$ is $\tan x$. The definite integral is obtained by subtracting the antiderivative at the lower limit from its value at the upper limit.

Step 1: Simplify the integrand Let the required integral be I. Write the numerator as the denominator minus one. Then

$$\begin{aligned}\frac{\cos 2x}{1+\cos 2x} &= 1 - \frac{1}{1+\cos 2x}\\ &= 1 - \frac{1}{2\cos^2 x}\\ &= 1 - \frac{1}{2}\sec^2 x.\end{aligned}$$

The denominator stays positive throughout the interval, so this simplification introduces no singularity. It also reduces the integrand to two standard terms.

Quick Tip to Solve: Write the numerator as $(1+\cos 2x) - 1$ before using the double-angle identity; this avoids an unnecessary substitution.

Step 2: Integrate and substitute both limits Therefore

$$\begin{aligned} I &= \int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2}\sec^2 x\right) dx \\ &= \left[x - \frac{1}{2}\tan x\right]_{-\pi/4}^{\pi/4} \\ &= \left(\frac{\pi}{4} - \frac{1}{2}\right) - \left(-\frac{\pi}{4} + \frac{1}{2}\right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Here $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$. As a sign check, $\cos 2x \geq 0$ on the stated interval; the integrand and the resulting value are nonnegative.

Step 3: Conclusion The required value is $\boxed{\frac{\pi}{2} - 1}$.

Quick Tip to Solve: Write the numerator as $(1 + \cos 2x) - 1$ before using the double-angle identity; this avoids an unnecessary substitution.

OR

28(b). Find: $\int e^{x^2}\,(x^5 + 2x^3)\,dx$. [3 marks]

Solution:

Concept:

- For an integrand containing e^{x^2} and odd powers of x, use $t = x^2$, so $dt = 2x\,dx$.
- A polynomial multiplied by e^t can be integrated by parts: $\int u\,dv = uv - \int v\,du$. The final arbitrary constant is denoted by C.

Step 1: Reduce the integral by substitution Write $x^5 + 2x^3 = x(x^4 + 2x^2)$. Setting $t = x^2$ gives $x^4 = t^2$ and $x\,dx = dt/2$. Thus

$$I = \int e^{x^2}(x^5 + 2x^3)\,dx = \frac{1}{2}\int e^t(t^2 + 2t)\,dt.$$

This substitution absorbs the odd factor x into the differential and leaves a polynomial in t.

Step 2: Apply integration by parts Take $u = t^2 + 2t$ and $dv = e^t\,dt$; then $du = (2t + 2)\,dt$ and $v = e^t$. Hence

$$I = \frac{1}{2}\left[e^t(t^2 + 2t) - \int e^t(2t + 2)\,dt\right].$$

Since $\frac{d}{dt}(2te^t) = e^t(2t + 2)$, the remaining integral is $2te^t$. Therefore

$$I = \frac{1}{2}e^t(t^2 + 2t) - te^t + C = \frac{1}{2}t^2e^t + C.$$

Substitution back to x gives $I = \frac{1}{2}x^4e^{x^2} + C$.

Quick Tip to Solve: After setting $t = x^2$, look for $e^tP(t)$: its derivative is $e^t(P + P\prime)$, and here $P = t^2$.

Step 3: Check by differentiation Differentiating the obtained expression with respect to x,

$$\frac{d}{dx}\left(\frac{1}{2}x^4e^{x^2}\right) = 2x^3e^{x^2} + \frac{1}{2}x^4e^{x^2}(2x) = e^{x^2}(x^5 + 2x^3).$$

This exactly reproduces the printed integrand, verifying both the coefficient and the powers of x.

Step 4: Conclusion The required indefinite integral is $\boxed{\frac{1}{2}x^4e^{x^2} + C}$.

Quick Tip to Solve: After setting $t = x^2$, look for $e^tP(t)$: its derivative is $e^t(P + P\prime)$, and here $P = t^2$.

---

29. Find the area of the minor segment of the circle $x^2 + y^2 = 4$ cut off by the line $x = 1$, using integration. [3 marks]

Solution:

Concept:

- The circle $x^2 + y^2 = 4$ has centre $(0, 0)$ and radius 2. Its upper and lower branches are $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$.
- The area between two curves over an interval is $\int (y_{\text{upper}} - y_{\text{lower}})\, dx$. For $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ removes the radical.

Step 1: Identify the segment and set up its area The line $x = 1$ meets the circle at $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. Since the centre lies to the left of the line, the smaller segment lies to its right, from $x = 1$ to the rightmost point $x = 2$. A vertical strip in this segment has height $2\sqrt{4 - x^2}$. Consequently its area is

$$A = \int_1^2 \left(\sqrt{4 - x^2} - (-\sqrt{4 - x^2})\right) dx = 2\int_1^2 \sqrt{4 - x^2}\, dx.$$

Step 2: Evaluate the definite integral Put $x = 2\sin\theta$. For $1 \le x \le 2$, the transformed limits are $\theta = \pi/6$ and $\theta = \pi/2$; throughout this range $\cos\theta \ge 0$. Thus $dx = 2\cos\theta\, d\theta$ and $\sqrt{4 - x^2} = 2\cos\theta$. Therefore

$$\begin{aligned} A &= 8\int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta \\ &= 4\int_{\pi/6}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\ &= [4\theta + 2\sin 2\theta]_{\pi/6}^{\pi/2} \\ &= 2\pi - \left(\frac{2\pi}{3} + \sqrt{3}\right). \end{aligned}$$

The factor two in the original area accounts for the equal portions above and below the x-axis.

Step 3: Conclusion The area of the minor segment is $\boxed{\frac{4\pi}{3} - \sqrt{3}}$ square units.

Quick Tip to Solve: Use vertical strips from $x = 1$ to $x = 2$, and double the upper-branch integral to include the lower half of the segment.

30. Find the distance between the lines:

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$ [3 marks]

Solution:

Concept:

- For parallel lines through position vectors $\vec{a}$ and $\vec{b}$, with common direction vector $\vec{d}$, the perpendicular distance is $D = \dfrac{|(\vec{b} - \vec{a}) \times \vec{d}|}{|\vec{d}|}$.
- The cross-product magnitude is the area of the parallelogram formed by the joining vector and the direction vector. Dividing this area by its base length $|\vec{d}|$ gives the perpendicular height.

Step 1: Check parallelism and choose the vectors The first line has direction vector $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. The second has direction vector $4\hat{i} + 6\hat{j} + 12\hat{k} = 2\vec{d}$; hence the lines are parallel. Taking their parameter values to be zero gives points with position vectors

$$\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}, \qquad \vec{b} = 3\hat{i} + 3\hat{j} - 5\hat{k}.$$

The joining vector is therefore

$$\vec{b} - \vec{a} = 2\hat{i} + \hat{j} - \hat{k}.$$

Its length alone is not the distance, because it is not perpendicular to the lines.

Step 2: Compute the cross product and its magnitude Using the common direction vector,

$$(\vec{b} - \vec{a}) \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix}$$
$$= 9\hat{i} - 14\hat{j} + 4\hat{k}.$$

Thus

$$|(\vec{b} - \vec{a}) \times \vec{d}| = \sqrt{9^2 + (-14)^2 + 4^2} = \sqrt{293},$$

while $|\vec{d}| = \sqrt{2^2 + 3^2 + 6^2} = 7$.

Step 3: Obtain the perpendicular distance The height of this parallelogram is

$$D = \frac{\sqrt{293}}{7}.$$

As a check, $|\vec{b} - \vec{a}|^2 = 6$ and $(\vec{b} - \vec{a}) \cdot \vec{d} = 1$. Removing the component parallel to $\vec{d}$ gives $D^2 = 6 - 1/49 = 293/49$, which agrees with the cross-product calculation.

Step 4: Conclusion The distance between the two parallel lines is $\boxed{\dfrac{\sqrt{293}}{7}}$ units.

Quick Tip to Solve: First notice that the second direction vector is twice the first; use the simpler common direction $(2, 3, 6)$ in the distance formula.

31(a). Find the coordinates of the foot of the perpendicular drawn from the point $P(0, 2, 3)$ to the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. [3 marks]

Solution:

Concept:

- A line in symmetric form determines a point on the line and a direction vector. Here a point is $A(-3, 1, -4)$ and the direction vector is $\vec{d} = 5\hat{i} + 2\hat{j} + 3\hat{k}$.
- The foot H lies on the line, and $\overrightarrow{PH}$ is perpendicular to its direction. Therefore $\overrightarrow{PH} \cdot \vec{d} = 0$. The dot product of perpendicular vectors is zero.

Step 1: Express the foot using one parameter Let the common value of the three symmetric ratios be t. Then the coordinates of any point on the line, and hence of the foot, are

$$H = (-3 + 5t,\ 1 + 2t,\ -4 + 3t).$$

Since $P = (0, 2, 3)$, the vector joining P to this point is

$$\overrightarrow{PH} = (-3 + 5t)\hat{i} + (-1 + 2t)\hat{j} + (-7 + 3t)\hat{k}.$$

This parametrisation ensures that the unknown foot is on the specified line.

Step 2: Apply perpendicularity and verify the point The perpendicularity condition gives

$$\begin{aligned} 5(-3 + 5t) + 2(-1 + 2t) + 3(-7 + 3t) &= 0, \\ -15 + 25t - 2 + 4t - 21 + 9t &= 0, \\ 38t - 38 &= 0, \\ t &= 1. \end{aligned}$$

Substitution in the coordinates of H gives $(-3 + 5, 1 + 2, -4 + 3)$. To verify both conditions, its three line ratios are $5/5 = 2/2 = 3/3 = 1$, and the joining vector from P is $(2, 1, -4)$. Its dot product with $(5, 2, 3)$ is $10 + 2 - 12 = 0$, as required.

Step 3: Conclusion The foot of the perpendicular is $\boxed{H(2, 3, -1)}$.

Quick Tip to Solve: Use one common parameter for the line, then set the dot product of the joining vector with $(5, 2, 3)$ equal to zero.

OR

31(b). Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c} = \vec{0}$. Evaluate the quantity $\mu = \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$. [3 marks]

Solution:

Concept:

- For any vector $\vec{v}$, its self dot product is $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. The zero vector has squared magnitude zero.

Quick Tip to Solve: Square the zero-vector sum directly; remember each pairwise dot product appears twice, so divide the negative sum of squared magnitudes by two.

- The dot product is distributive and commutative. Squaring a sum of three vectors produces their three squared magnitudes and twice the sum of the three pairwise dot products.

Step 1: Take the dot product of the given sum with itself The given relation is $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Taking the dot product of both sides with themselves gives

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0.$$

On expanding by distributivity and pairing the equal cross terms,

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0.$$

The pairwise dot products are scalars; no angles or vector coordinates are needed for their sum.

Step 2: Insert the magnitudes and isolate the requested sum Writing $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, the identity becomes

$$\begin{aligned} 3^2+4^2+2^2+2\mu&=0,\\ 9+16+4+2\mu&=0,\\ 2\mu&=-29,\\ \mu&=-\frac{29}{2}. \end{aligned}$$

The factor of two is essential: each distinct pair occurs twice in the expansion, once in each order. As a check, substituting this value makes the squared magnitude of the given sum $29+2(-29/2)=0$, consistent with the zero-vector condition.

Step 3: Conclusion The required value is $\boxed{\mu=-\frac{29}{2}}$.

Quick Tip to Solve: Square the zero-vector sum directly; remember each pairwise dot product appears twice, so divide the negative sum of squared magnitudes by two.

Section D (20 marks)

This section comprises of Long Answer (LA) type questions of 5 marks each.

32. Evaluate: $\displaystyle\int_0^{\pi}\frac{x}{1+\sin x}\,dx$**.** [5 marks]

Solution:

Concept:

- The reflection property of a definite integral is $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$. Here $\sin(\pi-x)=\sin x$, so reflection leaves the denominator unchanged.
- The substitution $t=\tan(x/2)$ gives $\sin x=2t/(1+t^2)$ and $dx=2\,dt/(1+t^2)$ for $0\le x<\pi$. It converts the remaining trigonometric integral into a rational integral.

Quick Tip to Solve: First replace x by $\pi-x$ and add; then use $t=\tan(x/2)$ to evaluate the remaining integral without a false singularity at $\pi/2$.

Step 1: Reflect the integral about the midpoint Let

$$I = \int_0^{\pi} \frac{x}{1+\sin x}\, dx.$$

Use the substitution $u = \pi - x$. Then $dx = -du$, the limits $x = 0, \pi$ become $u = \pi, 0$, respectively, and $\sin(\pi - u) = \sin u$. Thus

$$I = \int_0^{\pi} \frac{\pi - u}{1+\sin u}\, du.$$

Rename the dummy variable as x.

Step 2: Add the two equivalent expressions Adding the integrands over the same limits cancels their variable terms, since the numerator becomes $x + (\pi - x) = \pi$. Therefore

$$2I = \pi \int_0^{\pi} \frac{1}{1+\sin x}\, dx.$$

Denote the last integral by J. We therefore need only calculate J, after which $I = \pi J/2$.

Step 3: Evaluate the remaining integral with a half-angle substitution Put $t = \tan(x/2)$. As x increases from 0 to π, t increases from 0 to $+\infty$. Also,

$$1 + \sin x = 1 + \frac{2t}{1+t^2} = \frac{(t+1)^2}{1+t^2}.$$

Using the accompanying expression for dx,

$$\begin{aligned} J &= \int_0^{\infty} \frac{1+t^2}{(t+1)^2} \frac{2\, dt}{1+t^2} \\ &= \lim_{R\to\infty} \int_0^{R} \frac{2}{(t+1)^2}\, dt \\ &= \lim_{R\to\infty} \left[-\frac{2}{t+1} \right]_0^{R} = 2. \end{aligned}$$

On the original interval the denominator is at least one, so the original integral is continuous and convergent. Thus $2I = \pi J = 2\pi$.

Step 4: Conclusion Therefore, $\boxed{\int_0^{\pi} \frac{x}{1+\sin x}\, dx = \pi}$.

Quick Tip to Solve: First replace x by $\pi - x$ and add; then use $t = \tan(x/2)$ to evaluate the remaining integral without a false singularity at $\pi/2$.

33(a). The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. [5 marks]

Solution:

Concept:

- In an equilateral triangle of side s, the median is also an altitude and bisects the opposite side. Its length m is related to s by the Pythagorean theorem.
- A rate of change with time is a derivative with respect to time t. Differentiate the relation between the changing lengths with respect to t, keeping constant numerical factors unchanged.

Step 1: Express the median in terms of the side Let s cm be the side and m cm be the median at time t seconds. The median divides the equilateral triangle into two congruent right triangles. Each has hypotenuse s, one leg $s/2$, and the other leg m. Thus

$$\begin{aligned} m^2 + \left(\frac{s}{2}\right)^2 &= s^2, \\ m^2 &= \frac{3s^2}{4}, \\ m &= \frac{\sqrt{3}}{2}s. \end{aligned}$$

We take the positive square root because both quantities represent lengths. The same proportional relation holds throughout the increase, since the triangle remains equilateral.

Step 2: Differentiate the geometric relation Both m and s depend on time. Differentiating the length relation, rather than treating either length as a constant, gives

$$\frac{dm}{dt} = \frac{\sqrt{3}}{2}\frac{ds}{dt}.$$

The given rate is $dm/dt = 2\sqrt{3}$ cm/s. Substituting it and isolating the required derivative gives

$$\begin{aligned} 2\sqrt{3} &= \frac{\sqrt{3}}{2}\frac{ds}{dt}, \\ \frac{ds}{dt} &= \frac{2}{\sqrt{3}}\left(2\sqrt{3}\right). \end{aligned}$$

The derivative is positive, consistent with the stated increase. Since $s = 2m/\sqrt{3}$, the side must increase faster than the median; the calculated rate respects this proportionality. No current side length is needed: the relation between the two rates depends only on the fixed shape of an equilateral triangle. The units remain centimetres per second because the proportional factor is dimensionless.

Step 3: Conclusion The side of the equilateral triangle is increasing at $\boxed{4\,\text{cm/s}}$.

Quick Tip to Solve: Write median $= \sqrt{3} \times$ side$/2$ first; their time rates have exactly the same constant ratio.

OR

33(b). Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. [5 marks]

Solution:

Concept:

- When two numbers have a fixed sum, express one in terms of the other to obtain an objective function of a single variable.
- At a differentiable interior minimum, the first derivative is zero. A positive second derivative verifies a strict local minimum; completing the square can also establish the global minimum.

Step 1: Form the sum of cubes Let the two numbers be x and $5 - x$, so that their sum is exactly 5. Their sum of cubes is the function

$$\begin{aligned} f(x) &= x^3 + (5 - x)^3 \\ &= x^3 + 125 - 75x + 15x^2 - x^3 \\ &= 15x^2 - 75x + 125. \end{aligned}$$

The cubic terms cancel because the sum of the numbers is fixed. The question does not impose a positivity restriction, so we may minimize this function over all real x. The value found below is positive for each number and therefore also applies if positive numbers are intended.

Step 2: Find and verify the minimum Differentiate with respect to x:

$$f'(x) = 30x - 75, \qquad f''(x) = 30.$$

The stationary-point equation is $30x - 75 = 0$, giving $x = 5/2$. Since $f''(5/2) = 30 > 0$, this point gives a strict local minimum. To confirm that it is the least value over the whole domain, rewrite the objective as

$$f(x) = 15\left(x - \frac{5}{2}\right)^2 + \frac{125}{4}.$$

The square is nonnegative for every real x, and vanishes only at $x = 5/2$. Hence this is the unique global minimum of the sum of cubes. At that point the other number is also $5 - 5/2 = 5/2$.

Step 3: Compute the requested sum of squares Use the numbers at which the cubes have least sum. The requested expression is a different quantity from the minimum value of f:

$$x^2 + (5 - x)^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = 2\left(\frac{25}{4}\right).$$

As a check, the product of the two numbers is $25/4$, so the identity $x^2 + y^2 = (x + y)^2 - 2xy$ gives the same expression, $25 - 2(25/4)$.

Step 4: Conclusion When the sum of cubes is least, the numbers are both $5/2$, and their sum of squares is $\boxed{25/2}$.

Quick Tip to Solve: First minimize the sum of cubes under the fixed-sum constraint; then substitute those numbers into the sum of squares.

34(a). **In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$. What is the probability that the student knows the answer, given that he answered it correctly?** [5 marks]

Solution:

Concept:

- Conditional probability is $P(K \mid C) = P(K \cap C)/P(C)$, where K is the event that the student knows the answer and C is the event that his answer is correct.
- The events K and G, where G denotes guessing, are mutually exclusive and exhaustive. The total probability of a correct answer is $P(C) = P(K)P(C \mid K) + P(G)P(C \mid G)$.
- Bayes' theorem reverses the conditioning: $P(K \mid C) = \dfrac{P(K)P(C \mid K)}{P(K)P(C \mid K) + P(G)P(C \mid G)}$.

Step 1: Define the events and list the given probabilities Let K mean that the student knows the answer, G mean that he guesses, and C mean that he answers correctly. Since he either knows or guesses, $K \cap G = \varnothing$ and $K \cup G$ is the sample space. From the question,

$$P(K) = \frac{3}{5}, \qquad P(G) = \frac{2}{5}, \qquad P(C \mid G) = \frac{1}{3}.$$

A student who knows the answer gives the correct answer, so $P(C \mid K) = 1$. The desired probability is $P(K \mid C)$; it is different from the prior probability $P(K)$, since we now have the extra information that the response was correct.

Step 2: Find the probability of a correct response A correct response can arise in exactly two disjoint ways: the student knows the answer and is correct, or he guesses and happens to be correct. Multiplying along each route gives

$$P(K \cap C) = P(K)P(C \mid K) = \frac{3}{5},$$
$$P(G \cap C) = P(G)P(C \mid G) = \frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}.$$

Therefore, by the theorem of total probability,

$$P(C) = \frac{3}{5} + \frac{2}{15} = \frac{9+2}{15} = \frac{11}{15}.$$

Including the second route is essential: a correct answer does not by itself prove that the student knew it.

Step 3: Apply Bayes' theorem to the observed correct answer Among all correct responses, the fraction arising from knowledge is obtained by dividing the probability of the knowledge-and-correct route by the total probability of correctness:

$$P(K \mid C) = \frac{P(K \cap C)}{P(C)}$$
$$= \frac{3/5}{11/15} = \frac{3}{5} \cdot \frac{15}{11} = \frac{9}{11}.$$

Quick Tip to Solve: First calculate both routes to a correct answer; divide the knows-and-correct route by their sum.

As a check, the complementary conditional probability of guessing is $(2/15)/(11/15) = 2/11$; the two conditional probabilities sum to 1, as required.

Step 4: Conclusion The probability that the student knows the answer, given that he answered correctly, is $\boxed{\frac{9}{11}}$.

Quick Tip to Solve: First calculate both routes to a correct answer; divide the knows-and-correct route by their sum.

OR

34(b). A box contains 10 tickets, 2 of which carry a prize of Rs. 8 each, 5 of which carry a prize of Rs. 4 each, and remaining 3 carry a prize of Rs. 2 each. If one ticket is drawn at random, find the mean value of the prize. [5 marks]

Solution:

Concept:

- A random variable assigns a numerical value to each outcome. Here X is the prize amount, in rupees, on the ticket drawn.
- For equally likely tickets, the probability of a given prize is the number of tickets carrying that prize divided by the total number of tickets.
- The mean of a discrete random variable is its expectation, $E(X) = \sum_i x_i P(X = x_i)$, where x_i runs over the possible prize amounts.

Step 1: Identify the random variable and its possible values Define X to be the value of the prize obtained on drawing one ticket. There are 10 tickets in the box, and each individual ticket has probability $1/10$ of being drawn. The tickets form three disjoint groups: 2 tickets have a prize of Rs. 8, 5 tickets have a prize of Rs. 4, and the remaining 3 tickets have a prize of Rs. 2. Thus the possible values of X are $2, 4, 8$. Every ticket carries one of these prizes, so there is no additional zero-prize outcome.

Step 2: Construct and check the probability distribution Count the tickets for each value of X and divide by 10:

x	2	4	8
$P(X = x)$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{2}{10}$

All probabilities are nonnegative, and their sum is

$$\frac{3}{10} + \frac{5}{10} + \frac{2}{10} = 1.$$

Therefore this is a valid probability distribution. The values $2, 4, 8$ are not equally likely: their probabilities depend on how many tickets carry each amount. This distinction determines the correct averaging method.

Step 3: Calculate the mean prize using expectation Multiply each prize by its probability and add the

Quick Tip to Solve: Weight each prize by its ticket count, add the weighted amounts, and divide by the total ten tickets.

contributions:

$$\begin{aligned} E(X) &= \sum_x xP(X=x) \\ &= 2\left(\frac{3}{10}\right) + 4\left(\frac{5}{10}\right) + 8\left(\frac{2}{10}\right) \\ &= \frac{6+20+16}{10} = \frac{42}{10} = 4.2. \end{aligned}$$

Equivalently, the total value of prizes on all ten tickets is Rs. 42, and dividing by ten gives the average value per ticket. The result lies between the smallest and largest prize amounts, as a weighted mean must. Although Rs. 4.2 is not a prize on any single ticket, an expected value need not itself be one of the possible outcomes; it describes the average prize over repeated random draws with the same distribution.

Step 4: Conclusion The mean value of the prize is $\boxed{\text{Rs. } 4.20}$.

Quick Tip to Solve: Weight each prize by its ticket count, add the weighted amounts, and divide by the total ten tickets.

35. Solve the following Linear Programming Problem graphically:

Maximize $P = 70x + 40y$ subject to the constraints $3x + 2y \leq 9$, $3x + y \leq 9$, $x \geq 0,\ y \geq 0$. [5 marks]

Solution:

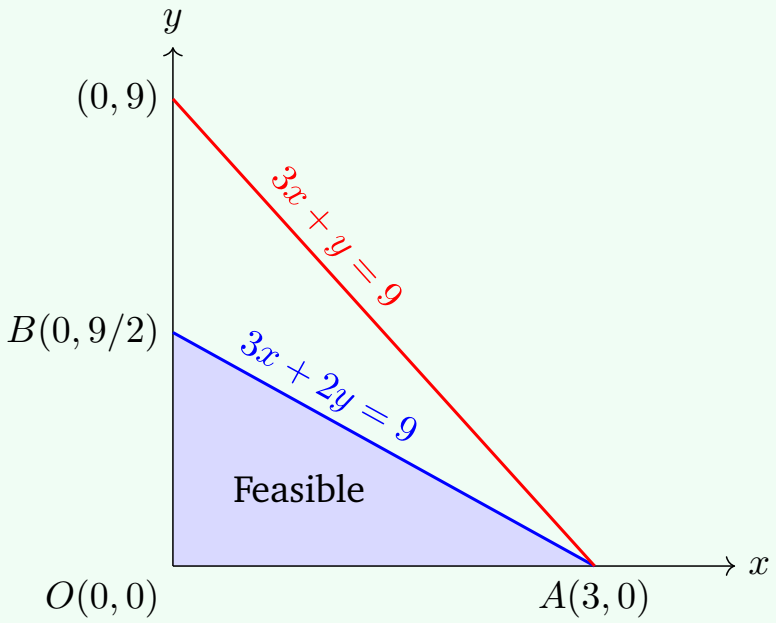
Concept:

- The feasible region is the common region satisfying all inequalities; $x \geq 0$ and $y \geq 0$ restrict it to the first quadrant.
- For a bounded polygonal feasible region, a linear objective function attains its maximum at a corner point. Here $P = 70x + 40y$.

Step 1: Draw the boundary lines For $3x + 2y = 9$, the intercepts are $(3, 0)$ and $(0, 9/2)$. For $3x + y = 9$, the intercepts are $(3, 0)$ and $(0, 9)$. Draw each line through its two intercepts, using the same scale on each axis. These intercepts determine the boundary lines exactly; the origin is not on either line.

Step 2: Identify and shade the feasible region At $(0, 0)$, both left sides are zero, so the half-plane containing the origin satisfies each inequality. Retain only their common part in the first quadrant. Since $y \geq 0$, $3x + y \leq 3x + 2y \leq 9$; hence every point satisfying the first inequality already satisfies the second. Thus the second constraint does not cut the feasible region further. The common region is the closed triangle with vertices $O(0, 0)$, $A(3, 0)$ and $B(0, 9/2)$, shaded in the graph. All three vertices satisfy both inequalities.

Quick Tip to Solve: For nonnegative y, the constraint with 2y implies the one with y; eliminate that redundant constraint after drawing both lines.



Boundary lines and the shaded feasible triangle OAB

Step 3: Evaluate the objective at every corner The feasible triangle is nonempty and bounded, so the corner-point method applies. Compute

Corner point	$P = 70x + 40y$
$(0, 0)$	0
$(3, 0)$	210
$(0, 9/2)$	180

The objective values are compared over all vertices of the common feasible region, rather than over the intercepts of the second line: $(0, 9)$ violates $3x + 2y \leq 9$. The largest listed value is 210. The adjacent vertices have strictly smaller values, so no entire boundary edge gives the same maximum.

Step 4: Conclusion The maximum value is $\boxed{P_{\max} = 210}$, attained at $\boxed{x = 3,\ y = 0}$.

Quick Tip to Solve: For nonnegative y, the constraint with 2y implies the one with y; eliminate that redundant constraint after drawing both lines.

Section E (12 marks)

This section comprises of 3 case study/passage-based questions of 4 marks each with two sub-parts. First two case study questions have three sub – parts (I), (II), (III) of marks 1, 1, 2 respectively. The third case study question has two sub – parts (I) and (II) of marks 2 each.

36. Gautam buys 5 pens, 3 bags and 1 instrument box and pays a sum of ₹ 160. From the same shop, Vikram buys 2 pens, 1 bag and 3 instrument boxes and pays a sum of ₹ 190. Also Ankur buys 1 pen, 2 bags and 4 instrument boxes and pays a sum of ₹ 250.

Based on the above information, answer the following questions. [4 marks]

36(I). Convert the given above situation into a matrix equation of the form $AX = B$. [1 mark]

Solution:

Concept:

- A matrix equation $AX = B$ represents a linear system: the rows of A contain coefficients, X contains the unknowns in a fixed order and B contains the totals.

Step 1: Define the unknowns and equations Let x, y, z be the prices, in rupees, of one pen, one bag and one instrument box, respectively. The three purchases give $5x + 3y + z = 160$, $2x + y + 3z = 190$ and $x + 2y + 4z = 250$.

Step 2: Arrange the coefficients Keep the order pen, bag, instrument box in every row, and the order Gautam, Vikram, Ankur in the rows and totals. Then

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 160 \\ 190 \\ 250 \end{pmatrix}.$$

Multiplying each row of A by X recovers its corresponding purchase equation.

Step 3: Conclusion The required equation is $\boxed{AX = B}$, with A, X, B as displayed above.

Quick Tip to Solve: Fix the unknown order before recording any row; totals must follow the same purchaser order as the coefficient rows.

36(II). Find $|A|$. [1 mark]

Solution:

Concept:

- Expand a 3×3 determinant along the first row using signs $+, -, +$; each minor is a 2×2 determinant.

Step 1: Write the coefficient determinant With the unknowns ordered as pen, bag and instrument box, the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

Quick Tip to Solve: Write the alternating signs before evaluating minors to avoid losing the negative middle term.

Its determinant uses only the purchase coefficients, not the payment totals.

Step 2: Expand and simplify Expanding along the first row,

$$\begin{aligned}|A| &= 5(1\cdot 4 - 3\cdot 2) - 3(2\cdot 4 - 3\cdot 1) + (2\cdot 2 - 1\cdot 1)\\ &= 5(-2) - 3(5) + 3\\ &= -10 - 15 + 3 = -22.\end{aligned}$$

The minus sign on the middle term is the cofactor sign, so that term is negative even though its minor equals positive five.

Step 3: Conclusion $\boxed{|A| = -22}$.

Quick Tip to Solve: Write the alternating signs before evaluating minors to avoid losing the negative middle term.

36(III)(a). Find A^{-1}. [2 marks]

Solution:

Concept:

- If $|A| \neq 0$, the inverse exists and $A^{-1} = \operatorname{adj} A/|A|$. The adjugate is the transpose of the cofactor matrix.
- The cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j.

Step 1: Compute the determinant and cofactors For $A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$, $|A| = 5(-2) - 3(5) + 3 = -22 \neq 0$, so its inverse exists. The first row of cofactors is $(-2, -5, 3)$. The second row is $(-(12-2), 20-1, -(10-3)) = (-10, 19, -7)$. The third row is $(9-1, -(15-2), 5-6) = (8, -13, -1)$. Thus

$$C = \begin{pmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{pmatrix}.$$

Step 2: Transpose and apply the inverse formula Transposing C gives

$$\operatorname{adj} A = \begin{pmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{pmatrix}.$$

Divide every entry by -22, or equivalently change all signs and divide by 22. To check the construction, multiplying A by $\begin{pmatrix} 2 & 10 & -8 \\ 5 & -19 & 13 \\ -3 & 7 & 1 \end{pmatrix}$ gives $22I_3$; division by 22 therefore gives the identity matrix.

Quick Tip to Solve: After finding cofactors, transpose them before dividing by the determinant; the cofactor matrix itself is not the adjugate.

Step 3: Conclusion

$$\boxed{A^{-1} = \frac{1}{22}\begin{pmatrix} 2 & 10 & -8 \\ 5 & -19 & 13 \\ -3 & 7 & 1 \end{pmatrix}.}$$

Quick Tip to Solve: After finding cofactors, transpose them before dividing by the determinant; the cofactor matrix itself is not the adjugate.

OR

36(III)(b). Determine $P = A^2 - 5A$. [2 marks]

Solution:

Concept:

- Matrix multiplication uses the row-column rule: the entry in position (i, j) of A^2 is the dot product of row i of A and column j of A.
- The coefficient matrix follows the order pens, bags, instrument boxes in each purchase. Scalar multiplication and subtraction are then performed entry by entry.

Step 1: Write the coefficient matrix From the three purchases, the coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

The rows correspond to Gautam, Vikram and Ankur, respectively; the columns keep the same order of articles throughout.

Step 2: Multiply the matrix by itself Apply the row-column rule to all nine entries:

$$A^2 = \begin{pmatrix} 25+6+1 & 15+3+2 & 5+9+4 \\ 10+2+3 & 6+1+6 & 2+3+12 \\ 5+4+4 & 3+2+8 & 1+6+16 \end{pmatrix} = \begin{pmatrix} 32 & 20 & 18 \\ 15 & 13 & 17 \\ 13 & 13 & 23 \end{pmatrix}.$$

For instance, the first off-diagonal entry is $5(3) + 3(1) + 1(2) = 20$; squaring a matrix does not mean squaring its entries.

Step 3: Subtract five times the original matrix First calculate

$$5A = \begin{pmatrix} 25 & 15 & 5 \\ 10 & 5 & 15 \\ 5 & 10 & 20 \end{pmatrix}.$$

Subtract corresponding entries of this matrix from those of A^2. For example, the top-left entry becomes $32 - 25 = 7$ and the bottom-right entry becomes $23 - 20 = 3$.

Quick Tip to Solve: Keep the article order fixed in all rows, and compute A squared by row-column multiplication before subtracting 5A.

Step 4: Conclusion

$$\boxed{P = A^2 - 5A = \begin{pmatrix} 7 & 5 & 13 \\ 5 & 8 & 2 \\ 8 & 3 & 3 \end{pmatrix}.}$$

Quick Tip to Solve: Keep the article order fixed in all rows, and compute A squared by row-column multiplication before subtracting 5A.

---

37. An organization conducted a bike race under two different categories - Boys and Girls. There were 28 participants in all. Among all of them, finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project. Let $B = \{b_1, b_2, b_3\}$ and $G = \{g_1, g_2\}$, where B represents the set of Boys selected and G the set of Girls selected for the final race.

Based on the above information, answer the following questions. [4 marks]

37(I). How many relations are possible from B to G? [1 mark]

Solution:

Concept:

- A relation from B to G is any subset of the Cartesian product $B \times G$.
- A set with n elements has 2^n subsets, including the empty set and the whole set.

Step 1: Count the available ordered pairs There are 3 selected boys and 2 selected girls. Each boy can be paired with either girl, so

$$|B \times G| = |B|\,|G| = 3 \times 2 = 6.$$

The 28 original race participants do not affect this count because the sets contain only the finalists.

Step 2: Count all subsets Each of the six ordered pairs can independently be included in or excluded from a relation. Thus the number of relations is

$$2^6 = 64.$$

Merely multiplying 3 by 2 counts ordered pairs rather than relations; a relation may contain any number of those pairs.

Step 3: Conclusion $\boxed{64}$ relations are possible from B to G.

Quick Tip to Solve: First count the ordered pairs in the Cartesian product, then use 2 raised to that count for relations.

37(II). Among all the possible relations from B to G, how many functions can be formed from B to G? [1 mark]

Solution:

Concept:

- A function from B to G assigns exactly one image in G to every element of B.
- If the domain has m elements and the codomain has n elements, the number of functions is n^m.

Step 1: Count the choices for each boy Each of b_1, b_2, b_3 must have precisely one image, which can be either g_1 or g_2. Thus each of the three domain elements has two choices. Different boys are allowed to have the same image, since injectivity is not required.

Step 2: Apply the multiplication principle The image choices for the three boys are independent, so the number of functions is

$$2 \times 2 \times 2 = 2^3 = 8.$$

The count $2^6 = 64$ includes all relations, many of which omit a boy or give a boy two images; such relations are not functions.

Step 3: Conclusion $\boxed{8}$ functions can be formed from B to G.

Quick Tip to Solve: For functions, count one codomain choice for every domain element: codomain size raised to domain size.

37(III)(a). Let $R : B \to B$ be defined by $R = \{(x, y) : x$ and y are students of the same sex$\}$. Check if R is an equivalence relation. [2 marks]

Solution:

Concept:

- An equivalence relation must be reflexive, symmetric and transitive on its specified set.
- Every element of $B = \{b_1, b_2, b_3\}$ represents a boy. Consequently, every pair of elements of B consists of students of the same sex.

Step 1: Identify the relation and verify reflexivity Since all students in B are boys, $R = B \times B$. For any $x \in B$, the student x has the same sex as himself, so $(x, x) \in R$. In particular all three diagonal pairs are present. Therefore R is reflexive on B.

Step 2: Verify symmetry If $(x, y) \in R$, then x and y have the same sex. Reversing their order does not alter this fact, so $(y, x) \in R$. This holds for every pair in R, and hence R is symmetric.

Step 3: Verify transitivity Let $(x, y) \in R$ and $(y, z) \in R$. Then x has the same sex as y, and y has the same sex as z. Thus x and z have the same sex, so $(x, z) \in R$. Equivalently, because all three students are in the all-boy set B, this ordered pair belongs to $B \times B$. Therefore R is transitive.

Step 4: Conclusion $\boxed{R \text{ is an equivalence relation on } B}$, since it is reflexive, symmetric and transitive.

Quick Tip to Solve: Notice that B contains only boys, making the same-sex relation the whole Cartesian product B times B.

OR

37(III)(b). A function $f : B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$. Check if f is bijective. Justify your answer. [2 marks]

Solution:

Concept:

- A function is injective if distinct domain elements have distinct images, and surjective if every codomain element is an image.
- A bijective function must satisfy both injectivity and surjectivity. Repeated images disprove injectivity, even when the whole codomain is covered.

Step 1: Check the given assignment and injectivity The ordered pairs specify $f(b_1) = g_1$, $f(b_2) = g_2$, and $f(b_3) = g_1$. Every member of B has exactly one image, so the assignment is a function. However,

$$b_1 \neq b_3 \quad \text{while} \quad f(b_1) = f(b_3) = g_1.$$

This supplies a direct counterexample to injectivity: two distinct boys have the same image.

Step 2: Check surjectivity and combine the conditions The element g_1 is the image of b_1 (also of b_3), and g_2 is the image of b_2. Hence

$$f(B) = \{g_1, g_2\} = G,$$

so the function is surjective. Covering both girls is sufficient for surjectivity, but does not remove the repeated image of b_1 and b_3. The two required conditions must hold together for a bijection.

Step 3: Conclusion $\boxed{f \text{ is not bijective}}$: it is surjective but not injective.

Quick Tip to Solve: List all images first: a repeated image disproves injectivity, while covering the codomain proves surjectivity.

---

38. An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation. A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be homogeneous if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$, we make the substitution $y = vx$ and then separate the variables.

Based on the above, answer the following questions. [4 marks]

38(I). Show that $(x^2 - y^2)\,dx + 2xy\,dy = 0$ is a differential equation of the type $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. [2 marks]

Solution:

Concept:

- A differential equation has the requested homogeneous form when its right-hand side depends only on the ratio y/x.

Quick Tip to Solve: Divide both numerator and denominator by x squared to expose the ratio y over x without changing the sign.

- A quotient of homogeneous polynomials of equal degree is homogeneous of degree zero, on the domain where its denominator is nonzero.

Step 1: Express the derivative explicitly The given equation is

$$(x^2 - y^2)\,dx + 2xy\,dy = 0.$$

On a portion of a curve where $xy \neq 0$, divide by dx and rearrange:

$$2xy\frac{dy}{dx} = y^2 - x^2, \qquad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

The sign changes because the first term is moved to the other side.

Step 2: Rewrite the right-hand side using the ratio Divide the numerator and denominator by x^2. This gives

$$\frac{dy}{dx} = \frac{(y/x)^2 - 1}{2(y/x)}.$$

With $u = y/x$, define

$$g(u) = \frac{u^2 - 1}{2u} \quad (u \neq 0).$$

This expression contains no separate occurrence of x or y, only their ratio. The restriction arises from the explicit derivative form; the original differential form is the starting equation.

Step 3: Conclusion Hence

$$\boxed{\frac{dy}{dx} = g\left(\frac{y}{x}\right), \qquad g(u) = \frac{u^2 - 1}{2u}}$$

on the domain $xy \neq 0$, so the given equation is of the requested homogeneous type.

Quick Tip to Solve: Divide both numerator and denominator by x squared to expose the ratio y over x without changing the sign.

38(II). Solve the above equation to find its general solution. [2 marks]

Solution:

Concept:

- For a homogeneous differential equation, substitute $y = vx$, where $v = y/x$, so that $dy/dx = v + x\,dv/dx$.
- Separate the variables and integrate both sides; the arbitrary constant gives the general solution on an interval where the differential equation is defined.

Step 1: Substitute into the given equation The equation from part (I) is $(x^2 - y^2)\,dx + 2xy\,dy = 0$. Writing it in derivative form gives

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Quick Tip to Solve: After substituting $y = vx$, retain both terms in $dy/dx = v + x\,dv/dx$; omitting v changes the separated equation.

Put $y = vx$. On an interval with $x \neq 0$ and $y \neq 0$, substitution gives

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}, \qquad x\frac{dv}{dx} = -\frac{v^2 + 1}{2v}.$$

Thus the derivative of v can be separated from the variable x.

Step 2: Separate, integrate and return to the original variables Rearranging and integrating,

$$\int \frac{2v}{1 + v^2}\, dv = -\int \frac{dx}{x}, \qquad \ln(1 + v^2) = -\ln|x| + C_1.$$

Hence $|x|(1 + v^2)$ is a positive constant. The sign of x is fixed on the interval, so it can be absorbed into an arbitrary constant C. Therefore

$$x\left(1 + \frac{y^2}{x^2}\right) = C.$$

As a check, differentiating the equivalent relation gives $2x + 2y\, dy/dx = C$. Substituting $C = (x^2 + y^2)/x$ yields $2xy\, dy/dx = y^2 - x^2$, exactly the given equation.

Step 3: Conclusion The general solution is $\boxed{x^2 + y^2 = Cx}$, where C is an arbitrary constant, with solution branches taken on intervals where the derivative form is defined.

Quick Tip to Solve: After substituting $y = vx$, retain both terms in $dy/dx = v + x\, dv/dx$; omitting v changes the separated equation.