

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of 1 mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ (where $\mathbb{R}_+$ is the set of all non-negative real numbers) defined by $f(x) = 4x + 3$ is: [1 mark]

- (A) one-one but not onto
- (B) onto but not one-one
- (C) both one-one and onto
- (D) neither one-one nor onto

Correct Answer: (A) one-one but not onto

Solution:

Concept:

- A function is one-one if equal images imply equal inputs. It is onto if its range equals its stated codomain.
- For a linear function $f(x) = mx + c$, a nonzero slope m gives injectivity on any interval.

Step 1: Test whether the function is one-one Let $u, v \geq 0$ and suppose $f(u) = f(v)$. Then $4u + 3 = 4v + 3$, so $4(u - v) = 0$ and hence $u = v$. Distinct non-negative inputs therefore have distinct images.

Step 2: Determine the range and reject the other options Since $x \geq 0$, the image $4x + 3$ is at least 3. Conversely, every $y \geq 3$ has the preimage $x = (y - 3)/4 \geq 0$. Thus the range is $[3, \infty)$, which omits real numbers such as 0 from the codomain $\mathbb{R}$. The onto choices (B) and (C) fail, and (D) fails because injectivity holds.

Step 3: Conclusion The function is one-one but not onto. Hence, the correct option is (A).

Quick Tip to Solve: Check the stated codomain: a nonzero slope alone does not prove that a restricted-domain function is onto.

2. If a matrix has 36 elements, the number of possible orders it can have, is: [1 mark]

- (A) 13
- (B) 3
- (C) 5
- (D) 9

Correct Answer: (D) 9

Solution:

Concept:

- An $m \times n$ matrix has mn entries, where its numbers of rows m and columns n are positive integers.
- Matrix orders are ordered pairs: $m \times n$ and $n \times m$ are distinct when $m \neq n$.

Step 1: Translate the entry count into a factor condition For exactly 36 elements, the order must

Quick Tip to Solve: Count positive divisors of 36, keeping transposed rectangular orders separate and counting the square order once.

satisfy $mn = 36$. Each positive divisor m of 36 gives exactly one valid column count $n = 36/m$.

Step 2: List and count all orders The possible orders are

$$\begin{aligned} 1 \times 36, \quad 2 \times 18, \quad 3 \times 12, \\ 4 \times 9, \quad 6 \times 6, \quad 9 \times 4, \\ 12 \times 3, \quad 18 \times 2, \quad 36 \times 1. \end{aligned}$$

There are nine orders. Counting only unordered factor pairs would give five, option (C), but would wrongly identify transposed rectangular orders. The counts thirteen and three in (A) and (B) also disagree with the complete list.

Step 3: Conclusion There are 9 possible orders. Hence, the correct option is (D).

Quick Tip to Solve: Count positive divisors of 36, keeping transposed rectangular orders separate and counting the square order once.

3. Which of the following statements is true for the function

$$f(x) = \begin{cases} x^2 + 3, & x \neq 0 \\ 1, & x = 0 \end{cases} ?$$

[1 mark]

- (A) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R}$
(B) $f(x)$ is continuous $\forall x \in \mathbb{R}$
(C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$
(D) $f(x)$ is discontinuous at infinitely many points

Correct Answer: (C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$

Solution:

Concept:

- Continuity at c requires $\lim_{x \rightarrow c} f(x) = f(c)$. Differentiability at a point implies continuity there.
- A polynomial is continuous and differentiable at every real argument.

Step 1: Check the exceptional point For $x \neq 0$, $f(x) = x^2 + 3$. Therefore $\lim_{x \rightarrow 0} f(x) = 3$, whereas $f(0) = 1$. Since these are unequal, the function is discontinuous at zero and cannot be differentiable there.

Step 2: Check all remaining points and compare the choices At any fixed $c \neq 0$, a sufficiently small neighbourhood excludes zero. In that neighbourhood the function is the polynomial $x^2 + 3$, hence is continuous and differentiable, with derivative $2x$. Options (A) and (B) incorrectly include continuity at zero. Option (D) is false because there is exactly one discontinuity, rather than infinitely many.

Step 3: Conclusion The function is continuous and differentiable for all $x \in \mathbb{R} \setminus \{0\}$. Hence, the correct option is (C).

Quick Tip to Solve: For a piecewise function, first compare its limiting value with its assigned value at the exceptional point.

4. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) . Then, this function $f(x)$ is strictly increasing in (a, b) if: [1 mark]

(A) $f'(x) < 0, \forall x \in (a, b)$

(B) $f'(x) > 0, \forall x \in (a, b)$

(C) $f'(x) = 0, \forall x \in (a, b)$

(D) $f(x) > 0, \forall x \in (a, b)$

Correct Answer: (B) $f'(x) > 0, \forall x \in (a, b)$

Solution:

Concept:

- A positive derivative throughout an interval is a sufficient condition for strict increase there.
- The mean value theorem gives $f(v) - f(u) = f'(c)(v - u)$ for some $c \in (u, v)$, under continuity and differentiability.

Step 1: Apply the derivative criterion Take any $a < u < v < b$. The stated assumptions allow the mean value theorem on $[u, v]$, so $f(v) - f(u) = f'(c)(v - u)$ for some $c \in (u, v)$. If $f'(x) > 0$ throughout (a, b) , both factors are positive and $f(v) > f(u)$.

Step 2: Check the remaining conditions Option (A), a negative derivative, produces strict decrease. Option (C), zero derivative everywhere, produces a constant function. Option (D), $f(x) > 0$, concerns the values of the function and does not control its increase: for example $f(x) = 2 - x$ is positive but decreasing on $(0, 1)$. Thus only the positive-derivative condition among the printed choices ensures strict increase.

Step 3: Conclusion The sufficient condition is $f'(x) > 0$ for every $x \in (a, b)$. Hence, the correct option is (B).

Quick Tip to Solve: Use the sign of the first derivative to determine increase; positive function values do not determine its direction.

5. If $\begin{pmatrix} x+y & 2 \\ 5 & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$, then the value of $\frac{24}{x} + \frac{24}{y}$ is: [1 mark]

(A) 7

(B) 6

(C) 8

(D) 18

Correct Answer: (D) 18

Solution:

Concept:

- Equal matrices of the same order have equal corresponding entries.
- For nonzero x, y , $1/x + 1/y = (x + y)/(xy)$; this avoids finding the individual variables.

Step 1: Compare the corresponding matrix entries From the top-left entries, $x + y = 6$. From the

Quick Tip to Solve: Extract the sum and product directly from matrix equality, then combine the reciprocal terms before substitution.

bottom-right entries, $xy = 8$. The remaining entries, 2 and 5, already agree. Since $xy \neq 0$, neither variable is zero and the required reciprocals exist.

Step 2: Evaluate the requested expression Combine the fractions using their common denominator:

$$\frac{24}{x} + \frac{24}{y} = \frac{24(x+y)}{xy} = \frac{24 \cdot 6}{8} = 18.$$

Thus option (D) matches the expression; the alternatives 7, 6 and 8 in (A), (B) and (C) do not. As an independent check, x, y are 2, 4 in either order, giving $12 + 6$.

Step 3: Conclusion The value is 18. Hence, the correct option is (D).

Quick Tip to Solve: Extract the sum and product directly from matrix equality, then combine the reciprocal terms before substitution.

6. $\int_a^b f(x) dx$ is equal to:

[1 mark]

(A) $\int_a^b f(a-x) dx$

(B) $\int_a^b f(a+b-x) dx$

(C) $\int_a^b f(x-(a+b)) dx$

(D) $\int_a^b f((a-x) + (b-x)) dx$

Correct Answer: (B) $\int_a^b f(a+b-x) dx$

Solution:

Concept:

- Definite integration is unchanged by reflecting the variable in the midpoint of the interval.
- Under $u = a + b - x$, one has $dx = -du$; reversing the limits removes this minus sign.

Step 1: Make the reflection substitution Write $I = \int_a^b f(x) dx$ and put $u = a + b - x$. At $x = a$, $u = b$, and at $x = b$, $u = a$. Hence

$$I = - \int_b^a f(a+b-u) du = \int_a^b f(a+b-u) du.$$

Step 2: Compare the transformed integrands The integration variable is a dummy variable, so replacing u by x gives option (B). The arguments in (A) and (C) shift or reflect into a different interval in general. In (D), the argument is $a + b - 2x$, whose derivative is -2 ; its scale and endpoints differ from the required reflection, so it is not a general identity.

Step 3: Conclusion $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$. Hence, the correct option is (B).

Quick Tip to Solve: For reflection of a definite integral, add its two limits and subtract the variable, then reverse the new limits.

7. Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin \theta = \frac{3}{5}$. Then, $\hat{a} \cdot \hat{b}$ is equal to: [1 mark]

- (A) $\pm \frac{3}{5}$
(B) $\pm \frac{3}{4}$
(C) $\pm \frac{4}{5}$
(D) $\pm \frac{4}{3}$

Correct Answer: (C) $\pm \frac{4}{5}$

Solution:

Concept:

- The dot product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$; for unit vectors it equals $\cos \theta$.
- The identity $\sin^2 \theta + \cos^2 \theta = 1$ determines the magnitude of the cosine, but not its sign.

Step 1: Find the possible cosine values Given $\sin \theta = 3/5$,

$$\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}, \quad \cos \theta = \pm \frac{4}{5}.$$

The angle between vectors lies in $[0, \pi]$. A positive sine allows either an acute or an obtuse angle, so both cosine signs remain possible.

Step 2: Use the unit magnitudes and compare the choices Since both vectors have magnitude one, $\hat{a} \cdot \hat{b} = 1 \cdot 1 \cdot \cos \theta$. Thus its magnitude is $4/5$. Option (A) incorrectly uses sine as the dot product, and option (B) gives the wrong magnitude. Option (D) has magnitude greater than one and is impossible for a dot product of unit vectors.

Step 3: Conclusion $\hat{a} \cdot \hat{b} = \pm \frac{4}{5}$. Hence, the correct option is (C).

Quick Tip to Solve: Keep both cosine signs when only a positive sine is given and the vector angle is not stated to be acute.

8. The integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, $-1 < x < 1$, is: [1 mark]

- (A) $\frac{1}{x^2 - 1}$
(B) $\frac{1}{\sqrt{x^2 - 1}}$
(C) $\frac{1}{1 - x^2}$
(D) $\frac{1}{\sqrt{1 - x^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1 - x^2}}$

Solution:

Concept:

- For a linear equation $y' + P(x)y = Q(x)$, the integrating factor is $e^{\int P(x) dx}$.
- The interval $-1 < x < 1$ gives $1 - x^2 > 0$, so its real square root is defined.

Step 1: Reduce to linear form Dividing by the nonzero coefficient $1 - x^2$ gives

$$\frac{dy}{dx} + \frac{x}{1 - x^2}y = \frac{ax}{1 - x^2}.$$

Thus the coefficient used in the integrating-factor formula is $P(x) = x/(1 - x^2)$; the right-hand side does not determine the integrating factor.

Step 2: Integrate the coefficient and compare options With $u = 1 - x^2$, $du = -2x dx$, so

$$\int \frac{x}{1 - x^2} dx = -\frac{1}{2} \ln(1 - x^2).$$

Exponentiation gives $(1 - x^2)^{-1/2}$. Options (A) and (C) have exponent -1 , which doubles the required logarithmic derivative. Option (B) is not real on the stated interval.

Step 3: Conclusion The integrating factor is $\boxed{1/\sqrt{1 - x^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: First divide by the coefficient of dy/dx ; then use $d(1 - x^2) = -2x dx$ to obtain the exponent.

9. If the direction cosines of a line are $\sqrt{3}k$, $\sqrt{3}k$, $\sqrt{3}k$, then the value of k is:

[1 mark]

- (A) ± 1
- (B) $\pm\sqrt{3}$
- (C) ± 3
- (D) $\pm\frac{1}{3}$

Correct Answer: (D) $\pm\frac{1}{3}$

Solution:

Concept:

- Direction cosines l, m, n satisfy $l^2 + m^2 + n^2 = 1$.
- Reversing the orientation of a line reverses all three direction cosines, so both signs may occur.

Step 1: Apply the normalization identity Substituting $l = m = n = \sqrt{3}k$ gives

$$(\sqrt{3}k)^2 + (\sqrt{3}k)^2 + (\sqrt{3}k)^2 = 1, \quad 9k^2 = 1.$$

The squares add; one must not add the direction cosines themselves.

Step 2: Solve and check the choices Therefore $k^2 = 1/9$, giving $k = \pm 1/3$. For these values, each cosine is $\pm 1/\sqrt{3}$, and the sum of their squares is exactly one. For options (A), (B) and (C), the sums would instead be 9, 27 and 81, respectively, so none satisfies the normalization identity.

Step 3: Conclusion $k = \pm\frac{1}{3}$. Hence, the correct option is (D).

Quick Tip to Solve: Square each direction cosine before adding; three identical entries $\sqrt{3}k$ give $9k^2$, not $3k^2$.

10. A linear programming problem deals with the optimization of a/an:

[1 mark]

- (A) logarithmic function
- (B) linear function
- (C) quadratic function
- (D) exponential function

Correct Answer: (B) linear function

Solution:

Concept:

- A linear programming problem maximizes or minimizes a linear objective function subject to linear constraints.
- For two variables, the objective has the form $Z = ax + by$, where a, b are fixed coefficients.

Step 1: Identify the function being optimized The word optimization means finding the maximum or minimum permitted value of the objective function. In linear programming this function is linear in the decision variables, such as $Z = 3x + 5y$. The feasible choices are restricted by linear equalities or

Quick Tip to Solve: Look at the objective being maximized or minimized: in an LPP every decision variable occurs only to the first power.

inequalities and any stated non-negativity conditions.

Step 2: Compare the listed types A logarithmic objective involves a logarithm of a variable; a quadratic objective contains products or squares of variables; an exponential objective has a variable in an exponent. These are not linear objectives and therefore do not describe the defining objective of a linear programming problem. The listed linear function matches the definition directly.

Step 3: Conclusion A linear programming problem optimizes a linear function. Hence, the correct option is (B).

Quick Tip to Solve: Look at the objective being maximized or minimized: in an LPP every decision variable occurs only to the first power.

11. If $P(A | B) = P(A' | B)$, then which of the following statements is true?

[1 mark]

- (A) $P(A) = P(A')$
- (B) $P(A) = 2P(B)$
- (C) $P(A \cap B) = \frac{1}{2}P(B)$
- (D) $P(A \cap B) = 2P(B)$

Correct Answer: (C) $P(A \cap B) = \frac{1}{2}P(B)$

Solution:

Concept:

- For $P(B) > 0$, conditional probability is $P(A | B) = P(A \cap B)/P(B)$.
- Within B , the events $A \cap B$ and $A' \cap B$ partition B , so their conditional probabilities add to one.

Step 1: Use the conditional complement relation Let the common value of $P(A | B)$ and $P(A' | B)$ be p . Since $P(A | B) + P(A' | B) = 1$, we get $2p = 1$, or $p = 1/2$. The given condition concerns probabilities within event B , not necessarily in the whole sample space.

Step 2: Convert to an intersection probability Using the definition,

$$\frac{P(A \cap B)}{P(B)} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{2}P(B).$$

Option (A) would require unconditional probabilities to be equal, which need not follow. Option (B) similarly asserts an unsupported unconditional relationship. Option (D) contradicts both the derived half-factor and the fact that $A \cap B \subseteq B$.

Step 3: Conclusion The intersection probability is half the conditioning-event probability. Hence, the correct option is (C).

Quick Tip to Solve: When complementary events have equal conditional probabilities, each is one half; multiply by the conditioning probability.

12. $\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}$ is equal to:

[1 mark]

- (A) $2x^3$
- (B) 2
- (C) 0
- (D) $2x^3 - 2$

Correct Answer: (B) 2

Solution:

Concept:

- For a two by two determinant, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- The identities $(x+1)(x^2-x+1) = x^3+1$ and $(x-1)(x^2+x+1) = x^3-1$ simplify the products.

Step 1: Expand along the first row Apply the determinant rule in the printed order:

$$D = (x+1)(x^2-x+1) - (x-1)(x^2+x+1).$$

The second product is subtracted. This sign is essential because the bottom entries have opposite signs in their middle terms.

Step 2: Simplify the products and compare options The first product expands to x^3+1 , while the second expands to x^3-1 . Hence

$$D = (x^3+1) - (x^3-1) = 2.$$

The cubic terms cancel for every real x , so the result is constant. Options (A) and (D) depend on x and cannot be identities; option (C) misses the difference of the two constant terms.

Step 3: Conclusion The determinant equals $\boxed{2}$. Hence, the correct option is (B).

Quick Tip to Solve: Recognize the sum and difference of cubes in the two diagonal products; subtract x^3-1 with brackets intact.

13. The derivative of $\sin(x^2)$ w.r.t. x , at $x = \sqrt{\pi}$, is:

[1 mark]

- (A) 1
- (B) -1
- (C) $-2\sqrt{\pi}$
- (D) $2\sqrt{\pi}$

Correct Answer: (C) $-2\sqrt{\pi}$

Solution:

Concept:

- The chain rule gives $d[\sin u]/dx = \cos u \, du/dx$.

Quick Tip to Solve: Keep the inner derivative $2x$ when differentiating $\sin(x^2)$, then use $\cos \pi = -1$.

- Here $u = x^2$, so $du/dx = 2x$, and trigonometric arguments are in radians.

Step 1: Differentiate before substituting For $f(x) = \sin(x^2)$, the outer sine contributes $\cos(x^2)$ and the inner square contributes $2x$. Therefore

$$f'(x) = 2x \cos(x^2).$$

Substituting into the original sine first would give only a function value, not its derivative.

Step 2: Evaluate at the specified point At $x = \sqrt{\pi}$, we have $x^2 = \pi$ and $\cos \pi = -1$. Consequently

$$f'(\sqrt{\pi}) = 2\sqrt{\pi}(-1) = -2\sqrt{\pi}.$$

Options (A) and (B) omit the inner derivative factor. Option (D) has the wrong sign because the cosine at π is negative.

Step 3: Conclusion The derivative at the point is $\boxed{-2\sqrt{\pi}}$. Hence, the correct option is (C).

Quick Tip to Solve: Keep the inner derivative $2x$ when differentiating $\sin(x^2)$, then use $\cos \pi = -1$.

14. The order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$ respectively are: [1 mark]

- (A) 1, 2
- (B) 2, 3
- (C) 2, 1
- (D) 2, 6

Correct Answer: (C) 2, 1

Solution:

Concept:

- The order is the order of the highest derivative present in a differential equation.
- When an equation is polynomial in its derivatives, its degree is the power of the highest-order derivative.

Step 1: Identify the highest derivative The given equation contains dy/dx and d^2y/dx^2 . The highest derivative is the second derivative, so the order is two. The outside exponent three changes powers of the first derivative but does not introduce a third derivative.

Step 2: Find the power of the highest-order derivative Writing the equation in polynomial form gives

$$\frac{d^2y}{dx^2} - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0.$$

The second derivative occurs to the first power, so the degree is one. Option (A) has the wrong order. Options (B) and (D) confuse the outside power or the resulting sixth power of the first derivative with

Quick Tip to Solve: Locate the highest-order derivative first; its own exponent determines degree, even if a lower derivative has a larger power.

the degree of the equation.

Step 3: Conclusion The order and degree are $\boxed{2, 1}$, respectively. Hence, the correct option is (C).

Quick Tip to Solve: Locate the highest-order derivative first; its own exponent determines degree, even if a lower derivative has a larger power.

15. The vector with terminal point $A(2, -3, 5)$ and initial point $B(3, -4, 7)$ is:

[1 mark]

- (A) $\hat{i} - \hat{j} + 2\hat{k}$
- (B) $\hat{i} + \hat{j} + 2\hat{k}$
- (C) $-\hat{i} - \hat{j} - 2\hat{k}$
- (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Solution:

Concept:

- A directed vector equals the position vector of its terminal point minus the position vector of its initial point.
- For initial point B and terminal point A , $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$.

Step 1: Subtract coordinates in the stated direction The vector starts at $B(3, -4, 7)$ and ends at $A(2, -3, 5)$. Therefore

$$\overrightarrow{BA} = (2 - 3)\hat{i} + (-3 + 4)\hat{j} + (5 - 7)\hat{k} = -\hat{i} + \hat{j} - 2\hat{k}.$$

Step 2: Check the four options Option (D) has exactly the components $(-1, 1, -2)$. Option (A) reverses the direction and represents $\overrightarrow{AB}$; option (B) has the wrong signs for the first and third components; option (C) has the wrong sign for the second component.

Step 3: Conclusion The required vector is $\boxed{-\hat{i} + \hat{j} - 2\hat{k}}$. Hence, the correct option is (D).

Quick Tip to Solve: Write terminal minus initial above the coordinates before subtracting; reversing the order reverses all signs.

16. The distance of point $P(a, b, c)$ from y -axis is:

[1 mark]

- (A) b
- (B) b^2
- (C) $\sqrt{a^2 + c^2}$
- (D) $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$

Solution:

Concept:

- Every point of the y -axis has coordinates $(0, t, 0)$, where t is real.
- The perpendicular distance to an axis is the shortest distance to any point on that axis.

Step 1: Find the nearest point on the axis For $Q(0, t, 0)$ on the y -axis,

$$PQ^2 = a^2 + (b - t)^2 + c^2.$$

The term $(b - t)^2$ is non-negative and is smallest when $t = b$. Thus the perpendicular foot is $Q(0, b, 0)$.

Step 2: Evaluate and compare The minimum distance is $PQ = \sqrt{a^2 + c^2}$. Options (A) and (B) use the coordinate along the axis, which does not determine the perpendicular distance. Option (D) gives the square of the distance rather than the distance itself.

Step 3: Conclusion The distance is $\sqrt{a^2 + c^2}$. Hence, the correct option is (C).

Quick Tip to Solve: For distance to a coordinate axis, omit the coordinate along that axis and take the square root of the other two squares.

17. The number of corner points of the feasible region determined by constraints $x \geq 0$, $y \geq 0$, $x + y \geq 4$ is:

[1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (C) 2

Solution:

Concept:

- A corner point is a vertex of the feasible region, formed where active boundary lines meet.
- The constraints $x \geq 0$, $y \geq 0$ restrict the region to the first quadrant; $x + y \geq 4$ selects points on or beyond the line.

Step 1: Identify the feasible boundary intersections The line $x + y = 4$ meets the axes at $(4, 0)$ and $(0, 4)$. Both satisfy all three inequalities. The axes meet at $(0, 0)$, but this point violates $x + y \geq 4$, so it is not feasible.

Quick Tip to Solve: Check each boundary intersection against every inequality; the origin is excluded here because its coordinate sum is less than 4.

Step 2: Count the vertices The feasible boundary consists of the segment joining $(4, 0)$ and $(0, 4)$, and rays along the axes extending away from these points. Only the two endpoints are corners. The region is unbounded, but this does not prevent it from having vertices. Hence counts 0, 1 and 3 are incorrect.

Step 3: Conclusion There are $\boxed{2}$ corner points. Hence, the correct option is (C).

Quick Tip to Solve: Check each boundary intersection against every inequality; the origin is excluded here because its coordinate sum is less than 4.

18. If A and B are two non-zero square matrices of same order such that $(A + B)^2 = A^2 + B^2$, then: [1 mark]

- (A) $AB = O$
- (B) $AB = -BA$
- (C) $BA = O$
- (D) $AB = BA$

Correct Answer: (B) $AB = -BA$

Solution:

Concept:

- Matrix multiplication need not commute: generally $AB \neq BA$.
- Distributivity gives $(A + B)^2 = A^2 + AB + BA + B^2$.

Step 1: Expand without assuming commutativity The given equality yields

$$A^2 + AB + BA + B^2 = A^2 + B^2.$$

Subtracting the common terms from both sides gives $AB + BA = O$, where O is the zero matrix.

Step 2: Compare the proposed conditions Thus $AB = -BA$, which is option (B). Neither $AB = O$ nor $BA = O$ is forced. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have nonzero products with $AB = -BA$. They also show that $AB = BA$ is not a necessary conclusion.

Step 3: Conclusion $\boxed{AB = -BA}$. Hence, the correct option is (B).

Quick Tip to Solve: Expand a matrix square as four ordered products; do not replace the cross terms by twice one product.

19. Assertion (A): For matrix $A = \begin{pmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{pmatrix}$, where $\theta \in [0, 2\pi]$, $|A| \in [2, 4]$.

Reason (R): $\cos \theta \in [-1, 1]$, $\forall \theta \in [0, 2\pi]$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- Expand a determinant along a row using alternating cofactor signs.
- If $-1 \leq u \leq 1$, then $0 \leq u^2 \leq 1$.

Step 1: Judge the assertion Put $u = \cos \theta$. Expansion along the first row gives

$$\begin{aligned} |A| &= 1(1 + u^2) - u(-u + u) + 1(u^2 + 1) \\ &= 2 + 2u^2. \end{aligned}$$

Since $0 \leq \cos^2 \theta \leq 1$, the determinant lies in $[2, 4]$. Both endpoints are attained, so Assertion (A) is true.

Step 2: Judge the reason For every real angle, cosine lies between -1 and 1 . In particular, $\cos \theta \in [-1, 1]$ for all $\theta \in [0, 2\pi]$. Thus Reason (R) is true.

Step 3: Decide the explanatory link The stated cosine range implies $\cos^2 \theta \in [0, 1]$; substituting in $|A| = 2 + 2\cos^2 \theta$ produces precisely the asserted determinant range. Therefore the reason correctly explains the assertion. This rules out (B), while (C) and (D) incorrectly declare a true statement false.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Simplify the determinant using one symbol for cosine, then apply the range to its square rather than to cosine itself.

20. Assertion (A): A line in space cannot be drawn perpendicular to x , y and z axes simultaneously.

Reason (R): For any line making angles α, β, γ with the positive directions of x , y and z axes respectively, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The direction cosines $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$ of a line satisfy $l^2 + m^2 + n^2 = 1$.
- Perpendicular directions make an angle of 90° , whose cosine is zero.

Step 1: Judge the assertion If a line were perpendicular to all three coordinate axes, all its direction cosines would be zero. Its direction vector would then have zero component in every coordinate direction, so it could not define a line. Thus Assertion (A) is true.

Step 2: Judge the reason For any nonzero direction vector (a, b, c) , divide its components by $\sqrt{a^2 + b^2 + c^2}$. The resulting direction cosines satisfy

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = 1.$$

Reason (R) is therefore true.

Step 3: Use the reason to explain the assertion Simultaneous perpendicularity would give $\alpha = \beta = \gamma = 90^\circ$, making the sum in the reason equal to zero. This contradicts its required value 1. The reason therefore explains why such a line cannot exist; options (B), (C) and (D) do not describe the truth and explanatory relation correctly.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Test simultaneous perpendicularity by setting all three direction cosines to zero and checking their squared-sum identity.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Check whether the function $f(x) = x^2|x|$ is differentiable at $x = 0$ or not.

[2 marks]

Solution:

Concept:

- A function is differentiable at 0 when $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ exists as a finite value.
- The modulus satisfies $|h| = h$ for $h > 0$ and $|h| = -h$ for $h < 0$. Equal one-sided derivative limits establish differentiability.

Step 1: Form the difference quotient Here $f(0) = 0^2|0| = 0$. For $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \frac{h^2|h|}{h} = h|h|.$$

We must evaluate this quotient from both sides; the mere presence of a modulus does not establish failure of differentiability.

Step 2: Compute the right-hand derivative For $h > 0$, $|h| = h$. Hence

$$f'_+(0) = \lim_{h \rightarrow 0^+} h^2 = 0.$$

Equivalently, on the nonnegative side the function is x^3 , whose slope tends to zero at the origin.

Step 3: Compute the left-hand derivative and compare For $h < 0$, $|h| = -h$. Hence

$$f'_-(0) = \lim_{h \rightarrow 0^-} (-h^2) = 0.$$

On the negative side the function is $-x^3$, also approaching the origin with slope zero. The two one-sided limits are equal and finite, so the two-sided derivative exists. Also $f(x) \rightarrow 0 = f(0)$, consistent with the continuity required by differentiability.

Step 4: Conclusion The function $f(x) = x^2|x|$ is differentiable at $x = 0$, with $f'(0) = 0$.

Quick Tip to Solve: Use the difference quotient at the modulus junction; the factor x squared makes both one-sided derivative limits zero.

OR

21(b). If $y = \sqrt{\tan \sqrt{x}}$, prove that $\sqrt{x} \frac{dy}{dx} = \frac{1+y^4}{4y}$.

[2 marks]

Solution:

Concept:

- The chain rule differentiates a composite function one layer at a time: $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$.

Quick Tip to Solve: Square the outer radical first, then replace $\tan \sqrt{x}$ by y^2 after differentiating.

- The identity $\sec^2 u = 1 + \tan^2 u$ converts the trigonometric expression into an expression in y .

Step 1: Square the given relation From $y = \sqrt{\tan \sqrt{x}}$, we have $y^2 = \tan \sqrt{x}$. Work at points of the real domain where the displayed derivative and the requested expression are defined, so $x > 0$, $y > 0$, and $\tan \sqrt{x}$ is finite. Squaring removes the outer radical and makes implicit differentiation convenient.

Step 2: Differentiate and use the identity Differentiating both sides with respect to x ,

$$2y \frac{dy}{dx} = \sec^2 \sqrt{x} \frac{1}{2\sqrt{x}}.$$

Multiplying by $\sqrt{x}$ and dividing by $2y$ gives

$$\sqrt{x} \frac{dy}{dx} = \frac{\sec^2 \sqrt{x}}{4y}.$$

Since $\tan \sqrt{x} = y^2$, we obtain $\sec^2 \sqrt{x} = 1 + \tan^2 \sqrt{x} = 1 + y^4$.

Step 3: Conclusion Therefore, $\sqrt{x} \frac{dy}{dx} = \frac{1 + y^4}{4y}$, as required.

Quick Tip to Solve: Square the outer radical first, then replace $\tan \sqrt{x}$ by y^2 after differentiating.

22. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima. [2 marks]

Solution:

Concept:

- A differentiable function has a local extremum at an interior stationary point only if its derivative changes sign appropriately.
- If $f'(x) > 0$ on both sides of a stationary point, the function continues to increase through that point.

Step 1: Find the stationary point For $f(x) = 4x^3 - 18x^2 + 27x - 7$, differentiate:

$$f'(x) = 12x^2 - 36x + 27 = 3(2x - 3)^2.$$

Thus $f'(x) = 0$ only at $x = \frac{3}{2}$. At every other real value of x , the squared factor is strictly positive, so $f'(x) > 0$.

Step 2: Check the sign on both sides For $x < \frac{3}{2}$, $f'(x) > 0$; for $x > \frac{3}{2}$, it is again positive. Hence the function increases both before and after $x = \frac{3}{2}$; the derivative has no sign change there. A local maximum would require a change from positive to negative, while a local minimum would require negative to positive. Neither occurs. Since a polynomial is differentiable everywhere, there are no other possible interior extremum points on its real domain.

Step 3: Conclusion Therefore, $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither a local maximum nor a local minimum.

Quick Tip to Solve: Factor the first derivative into $3(2x - 3)^2$; its unchanged positive sign settles the question.

23(a). Find: $\int x \sqrt{1+2x} dx$

[2 marks]

Solution:

Concept:

- For substitution $u = 1 + 2x$, use $x = (u - 1)/2$ and $dx = du/2$ together.
- The power rule is $\int u^n du = u^{n+1}/(n+1) + C$, for $n \neq -1$; C is an arbitrary constant.

Step 1: Change the variable completely Let $I = \int x \sqrt{1+2x} dx$. Set $u = 1 + 2x$. Then $du = 2 dx$, $dx = \frac{1}{2} du$, and $x = \frac{u-1}{2}$. Substituting each occurrence of x and dx ,

$$I = \frac{1}{4} \int (u-1)u^{1/2} du = \frac{1}{4} \int (u^{3/2} - u^{1/2}) du.$$

The factor $1/4$ includes one factor $1/2$ from x and another from dx .

Step 2: Integrate and substitute back Applying the power rule to both terms,

$$I = \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C.$$

Returning to $u = 1 + 2x$,

$$I = \frac{(1+2x)^{5/2}}{10} - \frac{(1+2x)^{3/2}}{6} + C.$$

Factoring $(1+2x)^{3/2}$, the remaining factor is $\frac{1+2x}{10} - \frac{1}{6} = \frac{3x-1}{15}$.

Step 3: Conclusion

$$\int x \sqrt{1+2x} dx = \frac{(3x-1)(1+2x)^{3/2}}{15} + C.$$

Quick Tip to Solve: When using $u = 1 + 2x$, replace the extra factor x as well as the radical and differential.

OR

23(b). Evaluate: $\int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

[2 marks]

Solution:

Concept:

- Use $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t dt$; this cancels the square root in the denominator.
- In a definite integral, transform both limits along with the variable. Also $\int \sin t dt = -\cos t$.

Step 1: Substitute and change the limits Let $I = \int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$. Set $t = \sqrt{x}$, giving $dx = 2t dt$. The lower limit $x = 0$ becomes $t = 0$, and the upper limit $x = \pi^2/4$ becomes $t = \pi/2$. For $t > 0$, the factor t cancels:

$$I = 2 \int_0^{\pi/2} \sin t dt.$$

Quick Tip to Solve: Set $t = \sqrt{x}$ and immediately convert the upper limit to $\pi/2$ to avoid mixing variables.

At the original lower endpoint, $\sin \sqrt{x}/\sqrt{x}$ has limiting value 1, so its removable endpoint singularity does not affect the value of the integral.

Step 2: Evaluate the antiderivative at the limits Integrate the transformed expression:

$$I = 2[-\cos t]_0^{\pi/2} = 2\left(-\cos \frac{\pi}{2} + \cos 0\right).$$

Since $\cos(\pi/2) = 0$ and $\cos 0 = 1$, the bracket equals 1. Both limits have already been expressed in t , so the evaluation is performed directly without substituting back to x .

Step 3: Conclusion

$$\int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx = 2.$$

Quick Tip to Solve: Set $t = \sqrt{x}$ and immediately convert the upper limit to $\pi/2$ to avoid mixing variables.

24. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}|\vec{a}|$. [2 marks]

Solution:

Concept:

- For perpendicular vectors $\vec{u}$ and $\vec{v}$, their scalar product satisfies $\vec{u} \cdot \vec{v} = 0$.
- The scalar product is distributive, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Step 1: Use the first perpendicularity condition Since $(\vec{a} + \vec{b}) \perp \vec{a}$, take the scalar product with $\vec{a}$:

$$(\vec{a} + \vec{b}) \cdot \vec{a} = 0.$$

By distributivity,

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} = 0, \quad \vec{a} \cdot \vec{b} = -|\vec{a}|^2.$$

This expresses the mixed scalar product entirely in terms of the magnitude of $\vec{a}$.

Step 2: Use the second condition and eliminate the mixed product The condition $(2\vec{a} + \vec{b}) \perp \vec{b}$ gives

$$(2\vec{a} + \vec{b}) \cdot \vec{b} = 0, \quad 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 0.$$

Substitute $\vec{a} \cdot \vec{b} = -|\vec{a}|^2$ from the first condition:

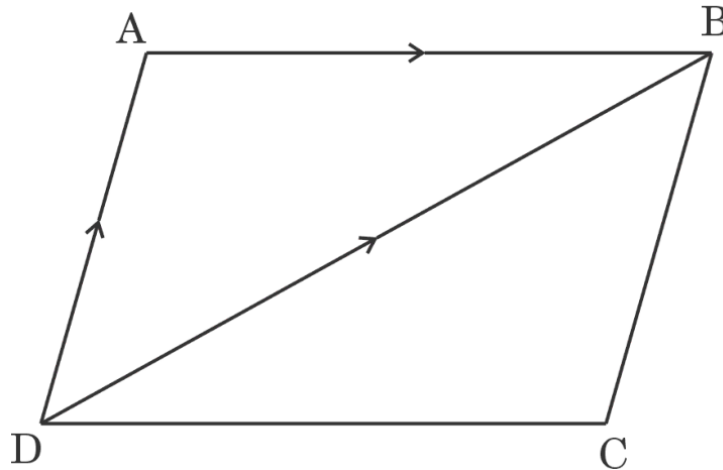
$$-2|\vec{a}|^2 + |\vec{b}|^2 = 0.$$

Thus $|\vec{b}|^2 = 2|\vec{a}|^2$. Magnitudes are non-negative, and the given vectors are non-zero, so taking the positive square root is justified.

Step 3: Conclusion Hence, $|\vec{b}| = \sqrt{2}|\vec{a}|$, as required.

Quick Tip to Solve: Turn each perpendicularity statement into a zero scalar product, then eliminate $\vec{a} \cdot \vec{b}$.

25. In the given figure, $ABCD$ is a parallelogram. If $\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\overrightarrow{AD}$ and hence find the area of parallelogram $ABCD$. [2 marks]



Solution:

Concept:

- For successive directed segments, $\overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB}$. Reversing a vector changes its sign.
- The area of a parallelogram with adjacent side vectors $\vec{u}, \vec{v}$ is $|\vec{u} \times \vec{v}|$.

Step 1: Find the adjacent side vector The diagonal is directed from D to B . Thus $-\overrightarrow{AD} + \overrightarrow{AB} = \overrightarrow{DB}$, and hence

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} - \overrightarrow{DB} \\ &= (2 - 3)\hat{i} + (-4 + 6)\hat{j} + (5 - 2)\hat{k} \\ &= -\hat{i} + 2\hat{j} + 3\hat{k}.\end{aligned}$$

Using the sum instead would give the wrong direction for the side.

Step 2: Calculate the cross product and its magnitude The vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ have the common initial point A , so they are the required adjacent sides.

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ -1 & 2 & 3 \end{vmatrix} \\ &= (-12 - 10)\hat{i} - (6 + 5)\hat{j} + (4 - 4)\hat{k} \\ &= -22\hat{i} - 11\hat{j}.\end{aligned}$$

Therefore the area is $\sqrt{(-22)^2 + (-11)^2} = \sqrt{605} = 11\sqrt{5}$ square units. The cross-product magnitude, rather than the scalar product, measures the area.

Step 3: Conclusion $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 3\hat{k}$, and the area of $ABCD$ is $11\sqrt{5}$ square units.

Quick Tip to Solve: Follow the directed path D to A to B before using the cross product; it fixes the subtraction sign.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as $R = \{(x, y) : |x^2 - y^2| < 8\}$. Check whether the relation R is reflexive, symmetric and transitive. [3 marks]

Solution:

Concept:

- A relation on A is reflexive when $(x, x) \in R$ for every $x \in A$, and symmetric when $(x, y) \in R$ implies $(y, x) \in R$.
- Transitivity requires $(x, y), (y, z) \in R$ to imply $(x, z) \in R$ for every triple in A . One counterexample disproves it.

Step 1: Check reflexivity For every $x \in A = \{1, 2, 3, 4, 5\}$,

$$|x^2 - x^2| = 0 < 8.$$

Thus all five diagonal pairs $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ belong to R . The condition holds for every member of the set, so R is reflexive.

Step 2: Check symmetry Let $(x, y) \in R$. By definition, $|x^2 - y^2| < 8$. Absolute value is unchanged when the sign of its argument is reversed, so

$$|y^2 - x^2| = |-(x^2 - y^2)| = |x^2 - y^2| < 8.$$

Consequently $(y, x) \in R$. This argument applies to every related pair, and proves that R is symmetric.

Step 3: Test transitivity with a triple Take $x = 1, y = 2, z = 3$, all of which lie in A . We have

$$|1^2 - 2^2| = 3 < 8, \quad |2^2 - 3^2| = 5 < 8.$$

Hence $(1, 2) \in R$ and $(2, 3) \in R$. However,

$$|1^2 - 3^2| = 8,$$

which does not satisfy the strict inequality < 8 . Thus $(1, 3) \notin R$, although the two successive pairs are related. This violates transitivity. In particular, replacing the strict inequality by ≤ 8 would invalidate this counterexample, so the boundary matters.

Step 4: Conclusion The relation R is reflexive and symmetric, but not transitive. Therefore it is not an equivalence relation.

Quick Tip to Solve: Try the consecutive triple 1, 2, 3: the squared differences are 3, 5 and 8, exposing the strict boundary.

26(b). A function f is defined from $\mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find function $f(x)$. Hence, check whether function $f(x)$ is one-one and onto or not. [3 marks]

Solution:

Concept:

- The values of a linear function $f(x) = ax + b$ at two distinct inputs determine a and b .
- One-one means equal outputs force equal inputs. Onto $\mathbb{R}$ means every real target y has a real preimage x satisfying $f(x) = y$.

Step 1: Determine the function Using the two supplied values gives

$$a + b = 1, \quad 2a + b = 3.$$

Subtracting the first equation from the second yields $a = 2$; substituting this into $a + b = 1$ gives $b = -1$. Thus $f(x) = 2x - 1$. A check gives $f(1) = 2 - 1 = 1$ and $f(2) = 4 - 1 = 3$, as required.

Step 2: Prove that the function is one-one Let $x_1, x_2 \in \mathbb{R}$ and suppose their images are equal. Then

$$\begin{aligned} f(x_1) &= f(x_2) \\ 2x_1 - 1 &= 2x_2 - 1 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2. \end{aligned}$$

Thus no two distinct real inputs have the same output, so f is one-one. Division by 2 is legitimate because the slope is nonzero.

Step 3: Prove that the function is onto Choose an arbitrary real number y in the codomain. Solving $2x - 1 = y$ gives

$$x = \frac{y + 1}{2}.$$

Since y is real, this chosen x is also real and belongs to the domain. Moreover,

$$f\left(\frac{y + 1}{2}\right) = 2\left(\frac{y + 1}{2}\right) - 1 = y.$$

Every real target therefore has a preimage, proving surjectivity onto the specified codomain $\mathbb{R}$. The domain and codomain are essential to this check.

Step 4: Conclusion $f(x) = 2x - 1$. The function is both one-one and onto, hence bijective.

Quick Tip to Solve: For onto, start with an arbitrary target y and exhibit the real preimage $(y+1)/2$ explicitly.

27(a). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[3 marks]

Solution:

Concept:

- Implicit differentiation uses $\frac{d}{dx}\sqrt{1-y^2} = -\frac{y}{\sqrt{1-y^2}}\frac{dy}{dx}$. The parameter a is constant.
- The stated derivative is considered on a differentiable branch where $|x| < 1$ and $|y| < 1$; here both square roots are positive.

Step 1: Differentiate the given equation Write $U = \sqrt{1-x^2}$, $V = \sqrt{1-y^2}$ and $y' = dy/dx$. The given equation becomes $U + V = a(x-y)$. Differentiating with respect to x , including the chain rule for the term containing y , gives

$$-\frac{x}{U} - \frac{y}{V}y' = a(1-y').$$

Collecting the derivative terms on one side yields

$$\left(a - \frac{y}{V}\right)y' = a + \frac{x}{U}.$$

We must simplify this expression using the original equation, since a, x, y are linked by that equation.

Step 2: Eliminate the constant and simplify Since $U + V > 0$, the original equation implies $x - y \neq 0$. Therefore $a = (U + V)/(x - y)$. Using $U^2 = 1 - x^2$ and $V^2 = 1 - y^2$, we obtain

$$\begin{aligned} a + \frac{x}{U} &= \frac{U(U + V) + x(x - y)}{U(x - y)} \\ &= \frac{1 - xy + UV}{U(x - y)}, \\ a - \frac{y}{V} &= \frac{V(U + V) - y(x - y)}{V(x - y)} \\ &= \frac{1 - xy + UV}{V(x - y)}. \end{aligned}$$

For interior points $|xy| < 1$, so $1 - xy + UV > 0$; the common numerator is nonzero. The second expression is therefore nonzero and division is valid. Substituting into the differentiated equation gives

$$y' = \frac{(1 - xy + UV)/(U(x - y))}{(1 - xy + UV)/(V(x - y))} = \frac{V}{U}.$$

Both U and V are positive, so their ratio is the positive square root of $(1 - y^2)/(1 - x^2)$.

Step 3: Conclusion Hence, wherever the stated derivative is defined on the differentiable branch,

$$\boxed{\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}}.$$

Quick Tip to Solve: Eliminate a with the original equation; both derivative coefficients then share the factor $1 - xy + UV$.

27(b). If $y = (\tan x)^x$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- When both the base and exponent depend on x , logarithmic differentiation converts the power into a product. For $u > 0$, $\log(u^v) = v \log u$.
- The product rule is $(uv)' = u'v + uv'$, and the chain rule gives $\frac{d}{dx} \log u = \frac{u'}{u}$. Here $\frac{d}{dx} \tan x = \sec^2 x$.

Step 1: Take logarithms Work on an interval where $\tan x > 0$, so that the given power and its natural logarithm are real and differentiable. Since $y = (\tan x)^x$, taking natural logarithms on both sides gives

$$\log y = x \log(\tan x).$$

This step is useful because the variable exponent x is now a factor. The ordinary power rule alone cannot differentiate the original expression: the base $\tan x$ also varies with x .

Step 2: Differentiate the product Differentiating both sides with respect to x , use the chain rule on the left and the product rule on the right:

$$\frac{1}{y} \frac{dy}{dx} = 1 \cdot \log(\tan x) + x \frac{1}{\tan x} \sec^2 x.$$

The first term comes from differentiating the exponent factor x ; the second comes from differentiating $\log(\tan x)$. Both terms must be retained.

Step 3: Substitute the original function Multiply by y and substitute $y = (\tan x)^x$. The trigonometric factor can also be written as

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x}.$$

Either form gives the same derivative on the stated intervals, where the denominators are nonzero.

Step 4: Conclusion

$$\frac{dy}{dx} = (\tan x)^x \left[\log(\tan x) + \frac{x \sec^2 x}{\tan x} \right].$$

Quick Tip to Solve: For a variable base raised to a variable exponent, take logarithms first and retain both product-rule terms.

28(a). Find: $\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$

[3 marks]

Solution:

Concept:

- A rational function with distinct quadratic factors can be decomposed into simpler fractions by equating coefficients after multiplying through by the denominator.
- For $a > 0$, the standard integral is $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$, where C is a constant.

Step 1: Decompose the rational function Because the numerator and both factors are even functions of x , try constant numerators:

$$\frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{A}{x^2 + 4} + \frac{B}{x^2 + 9}.$$

Multiplying by $(x^2 + 4)(x^2 + 9)$ gives $x^2 = A(x^2 + 9) + B(x^2 + 4)$. Equating the coefficient of x^2 and the constant term,

$$A + B = 1, \quad 9A + 4B = 0.$$

Subtracting four times the first equation from the second yields $5A = -4$; hence $A = -4/5$ and $B = 9/5$.

Step 2: Integrate the two standard terms Using the decomposition and the standard inverse-tangent integral,

$$\begin{aligned} I &= -\frac{4}{5} \int \frac{dx}{x^2 + 2^2} + \frac{9}{5} \int \frac{dx}{x^2 + 3^2} \\ &= -\frac{4}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C. \end{aligned}$$

The factors $1/2$ and $1/3$ are essential: they arise from the scale in each quadratic denominator.

Step 3: Verify the coefficients Differentiating the two inverse-tangent terms gives $-4/[5(x^2 + 4)] + 9/[5(x^2 + 9)]$. On a common denominator its numerator is $\{-4(x^2 + 9) + 9(x^2 + 4)\}/5 = x^2$, confirming the original integrand. Only one arbitrary constant is needed for the entire antiderivative.

Step 4: Conclusion

$$I = -\frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5} \tan^{-1}\left(\frac{x}{3}\right) + C.$$

Quick Tip to Solve: Equate the constant term as well as the coefficient of x squared, then include the inverse-tangent scale factors.

28(b). Evaluate: $\int_1^3 (|x-1| + |x-2| + |x-3|) dx$

[3 marks]

Solution:

Concept:

- The modulus satisfies $|u| = u$ for $u \geq 0$ and $|u| = -u$ for $u < 0$. Split an integral at any interior point where a modulus argument changes sign.
- Definite integrals are additive across a split point: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

Step 1: Identify the signs on the interval On the complete interval $[1, 3]$, $x-1 \geq 0$ and $x-3 \leq 0$. Therefore $|x-1| = x-1$ and $|x-3| = 3-x$; their sum is the constant 2. The middle term $|x-2|$ changes its expression at $x=2$, which lies inside the interval, so this is the necessary split point. At the endpoints the zero arguments cause no difficulty.

Step 2: Remove the modulus on each part For $1 \leq x \leq 2$, $|x-2| = 2-x$, giving the total integrand $4-x$. For $2 \leq x \leq 3$, $|x-2| = x-2$, giving the total integrand x . Thus

$$I = \int_1^2 (4-x) dx + \int_2^3 x dx.$$

Both expressions agree at $x=2$, where the total is 2, so the split correctly represents the continuous original integrand.

Step 3: Evaluate both contributions Apply the limits to each elementary antiderivative:

$$\begin{aligned} I &= \left[4x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} \right]_2^3 \\ &= \left(8 - 2 - 4 + \frac{1}{2} \right) + \left(\frac{9}{2} - 2 \right) \\ &= \frac{5}{2} + \frac{5}{2}. \end{aligned}$$

As a check, the constant part 2 contributes 4 over an interval of length 2, while $\int_1^3 |x-2| dx = 1$; these contributions reproduce the calculation.

Step 4: Conclusion

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx = 5.$$

Quick Tip to Solve: Combine the two outer modulus terms into the constant 2 before splitting only the middle term at x equals 2.

29. Find the particular solution of the differential equation given by $x^2 \frac{dy}{dx} - xy = x^2 \cos^2\left(\frac{y}{2x}\right)$, given that when $x = 1, y = \frac{\pi}{2}$. [3 marks]

Solution:

Concept:

- A homogeneous differential equation involving y/x is reduced by setting $v = y/x$, so that $y = xv$ and $dy/dx = v + x dv/dx$.
- After separating variables, use $\int \sec^2(v/2) dv = 2 \tan(v/2) + C$. The initial condition determines the integration constant.

Step 1: Reduce the homogeneous equation For $x \neq 0$, divide the given equation by x^2 to obtain

$$\frac{dy}{dx} - \frac{y}{x} = \cos^2\left(\frac{y}{2x}\right).$$

Put $y = xv$. Substituting the derivative and cancelling v gives

$$x \frac{dv}{dx} = \cos^2(v/2).$$

The argument is $v/2$, which must be retained when integrating.

Step 2: Separate and integrate Near the initial point $v = \pi/2$, the cosine is nonzero, so division is valid. Thus

$$\sec^2(v/2) dv = \frac{dx}{x}.$$

Integrating both sides gives

$$2 \tan(v/2) = \ln|x| + C.$$

The factor 2 on the left follows from the chain rule. The solution interval containing $x = 1$ lies in $x > 0$, so $\ln|x| = \ln x$ there.

Step 3: Use the initial condition and verify At $x = 1, v = y/x = \pi/2$, giving $2 \tan(\pi/4) = 0 + C$, hence $C = 2$. Therefore

$$2 \tan\left(\frac{y}{2x}\right) = \ln x + 2.$$

Taking the continuous branch through the initial point gives $y = 2x \tan^{-1}(1 + \frac{1}{2} \ln x)$. Indeed, for this expression $v = 2 \tan^{-1}(1 + \frac{1}{2} \ln x)$, so $x \frac{dv}{dx} = 1/[1 + (1 + \frac{1}{2} \ln x)^2] = \cos^2(v/2)$, verifying the reduced equation.

Step 4: Conclusion The particular solution is

$$y = 2x \tan^{-1}\left(1 + \frac{1}{2} \ln x\right), \quad x > 0.$$

Equivalently, $2 \tan(y/(2x)) = \ln x + 2$, on the branch containing the initial point.

Quick Tip to Solve: Use $y=xv$ first, and preserve the 2 in the cosine argument: integrating secant squared of $v/2$ produces $2 \tan(v/2)$.

30. Solve the following linear programming problem graphically:

Maximise $z = 500x + 300y$, **subject to constraints**

$$x + 2y \leq 12$$

$$2x + y \leq 12$$

$$4x + 5y \geq 20$$

$$x \geq 0, y \geq 0$$

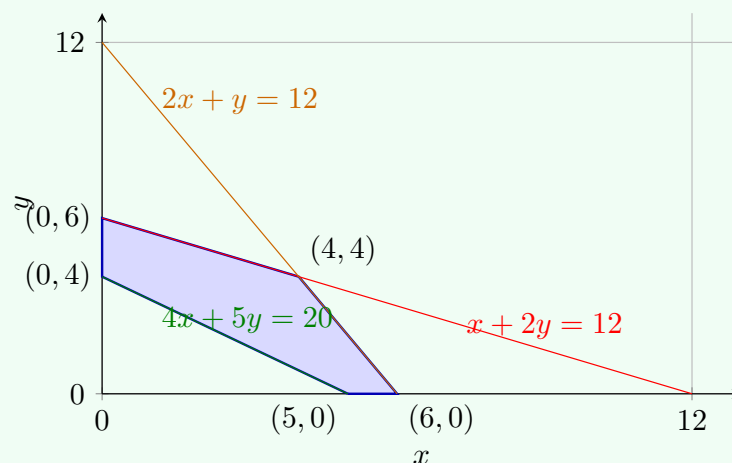
[3 marks]

Solution:

Concept:

- Each linear inequality represents a closed half-plane. Their common region in the first quadrant is the feasible region.
- For a nonempty bounded polygon, a linear objective function attains its maximum at a corner point. Evaluate $z = 500x + 300y$ at every feasible vertex.

Step 1: Plot the boundary lines and choose half-planes The intercepts of $x + 2y = 12$ are $(12, 0)$ and $(0, 6)$; those of $2x + y = 12$ are $(6, 0)$ and $(0, 12)$. Both inequalities select the sides containing the origin. The line $4x + 5y = 20$ has intercepts $(5, 0)$ and $(0, 4)$, and its inequality selects the side away from the origin. Restrict to $x \geq 0$ and $y \geq 0$. The shaded polygon in the graph is the common feasible region.



Feasible polygon and the three constraint boundaries

Step 2: Identify the feasible corner points On the x -axis, the feasible interval is $5 \leq x \leq 6$, giving $(5, 0)$ and $(6, 0)$. On the y -axis, it is $4 \leq y \leq 6$, giving $(0, 4)$ and $(0, 6)$. Solving $x + 2y = 12$ and $2x + y = 12$ gives $x = y = 4$. This intersection satisfies $4x + 5y = 36 \geq 20$. Thus all five vertices are $(5, 0)$, $(6, 0)$, $(4, 4)$, $(0, 6)$, $(0, 4)$. The lower boundary meets the upper boundaries outside the first quadrant, so it adds no further feasible vertex.

Step 3: Evaluate the objective at all vertices The feasible polygon is closed and bounded, so the

Quick Tip to Solve: Shade the side away from the origin for $4x + 5y \geq 20$, then test all five corners of the remaining first-quadrant polygon.

corner-point method applies. The objective values are

(x, y)	$z = 500x + 300y$
$(5, 0)$	2500
$(6, 0)$	3000
$(4, 4)$	3200
$(0, 6)$	1800
$(0, 4)$	1200

The largest value in this complete list is 3200. At its vertex, the first two constraints hold as equalities and the third holds as $36 \geq 20$.

Step 4: Conclusion The maximum is $z_{\max} = 3200$, attained at $x = 4, y = 4$.

Quick Tip to Solve: Shade the side away from the origin for $4x + 5y \geq 20$, then test all five corners of the remaining first-quadrant polygon.

31. E and F are two independent events such that $P(\bar{E}) = 0.6$ and $P(E \cup F) = 0.6$. Find $P(F)$ and $P(\bar{E} \cup \bar{F})$. [3 marks]

Solution:

Concept:

- For any event E , $P(E) = 1 - P(\bar{E})$. For independent events, $P(E \cap F) = P(E)P(F)$.
- The addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. De Morgan's law gives $\bar{E} \cup \bar{F} = \overline{E \cap F}$.

Step 1: Obtain the probability of E The given value is the probability of the complement of E . Therefore

$$P(E) = 1 - P(\bar{E}) = 1 - 0.6 = 0.4.$$

Write $p = P(F)$. Since the events are independent, their intersection has probability

$$P(E \cap F) = 0.4p.$$

Independence is essential for this multiplication; the events need not be mutually exclusive.

Step 2: Find the probability of F Apply the addition rule to the given union:

$$0.6 = 0.4 + p - 0.4p,$$

$$0.2 = 0.6p,$$

$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

This lies between 0 and 1, as a probability must. A check gives $0.4 + 1/3 - (0.4)(1/3) = 3/5 = 0.6$, agreeing with the printed condition.

Step 3: Find the union of the complements An outcome belongs to $\bar{E} \cup \bar{F}$ unless it belongs to both E

Quick Tip to Solve: Convert the given complement probability to $P(E)$ before using independence; for the requested union of complements, subtract $P(E \cap F)$ from 1.

and F . By De Morgan's law,

$$\begin{aligned}P(\overline{E} \cup \overline{F}) &= 1 - P(E \cap F) \\&= 1 - \frac{2}{5} \cdot \frac{1}{3} \\&= 1 - \frac{2}{15} = \frac{13}{15}.\end{aligned}$$

Notice that the complement of an intersection is a union of complements; using $1 - P(E \cup F)$ would instead give the intersection of the complements.

Step 4: Conclusion

$$P(F) = \frac{1}{3}, \quad P(\overline{E} \cup \overline{F}) = \frac{13}{15}.$$

Quick Tip to Solve: Convert the given complement probability to $P(E)$ before using independence; for the requested union of complements, subtract $P(E \text{ intersection } F)$ from 1.

Section D (20 marks)

This section comprises long answer type questions (LA) of 5 marks each.

32(a). If $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix}$, find A^{-1} and use it to solve the following system of equations: $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$. [5 marks]

Solution:

Concept:

- For an invertible square matrix, $A^{-1} = \text{adj } A / |A|$. The adjugate is the transpose of the cofactor matrix.
- The cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$, where the minor M_{ij} is obtained by deleting row i and column j .
- Writing a linear system as $AX = B$ gives $X = A^{-1}B$, where X is the column of unknowns and B is the column of constants.

Step 1: Check invertibility Expand along the first row, remembering the minus sign in the second cofactor:

$$|A| = 1 \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = (-1 - 2) + 2(2) = 1.$$

Since this determinant is nonzero, the inverse exists and the system has a unique solution. Division by the determinant will leave the entries of the adjugate unchanged.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constants column, and check the resulting values in each original equation.

Step 2: Find the cofactors and inverse The cofactors, including their alternating signs, are

$$\begin{aligned}C_{11} &= -3, & C_{12} &= -2, & C_{13} &= -4, \\C_{21} &= 2, & C_{22} &= 1, & C_{23} &= 2, \\C_{31} &= 2, & C_{32} &= 1, & C_{33} &= 3.\end{aligned}$$

For example, $C_{12} = -\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = -2$, whereas $C_{33} = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -1 + 4 = 3$. Transpose the cofactor matrix to obtain

$$\text{adj } A = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

Thus $A^{-1} = \text{adj } A$. Multiplication of this matrix by A gives the identity matrix; this also checks the cofactor signs and the required transposition.

Step 3: Use the inverse to solve the system The coefficients occur in exactly the same order as the rows of A , hence

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}.$$

Multiplying on the left by the inverse gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}.$$

To verify directly, the original left sides become $0 - 2(-5) = 10$, $2(0) - (-5) - (-3) = 8$, and $-2(-5) + (-3) = 7$. All three constants are recovered, confirming the computed solution.

Step 4: Conclusion

$$A^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}, \quad \boxed{x = 0, y = -5, z = -3}.$$

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constants column, and check the resulting values in each original equation.

32(b). If $A = \begin{pmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{pmatrix}$, find the value of $(a + x) - (b + y)$. [5 marks]

Solution:

Concept:

- An inverse matrix satisfies $AA^{-1} = I$, where I has diagonal entries 1 and all other entries 0.
- An entry of a matrix product is the dot product of the corresponding row of the first matrix with the corresponding column of the second. Choose entries that involve only one unknown.

Step 1: Find b and y from the third row The third row of A , namely $(3, 1, 1)$, is completely known. The first two entries of the third row of AA^{-1} must both be zero. Consequently

$$3(1) + 1(-8) + 1(b) = 0, \quad 3(-1) + 1(7) + 1(y) = 0.$$

The first equation gives $b = 5$, and the second gives $y = -4$. The third entry provides an independent consistency check: $3(1) + 1(-5) + 1(3) = 1$, as required for the diagonal entry of the identity. Thus the entire third row of the product is correct.

Step 2: Find a and x using the third column The third column of the given inverse, $(1, -5, 3)^T$, is completely known. Its product with the first row of A gives the $(1, 3)$ entry, which is zero:

$$-1(1) + a(-5) + 2(3) = 0.$$

Therefore $-1 - 5a + 6 = 0$, so $a = 1$. The $(2, 3)$ entry is also zero, giving

$$1(1) + 2(-5) + x(3) = 0.$$

Hence $1 - 10 + 3x = 0$, and $x = 3$. Using off-diagonal entries avoids simultaneous equations involving products of two unknowns.

Step 3: Verify the matrices and evaluate the expression Substitution gives

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{pmatrix}.$$

The other entries of the product check as follows: the first row begins with $-1 - 8 + 10 = 1$ and $1 + 7 - 8 = 0$; the second row begins with $1 - 16 + 15 = 0$ and $-1 + 14 - 12 = 1$. Hence $AA^{-1} = I$. Substituting into the required expression:

$$(a + x) - (b + y) = (1 + 3) - \{5 + (-4)\} = 4 - 1.$$

Step 4: Conclusion

$$(a + x) - (b + y) = 3.$$

Quick Tip to Solve: Use the known third row to obtain b and y , then the known third column to obtain a and x ; retain parentheses around the negative y .

33(a). Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

[5 marks]

Solution:

Concept:

- In a definite integral, substitution changes both the integrand and its limits. Choose a function whose derivative matches the numerator.
- The identity $(\sin x - \cos x)^2 = 1 - \sin 2x$ converts the denominator into a quadratic in the new variable.
- Factor a difference of squares and use partial fractions; integrating the reciprocal of a linear function gives a logarithm.

Step 1: Choose a substitution using the numerator Let the required integral be I . The derivative of $\sin x - \cos x$ matches the numerator exactly. Therefore set

$$t = \sin x - \cos x, \quad dt = (\sin x + \cos x) dx.$$

Squaring and using $\sin^2 x + \cos^2 x = 1$ gives

$$t^2 = 1 - 2 \sin x \cos x = 1 - \sin 2x.$$

Hence $\sin 2x = 1 - t^2$, and the denominator becomes $9 + 16(1 - t^2) = 25 - 16t^2$. Thus both the numerator and the denominator are expressed completely in terms of the new variable, without leaving a trigonometric factor behind.

Step 2: Change the limits and obtain partial fractions At $x = 0$, $t = -1$. At $x = \pi/4$, the sine and cosine are equal, giving $t = 0$. Consequently,

$$I = \int_{-1}^0 \frac{dt}{25 - 16t^2}.$$

Factor the denominator as $(5 + 4t)(5 - 4t)$. The corresponding partial-fraction decomposition is

$$\frac{1}{25 - 16t^2} = \frac{1}{10} \left(\frac{1}{5 + 4t} + \frac{1}{5 - 4t} \right).$$

Indeed, combining these two terms produces the numerator $(5 - 4t) + (5 + 4t) = 10$, cancelling the outside factor $1/10$. Also, $25 - 16t^2 \geq 9$ on the integration interval, so no singularity occurs.

Step 3: Integrate the linear factors and substitute the endpoints Integrating each term, remembering the opposite signs of their inner derivatives, gives

$$\begin{aligned} I &= \frac{1}{40} [\ln(5 + 4t) - \ln(5 - 4t)]_{-1}^0 \\ &= \frac{1}{40} \left[\ln \left(\frac{5 + 4t}{5 - 4t} \right) \right]_{-1}^0. \end{aligned}$$

Both linear factors are positive for $-1 \leq t \leq 0$, which permits the displayed logarithms without

Quick Tip to Solve: For a numerator $\sin x + \cos x$, try $t = \sin x - \cos x$; squaring this substitution also replaces $\sin 2x$.

absolute-value signs. At $t = 0$ the ratio is 1; at $t = -1$ it is $1/9$. Thus the endpoint difference is $\frac{1}{40}(\ln 1 - \ln(1/9)) = \frac{\ln 9}{40}$. Its positive sign also agrees with the positive original integrand.

Step 4: Conclusion The required integral is $\boxed{\frac{\ln 9}{40} = \frac{\ln 3}{20}}$.

Quick Tip to Solve: For a numerator $\sin x + \cos x$, try $t = \sin x - \cos x$; squaring this substitution also replaces $\sin 2x$.

OR

33(b). Evaluate: $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) dx$

[5 marks]

Solution:

Concept:

- The double-angle identity $\sin 2x = 2 \sin x \cos x$ suggests the substitution $t = \sin x$, with $dt = \cos x dx$.
- Integration by parts states $\int u dv = uv - \int v du$. For an inverse tangent factor, its derivative $1/(1+t^2)$ simplifies the remaining integral.
- Divide a rational integrand before integrating: $t^2/(1+t^2) = 1 - 1/(1+t^2)$.

Step 1: Remove the trigonometric factor by substitution Denote the required integral by I . First use the double-angle identity to write

$$I = \int_0^{\pi/2} 2 \sin x \cos x \tan^{-1}(\sin x) dx.$$

Set $t = \sin x$, so that $dt = \cos x dx$. The lower limit becomes $t = 0$, and the upper limit becomes $t = 1$. Hence

$$I = \int_0^1 2t \tan^{-1} t dt.$$

Here $\tan^{-1} t$ denotes the principal inverse tangent. The substitution maps the original interval monotonically onto $[0, 1]$, so the transformed limits are in the same order.

Step 2: Apply integration by parts Choose the inverse tangent as the factor to differentiate, because its derivative is rational. Specifically, take

$$u = \tan^{-1} t, \quad dv = 2t dt, \quad du = \frac{dt}{1+t^2}, \quad v = t^2.$$

The integration-by-parts formula now gives

$$I = [t^2 \tan^{-1} t]_0^1 - \int_0^1 \frac{t^2}{1+t^2} dt.$$

At $t = 1$, $\tan^{-1} 1 = \pi/4$, and at $t = 0$ the product vanishes. Thus the boundary term is $\pi/4$. This step has reduced the inverse-trigonometric product to an elementary rational integral.

Quick Tip to Solve: Replace $\sin 2x dx$ by $2t dt$ using $t = \sin x$, then differentiate the inverse tangent during integration by parts.

Step 3: Evaluate the remaining rational integral Write the numerator as $t^2 = (1 + t^2) - 1$. Therefore

$$\begin{aligned}\int_0^1 \frac{t^2}{1+t^2} dt &= \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt \\ &= [t - \tan^{-1} t]_0^1 \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

Substituting this value into the integration-by-parts expression gives $I = \frac{\pi}{4} - (1 - \frac{\pi}{4})$. The minus sign applies to the entire remaining integral, so the two inverse-tangent contributions add. As a check on the working, differentiating $(t^2 + 1) \tan^{-1} t - t$ gives $2t \tan^{-1} t$, exactly the transformed integrand.

Step 4: Conclusion The required definite integral is $\boxed{\frac{\pi}{2} - 1}$.

Quick Tip to Solve: Replace $\sin 2x dx$ by $2t dt$ using $t = \sin x$, then differentiate the inverse tangent during integration by parts.

34. Using integration, find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$, included between the lines $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- For vertical strips, the area between an upper curve y_u and lower curve y_l is $A = \int_a^b (y_u - y_l) dx$, where a, b are the bounding abscissae.
- The ellipse is symmetric about both coordinate axes. The area sought is four times the first-quadrant area between $x = 0$ and $x = 2$.
- Use $x = 4 \sin \theta$ and $\cos^2 \theta = (1 + \cos 2\theta)/2$ to evaluate the radical integral.

Step 1: Determine the upper and lower ordinates The printed ellipse gives

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}, \quad y = \pm \frac{1}{2} \sqrt{16 - x^2}.$$

The positive root is the upper boundary, and the negative root is the lower boundary. Thus the height of a vertical strip in the central portion is $\sqrt{16 - x^2}$. This height is positive for the entire interval from -2 to 2 , so there is no change of boundaries within the region. The limits must be the given vertical lines, rather than the endpoints -4 and 4 of the whole ellipse.

Step 2: Form the definite integral using symmetry The required area is

$$\begin{aligned}A &= \int_{-2}^2 \left(\frac{1}{2} \sqrt{16 - x^2} + \frac{1}{2} \sqrt{16 - x^2} \right) dx \\ &= 2 \int_0^2 \sqrt{16 - x^2} dx.\end{aligned}$$

The second equality uses the evenness of the integrand. Here the strip height has already included

Quick Tip to Solve: Subtract the lower ordinate from the upper one first, then use evenness to reduce the integration interval without losing a factor of two.

both upper and lower halves, so this last factor of two accounts only for the left and right halves. Equivalently, multiplying the integral of the positive ordinate from zero to two by four gives the same result.

Step 3: Evaluate the integral Put $x = 4 \sin \theta$, giving $dx = 4 \cos \theta d\theta$. When $x = 0$, $\theta = 0$; when $x = 2$, $\theta = \pi/6$. Cosine is positive on this interval, hence $\sqrt{16 - x^2} = 4 \cos \theta$. Consequently,

$$\begin{aligned} A &= 32 \int_0^{\pi/6} \cos^2 \theta d\theta \\ &= 16 \int_0^{\pi/6} (1 + \cos 2\theta) d\theta \\ &= 16 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/6} \\ &= 16 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right). \end{aligned}$$

Both terms are positive as required for area. As a magnitude check, this central area is smaller than the total ellipse area $\pi(4)(2) = 8\pi$, consistently with the fact that the two outer portions are excluded.

Step 4: Conclusion The required area is $\boxed{\frac{8\pi}{3} + 4\sqrt{3}}$ square units.

Quick Tip to Solve: Subtract the lower ordinate from the upper one first, then use evenness to reduce the integration interval without losing a factor of two.

35. The image of point $P(x, y, z)$ with respect to line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is $P'(1, 0, 7)$. Find the coordinates of point P . [5 marks]

Solution:

Concept:

- The image of a point in a line is obtained by reflection: the line meets the segment joining the point and its image at its midpoint and is perpendicular to that segment.
- A point on the given line has coordinates $H(t) = (t, 1 + 2t, 2 + 3t)$, and the line has direction vector $\mathbf{d} = (1, 2, 3)$.
- If H is the perpendicular foot from the image P' , then $(P' - H) \cdot \mathbf{d} = 0$ and the original point is $P = 2H - P'$.

Step 1: Locate the midpoint on the reflecting line Let H be the midpoint of PP' . For reflection in the line, H lies on the line and the joining segment is perpendicular to it. Writing the common ratio in the symmetric line equation as t , we get

$$x = t, \quad y = 1 + 2t, \quad z = 2 + 3t.$$

Hence $H = (t, 1 + 2t, 2 + 3t)$. The direction ratios of the line are 1, 2, 3. Since $P' = (1, 0, 7)$ is known,

Quick Tip to Solve: Project the known image onto the line first; twice this perpendicular foot minus the image immediately gives the original point.

the foot of the perpendicular from this known image determines the midpoint directly, without first introducing three unknown coordinates for P .

Step 2: Use the perpendicularity condition The vector from H to the image is

$$\overrightarrow{HP'} = (1 - t, -1 - 2t, 5 - 3t).$$

Its scalar product with the direction vector of the line must vanish. Therefore,

$$\begin{aligned}(1 - t) + 2(-1 - 2t) + 3(5 - 3t) &= 0 \\ 1 - t - 2 - 4t + 15 - 9t &= 0 \\ 14 - 14t &= 0, \quad t = 1.\end{aligned}$$

Substituting into the parametric coordinates gives $H = (1, 3, 5)$. The coefficient 14 is the squared length of the direction vector, so it is nonzero and the perpendicular foot is uniquely determined.

Step 3: Recover the original point and check the reflection Since H is the midpoint,

$$\begin{aligned}\frac{x + 1}{2} &= 1, & \frac{y + 0}{2} &= 3, & \frac{z + 7}{2} &= 5 \\ x &= 1, & y &= 6, & z &= 3.\end{aligned}$$

To verify, the midpoint of $(1, 6, 3)$ and $(1, 0, 7)$ is indeed $(1, 3, 5)$, whose three line ratios are all one. The joining vector is $P' - P = (0, -6, 4)$, and

$$(0, -6, 4) \cdot (1, 2, 3) = 0 - 12 + 12 = 0.$$

Thus the joining segment is perpendicular to the reflecting line and is bisected by it. Also, the displacements from H to the two points are negatives of one another, confirming that their distances from the foot are equal.

Step 4: Conclusion The coordinates of the original point are $P = (1, 6, 3)$.

Quick Tip to Solve: Project the known image onto the line first; twice this perpendicular foot minus the image immediately gives the original point.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.

A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ .

On the basis of the above information, answer the following questions:

[4 marks]

36(i). Express θ in terms of height of the camera installed on the pole and x .

[1 mark]

Solution:

Concept:

- For a right triangle, $\tan \theta$ is the opposite side divided by the adjacent side.
- The inverse tangent gives an acute angle when its argument is positive. Here the pole height is 5 m and the horizontal distance is $x > 0$.

Step 1: Identify the sides The vertical pole and level ground form a right angle. Viewed from the car, the camera height 5 m is the opposite side and the ground distance x m is the adjacent side. The line of sight to the camera is the hypotenuse.

Step 2: Apply the tangent ratio Therefore

$$\tan \theta = \frac{5}{x}.$$

Since an angle of elevation lies between 0 and $\pi/2$, take the principal inverse tangent. A sine ratio would require the slant distance, whereas the given distance is measured along the ground.

Step 3: Conclusion $\theta = \tan^{-1}(5/x), \quad x > 0$.

Quick Tip to Solve: Use the ground distance as the adjacent side, not as the hypotenuse.

36(ii). Find $\frac{d\theta}{dx}$.

[1 mark]

Solution:

Concept:

- The camera geometry gives $\tan \theta = 5/x$, where x is horizontal distance in metres.
- Implicit differentiation uses $d(\tan \theta)/dx = \sec^2 \theta d\theta/dx$; angles are in radians.

Quick Tip to Solve: Replace secant squared by one plus tangent squared before simplifying.

Step 1: Differentiate the relation Differentiating both sides of $\tan \theta = 5/x$ with respect to x gives

$$\sec^2 \theta \frac{d\theta}{dx} = -\frac{5}{x^2}.$$

The height is constant, so only the distance changes.

Step 2: Eliminate the angle Use $\sec^2 \theta = 1 + \tan^2 \theta = 1 + 25/x^2 = (x^2 + 25)/x^2$. Thus

$$\frac{d\theta}{dx} = -\frac{5}{x^2} \frac{x^2}{x^2 + 25}.$$

The negative sign agrees with the decreasing angle as the car moves farther away.

Step 3: Conclusion $\frac{d\theta}{dx} = -\frac{5}{x^2 + 25}$ radians per metre.

Quick Tip to Solve: Replace secant squared by one plus tangent squared before simplifying.

36(iii)(a). Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole. [2 marks]

Solution:

Concept:

- The geometry is $\tan \theta = 5/x$, so $d\theta/dx = -5/(x^2 + 25)$.
- The chain rule is $d\theta/dt = (d\theta/dx)(dx/dt)$, where t is time in seconds. Movement away gives positive dx/dt .

Step 1: Relate the angular and linear rates Differentiate $\tan \theta = 5/x$ with respect to time:

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{5}{x^2} \frac{dx}{dt}.$$

Substituting $\sec^2 \theta = (x^2 + 25)/x^2$ yields

$$\frac{d\theta}{dt} = -\frac{5}{x^2 + 25} \frac{dx}{dt}.$$

This connects the changing line-of-sight angle to the speed along the ground.

Step 2: Insert the instantaneous distance and speed At the specified instant, $x = 50$ m and $dx/dt = 20$ m/s, since the car travels away from the pole. Hence

$$\frac{d\theta}{dt} = -\frac{5(20)}{50^2 + 25} = -\frac{100}{2525} = -\frac{4}{101} \text{ rad/s}.$$

The distance is substituted only after differentiation because it varies with time. The negative angular rate means the angle decreases.

Step 3: Conclusion The rate of change is $-\frac{4}{101}$ rad/s; the angle decreases at 4/101 rad/s.

Quick Tip to Solve: Differentiate with a variable distance first, then substitute 50 m and the outward speed.

36(iii)(b). If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car. [2 marks]

Solution:

Concept:

- For the camera at height 5 m, $\tan \theta = 5/x$ and $d\theta/dt = -5(dx/dt)/(x^2 + 25)$.
- Speed is the nonnegative magnitude $v = |dx/dt|$. Thus $|d\theta/dt| = 5v/(x^2 + 25)$, regardless of the direction of travel.

Step 1: Set up the relation at the specified distance Implicit differentiation of $\tan \theta = 5/x$ gives

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{5}{x^2} \frac{dx}{dt}.$$

Use $\sec^2 \theta = 1 + 25/x^2$ and cancel the common factor to obtain the angular-rate relation. At $x = 50$ m, its magnitude is

$$\left| \frac{d\theta}{dt} \right| = \frac{5v}{50^2 + 25} = \frac{v}{505}.$$

Step 2: Use the given angular rate The printed rate is $3/101$ rad/s. Therefore

$$\frac{v}{505} = \frac{3}{101}, \quad v = 505 \frac{3}{101} = 15 \text{ m/s}.$$

If the angular rate is interpreted as signed positive, the relation gives $dx/dt = -15$ m/s: that car approaches the pole. Its speed is still the positive magnitude. The other car need not have the first car's speed of 20 m/s.

Step 3: Conclusion The speed of the other car is 15 m/s.

Quick Tip to Solve: Use the magnitude of the angular rate to find speed; its sign specifies direction.

37. Case Study - 2

According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that, an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.

On the basis of the above information, answer the following questions:

[4 marks]

37(i). Find the probability that an airplane reached its destination late.

[2 marks]

Solution:

Concept:

- Let S, M, T denote severe, moderate and light turbulence, respectively, and let L denote late arrival.

Quick Tip to Solve: With equally likely turbulence types, average the three conditional lateness probabilities.

- These three turbulence types form an exhaustive, mutually exclusive partition with $P(S) = P(M) = P(T) = 1/3$.
- The total probability theorem gives $P(L) = \sum_E P(E)P(L | E)$, summed over the turbulence types.

Step 1: Convert the conditional probabilities The stated percentages are probabilities of late arrival within each turbulence category:

$$P(L | S) = \frac{55}{100}, \quad P(L | M) = \frac{37}{100}, \quad P(L | T) = \frac{17}{100}.$$

They are conditional probabilities, so each must be weighted by the probability of its category. Equal category probabilities mean equal weights, rather than equal late-arrival probabilities.

Step 2: Add the disjoint contributions A late flight can fall in exactly one of the three categories. Hence

$$\begin{aligned} P(L) &= P(S)P(L | S) + P(M)P(L | M) \\ &\quad + P(T)P(L | T) \\ &= \frac{1}{3} \frac{55}{100} + \frac{1}{3} \frac{37}{100} + \frac{1}{3} \frac{17}{100} \\ &= \frac{55 + 37 + 17}{300} = \frac{109}{300}. \end{aligned}$$

The result lies between the smallest and largest conditional probabilities, as expected for a weighted average.

Step 3: Conclusion The required probability is $\boxed{\frac{109}{300}}$.

Quick Tip to Solve: With equally likely turbulence types, average the three conditional lateness probabilities.

37(ii). If the airplane reached its destination late, find the probability that it was due to moderate turbulence. [2 marks]

Solution:

Concept:

- Let M be moderate turbulence and L be late arrival. The required probability is $P(M | L)$.
- Bayes' theorem gives $P(M | L) = P(M)P(L | M)/P(L)$.
- The late-arrival probability is obtained by weighting each turbulence category by its prior probability $1/3$.

Step 1: Find the late-arrival probability The three categories are exhaustive and mutually exclusive. By total probability,

$$P(L) = \frac{1}{3} \left(\frac{55}{100} + \frac{37}{100} + \frac{17}{100} \right) = \frac{109}{300}.$$

This is the probability of the evidence supplied in the question: the airplane has arrived late. It includes late arrivals under all turbulence types.

Quick Tip to Solve: For equal prior probabilities, divide the moderate lateness percentage by their total.

Step 2: Apply Bayes theorem The probability of both moderate turbulence and late arrival is

$$P(M \cap L) = P(M)P(L | M) = \frac{1}{3} \frac{37}{100} = \frac{37}{300}.$$

Condition on the observed late arrival:

$$P(M | L) = \frac{P(M \cap L)}{P(L)} = \frac{37/300}{109/300} = \frac{37}{109}.$$

The given 37% describes lateness among moderately turbulent flights; reversing the condition requires this calculation, so it cannot be used directly as the answer.

Step 3: Conclusion The required probability is $\boxed{\frac{37}{109}}$.

Quick Tip to Solve: For equal prior probabilities, divide the moderate lateness percentage by their total.

38. Case Study - 3

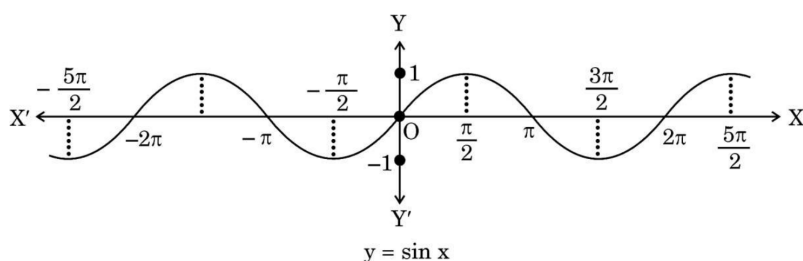
If a function $f : X \rightarrow Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \rightarrow X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Function g is called the inverse of function f .

The domain of sine function is $\mathbb{R}$ and function sine: $\mathbb{R} \rightarrow \mathbb{R}$ is neither one-one nor onto. The following graph shows the sine function.

Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A .

On the basis of the above information, answer the following questions:

[4 marks]



38(i). If A is the interval other than principal value branch, give an example of one such interval.

[1 mark]

Solution:

Concept:

- An inverse exists when the restricted sine function is one-one and onto $[-1, 1]$.
- The principal branch is $[-\pi/2, \pi/2]$; a different monotonic half-period can give another branch.

Step 1: Choose a different half-period Take $A = [\pi/2, 3\pi/2]$. On this interval, sine decreases strictly from $\sin(\pi/2) = 1$ to $\sin(3\pi/2) = -1$. Thus each value in $[-1, 1]$ occurs exactly once.

Step 2: Check invertibility Continuity gives every intermediate value between -1 and 1 , so the

Quick Tip to Solve: Choose adjacent maximum and minimum points of sine: their closed interval is a valid inverse branch.

restricted function is onto $[-1, 1]$. Strict decrease makes it one-one. This interval differs from the principal branch, while an entire period would repeat values and would not be one-one.

Step 3: Conclusion One suitable interval is $A = [\pi/2, 3\pi/2]$.

Quick Tip to Solve: Choose adjacent maximum and minimum points of sine: their closed interval is a valid inverse branch.

38(ii). If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$. [1 mark]

Solution:

Concept:

- The principal value of $\sin^{-1} t$ lies in $[-\pi/2, \pi/2]$.
- Choose the unique angle in that interval whose sine equals the given argument.

Step 1: Evaluate the two principal values Since $\sin(-\pi/6) = -1/2$ and $-\pi/6$ belongs to the principal branch, $\sin^{-1}(-1/2) = -\pi/6$. Also $\sin(\pi/2) = 1$, so $\sin^{-1}(1) = \pi/2$.

Step 2: Subtract in the printed order

$$\sin^{-1}(-1/2) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{6} - \frac{3\pi}{6} = -\frac{4\pi}{6}.$$

Angles such as $7\pi/6$, although having sine $-1/2$, cannot be used because they lie outside the principal branch.

Step 3: Conclusion The required value is $-2\pi/3$.

Quick Tip to Solve: Write both principal angles first; the subtraction of a positive angle makes the result more negative.

38(iii)(a). Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch. [2 marks]

Solution:

Concept:

- The principal inverse sine has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
- Its graph is obtained by reflecting the graph of sine restricted to $[-\pi/2, \pi/2]$ in the line $y = x$.

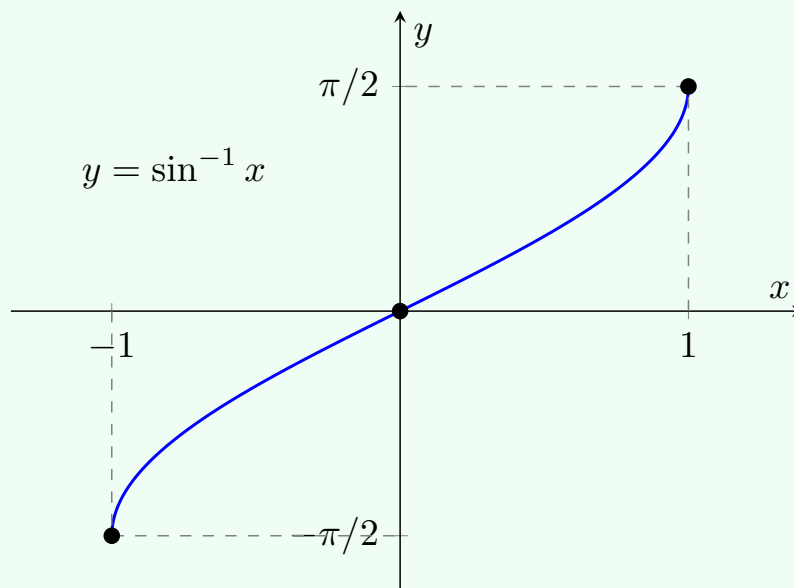
Step 1: Locate the essential points Write $y = \sin^{-1} x$, equivalently $x = \sin y$ with $-\pi/2 \leq y \leq \pi/2$. The endpoint and origin values are

$$\sin^{-1}(-1) = -\frac{\pi}{2}, \quad \sin^{-1}(0) = 0, \quad \sin^{-1}(1) = \frac{\pi}{2}.$$

Thus the curve passes through $(-1, -\pi/2)$, $(0, 0)$ and $(1, \pi/2)$.

Quick Tip to Solve: Mark the two included endpoints and the origin before drawing the increasing inverse-sine curve.

Step 2: Draw the principal inverse curve The restricted sine is continuous and strictly increasing, so its inverse is continuous and strictly increasing as well. Draw a smooth curve through these three points, including both endpoints. It is symmetric about the origin because $\sin^{-1}(-x) = -\sin^{-1} x$. The curve becomes steep near $x = -1$ and $x = 1$, and has no portion beyond these domain endpoints.



Principal graph of $y = \sin^{-1} x$

Step 3: Conclusion The displayed graph represents $y = \sin^{-1} x, -1 \leq x \leq 1$, with principal range $[-\pi/2, \pi/2]$.

Quick Tip to Solve: Mark the two included endpoints and the origin before drawing the increasing inverse-sine curve.

OR

38(iii)(b). Find the domain and range of $f(x) = 2 \sin^{-1}(1 - x)$.

[2 marks]

Solution:

Concept:

- A real inverse sine requires its argument to lie in $[-1, 1]$.
- For principal inverse sine, $-\pi/2 \leq \sin^{-1} t \leq \pi/2$. Multiplication by the positive number 2 doubles both range endpoints.

Step 1: Find the permissible inputs For $f(x) = 2 \sin^{-1}(1 - x)$, impose

$$-1 \leq 1 - x \leq 1.$$

Subtracting 1 gives $-2 \leq -x \leq 0$. Multiplying by -1 reverses the inequalities, yielding $0 \leq x \leq 2$. Both endpoints are allowed because inverse sine is defined at arguments 1 and -1 .

Quick Tip to Solve: First constrain the inner argument; then use its full coverage of $[-1, 1]$ to determine the exact range.

Step 2: Find the complete output interval As x increases from 0 to 2, the argument $1 - x$ runs through every value from 1 to -1 . Its principal inverse sine therefore runs through the complete interval $[-\pi/2, \pi/2]$. Consequently

$$-\pi \leq 2 \sin^{-1}(1 - x) \leq \pi.$$

The endpoints are attained: $f(0) = \pi$ and $f(2) = -\pi$. Continuity ensures that every output between them is attained.

Step 3: Conclusion The domain is $[0, 2]$ and the range is $[-\pi, \pi]$.

Quick Tip to Solve: First constrain the inner argument; then use its full coverage of $[-1, 1]$ to determine the exact range.