CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-5-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are compulsory.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Question Number **1 to 18** are Multiple Choice Questions (MCQ) type and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Question Number **21 to 25** are Very Short Answer (VSA) type questions of **2** marks each.

(v) In **Section C** – Question Number **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Question Number **32 to 35** are Long Answer (LA) type questions carrying **5** marks each.

(vii) In **Section E** – Question Number **36 to 38** are case study based questions carrying **4** marks each where **2 VSA** type questions are of **1** mark each and 1 SA type question is of **2** marks. Internal choice is provided in **2** marks question in each case-study.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

(Multiple Choice Questions)
Each question carries 1 mark.
Select the correct option out of the four given options :

1. Let R be a relation in the set $\mathbb{N}$ given by $R = \{(a, b) : \ a = b - 2, \ b > 6\}$. Then: [1 mark]

(A) $(8, 7) \in R$
(B) $(6, 8) \in R$
(C) $(3, 8) \in R$
(D) $(2, 4) \in R$

Correct Answer: (B) $(6, 8) \in R$

Solution:

Concept:

- A pair (a, b) belongs to R only if both conditions $a = b - 2$ and $b > 6$ hold.
- The first coordinate is a; the second is b, so order matters.

Step 1: Check the defining conditions For $(6, 8)$, take $a = 6$ and $b = 8$. Then $b - 2 = 8 - 2 = 6 = a$, and $8 > 6$. Thus this pair satisfies both conditions simultaneously.

Step 2: Eliminate the remaining pairs For $(8, 7)$, $7 - 2 = 5 \neq 8$. For $(3, 8)$, $8 - 2 = 6 \neq 3$. For $(2, 4)$, although $2 = 4 - 2$, the restriction $4 > 6$ fails. Hence none of these three pairs belongs to the relation.

Step 3: Conclusion $(6, 8) \in R$. Hence, the correct option is (B).

Quick Tip to Solve: Test the second-coordinate restriction first, then subtract 2 and compare with the first coordinate.

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2. If $A = \begin{pmatrix} 5 & x \\ y & 0 \end{pmatrix}$ and $A = A^T$, where A^T is the transpose of the matrix A, then: [1 mark]

(A) $x = 0, \ y = 5$
(B) $x = y$
(C) $x + y = 5$
(D) $x = 5, \ y = 0$

Correct Answer: (B) $x = y$

Solution:

Concept:

- Transposition interchanges the row and column positions: the entry a_{ij} becomes a_{ji}.
- A matrix satisfying $A = A^T$ is symmetric; its corresponding off-diagonal entries must be equal.

Quick Tip to Solve: For a two by two symmetric matrix, compare the two entries across the main diagonal.

Step 1: Write the transpose Transposing the given matrix gives

$$A^T = \begin{pmatrix} 5 & y \\ x & 0 \end{pmatrix}.$$

The diagonal entries remain 5 and 0, while the entries x and y interchange their positions.

Step 2: Compare corresponding entries The equality $A = A^T$ therefore requires $x = y$. No condition on their common value is imposed. Options (A) and (D) make the off-diagonal entries unequal. Option (C) fixes their sum, but equal entries need not sum to 5; for example, $x = y = 0$ also gives a symmetric matrix.

Step 3: Conclusion $x = y$. Hence, the correct option is (B).

Quick Tip to Solve: For a two by two symmetric matrix, compare the two entries across the main diagonal.

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3. $\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$ **is equal to:** [1 mark]

(A) 1
(B) $\frac{1}{2}$
(C) $\frac{1}{3}$
(D) $\frac{1}{4}$

Correct Answer: (A) 1

Solution:

Concept:

- The principal value of $\sin^{-1} u$ lies in $[-\pi/2, \pi/2]$.
- Use $\sin(\pi/6) = 1/2$ and $\sin(\pi/2) = 1$ to evaluate the expression.

Step 1: Evaluate the inverse sine Since $\pi/6$ is in the principal range and $\sin(\pi/6) = 1/2$, we have

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Here inverse sine denotes an angle, not the reciprocal of sine.

Step 2: Add the angles and evaluate Substitution gives

$$\sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\left(\frac{\pi}{2}\right) = 1.$$

The values $1/2$, $1/3$ and $1/4$ do not equal the sine of this combined angle; in particular, $1/2$ is the input of the inverse sine, not the requested final value.

Step 3: Conclusion The value is 1. Hence, the correct option is (A).

Quick Tip to Solve: Convert the inverse sine to its principal angle before combining the two angles.

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4. For a square matrix A, if $A^2 - A + I = O$, then A^{-1} equals: [1 mark]

(A) A

(B) $A + I$

(C) $I - A$

(D) $A - I$

Correct Answer: (C) $I - A$

Solution:

Concept:

- The identity matrix I satisfies $AI = IA = A$, and O is the zero matrix.
- A matrix B is the inverse of A when $AB = BA = I$.

Step 1: Rearrange the given identity From $A^2 - A + I = O$, move the first two terms to obtain

$$I = A - A^2 = A(I - A).$$

Since $I - A$ is a polynomial in A, it also commutes with A, so $(I - A)A = A - A^2 = I$.

Step 2: Identify the inverse and compare Thus $I - A$ satisfies both inverse identities. The expression $A - I$ has the opposite sign and its product with A is $-I$. The given identity does not establish that either A or $A + I$ has product I with A; the factor obtained directly is $I - A$.

Step 3: Conclusion $A^{-1} = I - A$. Hence, the correct option is (C).

Quick Tip to Solve: Rearrange a polynomial identity into a product equal to the identity matrix to read off the inverse.

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5. If $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$, then the value of α is: [1 mark]

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- A row replacement $R_i \to R_i + kR_j$ does not change the determinant.
- Expanding along a row with two zeros reduces a three by three determinant to one cofactor.

Quick Tip to Solve: Subtract the second row from the third row before expanding; only one cofactor then remains.

Step 1: Create zeros in the third row Apply $R_3 \to R_3 - R_2$:

$$\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{vmatrix}.$$

This operation preserves the given value of the determinant.

Step 2: Expand and solve the equation Expansion along the third row gives

$$D = 2(-1)^{3+2} \begin{vmatrix} \alpha & 4 \\ 1 & 1 \end{vmatrix} = -2(\alpha - 4).$$

Therefore $D = 0$ implies $\alpha = 4$. At the other listed values $1, 2, 3$, the determinant equals $6, 4, 2$, respectively, so none satisfies the equation.

Step 3: Conclusion $\alpha = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Subtract the second row from the third row before expanding; only one cofactor then remains.

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6. If $f(x) = |\cos x|$, then $f\left(\frac{3\pi}{4}\right)$ is: [1 mark]

(A) 1

(B) -1

(C) $\frac{-1}{\sqrt{2}}$

(D) $\frac{1}{\sqrt{2}}$

Correct Answer: (D) $\frac{1}{\sqrt{2}}$

Solution:

Concept:

- Absolute value makes a negative real number positive: $|u| = -u$ for $u < 0$.
- In the second quadrant, cosine is negative; $\cos(\pi - \theta) = -\cos\theta$.

Step 1: Find the cosine value The angle $3\pi/4 = \pi - \pi/4$ lies in the second quadrant. Hence

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

This is the value inside the absolute-value signs.

Step 2: Apply absolute value Using the definition of the given function,

$$f\left(\frac{3\pi}{4}\right) = \left|-\frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}.$$

Quick Tip to Solve: Evaluate cosine with its quadrant sign first, then apply the outer absolute value.

Option (C) omits the absolute-value operation. Options (A) and (B) have magnitude 1, whereas the magnitude here is $1/\sqrt{2}$. The requested quantity is the function value itself.

Step 3: Conclusion $f(3\pi/4) = 1/\sqrt{2}$. Hence, the correct option is (D).

Quick Tip to Solve: Evaluate cosine with its quadrant sign first, then apply the outer absolute value.

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7. If $x = A\cos 4t + B\sin 4t$, then $\dfrac{d^2x}{dt^2}$ is equal to: [1 mark]

(A) x

(B) $-x$

(C) $16x$

(D) $-16x$

Correct Answer: (D) $-16x$

Solution:

Concept:

- The constants A and B do not vary with t.
- The chain rule gives $\frac{d}{dt}\sin 4t = 4\cos 4t$ and $\frac{d}{dt}\cos 4t = -4\sin 4t$.

Step 1: Differentiate once Apply the chain rule to each term:

$$\frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t.$$

Each derivative includes a factor 4 from the derivative of the inner angle $4t$.

Step 2: Differentiate again and factor A second differentiation yields

$$\frac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t = -16(A\cos 4t + B\sin 4t).$$

The bracket equals the original x. Options x and $-x$ miss the squared chain-rule factor, and $16x$ has the wrong sign because the second derivative of sine or cosine changes its sign.

Step 3: Conclusion $\dfrac{d^2x}{dt^2} = -16x$. Hence, the correct option is (D).

Quick Tip to Solve: For a sine-cosine combination with angular factor 4, the second derivative multiplies it by minus 4 squared.

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8. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at: [1 mark]

(A) $x = 1$

(B) $x = 1.5$

(C) $x = -2$

(D) $x = 4$

Correct Answer: (B) $x = 1.5$

Solution:

Concept:

- The greatest integer function satisfies $[x] = n$ throughout $n \leq x < n + 1$, for every integer n.
- Continuity at a requires both one-sided limits to agree with $f(a)$.

Step 1: Check the noninteger point For every x sufficiently close to 1.5, we have $1 < x < 2$, so $[x] = 1$. Therefore

$$\lim_{x \to 1.5} [x] = 1 = [1.5],$$

and the function is continuous at $x = 1.5$.

Step 2: Check the integer alternatives At any integer n, numbers just below n have greatest integer $n - 1$, while numbers just above n have greatest integer n. Thus the left and right limits differ. Each of $1, -2, 4$ is an integer, so the function is discontinuous at all three of these listed points.

Step 3: Conclusion The function is continuous at $x = 1.5$. Hence, the correct option is (B).

Quick Tip to Solve: For the greatest integer function, reject integer points immediately and test the interval containing a noninteger point.

9. The function $f(x) = x^3 + 3x$ is increasing in interval: [1 mark]

(A) $(-\infty, 0)$

(B) $(0, \infty)$

(C) $\mathbb{R}$

(D) $(0, 1)$

Correct Answer: (C) $\mathbb{R}$

Solution:

Concept:

- A differentiable function is strictly increasing on an interval when its derivative is positive throughout that interval.
- For a polynomial, the domain is all real numbers; differentiate each power using the power rule.

Step 1: Differentiate the function For $f(x) = x^3 + 3x$, the power rule gives

$$f\prime(x) = 3x^2 + 3 = 3(x^2 + 1).$$

Quick Tip to Solve: Factor the derivative and check its sign on the entire domain before selecting a smaller interval.

Since $x^2 \geq 0$ for every real number, this derivative is at least 3, and is always positive.

Step 2: Identify the full increasing interval There is no real value where the derivative vanishes or becomes negative. Thus the function increases throughout its domain. Options (A), (B) and (D) describe smaller intervals on which it also increases, but omit other real values; option (C) gives the complete interval.

Step 3: Conclusion The full interval of increase is $\mathbb{R}$. Hence, the correct option is (C).

Quick Tip to Solve: Factor the derivative and check its sign on the entire domain before selecting a smaller interval.

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10. The value of $\int_{-1}^{1} \frac{|x-2|}{x-2}\,dx,\ x \neq 2$, is: [1 mark]

(A) 1

(B) -1

(C) 2

(D) -2

Correct Answer: (D) -2

Solution:

Concept:

- Absolute value satisfies $|u| = -u$ when $u < 0$ and $|u| = u$ when $u \geq 0$.
- The definite integral of a constant k over $[a, b]$ is $k(b-a)$.

Step 1: Determine the sign over the interval For every $x \in [-1, 1]$, we have $x - 2 < 0$. Therefore $|x-2| = -(x-2)$, and

$$\frac{|x-2|}{x-2} = -1.$$

The excluded point $x = 2$ lies outside the interval, so the integrand is defined throughout the interval of integration.

Step 2: Evaluate the constant integral Consequently,

$$\int_{-1}^{1} \frac{|x-2|}{x-2}\,dx = \int_{-1}^{1} (-1)\,dx = [-x]_{-1}^{1} = -1 - 1 = -2.$$

Options (A) and (C) have the wrong sign, while option (B) overlooks that the interval has length 2.

Step 3: Conclusion The integral equals -2. Hence, the correct option is (D).

Quick Tip to Solve: Check whether the zero of the absolute-value expression lies inside the limits before splitting an integral.

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11. $\int \frac{\sec x}{\sec x - \tan x}\, dx$ **equals:** [1 mark]

(A) $\sec x - \tan x + c$

(B) $\sec x + \tan x + c$

(C) $\tan x - \sec x + c$

(D) $-(\sec x + \tan x) + c$

Correct Answer: (B) $\sec x + \tan x + c$

Solution:

Concept:

- The identity $\sec^2 x - \tan^2 x = 1$ follows from $1 + \tan^2 x = \sec^2 x$.
- The standard derivatives are $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\sec x = \sec x \tan x$.

Step 1: Rationalise the denominator Multiply numerator and denominator by $\sec x + \tan x$. On any interval where the original expression is defined,

$$\frac{\sec x}{\sec x - \tan x} = \frac{\sec x(\sec x + \tan x)}{\sec^2 x - \tan^2 x} = \sec^2 x + \sec x \tan x.$$

Step 2: Integrate and check the signs Integrating the two terms gives

$$\int (\sec^2 x + \sec x \tan x)\, dx = \tan x + \sec x + c.$$

Differentiating verifies the original integrand. Options (A) and (C) give a difference of these derivative terms; option (D) gives their negative sum. None has the required positive sum.

Step 3: Conclusion The integral is $\sec x + \tan x + c$. Hence, the correct option is (B).

Quick Tip to Solve: For a denominator containing secant minus tangent, multiply by secant plus tangent to obtain denominator one.

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12. The order and the degree of the differential equation $\left(1 + 3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$ **respectively are:** [1 mark]

(A) $1,\ \frac{2}{3}$

(B) $3,\ 1$

(C) $3,\ 3$

(D) $1,\ 2$

Correct Answer: (B) $3,\ 1$

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.
- The degree is the power of the highest-order derivative when the equation is polynomial in its

Quick Tip to Solve: First select the highest-order derivative; then inspect only its exponent when finding the degree.

derivatives.

Step 1: Locate the highest derivative The equation contains dy/dx on the left and d^3y/dx^3 on the right. The highest derivative is the third derivative, so the order is 3. The square on the first-derivative expression does not change its order.

Step 2: Find the degree after expanding Expansion gives

$$1 + 6\frac{dy}{dx} + 9\left(\frac{dy}{dx}\right)^2 - 4\frac{d^3y}{dx^3} = 0.$$

This is polynomial in the derivatives, and the third derivative occurs to power 1. Hence the degree is 1, not the power 2 of the first derivative. Options (A) and (D) give the wrong order, while (C) gives the wrong degree.

Step 3: Conclusion The order and degree are $3, 1$, respectively. Hence, the correct option is (B).

Quick Tip to Solve: First select the highest-order derivative; then inspect only its exponent when finding the degree.

13. If $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$, then $\vec{a}$ is: [1 mark]

(A) $\hat{k}$

(B) $\hat{i}$

(C) $\hat{j}$

(D) $\hat{i} + \hat{j} + \hat{k}$

Correct Answer: (B) $\hat{i}$

Solution:

Concept:

- Write a vector in Cartesian form as $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$.
- The unit vectors are orthonormal: their self-dot-products are one, and their mutual dot-products are zero.

Step 1: Convert the dot products into equations Using the Cartesian components, the three given conditions become

$$a_x = 1, \qquad a_x + a_y = 1, \qquad a_x + a_y + a_z = 1.$$

Each condition must hold separately; the common value does not mean that every component is one.

Step 2: Determine all components The first equation gives $a_x = 1$. Subtracting it from the second gives $a_y = 0$; subtracting the second from the third gives $a_z = 0$. The vectors $\hat{j}$ and $\hat{k}$ fail the first condition. For $\hat{i} + \hat{j} + \hat{k}$, the second and third dot products would be 2 and 3, so option (D) fails.

Step 3: Conclusion We obtain $\vec{a} = \hat{i}$. Hence, the correct option is (B).

Quick Tip to Solve: Subtract consecutive dot-product equations to isolate the new Cartesian component immediately.

14. Five fair coins are tossed simultaneously. The probability of the event that at least one head comes up is: [1 mark]

(A) $\frac{27}{32}$

(B) $\frac{5}{32}$

(C) $\frac{31}{32}$

(D) $\frac{1}{32}$

Correct Answer: (C) $\frac{31}{32}$

Solution:

Concept:

- For an event E, the complement rule is $P(E) = 1 - P(E^c)$.
- Independent fair coins each have tail probability $1/2$; multiply probabilities for a specified joint outcome.

Step 1: Identify the complementary event Let E be the event that at least one head occurs. Its complement is that no head occurs, which means all five coins show tails. There are $2^5 = 32$ equally likely outcomes, and only one of them has all tails.

Step 2: Apply the complement rule Thus

$$P(E^c) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}, \qquad P(E) = 1 - \frac{1}{32} = \frac{31}{32}.$$

Option (D) is the complement rather than the required event. Option (B), $5/32$, counts exactly one head, omitting outcomes with two or more heads. Option (A) also omits valid outcomes.

Step 3: Conclusion The required probability is $31/32$. Hence, the correct option is (C).

Quick Tip to Solve: For at least one success, subtract the probability of zero successes from one.

15. If for any two events A and B, $P(A) = \frac{4}{5}$ and $P(A \cap B) = \frac{7}{10}$, then $P(B/A)$ is equal to: [1 mark]

(A) $\frac{1}{10}$

(B) $\frac{1}{8}$

(C) $\frac{7}{8}$

(D) $\frac{17}{20}$

Correct Answer: (C) $\frac{7}{8}$

Solution:

Concept:

- Conditional probability is $P(B \mid A) = P(A \cap B)/P(A)$, provided $P(A) > 0$.
- The notation $P(B/A)$ means the probability of event B given that event A has occurred.

Step 1: Identify the conditioning event The event after the slash is A, so its probability belongs in the denominator. The given value $P(A) = 4/5$ is positive, and the intersection probability $7/10$ does not exceed it, so the conditional probability formula applies.

Step 2: Substitute and simplify Using the given probabilities,

$$P(B \mid A) = \frac{7/10}{4/5} = \frac{7}{10} \times \frac{5}{4} = \frac{7}{8}.$$

Options (A), (B) and (D) do not equal this ratio. In particular, $1/8$ is the probability of the complementary event B^c given A, rather than the event asked for.

Step 3: Conclusion We obtain $P(B/A) = 7/8$. Hence, the correct option is (C).

Quick Tip to Solve: Read the event after the conditional slash first: its probability is always the denominator.

16. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is: [1 mark]

(A) 0°

(B) 30°

(C) 45°

(D) 90°

Correct Answer: (D) 90°

Solution:

Concept:

- Direction ratios can be read by setting the common value in a symmetric line equation equal to a parameter.
- For direction vectors $\vec{d_1}, \vec{d_2}$, the acute angle satisfies $\cos\theta = |\vec{d_1} \cdot \vec{d_2}|/(|\vec{d_1}||\vec{d_2}|)$.

Quick Tip to Solve: Convert coefficients in a common-value equation to reciprocal direction ratios before using the dot product.

Step 1: Extract the two direction vectors Put $2x = 3y = -z = t$, giving $(x, y, z) = (t/2, t/3, -t)$. Direction ratios are therefore $(3, 2, -6)$. Similarly, put $6x = -y = -4z = s$, giving $(x, y, z) = (s/6, -s, -s/4)$; multiplying by 12 gives direction ratios $(2, -12, -3)$.

Step 2: Evaluate the dot product The dot product is

$$3(2) + 2(-12) + (-6)(-3) = 6 - 24 + 18 = 0.$$

Both direction vectors are nonzero, so their angle has cosine zero. The lines are perpendicular. The cosines for options (A), (B) and (C) are nonzero, so those angles cannot satisfy the calculated dot product.

Step 3: Conclusion The angle between the lines is $90°$. Hence, the correct option is (D).

Quick Tip to Solve: Convert coefficients in a common-value equation to reciprocal direction ratios before using the dot product.

17. If a line makes angles of $90°$, $135°$ and $45°$ with the x, y and z axes respectively, then its direction cosines are: [1 mark]

(A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

(B) $-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}$

(D) $0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Correct Answer: (A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Solution:

Concept:

- Direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, where the angles are measured with the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$; their order is the order of the x, y, z axes.

Step 1: Evaluate the three cosines Use the angles in their given order. The angle with the x-axis is a right angle, while the angle with the y-axis is obtuse. Thus

$$l = \cos 90° = 0, \quad m = \cos 135° = -\frac{1}{\sqrt{2}}, \quad n = \cos 45° = \frac{1}{\sqrt{2}}.$$

Step 2: Check the signs and options The sum of their squares is $0 + \frac{1}{2} + \frac{1}{2} = 1$. Options (B) and (C) incorrectly make the second cosine zero and the first nonzero. Option (D) has a positive second cosine, which would represent an acute angle with the positive y-axis rather than the given obtuse angle.

Step 3: Conclusion The direction cosines are $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Hence, the correct option is (A).

Quick Tip to Solve: Keep the axis order fixed and use the obtuse angle to identify the negative direction cosine.

18. The magnitude of the vector $6\hat{i} - 2\hat{j} + 3\hat{k}$ is: [1 mark]

(A) 1

(B) 5

(C) 7

(D) 12

Correct Answer: (C) 7

Solution:

Concept:

- For $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, its magnitude is $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$.
- The coordinate unit vectors are mutually perpendicular, so the square of the length is the sum of the squared components.

Step 1: Substitute the components Here the components are $a = 6$, $b = -2$ and $c = 3$. Squaring the negative component gives a positive contribution.

$$|\vec{v}|^2 = 6^2 + (-2)^2 + 3^2 = 36 + 4 + 9 = 49.$$

Step 2: Take the nonnegative square root Magnitude denotes length, so choose the nonnegative square root of the sum. The numbers in options (A), (B) and (D) have squares $1, 25, 144$, respectively, and therefore none equals the required squared length 49. Adding signed components instead of their squares does not calculate vector length.

Step 3: Conclusion $|6\hat{i} - 2\hat{j} + 3\hat{k}| = 7$. Hence, the correct option is (C).

Quick Tip to Solve: Square all three components before adding; the minus sign on a component disappears in the magnitude formula.

19. Assertion (A): $\displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx = 3.$

Reason (R): $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$ [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- For an integrable function, reflection of the integration variable gives $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

Quick Tip to Solve: When the limits sum to 10, replace x by $10 - x$ and add the two integrands before integrating.

- If $f(x) + f(a+b-x) = 1$, adding the reflected integral to the original gives twice the integral as $b-a$.

Step 1: Evaluate the assertion Let I denote the stated integral. On $[2, 8]$, both square roots are real and the denominator is positive. Reflecting about the midpoint 5 replaces x by $10 - x$, so

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x} + \sqrt{x}}\, dx.$$

Adding this to the original expression gives $2I = \int_2^8 1\, dx = 6$. Thus the asserted value is correct.

Step 2: Verify the reason Put $u = a + b - x$; then $dx = -du$. The limits $x = a, b$ become $u = b, a$. Reversing these limits cancels the minus sign, proving the identity in the reason. Hence (R) is true.

Step 3: Connect the reason with the assertion The reason is precisely the reflection property used with $a = 2$, $b = 8$. It produces complementary numerators whose sum equals the denominator. It therefore directly explains (A). Option (B) denies this explanation, while (C) and (D) incorrectly declare one of the true statements false.

Step 4: Conclusion $I = 3$. Both (A) and (R) are true, and (R) correctly explains (A). Hence, the correct option is (A).

Quick Tip to Solve: When the limits sum to 10, replace x by $10 - x$ and add the two integrands before integrating.

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20. Assertion (A): Two coins are tossed simultaneously. The probability of getting two heads, if it is known that at least one head comes up, is $\frac{1}{3}$.

Reason (R): Let E and F be two events with a random experiment, then $P(F/E) = \frac{P(E \cap F)}{P(E)}$.

[1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- Conditional probability is $P(F \mid E) = P(E \cap F)/P(E)$, provided $P(E) > 0$.
- For two fair coins tossed independently, HH, HT, TH, TT are equally likely outcomes.

Step 1: Evaluate the assertion Let E mean at least one head and F mean two heads. Then $E = \{HH, HT, TH\}$, $F = \{HH\}$, and $E \cap F = \{HH\}$. Thus

$$P(F \mid E) = \frac{1/4}{3/4} = \frac{1}{3}.$$

Quick Tip to Solve: Remove TT from the equally likely outcomes when at least one head is known, then count HH among the three remaining outcomes.

The assertion is true; the outcome TT is excluded by the given information.

Step 2: Verify the reason The reason states the definition of conditional probability. It is valid when the conditioning event has positive probability. In this experiment $P(E) = 3/4 > 0$, so the formula applies and (R) is true.

Step 3: Check the explanatory link Identifying E as at least one head and F as two heads in the reason gives exactly the calculation that establishes the assertion. Therefore the reason explains it. Option (B) wrongly denies the link; options (C) and (D) wrongly declare a statement false. The unconditional value $1/4$ is not appropriate once the information about a head is known.

Step 4: Conclusion The required conditional probability is $1/3$. Both statements are true and (R) correctly explains (A). Hence, the correct option is (A).

Quick Tip to Solve: Remove TT from the equally likely outcomes when at least one head is known, then count HH among the three remaining outcomes.

Section B (10 marks)

This section comprises of Very Short Answer (VSA) type questions of 2 marks each.

21(a). Find the value of k for which the function f given as $f(x) = \begin{cases} \dfrac{1-\cos x}{2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$. [2 marks]

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x\to 0} f(x) = f(0) = k$.
- Use $1 - \cos x = 2\sin^2(x/2)$ and $\lim_{u\to 0} \sin u/u = 1$, with angles in radians.

Step 1: Rewrite the expression near zero The formula for $x \neq 0$ has the indeterminate form $0/0$ at zero, so direct substitution does not determine its limiting value. Apply the half-angle identity before taking the limit:

$$\frac{1-\cos x}{2x^2} = \frac{2\sin^2(x/2)}{2x^2} = \frac{1}{4}\left(\frac{\sin(x/2)}{x/2}\right)^2.$$

The factor $1/4$ comes from $(x/2)^2 = x^2/4$.

Step 2: Evaluate the limit and impose continuity As $x \to 0$, the variable $u = x/2$ also tends to zero from either side. Therefore the standard sine limit gives

$$\lim_{x\to 0} f(x) = \frac{1}{4}(1)^2 = \frac{1}{4}.$$

The left and right limits are equal. Continuity requires the separately defined value $f(0) = k$ to equal this common limit. Any other choice leaves a removable discontinuity at zero; the nonzero branch alone cannot change the assigned value at zero.

Step 3: Conclusion The function is continuous at $x = 0$ exactly when $\boxed{k = \frac{1}{4}}$.

Quick Tip to Solve: Rewrite $1 - \cos x$ using the half-angle identity and carefully retain the factor from $x/2$.

OR

21(b). **If $x = a\cos t$ and $y = b\sin t$, then find $\dfrac{d^2y}{dx^2}$.** [2 marks]

Solution:

Concept:

- For a parametric curve, $dy/dx = (dy/dt)/(dx/dt)$, wherever $dx/dt \neq 0$.
- The second derivative is $d^2y/dx^2 = [d(dy/dx)/dt]/(dx/dt)$; it is not simply the ratio of second derivatives with respect to t.

Step 1: Compute the first derivative Differentiate both coordinates with respect to the same parameter t:

$$\frac{dx}{dt} = -a\sin t, \qquad \frac{dy}{dt} = b\cos t.$$

For $a\sin t \neq 0$, divide these expressions to obtain

$$\frac{dy}{dx} = -\frac{b}{a}\frac{\cos t}{\sin t} = -\frac{b}{a}\cot t.$$

The negative sign comes from the derivative of $\cos t$ in the denominator.

Step 2: Differentiate once more with respect to x First differentiate the slope with respect to t, using $d(\cot t)/dt = -\csc^2 t$:

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{b}{a}\csc^2 t.$$

Now divide by $dx/dt = -a\sin t$, as required by the chain rule:

$$\frac{d^2y}{dx^2} = \frac{(b/a)\csc^2 t}{-a\sin t} = -\frac{b}{a^2\sin^3 t}.$$

This expression applies on portions of the parametrised curve where $a \neq 0$ and $\sin t \neq 0$, so that x is locally a valid independent variable.

Step 3: Conclusion $\boxed{\dfrac{d^2y}{dx^2} = -\dfrac{b}{a^2\sin^3 t}}$, where $a\sin t \neq 0$.

Quick Tip to Solve: After differentiating the slope with respect to t, divide by $-a\sin t$ once again to obtain the second derivative.

---

22. **Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right] + \tan^{-1} 1$.** [2 marks]

Solution:

Concept:

- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$, so $\sin^{-1}(1/2) = \pi/6$.
- The principal range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; in particular, $\tan^{-1} 1 = \pi/4$.

Quick Tip to Solve: Simplify the inner trigonometric expression first; adding two known angles avoids inverse-tangent sum formulas at a zero denominator.

Step 1: Evaluate the inner cosine First simplify the expression inside the first inverse tangent. Since $\pi/6$ lies in the principal range of inverse sine,

$$2\cos\left(2\sin^{-1}\frac{1}{2}\right) = 2\cos\left(2\cdot\frac{\pi}{6}\right) = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1.$$

Thus the first inverse tangent has argument 1.

Step 2: Add the principal inverse-tangent values Substituting the simplified argument, the given expression becomes

$$\tan^{-1}1 + \tan^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Each inverse tangent is evaluated separately on its principal branch. Their sum is allowed to equal $\pi/2$; it need not itself lie in the range of an inverse-tangent function.

Step 3: Conclusion The required value is $\boxed{\pi/2}$.

Quick Tip to Solve: Simplify the inner trigonometric expression first; adding two known angles avoids inverse-tangent sum formulas at a zero denominator.

23. Find the vector and the cartesian equations of a line that passes through the point $A(1, 2, -1)$ and is parallel to the line $5x - 25 = 14 - 7y = 35z$. [2 marks]

Solution:

Concept:

- A line through position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.
- If direction ratios are l, m, n, its Cartesian form through (x_0, y_0, z_0) is $(x - x_0)/l = (y - y_0)/m = (z - z_0)/n$.

Step 1: Extract direction ratios of the given line Let the common value in the printed equation be t. Then

$$5x - 25 = t, \qquad 14 - 7y = t, \qquad 35z = t.$$

Hence $x = 5 + t/5$, $y = 2 - t/7$ and $z = t/35$. Its direction ratios are therefore $1/5, -1/7, 1/35$. Multiplying all three by 35, we obtain the equivalent direction ratios $7, -5, 1$. The negative sign in the second ratio comes from the term $-7y$.

Step 2: Use the given point to construct the parallel line The point $A(1, 2, -1)$ has position vector $\hat{i} + 2\hat{j} - \hat{k}$. Thus the required vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

Its parametric equations are $x = 1 + 7\lambda$, $y = 2 - 5\lambda$, $z = -1 + \lambda$. Eliminating λ gives

$$\frac{x-1}{7} = \frac{y-2}{-5} = \frac{z+1}{1}.$$

Quick Tip to Solve: Introduce a common parameter before reading direction ratios; the coefficients of x, y and z themselves are not direction ratios.

The point is recovered when $\lambda = 0$, and the direction vector is parallel to the given line.

Step 3: Conclusion Vector equation: $\boxed{\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(7\hat{i} - 5\hat{j} + \hat{k})}$. Cartesian equation: $\boxed{\frac{x-1}{7} = \frac{y-2}{-5} = z + 1}$.

Quick Tip to Solve: Introduce a common parameter before reading direction ratios; the coefficients of x, y and z themselves are not direction ratios.

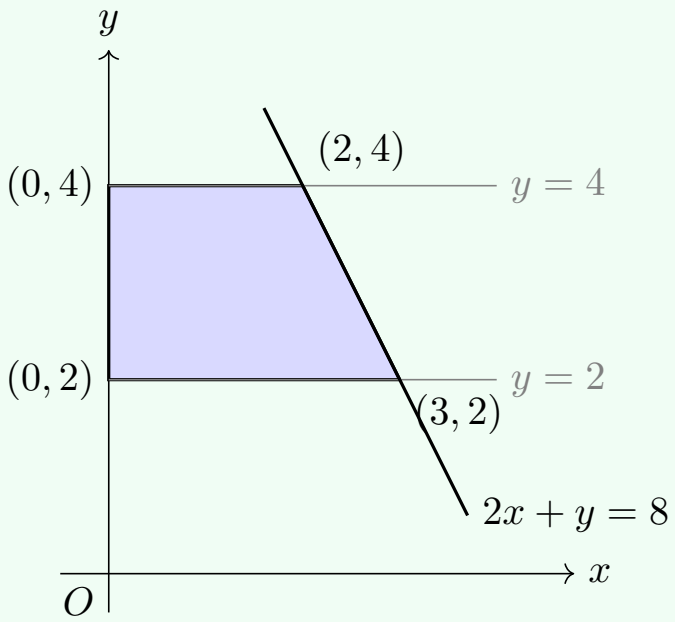
24. Sketch the region bounded by the lines $2x + y = 8$, $y = 2$, $y = 4$ and the y-axis. Hence, obtain its area using integration. [2 marks]

Solution:

Concept:

- Write the sloping boundary as $x = (8 - y)/2$. A horizontal strip has width $x_{\text{right}} - x_{\text{left}}$ and infinitesimal area $(x_{\text{right}} - x_{\text{left}})\, dy$.
- The total bounded area is $A = \int_{y_1}^{y_2} (x_{\text{right}} - x_{\text{left}})\, dy$, with $y_1 = 2$ and $y_2 = 4$ here.

Step 1: Locate and sketch the bounded region On the y-axis, the horizontal boundaries meet at $(0, 2)$ and $(0, 4)$. Substitution in $2x + y = 8$ gives $(3, 2)$ when $y = 2$ and $(2, 4)$ when $y = 4$. Thus the bounded region is the quadrilateral with consecutive vertices $(0, 2), (3, 2), (2, 4), (0, 4)$, shaded in the sketch. Throughout this region, $2 \le y \le 4$ and $0 \le x \le (8 - y)/2$.



Region bounded by the four given lines

Step 2: Integrate the horizontal strip widths The left boundary is $x = 0$, and the right boundary is

Quick Tip to Solve: Use horizontal strips because y=2 and y=4 are constant limits; vertical strips would require splitting the region.

$x = (8 - y)/2$. Therefore

$$A = \int_2^4 \left(\frac{8-y}{2} - 0\right) dy$$
$$= \left[4y - \frac{y^2}{4}\right]_2^4$$
$$= (16 - 4) - (8 - 1) = 12 - 7 = 5.$$

The strip width is positive over the entire interval, so this integral gives the geometric area directly without taking an absolute value.

Step 3: Conclusion The area of the shaded bounded region is $\boxed{5 \text{ square units}}$.

Quick Tip to Solve: Use horizontal strips because y=2 and y=4 are constant limits; vertical strips would require splitting the region.

25(a). If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. [2 marks]

Solution:

Concept:

- For nonzero vectors, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$, where $\theta \in [0, \pi]$ is the angle between the vectors.
- A unit vector has magnitude 1. On $[0, \pi]$, a positive sine generally corresponds to both an acute and its supplementary obtuse angle.

Step 1: Apply the magnitude of the cross product We are given $|\vec{a}| = 3$ and $|\vec{b}| = 2/3$, so both vectors are nonzero and the angle between them is defined. Since $\vec{a} \times \vec{b}$ is a unit vector, its magnitude equals 1. Substituting into the cross-product formula gives

$$1 = 3 \cdot \frac{2}{3} \sin\theta = 2\sin\theta, \qquad \sin\theta = \frac{1}{2}.$$

This uses the magnitude of the cross product; no direction of the resulting unit vector is specified.

Step 2: Find all angles in the angle-between-vectors range The angle between two vectors is in $[0, \pi]$. In this interval, $\sin\theta = 1/2$ holds at $\theta = \pi/6$ and $\theta = 5\pi/6$. Both give $2\sin\theta = 1$, as required. The given magnitudes do not specify the sign of $\vec{a} \cdot \vec{b}$, so there is no additional condition selecting only the acute angle. The supplementary obtuse angle must therefore also be included.

Step 3: Conclusion The possible angles are $\boxed{\theta = \pi/6 \text{ or } 5\pi/6}$, that is, $\boxed{30^\circ \text{ or } 150^\circ}$.

Quick Tip to Solve: A cross-product magnitude fixes sine, so check both supplementary angles unless an acute-angle condition or dot-product sign is given.

OR

25(b). Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$. [2 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$. The magnitude of the cross product equals the product of one side and the perpendicular height.
- For a vector $u\hat{i} + v\hat{j} + w\hat{k}$, its magnitude is $\sqrt{u^2 + v^2 + w^2}$.

Step 1: Form and evaluate the cross product Using the given vectors in the determinant formula,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} \\ &= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) \\ &= 20\hat{i} + 5\hat{j} - 5\hat{k}.\end{aligned}$$

The middle component includes the negative cofactor sign, so it is positive.

Step 2: Take the magnitude The magnitude gives a non-negative scalar area, independent of which adjacent side is taken first:

$$A = \sqrt{20^2 + 5^2 + (-5)^2} = \sqrt{450} = \sqrt{225 \cdot 2}.$$

Both side vectors are non-parallel, as their cross product is nonzero. Hence this magnitude represents the area of the parallelogram, with no extra factor of one-half.

Step 3: Conclusion The required area is $\boxed{15\sqrt{2}}$ square units.

Quick Tip to Solve: Use the cross-product magnitude directly for a parallelogram; a factor of one-half would give a triangle area.

Section C (18 marks)

This section comprises of Short Answer (SA) type questions of 3 marks each.

26. Show that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ. [3 marks]

Solution:

Concept:

- Expand a third-order determinant along its first row using the cofactor signs $+, -, +$. Each minor is a second-order determinant evaluated as $ad - bc$.

Quick Tip to Solve: Keep the negative cofactor sign on the sine term; the mixed sine-cosine terms must cancel before applying the identity.

- The identity $\sin^2\theta + \cos^2\theta = 1$ removes the angular terms after expansion. Here x is held fixed while θ varies.

Step 1: Expand along the first row Write $s = \sin\theta$ and $c = \cos\theta$ temporarily. For the given determinant D, expansion along the first row gives

$$D = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - s\begin{vmatrix} -s & 1 \\ c & x \end{vmatrix} + c\begin{vmatrix} -s & -x \\ c & 1 \end{vmatrix}.$$

The second term has a minus sign because its cofactor is in position $(1,2)$. The third term retains the plus sign for position $(1,3)$.

Step 2: Evaluate the minors and combine terms The three minors are respectively $-x^2 - 1$, $-sx - c$ and $-s + xc$. Therefore

$$\begin{aligned} D &= x(-x^2 - 1) - s(-sx - c) + c(-s + xc) \\ &= -x^3 - x + xs^2 + sc - sc + xc^2 \\ &= -x^3 - x + x(s^2 + c^2). \end{aligned}$$

The mixed products cancel exactly. Substituting $s^2 + c^2 = 1$ leaves $D = -x^3 - x + x$. This calculation makes no restriction on θ, since the identity holds for every real angle. It also involves no division by x, s or c, so zero values cause no exception.

Step 3: Conclusion Thus $\boxed{D = -x^3}$, which contains no θ. Hence the determinant is independent of θ.

Quick Tip to Solve: Keep the negative cofactor sign on the sine term; the mixed sine-cosine terms must cancel before applying the identity.

27. Using integration, find the area of the region bounded by $y = mx$ $(m > 0)$, $x = 1$, $x = 2$ and the x-axis. [3 marks]

Solution:

Concept:

- For a region between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, from $x = a$ to $x = b$, its area is $\int_a^b [f(x) - g(x)]\,dx$.
- Here $m > 0$ and $1 \le x \le 2$, so $mx > 0$. The line $y = mx$ is above the x-axis, whose equation is $y = 0$.

Step 1: Set up the area integral Take a thin vertical strip at a point x between the two vertical boundaries. Its upper end lies on $y = mx$, and its lower end lies on the x-axis. Thus its height is $mx - 0 = mx$. With width dx, its elemental area is $dA = mx\,dx$. The boundaries $x = 1$ and $x = 2$ supply the lower and upper limits, respectively. Adding these strips by integration gives

$$A = \int_1^2 mx\,dx.$$

Since the strip height stays positive throughout this interval, the integral already gives the geometric

Quick Tip to Solve: For this line, use upper minus lower with the stated limits 1 and 2; do not include the triangular region from x=0 to x=1.

area without a sign change or subdivision.

Step 2: Integrate and apply the limits The slope m is a constant with respect to x, so it can be taken outside the integral. Using the power rule,

$$\begin{aligned} A &= m\int_1^2 x\,dx \\ &= m\left[\frac{x^2}{2}\right]_1^2 \\ &= \frac{m}{2}(2^2-1^2) \\ &= \frac{m}{2}(4-1). \end{aligned}$$

Both limiting values are essential: integrating from the origin instead would include a region lying outside the specified left boundary. The resulting expression is positive because $m > 0$, as a geometric area should be.

Step 3: Conclusion The required area is $\boxed{\frac{3m}{2}}$ square units.

Quick Tip to Solve: For this line, use upper minus lower with the stated limits 1 and 2; do not include the triangular region from x=0 to x=1.

28(a). **Find the coordinates of the foot of the perpendicular drawn from point $(5, 7, 3)$ to the line $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.** [3 marks]

Solution:

Concept:

- A point on a line with symmetric equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$ has coordinates $(x_0 + at, y_0 + bt, z_0 + ct)$.
- If H is the foot from P, the vector $\overrightarrow{PH}$ is perpendicular to the direction vector $\vec{d}$ of the line, so $\overrightarrow{PH} \cdot \vec{d} = 0$.

Step 1: Express the foot as a point on the line Let the common value of the three ratios be t. Every point on the line therefore has coordinates

$$(x, y, z) = (15 + 3t,\ 29 + 8t,\ 5 - 5t).$$

The required foot H must be one such point. The direction vector of the line is $\vec{d} = (3, 8, -5)$, obtained from the denominators of its symmetric equation. With $P = (5, 7, 3)$, the vector from P to H is

$$\overrightarrow{PH} = (10 + 3t,\ 22 + 8t,\ 2 - 5t).$$

Step 2: Impose perpendicularity and determine the parameter The dot product of perpendicular

Quick Tip to Solve: Use the direction vector (3,8,-5) and one common parameter; retaining the negative sign in the z-coordinate is crucial.

vectors vanishes. Hence

$$\begin{aligned} 3(10+3t)+8(22+8t)-5(2-5t)&=0,\\ 30+9t+176+64t-10+25t&=0,\\ 196+98t&=0. \end{aligned}$$

Thus $t=-2$. Substitution into the line coordinates gives $H=(15-6,29-16,5+10)$. As a check, the three line ratios at this point are all -2. Also $\overrightarrow{PH}=(4,6,12)$, and $3(4)+8(6)-5(12)=12+48-60=0$. These checks establish both necessary properties: the point lies on the line and the segment to it is perpendicular to the line.

Step 3: Conclusion The foot of the perpendicular is $\boxed{(9,13,15)}$.

Quick Tip to Solve: Use the direction vector (3,8,-5) and one common parameter; retaining the negative sign in the z-coordinate is crucial.

OR

28(b). If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$, then find a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. [3 marks]

Solution:

Concept:

- The cross product of two non-parallel vectors is perpendicular to both. If $\vec{c}=\vec{u}\times\vec{v}\neq\vec{0}$, a unit perpendicular vector is $\vec{c}/|\vec{c}|$.
- A vector is a unit vector when its magnitude is 1. Its negative is also a unit vector perpendicular to the same two vectors.

Step 1: Form the two given combinations Using the coefficients of $\hat{i},\hat{j},\hat{k}$ separately, we obtain

$$\vec{u}=\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}, \qquad \vec{v}=\vec{a}-\vec{b}=-\hat{j}-2\hat{k}.$$

These are the vectors to which the required vector must be perpendicular; normalising either of these vectors alone would not satisfy both conditions.

Step 2: Calculate their cross product Expanding the determinant along the first row,

$$\begin{aligned} \vec{u}\times\vec{v}&=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&4\\0&-1&-2\end{vmatrix}\\ &=\hat{i}(-6+4)-\hat{j}(-4)+\hat{k}(-2)\\ &=-2\hat{i}+4\hat{j}-2\hat{k}. \end{aligned}$$

Its magnitude is $\sqrt{4+16+4}=2\sqrt{6}$, which is nonzero, so the two given vectors have a well-defined common normal direction.

Step 3: Normalise and verify Dividing the cross product by $2\sqrt{6}$ gives $\hat{n}=(-\hat{i}+2\hat{j}-\hat{k})/\sqrt{6}$. Its

Quick Tip to Solve: Take the cross product first and divide by its magnitude only at the end; check both dot products are zero.

squared magnitude is $(1+4+1)/6 = 1$. Also,

$$\hat{n}\cdot\vec{u} = \frac{-2+6-4}{\sqrt{6}} = 0, \qquad \hat{n}\cdot\vec{v} = \frac{-2+2}{\sqrt{6}} = 0.$$

The zero dot products check perpendicularity to both combinations independently.

Step 4: Conclusion A required unit vector is $\boxed{\dfrac{-\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}}$. Its negative is equally valid.

Quick Tip to Solve: Take the cross product first and divide by its magnitude only at the end; check both dot products are zero.

29. Find the distance between the lines:

$\vec{r} = (\hat{i}+2\hat{j}-4\hat{k}) + \lambda(2\hat{i}+3\hat{j}+6\hat{k})$

$\vec{r} = (3\hat{i}+3\hat{j}-5\hat{k}) + \mu(4\hat{i}+6\hat{j}+12\hat{k})$ [3 marks]

Solution:

Concept:

- For parallel lines through position vectors $\vec{a}_1, \vec{a}_2$ with a common direction $\vec{b}$, the perpendicular separation is $d = |(\vec{a}_2 - \vec{a}_1)\times\vec{b}|/|\vec{b}|$.
- The magnitude of a cross product equals the base length times the perpendicular height of the corresponding parallelogram. This explains why dividing by the direction-vector magnitude gives the shortest distance.

Step 1: Check that the lines are parallel The first line has direction vector $\vec{b} = 2\hat{i}+3\hat{j}+6\hat{k}$. The second has direction vector $4\hat{i}+6\hat{j}+12\hat{k} = 2\vec{b}$. Their directions are therefore parallel. Choose the points $A(1,2,-4)$ and $B(3,3,-5)$, obtained by setting the line parameters to zero. Their joining vector is $\overrightarrow{AB} = 2\hat{i}+\hat{j}-\hat{k}$.

Step 2: Compute the normal component using a cross product The required cross product is

$$\overrightarrow{AB}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}.$$

Hence

$$|\overrightarrow{AB}\times\vec{b}| = \sqrt{81+196+16} = \sqrt{293}, \qquad |\vec{b}| = \sqrt{4+9+36} = 7.$$

The nonzero cross product also confirms that the two parallel lines are distinct.

Step 3: Evaluate the separation and check it The distance formula gives $d = \sqrt{293}/7$. For an independent check, $|\overrightarrow{AB}|^2 = 6$ and $\overrightarrow{AB}\cdot\vec{b} = 4+3-6 = 1$. Removing the squared projection along the line gives

$$d^2 = 6 - \frac{1^2}{49} = \frac{293}{49},$$

Quick Tip to Solve: Compare direction vectors first: when one is a multiple of the other, use the parallel-line distance formula.

consistent with the cross-product calculation. The length $AB = \sqrt{6}$ itself is not the required distance because AB is not perpendicular to the lines.

Step 4: Conclusion The distance between the lines is $\boxed{\frac{\sqrt{293}}{7}}$ units.

Quick Tip to Solve: Compare direction vectors first: when one is a multiple of the other, use the parallel-line distance formula.

30(a). Differentiate $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ **w.r.t.** $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$**.** [3 marks]

Solution:

Concept:

- For differentiable functions $u(x)$ and $v(x)$ with $v'(x) \neq 0$, differentiation with respect to v gives $du/dv = u'(x)/v'(x)$.
- Principal inverse trigonometric functions require careful square-root signs: $\sqrt{x^2} = |x|$ and $\sqrt{(1-2x^2)^2} = |1-2x^2|$.

Step 1: Define the functions and their real domain Let

$$u = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), \qquad v = \sin^{-1}\left(2x\sqrt{1-x^2}\right).$$

Both are real for $-1 < x < 1$. The endpoints are excluded because the argument of the inverse secant has a zero denominator. The printed question specifies no smaller interval, so the signs arising from the principal branches must be retained.

Step 2: Differentiate the inverse secant Put $z = (1-x^2)^{-1/2} \geq 1$. For $x \neq 0$,

$$z' = \frac{x}{(1-x^2)^{3/2}}, \qquad \sqrt{z^2-1} = \frac{|x|}{\sqrt{1-x^2}}.$$

Using $d(\sec^{-1} z)/dx = z'/(|z|\sqrt{z^2-1})$, we get

$$\frac{du}{dx} = \frac{x}{|x|\sqrt{1-x^2}}.$$

Equivalently, $u = \sin^{-1}|x|$, which also exhibits the cusp at zero.

Step 3: Differentiate the inverse sine and take the ratio Writing $t = 2x\sqrt{1-x^2}$, we have

$$t' = \frac{2(1-2x^2)}{\sqrt{1-x^2}}, \qquad \sqrt{1-t^2} = |1-2x^2|.$$

Thus, away from $x = \pm 1/\sqrt{2}$,

$$\frac{dv}{dx} = \frac{2(1-2x^2)}{\sqrt{1-x^2}|1-2x^2|}.$$

Quick Tip to Solve: Before cancelling roots in inverse-trigonometric derivatives, replace square roots of squares by absolute values.

Dividing the two derivatives gives

$$\frac{du}{dv} = \frac{x\,|1-2x^2|}{2|x|(1-2x^2)}.$$

On the customary interval $0 < x < 1/\sqrt{2}$, this reduces to $1/2$. At zero, u has a cusp; at $x = \pm 1/\sqrt{2}$, v has a cusp, so a two-sided derivative with respect to v is not defined there.

Step 4: Conclusion $\boxed{\frac{du}{dv} = \frac{x\,|1-2x^2|}{2|x|(1-2x^2)}}$ for $-1 < x < 1$, excluding $0, \pm 1/\sqrt{2}$. In particular, it is $\boxed{1/2}$ when $0 < x < 1/\sqrt{2}$.

Quick Tip to Solve: Before cancelling roots in inverse-trigonometric derivatives, replace square roots of squares by absolute values.

OR

30(b). If $y = \tan x + \sec x$, then prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$. [3 marks]

Solution:

Concept:

- The derivative of a quotient f/g is $(f'g - fg')/g^2$, provided $g \neq 0$.
- The identity $\cos^2 x = 1 - \sin^2 x = (1-\sin x)(1+\sin x)$ can simplify a trigonometric quotient before differentiating.

Step 1: Rewrite the expression in a useful form For the given function,

$$y = \tan x + \sec x = \frac{\sin x + 1}{\cos x}.$$

The original expression is defined when $\cos x \neq 0$. On this domain, $1 - \sin x \neq 0$, and multiplying numerator and denominator by $1 - \sin x$ gives

$$y = \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)} = \frac{\cos x}{1-\sin x}.$$

This form makes the cancellation in the first derivative especially simple.

Step 2: Find the first derivative Apply the quotient rule with numerator $\cos x$ and denominator $1 - \sin x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(-\sin x)(1-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}.\end{aligned}$$

The middle simplification uses $\sin^2 x + \cos^2 x = 1$; keeping the denominator intact until this step avoids a sign error.

Quick Tip to Solve: Rewrite tan x plus sec x as cos x divided by one minus sin x before taking the two derivatives.

Step 3: Differentiate once more Write the first derivative as $(1-\sin x)^{-1}$. By the chain rule,

$$\frac{d^2y}{dx^2} = -1(1-\sin x)^{-2}(-\cos x) = \frac{\cos x}{(1-\sin x)^2}.$$

The two negative signs arise from differentiating the reciprocal and from differentiating $1-\sin x$. All transformations hold on every interval where the original function is defined.

Step 4: Conclusion Hence $\boxed{\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}}$, as required, for $\cos x \neq 0$.

Quick Tip to Solve: Rewrite tan x plus sec x as cos x divided by one minus sin x before taking the two derivatives.

31(a). Evaluate: $\displaystyle\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. [3 marks]

Solution:

Concept:

- Reflection property: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, obtained by the substitution $t = a+b-x$.
- For any real u, $\frac{1}{1+e^u} + \frac{1}{1+e^{-u}} = 1$. Also, $\sin(2\pi - x) = -\sin x$.

Step 1: Reflect the interval Let $I = \displaystyle\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. The denominator is positive throughout the interval, so this integral exists. Set $t = 2\pi - x$; then $dx = -dt$, and the endpoints $x = 0, 2\pi$ become $t = 2\pi, 0$, respectively. Hence

$$I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - t)}}\,dt = \int_0^{2\pi} \frac{1}{1+e^{-\sin t}}\,dt.$$

The letter used for the integration variable is immaterial, so we can write the last integral using x.

Step 2: Add the two representations Multiplying numerator and denominator of the reflected integrand by $e^{\sin x}$ gives

$$2I = \int_0^{2\pi} \left(\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right) dx = \int_0^{2\pi} 1\,dx = 2\pi.$$

The cancellation is pointwise: the paired values at x and $2\pi - x$ always sum to one. This avoids trying to find an elementary antiderivative of either integrand separately.

Step 3: Conclusion Dividing by two, the required definite integral is $\boxed{\pi}$.

Quick Tip to Solve: Pair the integrand at x with its value at 2 pi minus x; the sine changes sign and the two fractions sum to one.

OR

31(b). Find: $\int \frac{x^4}{(x-1)(x^2+1)}\,dx$. [3 marks]

Solution:

Concept:

- When the numerator degree is at least the denominator degree, first divide the polynomials to obtain a proper rational fraction.
- A proper fraction with denominator $(x-1)(x^2+1)$ has the form $\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$, where A, B, C are constants.

Step 1: Divide the polynomials Expanding the denominator gives $(x-1)(x^2+1) = x^3 - x^2 + x - 1$. Since

$$x^4 = (x+1)(x^3 - x^2 + x - 1) + 1,$$

we have $\frac{x^4}{(x-1)(x^2+1)} = x + 1 + \frac{1}{(x-1)(x^2+1)}$. This identity also explains why partial fractions should be applied only to the remaining proper fraction.

Step 2: Find the partial fractions Write $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$. Clearing denominators gives

$$1 = A(x^2+1) + (Bx+C)(x-1).$$

Comparing the coefficients of $x^2, x, 1$, respectively, yields $A+B=0$, $C-B=0$, and $A-C=1$. Thus $A = \frac{1}{2}$ and $B = C = -\frac{1}{2}$. Therefore the original integrand is

$$x + 1 + \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} - \frac{1}{2(x^2+1)}.$$

Step 3: Integrate each term Use $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$ and $\int \frac{1}{x^2+1}\,dx = \tan^{-1} x$. The linear-factor term gives $\frac{1}{2}\ln|x-1|$. The absolute value is necessary because $x-1$ can be negative, whereas x^2+1 is always positive. These formulae hold on any interval not containing the singular point $x = 1$.

Step 4: Conclusion The required integral is

$$\boxed{\frac{x^2}{2} + x + \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1} x + C}, \qquad x \neq 1.$$

Quick Tip to Solve: Divide first: the quotient is x plus one and the remainder is only one, making the partial fractions especially simple.

Section D (20 marks)

This section comprises of Long Answer (LA) type questions of 5 marks each.

32. Solve the following Linear Programming Problem graphically:

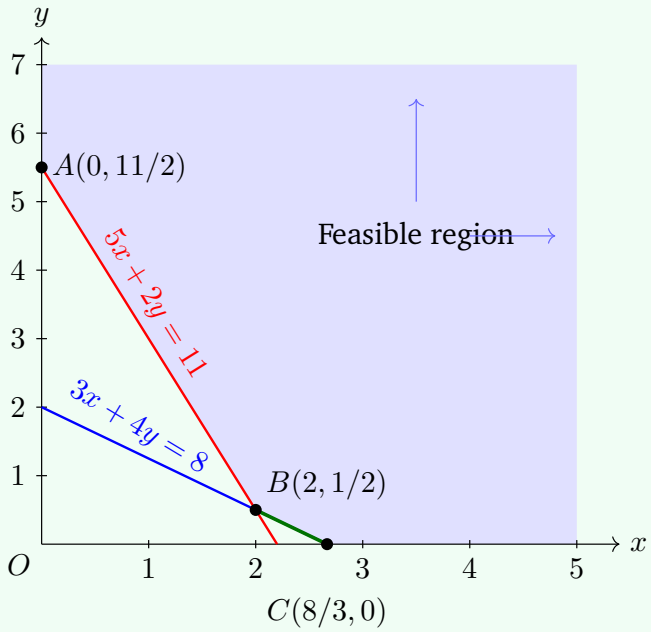
Minimise $Z = 60x + 80y$ subject to constraints $3x + 4y \geq 8$, $5x + 2y \geq 11$, $x, y \geq 0$. [5 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes representing all the constraints, restricted here to the first quadrant.
- A linear objective takes its optimum at a corner when an optimum exists. If the objective line coincides with a boundary segment, every point of that segment has the same optimum value.

Step 1: Draw the boundary lines and select the half-planes The line $3x + 4y = 8$ passes through $(0, 2)$ and $(8/3, 0)$. The line $5x + 2y = 11$ passes through $(0, 11/2)$ and $(11/5, 0)$. For each inequality, the origin gives a left side of zero and fails the required inequality. Thus choose the side away from the origin for both lines, and retain only $x, y \geq 0$. Their common region is unbounded above and to the right, as shaded in the graph.



The shaded feasible region is unbounded; the thick segment BC gives all minimum points.

Step 2: Determine the corner points Solve the two boundary equations simultaneously. Doubling $5x + 2y = 11$ and subtracting $3x + 4y = 8$ gives $7x = 14$; then $x = 2$ and $y = 1/2$. The common feasible boundary meets the y-axis at $A = (0, 11/2)$, the two lines at $B = (2, 1/2)$, and the x-axis at $C = (8/3, 0)$. The smaller axis intercepts $(0, 2)$ and $(11/5, 0)$ fail the other constraint and are not feasible vertices.

Quick Tip to Solve: Notice Z equals twenty times the first constraint's left side; its boundary immediately gives the minimum and the entire optimal segment.

Step 3: Compare objective values and establish the minimum At the three vertices,

$$Z(A) = 60(0) + 80(11/2) = 440,$$
$$Z(B) = 60(2) + 80(1/2) = 160,$$
$$Z(C) = 60(8/3) + 80(0) = 160.$$

Because the feasible region is unbounded, confirm the lower bound directly: $Z = 20(3x + 4y) \geq 20(8) = 160$ for every feasible point. Equality holds along the line $3x + 4y = 8$, but only its segment from B to C satisfies all the constraints. Indeed, on this line $y = 2 - 3x/4$, the second inequality becomes $(7/2)x + 4 \geq 11$, or $x \geq 2$, and $y \geq 0$ gives $x \leq 8/3$. The equal values at the two vertices therefore represent infinitely many optimal solutions.

Step 4: Conclusion The minimum is $\boxed{Z_{\min} = 160}$, attained at every point of the closed segment joining $(2, 1/2)$ and $(8/3, 0)$, namely $\boxed{2 \leq x \leq 8/3,\ y = 2 - 3x/4}$.

Quick Tip to Solve: Notice Z equals twenty times the first constraint's left side; its boundary immediately gives the minimum and the entire optimal segment.

33(a). The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. [5 marks]

Solution:

Concept:

- Related rates connect quantities that depend on the same time variable. If a length is measured in centimetres and time in seconds, its derivative with respect to time has unit cm/s.
- In an equilateral triangle, a median is also an altitude and bisects the opposite side. Thus, if the side is a and the median is m, Pythagoras gives $m = \dfrac{\sqrt{3}}{2}a$.

Step 1: Define the changing lengths Let $a(t)$ be the side length and $m(t)$ the median length at time t. The given rate is

$$\frac{dm}{dt} = 2\sqrt{3}\,\text{cm/s}.$$

We need $\dfrac{da}{dt}$, the rate of increase of the side. Since the triangle remains equilateral, the geometric relationship between these two lengths holds at every instant, even though both lengths change with time.

Step 2: Obtain the geometric relationship The median divides the equilateral triangle into two congruent right triangles. In either right triangle, the hypotenuse is a, one leg is $a/2$, and the other

Quick Tip to Solve: Write median = (sqrt(3)/2) times side before differentiating; the same constant ratio relates their rates.

leg is m. Therefore,

$$m^2 + \left(\frac{a}{2}\right)^2 = a^2,$$
$$m^2 = \frac{3}{4}a^2,$$
$$m = \frac{\sqrt{3}}{2}a.$$

We take the positive square root because the median and side are lengths. No particular side length is needed: their ratio is constant for every equilateral triangle.

Step 3: Differentiate and substitute the given rate Differentiate the relation with respect to time. The factor $\sqrt{3}/2$ is constant, so

$$\frac{dm}{dt} = \frac{\sqrt{3}}{2}\frac{da}{dt},$$
$$\frac{da}{dt} = \frac{2}{\sqrt{3}}\frac{dm}{dt},$$
$$= \frac{2}{\sqrt{3}}\left(2\sqrt{3}\right) = 4\,\text{cm/s}.$$

As a check, multiplying this rate by the median-to-side ratio gives $(\sqrt{3}/2) \times 4 = 2\sqrt{3}\,\text{cm/s}$, precisely the prescribed median rate. The positive sign agrees with the fact that the triangle is increasing in size.

Step 4: Conclusion The side of the equilateral triangle is increasing at $\boxed{4\,\text{cm/s}}$.

Quick Tip to Solve: Write median = (sqrt(3)/2) times side before differentiating; the same constant ratio relates their rates.

OR

33(b). Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. [5 marks]

Solution:

Concept:

- When the sum of two numbers is fixed, express one number in terms of the other to reduce an optimisation problem to one variable.
- For a differentiable function f, an interior minimum must satisfy $f'(x) = 0$. If $f''(x) > 0$ everywhere, the function is strictly convex, and its stationary point is its unique global minimum.

Step 1: Express the sum of cubes as a function Let the two numbers be x and y. The constraint is $x + y = 5$, so $y = 5 - x$. Hence their sum of cubes is

$$f(x) = x^3 + (5 - x)^3$$
$$= x^3 + 125 - 75x + 15x^2 - x^3$$
$$= 15x^2 - 75x + 125.$$

Quick Tip to Solve: Substitute y = 5 - x before differentiating, then evaluate the sum of squares at the minimising pair.

The cancellation of the cubic terms is useful: although the original expression contains cubes, the constrained function is quadratic. The question does not restrict the numbers to positive values, so we may optimise over all real x; the resulting numbers will in fact be positive.

Step 2: Find and classify the stationary point Differentiating the constrained function gives

$$f'(x) = 30x - 75, \qquad f''(x) = 30.$$

Set the first derivative equal to zero:

$$30x - 75 = 0 \quad \Longrightarrow \quad x = \frac{75}{30} = \frac{5}{2}.$$

Since $f''(x) = 30 > 0$ for every real x, this stationary point gives the unique global minimum. Equivalently, completing the square verifies this directly:

$$f(x) = 15\left(x - \frac{5}{2}\right)^2 + \frac{125}{4}.$$

The squared term is always non-negative and vanishes only at $x = 5/2$, so values outside the positive interval cannot give a smaller sum of cubes either.

Step 3: Calculate the requested sum of squares At the minimum, the second number is

$$y = 5 - \frac{5}{2} = \frac{5}{2}.$$

Thus both numbers are equal. The quantity asked for is their sum of squares, rather than the minimum sum of cubes:

$$\begin{aligned} x^2 + y^2 &= \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 \\ &= \frac{25}{4} + \frac{25}{4} = \frac{25}{2}. \end{aligned}$$

This can also be checked using $x^2 + y^2 = (x + y)^2 - 2xy$: with $x + y = 5$ and $xy = 25/4$, it gives $25 - 25/2 = 25/2$.

Step 4: Conclusion The sum of the cubes is least when both numbers are $5/2$. Their required sum of squares is $\boxed{\frac{25}{2}}$.

Quick Tip to Solve: Substitute y = 5 - x before differentiating, then evaluate the sum of squares at the minimising pair.

34. Evaluate: $\int_0^{\pi/2} \sin 2x \, \tan^{-1}(\sin x)\, dx$**.** [5 marks]

Solution:

Concept:

- The identity $\sin 2x = 2\sin x \cos x$ makes the substitution $t = \sin x$ suitable; then $dt = \cos x\, dx$.
- Integration by parts gives $\int u\, dv = uv - \int v\, du$. Here the inverse tangent is differentiated using $\dfrac{d}{dt}\tan^{-1} t = \dfrac{1}{1+t^2}$.
- For a definite integral, transform both limits with the substitution and evaluate every boundary term at the new limits.

Step 1: Transform the integral and its limits Denote the required integral by I. Using the double-angle identity,

$$I = \int_0^{\pi/2} 2\sin x \cos x \, \tan^{-1}(\sin x)\, dx.$$

Put $t = \sin x$, so $dt = \cos x\, dx$. As x increases from 0 to $\pi/2$, t increases from 0 to 1. Hence

$$I = \int_0^1 2t \tan^{-1} t\, dt.$$

The transformed integral has an algebraic factor and an inverse trigonometric factor, so integration by parts is convenient.

Step 2: Apply integration by parts Choose $u = \tan^{-1} t$ and $dv = 2t\, dt$. Then $du = \dfrac{dt}{1+t^2}$ and $v = t^2$. Thus

$$I = \left[t^2 \tan^{-1} t\right]_0^1 - \int_0^1 \frac{t^2}{1+t^2}\, dt.$$

At $t = 1$, the boundary expression is $\tan^{-1} 1 = \pi/4$, while at $t = 0$ it is zero. The remaining integral is rational and can be simplified before integration.

Step 3: Evaluate the remaining definite integral Write $t^2 = (1+t^2) - 1$, which gives

$$\begin{aligned}\int_0^1 \frac{t^2}{1+t^2}\, dt &= \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt \\ &= \left[t - \tan^{-1} t\right]_0^1 \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

Subtract this quantity from the integration-by-parts boundary term. As a check on the sign, the original integrand is nonnegative throughout the interval, so its integral must be positive.

Step 4: Conclusion Therefore, $I = \dfrac{\pi}{4} - \left(1 - \dfrac{\pi}{4}\right) = \boxed{\dfrac{\pi}{2} - 1}$.

Quick Tip to Solve: First replace $\sin 2x$ by $2\sin x \cos x$, then use $t = \sin x$; this turns the inverse sine argument into the integration variable.

35(a). **In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$. What is the probability that the student knows the answer, given that he answered it correctly?** [5 marks]

Solution:

Concept:

- Let K be the event that the student knows the answer, G the event that he guesses, and C the event that his answer is correct. Events K and G are mutually exclusive and exhaustive.
- The multiplication rule is $P(A \cap B) = P(A)P(B \mid A)$. The total probability theorem gives $P(C) = P(K)P(C \mid K) + P(G)P(C \mid G)$.
- Bayes' theorem reverses the conditioning: $P(K \mid C) = \dfrac{P(K)P(C \mid K)}{P(C)}$, provided $P(C) > 0$.

Step 1: Record the two ways of answering correctly The question gives

$$P(K) = \frac{3}{5}, \qquad P(G) = \frac{2}{5}, \qquad P(C \mid G) = \frac{1}{3}.$$

If the student knows the answer, he answers correctly, so $P(C \mid K) = 1$. A correct response may therefore result either from knowledge or from a successful guess. These two possibilities are disjoint because a student either knows the answer or guesses.

Step 2: Find the probability of a correct answer Using the multiplication rule for the two possibilities,

$$P(K \cap C) = \frac{3}{5} \times 1 = \frac{3}{5} = \frac{9}{15},$$
$$P(G \cap C) = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}.$$

Since these are all the possible ways to answer correctly, add their probabilities:

$$P(C) = P(K \cap C) + P(G \cap C) = \frac{9}{15} + \frac{2}{15} = \frac{11}{15}.$$

The denominator for the required conditional probability is thus the probability of a correct answer, rather than the probability of knowing the answer.

Step 3: Condition on the observed correct response We seek the fraction of correct responses that arise from knowledge. By Bayes' theorem,

$$P(K \mid C) = \frac{P(K)P(C \mid K)}{P(C)}$$
$$= \frac{\frac{3}{5} \times 1}{\frac{11}{15}}$$
$$= \frac{3}{5} \times \frac{15}{11}.$$

This posterior probability exceeds the initial probability $3/5$, as expected: learning that the answer was correct provides evidence favouring knowledge. It remains below one because some correct

Quick Tip to Solve: Separate correct answers into knowledge and successful guesses; divide the knowledge contribution by their total to reverse the conditioning.

responses are successful guesses.

Step 4: Conclusion The probability that the student knows the answer, given that he answered correctly, is $\boxed{\frac{9}{11}}$.

Quick Tip to Solve: Separate correct answers into knowledge and successful guesses; divide the knowledge contribution by their total to reverse the conditioning.

OR

35(b). A box contains 10 tickets, 2 of which carry a prize of ₹ 8 each, 5 of which carry a prize of ₹ 4 each, and remaining 3 carry a prize of ₹ 2 each. If one ticket is drawn at random, find the mean value of the prize. [5 marks]

Solution:

Concept:

- Let X denote the prize amount, in rupees, on the ticket drawn. A random variable associates a numerical value with each outcome of a random experiment.
- Every ticket is equally likely. Thus the probability of a given prize equals the number of tickets carrying it divided by the total number of tickets.
- The mean of a discrete random variable is $E(X) = \sum_i x_i P(X = x_i)$, where x_i are its distinct values. This is a probability-weighted average of the prizes.

Step 1: Identify the possible prizes The box contains ten tickets: two carry 8 rupees each, five carry 4 rupees each, and three carry 2 rupees each. Since one ticket is drawn at random, each ticket has probability $1/10$ of being selected. Therefore X can take exactly the values $2, 4, 8$. Tickets carrying the same amount are grouped together to determine the probability of that prize. The three group sizes add to $3 + 5 + 2 = 10$, as required.

Step 2: Form the probability distribution The probabilities of the three possible values are

$$P(X = 2) = \frac{3}{10}, \qquad P(X = 4) = \frac{5}{10}, \qquad P(X = 8) = \frac{2}{10}.$$

Each probability is nonnegative. Also,

$$\frac{3}{10} + \frac{5}{10} + \frac{2}{10} = 1.$$

Thus the distribution accounts for all possible draws. The prize 4 rupees receives the largest weight because five of the ten tickets carry it. It would be incorrect to give the three distinct prize amounts equal weights, since their ticket counts differ.

Quick Tip to Solve: Add the prizes on all ten equally likely tickets and divide by ten; do not average only the three distinct prize values.

Step 3: Compute and check the mean Using the definition of expectation,

$$\begin{aligned} E(X) &= 2\left(\frac{3}{10}\right) + 4\left(\frac{5}{10}\right) + 8\left(\frac{2}{10}\right) \\ &= \frac{6+20+16}{10} \\ &= \frac{42}{10} = 4.2. \end{aligned}$$

A direct check uses the sum of the prizes on all ten tickets. That sum is $3(2) + 5(4) + 2(8) = 42$ rupees, so dividing by the ten equally likely tickets gives the same value. The computed mean lies between the minimum prize 2 and the maximum prize 8, consistent with a weighted average. The mean represents the average prize over repeated random draws under this distribution; it need not be one of the prize amounts available on an individual ticket.

Step 4: Conclusion The mean value of the prize is $\boxed{4.20 \text{ rupees}}$.

Quick Tip to Solve: Add the prizes on all ten equally likely tickets and divide by ten; do not average only the three distinct prize values.

Section E (12 marks)

This section comprises of 3 case study/passage-based questions of 4 marks each with two sub-parts. First two case study questions have three sub – parts (I), (II), (III) of marks 1, 1, 2 respectively. The third case study question has two sub – parts (I) and (II) of marks 2 each.

36. An organization conducted a bike race under two different categories - Boys and Girls. There were 28 participants in all. Among all of them, finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.

Let $B = \{b_1, b_2, b_3\}$ and $G = \{g_1, g_2\}$, where B represents the set of Boys selected and G the set of Girls selected for the final race.

Based on the above information, answer the following questions. [4 marks]

36(I). How many relations are possible from B to G? [1 mark]

Solution:

Concept:

- A relation from B to G is any subset of the Cartesian product $B \times G$.
- A set with n elements has 2^n subsets, including the empty set and the set itself.

Step 1: Count the available ordered pairs The domain set contains three boys and the target set contains two girls. Thus $|B| = 3$, $|G| = 2$, and $|B \times G| = 3 \times 2 = 6$. Each pair has a boy as its first coordinate and a girl as its second coordinate; reversing them would count relations in the other direction.

Step 2: Count subsets of the Cartesian product For each of these six ordered pairs, a relation may either include that pair or exclude it. These choices are independent, so the number of possible

Quick Tip to Solve: Count ordered pairs first, then use $2^{|B||G|}$; do not use the function-count formula for arbitrary relations.

relations is

$$2^{|B\times G|} = 2^6 = 64.$$

A relation need not assign exactly one girl to each boy: that extra restriction belongs to functions.

Step 3: Conclusion There are $\boxed{64}$ relations from B to G.

Quick Tip to Solve: Count ordered pairs first, then use $2^{|B||G|}$; do not use the function-count formula for arbitrary relations.

36(II). Among all the possible relations from B to G, how many functions can be formed from B to G? [1 mark]

Solution:

Concept:

- A function from B to G must assign exactly one image in G to each element of B.
- For finite sets, the number of functions from a domain of size m to a codomain of size n is n^m.

Step 1: Identify the choice for each domain element Each of the three boys b_1, b_2, b_3 must have one image. For any boy this image can be either g_1 or g_2, giving two choices. Different boys are allowed to have the same image; injectivity is not required in this question.

Step 2: Apply the multiplication principle The image choices for the three distinct domain elements are independent. Hence the total is

$$2 \times 2 \times 2 = 2^3 = 8.$$

These are among the relations counted in part (I), but relations with a missing image or with two images for a boy are excluded because they are not functions.

Step 3: Conclusion Exactly $\boxed{8}$ functions can be formed from B to G.

Quick Tip to Solve: For functions, give each domain element one codomain choice: the codomain size is the base and the domain size is the exponent.

36(III)(a). Let $R : B \to B$ be defined by $R = \{(x, y) : x$ and y are students of the same sex$\}$. Check if R is an equivalence relation. [2 marks]

Solution:

Concept:

- An equivalence relation on a set is a relation that is reflexive, symmetric and transitive.
- Reflexivity means $(x, x) \in R$; symmetry means $(x, y) \in R \Rightarrow (y, x) \in R$; transitivity means $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.

Step 1: Identify the relation on the specified set The relation is on B, not on the combined set of all participants. Every member of $B = \{b_1, b_2, b_3\}$ is a boy. Therefore any two elements of B are students of the same sex, and $R = B \times B$. This includes all nine ordered pairs, with the three diagonal pairs

Quick Tip to Solve: Check the underlying set before testing properties: here every element is a boy, so the same-sex relation is the universal relation on B.

included.

Step 2: Check reflexivity and symmetry For every $x \in B$, the student x has the same sex as himself, so $(x, x) \in R$. Thus R is reflexive. If $(x, y) \in R$, then x and y have the same sex. The same statement holds with their order reversed, so $(y, x) \in R$, establishing symmetry.

Step 3: Check transitivity Let $x, y, z \in B$ with $(x, y) \in R$ and $(y, z) \in R$. Since all three students belong to the boys set, x and z have the same sex. Consequently $(x, z) \in R$, so the relation is transitive. Each required property holds for arbitrary elements, rather than only for a single example.

Step 4: Conclusion $\boxed{R \text{ is an equivalence relation on } B}$, since it is reflexive, symmetric and transitive.

Quick Tip to Solve: Check the underlying set before testing properties: here every element is a boy, so the same-sex relation is the universal relation on B.

OR

36(III)(b). A function $f : B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$. Check if f is bijective. Justify your answer. [2 marks]

Solution:

Concept:

- A function is one-one (injective) if distinct domain elements always have distinct images.
- A function is onto (surjective) if each element of its codomain occurs as an image. A bijective function must satisfy both properties.

Step 1: Read the images and confirm the function condition The given ordered pairs specify

$$f(b_1) = g_1, \qquad f(b_2) = g_2, \qquad f(b_3) = g_1.$$

Every member of B appears once as a first coordinate, and each image belongs to G, so this is indeed a function from B to G. Having a repeated image does not invalidate the function condition.

Step 2: Test whether the function is one-one The elements b_1 and b_3 are distinct, but $f(b_1) = f(b_3) = g_1$. Thus two different domain elements have the same image, contradicting injectivity. Therefore f is not one-one. A single such pair is enough to disprove this property.

Step 3: Test whether the function is onto The codomain is exactly $G = \{g_1, g_2\}$. The element g_1 has preimages b_1 and b_3, while g_2 has preimage b_2. Hence the range is $\{g_1, g_2\} = G$, and the function is onto. Since bijectivity also requires injectivity, being onto alone cannot establish it.

Step 4: Conclusion $\boxed{f \text{ is not bijective}}$: it is onto, but not one-one because $b_1 \neq b_3$ and $f(b_1) = f(b_3)$.

Quick Tip to Solve: Look for repeated images to disprove injectivity, then compare the set of images with the full codomain to test surjectivity.

37. Gautam buys 5 pens, 3 bags and 1 instrument box and pays a sum of ₹160. From the same shop, Vikram buys 2 pens, 1 bag and 3 instrument boxes and pays a sum of ₹190. Also Ankur buys 1 pen, 2 bags and 4 instrument boxes and pays a sum of ₹250.

Based on the above information, answer the following questions. [4 marks]

37(I). Convert the given above situation into a matrix equation of the form $AX = B$. [1 mark]

Solution:

Concept:

- The product AX forms the left sides of simultaneous linear equations when each row of A contains their coefficients.
- Use the same order of unknowns in every row: pen, bag, instrument box.

Step 1: Define the unknown prices Let x, y, z be the prices, in rupees, of one pen, one bag and one instrument box respectively. The three purchases give

$$5x + 3y + z = 160, \qquad 2x + y + 3z = 190, \qquad x + 2y + 4z = 250.$$

Each coefficient counts the corresponding article purchased.

Step 2: Arrange the coefficient and constant matrices Keeping the customers in the order Gautam, Vikram, Ankur, take

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 160 \\ 190 \\ 250 \end{pmatrix}.$$

Multiplying a row of A by X reproduces that customer's total cost.

Step 3: Conclusion The required matrix equation is

$$\begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 160 \\ 190 \\ 250 \end{pmatrix}, \quad \text{i.e. } AX = B.$$

Quick Tip to Solve: Fix the column order before entering coefficients; put each customer's bill in the matching row of B.

37(II). Find $|A|$. [1 mark]

Solution:

Concept:

- For a 3×3 matrix, expansion along the first row uses alternating signs $+, -, +$.
- The determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$.

Step 1: Identify the coefficient matrix The coefficients of the pen, bag and instrument-box prices, in

Quick Tip to Solve: Write the +, -, + signs before evaluating minors to avoid reversing the middle term.

that order, are

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

Use only the coefficients for the determinant; the payment amounts belong to B, not to A.

Step 2: Expand and evaluate Expanding along the first row gives

$$\begin{aligned} |A| &= 5(1 \cdot 4 - 3 \cdot 2) - 3(2 \cdot 4 - 3 \cdot 1) + 1(2 \cdot 2 - 1 \cdot 1) \\ &= 5(-2) - 3(5) + 3 \\ &= -10 - 15 + 3 = -22. \end{aligned}$$

The middle minor carries a negative sign, as required by the cofactor pattern.

Step 3: Conclusion The determinant is $\boxed{|A| = -22}$.

Quick Tip to Solve: Write the +, -, + signs before evaluating minors to avoid reversing the middle term.

37(III)(a). Find A^{-1}. [2 marks]

Solution:

Concept:

- If $|A| \neq 0$, then $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$.
- The adjugate is the transpose of the cofactor matrix; $C_{ij} = (-1)^{i+j} M_{ij}$.

Step 1: Find the determinant The coefficient matrix is

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

Its determinant, expanding along the first row, is

$$|A| = 5(4 - 6) - 3(8 - 3) + (4 - 1) = -22.$$

Thus the matrix is nonsingular and its inverse exists.

Step 2: Compute the cofactors and transpose The cofactor signs are $+, -, +$ in the first row, alternating thereafter. For example, $C_{12} = -(8 - 3) = -5$, while $C_{21} = -(12 - 2) = -10$. Evaluating all nine cofactors gives

$$C = \begin{pmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; cofactors alone are not the adjugate.

Hence

$$\operatorname{adj} A = C^T = \begin{pmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{pmatrix}.$$

Transposition interchanges the row and column positions of each cofactor.

Step 3: Apply the inverse formula Divide every entry of the adjugate by -22. Equivalently, change all its signs and use the positive denominator 22. As a check, the product of A and the resulting integer numerator is $22I$; its first diagonal entry is $5(2) + 3(5) - 3 = 22$.

Step 4: Conclusion

$$\boxed{A^{-1} = \frac{1}{22}\begin{pmatrix} 2 & 10 & -8 \\ 5 & -19 & 13 \\ -3 & 7 & 1 \end{pmatrix}}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; cofactors alone are not the adjugate.

OR

37(III)(b). Determine $P = A^2 - 5A$. [2 marks]

Solution:

Concept:

- Matrix squaring means $A^2 = A \cdot A$, using row-by-column multiplication.
- The matrix $5A$ multiplies each entry of A by 5; subtraction is entrywise.

Step 1: Calculate the square For

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix},$$

multiply each row of the first factor by each column of the second. The first row of A^2 is $(25 + 6 + 1,\ 15+3+2,\ 5+9+4) = (32, 20, 18)$. The second row is $(10+2+3,\ 6+1+6,\ 2+3+12) = (15, 13, 17)$, and the third row is $(5 + 4 + 4,\ 3 + 2 + 8,\ 1 + 6 + 16) = (13, 13, 23)$. Therefore

$$A^2 = \begin{pmatrix} 32 & 20 & 18 \\ 15 & 13 & 17 \\ 13 & 13 & 23 \end{pmatrix}.$$

Step 2: Subtract the scalar multiple Multiplying the original matrix by 5 gives

$$5A = \begin{pmatrix} 25 & 15 & 5 \\ 10 & 5 & 15 \\ 5 & 10 & 20 \end{pmatrix}.$$

Quick Tip to Solve: Keep A squared and 5A in separate matrices, then subtract corresponding entries; never square entries individually.

Now subtract entries in matching positions. In the first row the differences are $32-25, 20-15, 18-5$; the other two rows are obtained in exactly the same way. For example, the bottom-left entry is $13-5=8$, while the middle diagonal entry is $13-5=8$.

Step 3: Conclusion

$$\boxed{P = A^2 - 5A = \begin{pmatrix} 7 & 5 & 13 \\ 5 & 8 & 2 \\ 8 & 3 & 3 \end{pmatrix}}.$$

Quick Tip to Solve: Keep A squared and 5A in separate matrices, then subtract corresponding entries; never square entries individually.

38. An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation. A differential equation of the form $\frac{dy}{dx} = F(x,y)$ is said to be homogeneous if $F(x,y)$ is a homogeneous function of degree zero, whereas a function $F(x,y)$ is a homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x,y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x,y) = g\left(\frac{y}{x}\right)$, we make the substitution $y = vx$ and then separate the variables.

Based on the above, answer the following questions. [4 marks]

38(I). Show that $(x^2 - y^2)\,dx + 2xy\,dy = 0$ is a differential equation of the type $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. [2 marks]

Solution:

Concept:

- A first-order equation is homogeneous when $dy/dx = F(x,y)$ and $F(\lambda x, \lambda y) = F(x,y)$.
- A function depending only on y/x has degree zero, where the required denominators are nonzero.

Step 1: Express the derivative From

$$(x^2 - y^2)\,dx + 2xy\,dy = 0,$$

divide by dx along a differentiable solution and rearrange:

$$2xy\frac{dy}{dx} = y^2 - x^2, \qquad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

This derivative form is considered on a region with $xy \neq 0$, since division by $2xy$ requires that condition.

Step 2: Write the right side using the ratio Dividing numerator and denominator by x^2, we get

$$\frac{dy}{dx} = \frac{(y/x)^2 - 1}{2(y/x)}.$$

Define $g(u) = (u^2 - 1)/(2u)$ for $u \neq 0$. Then the right side is precisely $g(y/x)$. Also, replacing x, y by

Quick Tip to Solve: Divide the derivative ratio by x squared: matching degrees become a function of y divided by x.

$\lambda x, \lambda y$ multiplies both numerator and denominator of F by λ^2, so their ratio is unchanged for nonzero λ. This verifies degree zero directly.

Step 3: Conclusion Thus

$$\boxed{\frac{dy}{dx} = g\left(\frac{y}{x}\right), \quad g(u) = \frac{u^2 - 1}{2u}}$$

on regions where $xy \neq 0$; hence the differential equation is homogeneous.

Quick Tip to Solve: Divide the derivative ratio by x squared: matching degrees become a function of y divided by x.

38(II). Solve the above equation to find its general solution. [2 marks]

Solution:

Concept:

- For a homogeneous equation, put $y = vx$, so $dy/dx = v + x\,dv/dx$.
- After separating variables, integrate each side and combine logarithms using an arbitrary constant.

Step 1: Substitute the ratio variable The equation gives

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

On a region with $xy \neq 0$, let $y = vx$. By the product rule, $dy/dx = v + x\,dv/dx$. Substitution yields

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}, \qquad x\frac{dv}{dx} = -\frac{v^2 + 1}{2v}.$$

Step 2: Separate and integrate Collecting the variables gives

$$\frac{2v}{1 + v^2}\,dv = -\frac{dx}{x}.$$

Since $1 + v^2 > 0$ for real v, integration gives

$$\ln(1 + v^2) = -\ln|x| + C_1.$$

Therefore $|x|(1 + v^2) = K$. On an interval where x has fixed sign, absorb that sign into the arbitrary constant and write $x(1 + v^2) = C$.

Step 3: Return to the original variables and verify Replacing v by y/x and multiplying by x gives

$$x^2 + y^2 = Cx.$$

Differentiating gives $(2x - C)\,dx + 2y\,dy = 0$. Multiply this identity by x and use $Cx = x^2 + y^2$; it becomes

$$(x^2 - y^2)\,dx + 2xy\,dy = 0,$$

Quick Tip to Solve: After integrating, multiply x times one plus v squared and replace v by y divided by x before simplifying.

which is the original equation. This verifies the implicit family directly.

Step 4: Conclusion The general solution is $\boxed{x^2 + y^2 = Cx}$, where C is an arbitrary real constant.

Quick Tip to Solve: After integrating, multiply x times one plus v squared and replace v by y divided by x before simplifying.