CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin\theta = \frac{3}{5}$. Then, $\hat{a} \cdot \hat{b}$ is equal to: [1 mark]

(A) $\pm\frac{3}{5}$

(B) $\pm\frac{3}{4}$

(C) $\pm\frac{4}{5}$

(D) $\pm\frac{4}{3}$

Correct Answer: (C) $\pm\frac{4}{5}$

Solution:

Concept:

- The scalar product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between the vectors.
- Unit vectors have magnitude one, and $\sin^2\theta + \cos^2\theta = 1$.

Step 1: Determine the possible cosine values Substitute the given sine into the identity:

$$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}.$$

The angle between vectors lies in $[0, \pi]$. The given positive sine permits both an acute and an obtuse angle, so both signs of the cosine must be retained.

Step 2: Use the unit magnitudes and compare options Since both magnitudes are one, their product does not change the cosine. Option (A) incorrectly uses the sine, and (B) gives a value whose square does not equal $16/25$. Option (D) has magnitude greater than one and cannot be the dot product of two unit vectors.

Step 3: Conclusion $\hat{a} \cdot \hat{b} = \pm\frac{4}{5}$. Hence, the correct option is (C).

Quick Tip to Solve: When only the sine of the angle is given, retain both cosine signs unless the angle is known to be acute or obtuse.

2. The integrating factor of the differential equation $x\dfrac{dy}{dx} - y = x^4 - 3x$ is: [1 mark]

(A) x

(B) $-x$

(C) x^{-1}

(D) $\log(x^{-1})$

Correct Answer: (C) x^{-1}

Solution:

Concept:

- A first-order linear differential equation has standard form $dy/dx + P(x)y = Q(x)$.
- Its integrating factor is $e^{\int P(x)\,dx}$; multiplying an integrating factor by a nonzero constant gives another valid one.

Step 1: Put the equation in standard form On an interval where $x \neq 0$, divide the whole equation by x:

$$\frac{dy}{dx} - \frac{1}{x}y = x^3 - 3.$$

Thus the coefficient required in the integrating-factor formula is $P(x) = -1/x$, not the coefficient before division.

Step 2: Integrate the coefficient

$$e^{\int -\frac{1}{x}\,dx} = e^{-\log|x|} = \frac{1}{|x|}.$$

On either side of zero, $1/x$ differs from this factor by at most a constant sign and is valid. The factors x and $-x$ have logarithmic derivative $1/x$, with the wrong sign; $\log(x^{-1})$ omits the required exponential.

Step 3: Conclusion An integrating factor is x^{-1}. Hence, the correct option is (C).

Quick Tip to Solve: First make the coefficient of dy/dx equal to one; then integrate the coefficient of y and exponentiate.

3. If the direction cosines of a line are $\sqrt{3}\,k,\ \sqrt{3}\,k,\ \sqrt{3}\,k$, then the value of k is: [1 mark]

(A) ± 1

(B) $\pm\sqrt{3}$

(C) ± 3

(D) $\pm\dfrac{1}{3}$

Correct Answer: (D) $\pm\dfrac{1}{3}$

Solution:

Concept:

- Direction cosines l, m, n are the cosines of the angles made by a directed line with the positive coordinate axes.
- They satisfy the normalization identity $l^2 + m^2 + n^2 = 1$, unlike arbitrary direction ratios.

Step 1: Substitute the three equal direction cosines Here $l = m = n = \sqrt{3}\,k$. Squaring each complete expression gives

$$(\sqrt{3}\,k)^2 + (\sqrt{3}\,k)^2 + (\sqrt{3}\,k)^2 = 1, \qquad 9k^2 = 1.$$

The square of $\sqrt{3}\,k$ is $3k^2$, so all three terms must be included.

Step 2: Retain both orientations and check options The normalization gives $k^2 = 1/9$. Either sign is allowed: changing all three signs reverses the direction along the same line. For (A), (B) and (C), the sums of the squared direction cosines would instead be 9, 27 and 81, respectively, so none satisfies the identity.

Step 3: Conclusion $k = \pm\dfrac{1}{3}$. Hence, the correct option is (D).

Quick Tip to Solve: Square each full direction cosine and add; do not normalize the common factor alone or discard the negative orientation.

4. A linear programming problem deals with the optimization of a/an: [1 mark]

(A) logarithmic function

(B) linear function

(C) quadratic function

(D) exponential function

Correct Answer: (B) linear function

Solution:

Concept:

- Optimization means finding the maximum or minimum value of an objective function over the feasible set.
- A linear programming problem has a linear objective function, such as $Z = ax + by$, and linear constraints; a, b are constants and x, y are decision variables.

Quick Tip to Solve: In an LPP, check both the objective and constraints for linearity; optimization alone does not make a problem linear.

Step 1: Identify the defining feature In linear programming, quantities such as profit or cost are written as a linear expression in the decision variables. The constraints restrict which values of those variables are allowed. The objective is then maximized or minimized over those feasible values.

Step 2: Compare the four function types A linear expression contains variables only to the first power, with constant coefficients and no products of variables. Logarithmic, quadratic and exponential functions are nonlinear types, so options (A), (C) and (D) do not describe the objective function of a linear programming problem.

Step 3: Conclusion A linear programming problem optimizes a linear function. Hence, the correct option is (B).

Quick Tip to Solve: In an LPP, check both the objective and constraints for linearity; optimization alone does not make a problem linear.

5. If $P(A \mid B) = P(A' \mid B)$, then which of the following statements is true? [1 mark]

(A) $P(A) = P(A')$

(B) $P(A) = 2P(B)$

(C) $P(A \cap B) = \frac{1}{2}P(B)$

(D) $P(A \cap B) = 2P(B)$

Correct Answer: (C) $P(A \cap B) = \frac{1}{2}P(B)$

Solution:

Concept:

- For $P(B) > 0$, conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$.
- The complementary events A and A' partition B, so $P(A \mid B) + P(A' \mid B) = 1$.

Step 1: Use equality of the two conditional probabilities Let each of the given equal probabilities be p. Because they sum to one,

$$p + p = 1, \qquad p = \frac{1}{2}.$$

This equality describes probabilities within B; it does not by itself assert that A and its complement are equally likely in the whole sample space.

Step 2: Apply the conditional probability formula Substitute into the formula and multiply by $P(B)$:

$$\frac{P(A \cap B)}{P(B)} = \frac{1}{2}.$$

Option (A) concerns unconditional probabilities and need not follow. Option (B) has no basis in the given equality. Option (D) is impossible since $A \cap B \subseteq B$ and $P(B) > 0$.

Step 3: Conclusion $P(A \cap B) = \frac{1}{2}P(B)$. Hence, the correct option is (C).

Quick Tip to Solve: Equal conditional probabilities of complementary events are each one-half; multiply by the conditioning event's probability to find the intersection.

6. If a_{ij} and A_{ij} represent the $(i,j)^{\text{th}}$ element and its cofactor of $\begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$ respectively, then the value of $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$ is: [1 mark]

(A) 0

(B) -28

(C) 114

(D) -114

Correct Answer: (A) 0

Solution:

Concept:

- The cofactor of an entry is $A_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is its minor determinant.
- The sum of entries of one row multiplied by corresponding cofactors of a different row is zero: $\sum_j a_{ij}A_{kj} = 0$ when $i \neq k$.

Step 1: Recognize the different-row cofactor sum The entries have first row index 1, whereas the cofactors have row index 2. This expression is the expansion along the second row of a determinant in which the second row has been replaced by the first. That determinant has two identical rows and is therefore zero.

Step 2: Verify directly with the cofactors The required cofactors are

$$A_{21} = -[(-3)(-7) - 5(5)] = 4,$$
$$A_{22} = 2(-7) - 5(1) = -19,$$
$$A_{23} = -[2(5) - (-3)(1)] = -13.$$

Thus the indicated sum is $2(4) + (-3)(-19) + 5(-13) = 8 + 57 - 65$. The other three options are nonzero and fail both the row identity and this calculation.

Step 3: Conclusion The required value is 0. Hence, the correct option is (A).

Quick Tip to Solve: Before calculating minors, compare the row indices: entries of one row paired with cofactors of another always sum to zero.

7. The derivative of $\sin(x^2)$ w.r.t. x, at $x = \sqrt{\pi}$, is: [1 mark]

(A) 1

(B) -1

(C) $-2\sqrt{\pi}$

(D) $2\sqrt{\pi}$

Correct Answer: (C) $-2\sqrt{\pi}$

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$, with trigonometric arguments measured in radians.
- The inner function here is $u = x^2$, whose derivative is $2x$; also $\cos \pi = -1$.

Step 1: Differentiate before substituting Apply the chain rule to the outer sine and the inner square:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2).$$

Differentiating only the outer function would omit the factor $2x$. Substituting the fixed value first would instead turn the function into a constant and lose the requested derivative.

Step 2: Evaluate at the specified point At $x = \sqrt{\pi}$, the argument x^2 is π. Therefore the product to evaluate is $2\sqrt{\pi}\cos\pi$, with $\cos\pi = -1$. Options (A) and (B) omit the inner derivative's magnitude, while option (D) uses the wrong sign for the cosine.

Step 3: Conclusion The derivative at $x = \sqrt{\pi}$ is $-2\sqrt{\pi}$. Hence, the correct option is (C).

Quick Tip to Solve: For a derivative at a point, differentiate the composite function first, then substitute into both the inner derivative and the trigonometric factor.

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8. The order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$ respectively are: [1 mark]

(A) 1, 2

(B) 2, 3

(C) 2, 1

(D) 2, 6

Correct Answer: (C) 2, 1

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.
- For an equation polynomial in its derivatives, its degree is the exponent of the highest-order derivative.

Step 1: Identify the highest derivative The equation contains dy/dx and d^2y/dx^2. The latter is the highest-order derivative, so the order is 2; option (A) is excluded.

Quick Tip to Solve: Find the highest-order derivative first; only its power determines the degree.

Step 2: Inspect its polynomial form Writing $p = dy/dx$ and $q = d^2y/dx^2$, the equation becomes

$$q - (1 + 3p^2 + 3p^4 + p^6) = 0.$$

It is already a polynomial in the derivatives, and q occurs to power 1. The outer power 3 and the power 6 of the first derivative do not determine the degree, excluding (B) and (D).

Step 3: Conclusion The order and degree are $2, 1$, respectively. Hence, the correct option is (C).

Quick Tip to Solve: Find the highest-order derivative first; only its power determines the degree.

9. The vector with terminal point $A(2, -3, 5)$ and initial point $B(3, -4, 7)$ is: [1 mark]

(A) $\hat{i} - \hat{j} + 2\hat{k}$

(B) $\hat{i} + \hat{j} + 2\hat{k}$

(C) $-\hat{i} - \hat{j} - 2\hat{k}$

(D) $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Solution:

Concept:

- A directed vector equals the position vector of its terminal point minus that of its initial point.
- For initial point B and terminal point A, use $\overrightarrow{BA} = \vec{A} - \vec{B}$.

Step 1: Subtract coordinates in the given direction The vector starts at $B(3, -4, 7)$ and ends at $A(2, -3, 5)$. Therefore

$$\overrightarrow{BA} = (2 - 3)\hat{i} + [-3 - (-4)]\hat{j} + (5 - 7)\hat{k}.$$

Step 2: Check the signs and options The coordinate differences are $-1, +1, -2$. Option (A) reverses all three signs and gives $\overrightarrow{AB}$. Option (B) has incorrect first and third components, while option (C) has an incorrect second component.

Step 3: Conclusion The required vector is $-\hat{i} + \hat{j} - 2\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: Write terminal minus initial above the coordinates before subtracting negative numbers.

10. The distance of point $P(a, b, c)$ from y-axis is: [1 mark]

(A) b

(B) b^2

(C) $\sqrt{a^2 + c^2}$

(D) $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$

Solution:

Concept:

- A point on the y-axis has coordinates $(0, t, 0)$, with $t \in \mathbb{R}$.
- The distance from a point to an axis is the shortest distance to a point on that axis.

Step 1: Write the squared distance For $Q(0, t, 0)$ on the y-axis, the distance formula gives

$$PQ^2 = a^2 + (b - t)^2 + c^2.$$

Since $(b - t)^2 \geq 0$, this expression is smallest when $t = b$.

Step 2: Identify the perpendicular foot and reject distractors The foot is $Q(0, b, 0)$, so the shortest squared distance is $a^2 + c^2$. Option (D) gives the square of the distance. Options (A) and (B) use the coordinate along the axis, which does not measure perpendicular separation from it.

Step 3: Conclusion The distance is $\sqrt{a^2 + c^2}$. Hence, the correct option is (C).

Quick Tip to Solve: For distance from a coordinate axis, square and add the other two coordinates, then take the square root.

11. The number of corner points of the feasible region determined by constraints $x \geq 0,\ y \geq 0,\ x+y \geq 4$ is: [1 mark]

(A) 0

(B) 1

(C) 2

(D) 3

Correct Answer: (C) 2

Solution:

Concept:

- Corner points occur where boundary lines intersect and all constraints are satisfied.
- The constraints $x \geq 0$, $y \geq 0$ restrict the region to the first quadrant.

Step 1: Test the three boundary intersections The boundaries are $x = 0$, $y = 0$, and $x + y = 4$. Their pairwise intersections are $(0, 0)$, $(0, 4)$, and $(4, 0)$. The origin fails $x + y \geq 4$, while both intercepts satisfy every constraint.

Step 2: Count actual vertices The feasible region lies on and above $x + y = 4$ in the first quadrant and is unbounded. Its two finite vertices are $(0, 4)$ and $(4, 0)$. Unboundedness does not remove these

Quick Tip to Solve: Check each boundary intersection in every inequality; do not count the origin automatically.

vertices, so options (A) and (B) fail; counting the infeasible origin would incorrectly give option (D).

Step 3: Conclusion There are 2 corner points. Hence, the correct option is (C).

Quick Tip to Solve: Check each boundary intersection in every inequality; do not count the origin automatically.

12. If A and B are two non-zero square matrices of same order such that $(A+B)^2 = A^2 + B^2$, then: [1 mark]

(A) $AB = O$

(B) $AB = -BA$

(C) $BA = O$

(D) $AB = BA$

Correct Answer: (B) $AB = -BA$

Solution:

Concept:

- Matrix multiplication is distributive, but in general $AB \neq BA$.
- For square matrices of the same order, $(A+B)^2 = A^2 + AB + BA + B^2$.

Step 1: Expand without assuming commutativity Using the given equality,

$$A^2 + AB + BA + B^2 = A^2 + B^2.$$

Cancelling the common terms gives $AB + BA = O$, where O is the zero matrix.

Step 2: Check the alternative claims The cross-products need not separately vanish or be equal. For example, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -BA \neq O$. This satisfies the condition and disproves the necessity of (A), (C), and (D).

Step 3: Conclusion We must have $AB = -BA$. Hence, the correct option is (B).

Quick Tip to Solve: Expand matrix products in their original order; never replace the cross-terms by $2AB$ without justification.

13. A relation R defined on set $A = \{x : x \in \mathbb{Z}$ and $0 \leq x \leq 10\}$ as $R = \{(x, y) : x = y\}$ is given to be an equivalence relation. The number of equivalence classes is: [1 mark]

(A) 1

(B) 2

(C) 10

(D) 11

Correct Answer: (D) 11

Solution:

Concept:

- The equivalence class of $a \in A$ is the set of elements related to a.
- For the equality relation, each element is related only to itself, so $[a] = \{a\}$.

Step 1: Determine the classes The set is $A = \{0, 1, 2, \ldots, 10\}$. Since xRy means $x = y$, its equivalence classes are the distinct singleton sets $\{0\}, \{1\}, \ldots, \{10\}$. No two different elements belong to the same class.

Step 2: Count inclusively and compare options The number of integers from 0 through 10, including both endpoints, is $10 - 0 + 1 = 11$. Option (C) omits one endpoint. Options (A) and (B) would require grouping different elements together, which the equality relation does not permit.

Step 3: Conclusion The number of equivalence classes is 11. Hence, the correct option is (D).

Quick Tip to Solve: For the equality relation, count the elements of the set; each element forms its own class.

14. If a matrix has 36 elements, the number of possible orders it can have, is: [1 mark]

(A) 13

(B) 3

(C) 5

(D) 9

Correct Answer: (D) 9

Solution:

Concept:

- An $m \times n$ matrix has mn elements, where m, n are positive integers.
- The order records rows first and columns second, so reversed unequal factor pairs are different orders.

Step 1: List every ordered factor pair The condition is $mn = 36$. The possible orders are

$$1 \times 36, \quad 2 \times 18, \quad 3 \times 12, \quad 4 \times 9, \quad 6 \times 6,$$
$$9 \times 4, \quad 12 \times 3, \quad 18 \times 2, \quad 36 \times 1.$$

Step 2: Confirm the count and exclude distractors Each positive divisor m of $36 = 2^2 \cdot 3^2$ determines

Quick Tip to Solve: Count positive divisors of the number of elements; each divisor specifies a unique number of rows.

exactly one value $n = 36/m$. Thus the number is $(2+1)(2+1) = 9$. Option (C) counts unordered pairs and loses the reversed rectangular orders; neither (A) nor (B) agrees with the complete list.

Step 3: Conclusion The matrix has 9 possible orders. Hence, the correct option is (D).

Quick Tip to Solve: Count positive divisors of the number of elements; each divisor specifies a unique number of rows.

15. The number of points, where $f(x) = [x],\ 0 < x < 3$ ($[\cdot]$ denotes greatest integer function) is not differentiable is: [1 mark]

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

Concept:

- The greatest integer function $[x]$ is constant on every interval $(n, n+1)$, where n is an integer.
- Differentiability implies continuity; a jump discontinuity prevents a derivative from existing.

Step 1: Identify the possible exceptional points On each of $(0,1)$, $(1,2)$ and $(2,3)$, the function has a constant value, respectively $0, 1, 2$. Its derivative is therefore zero at every point inside these intervals. Only the integers 1 and 2 remain to be tested.

Step 2: Test the integers in the domain At an integer n, the left-hand limit is $n-1$, whereas the right-hand limit and $f(n)$ are n. Thus there is a jump at both 1 and 2, so neither point permits a derivative. The endpoints 0 and 3 are excluded by the strict inequalities and cannot be counted. Counting either or both endpoints would incorrectly suggest three or four points; counting only one internal integer would miss a jump.

Step 3: Conclusion There are $\boxed{2}$ points of non-differentiability. Hence, the correct option is (B).

Quick Tip to Solve: For $[x]$ on an open interval, count only the integers strictly between its endpoints.

16. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then, this function $f(x)$ is strictly increasing in (a, b) if: [1 mark]

(A) $f'(x) < 0,\ \forall x \in (a, b)$

(B) $f'(x) > 0,\ \forall x \in (a, b)$

(C) $f'(x) = 0,\ \forall x \in (a, b)$

(D) $f(x) > 0,\ \forall x \in (a, b)$

Correct Answer: (B) $f'(x) > 0,\ \forall x \in (a, b)$

Solution:

Concept:

- A positive derivative throughout an interval is a sufficient condition for a differentiable function to be strictly increasing there.
- The mean value theorem gives $f(v) - f(u) = f'(c)(v - u)$ for some $c \in (u, v)$, whenever $a < u < v < b$.

Step 1: Apply the increasing-function test Choose any two points $u < v$ in (a, b). Since $v - u > 0$, the condition $f'(x) > 0$ throughout the interval makes the mean value theorem expression positive. Hence $f(v) > f(u)$, which is exactly strict increase.

Step 2: Check the other conditions Option (A), a negative derivative everywhere, gives strict decrease. Option (C), a zero derivative everywhere, gives a constant function. Option (D) describes positive function values, not increasing values: for example $f(x) = 2 - x$ is positive on $(0, 1)$ but decreases there. Thus positivity of the function itself does not establish the required behaviour.

Step 3: Conclusion The sufficient condition is $\boxed{f'(x) > 0 \text{ for every } x \in (a, b)}$. Hence, the correct option is (B).

Quick Tip to Solve: Check the sign of the first derivative, not the sign of the function, when deciding increase or decrease.

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17. If $\begin{pmatrix} x+y & 2 \\ 5 & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$, then the value of $\left(\frac{24}{x} + \frac{24}{y}\right)$ is: [1 mark]

(A) 7

(B) 6

(C) 8

(D) 18

Correct Answer: (D) 18

Solution:

Concept:

- Equal matrices have equal corresponding entries, so their upper-left and lower-right entries can be compared separately.

Quick Tip to Solve: Use $(x + y)/(xy)$ for reciprocal sums; there is no need to solve separately for the two variables.

- For nonzero x, y, $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{x+y}{xy}$.

Step 1: Compare the corresponding entries The upper-left entries give $x+y=6$, and the lower-right entries give $xy=8$. The other two entries already agree and yield no further condition. Since the product is nonzero, both x and y are nonzero and the stated fractions are defined.

Step 2: Use the sum and product directly Combine the fractions before substituting:

$$\frac{24}{x}+\frac{24}{y}=24\left(\frac{x+y}{xy}\right)=24\left(\frac{6}{8}\right)=18.$$

As a check, the possible values are 2 and 4 in either order, giving $12+6=18$. Options (A), (B) and (C), namely $7, 6, 8$, do not equal the required sum; in particular, 6 and 8 are the sum and product of the variables, not the expression asked for.

Step 3: Conclusion The required value is $\boxed{18}$. Hence, the correct option is (D).

Quick Tip to Solve: Use $(x+y)/(xy)$ for reciprocal sums; there is no need to solve separately for the two variables.

18. If $\displaystyle\int_0^a \frac{1}{4+x^2}\,dx=\frac{\pi}{6}$, then the value of a is: [1 mark]

(A) $\dfrac{\sqrt{3}}{2}$

(B) $2\sqrt{3}$

(C) $\sqrt{3}$

(D) $\dfrac{1}{\sqrt{3}}$

Correct Answer: (B) $2\sqrt{3}$

Solution:

Concept:

- The standard integral is $\displaystyle\int \frac{dx}{b^2+x^2}=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+C$, for $b>0$.
- The principal range of $\tan^{-1}t$ is $(-\pi/2, \pi/2)$, so its value determines t uniquely.

Step 1: Evaluate the definite integral Here $4=2^2$, so the factor outside the inverse tangent is $1/2$:

$$\int_0^a \frac{dx}{4+x^2}=\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^a=\frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right).$$

The lower-limit term is zero.

Step 2: Solve the inverse-tangent equation The given equality therefore gives

$$\frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right)=\frac{\pi}{6},\qquad \tan^{-1}\left(\frac{a}{2}\right)=\frac{\pi}{3}.$$

This angle lies in the principal range, and taking tangent yields $a/2=\sqrt{3}$. Only option (B) has this

Quick Tip to Solve: For a denominator $4+x^2$, retain both factors of 2: the prefactor $1/2$ and the argument $x/2$.

ratio; each of (A), (C) and (D) gives a smaller positive value of $a/2$, and hence a smaller inverse tangent and integral.

Step 3: Conclusion $\boxed{a = 2\sqrt{3}}$. Hence, the correct option is (B).

Quick Tip to Solve: For a denominator $4 + x^2$, retain both factors of 2: the prefactor $1/2$ and the argument $x/2$.

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19. Assertion (A): A line in space cannot be drawn perpendicular to x, y and z axes simultaneously.
Reason (R): For any line making angles α, β, γ with the positive directions of x, y and z axes respectively, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The direction cosines l, m, n of a line satisfy $l^2 + m^2 + n^2 = 1$.
- A line perpendicular to a coordinate axis has direction cosine zero with respect to that axis.

Step 1: Judge the Assertion If a line were perpendicular to all three coordinate axes, its direction vector would have zero component along each axis. That would be the zero vector, which cannot specify a line direction. Thus the Assertion is true.

Step 2: Judge the Reason For a nonzero direction vector (p, q, r), divide its components by $\sqrt{p^2 + q^2 + r^2}$ to obtain the direction cosines. Their squared sum is $(p^2 + q^2 + r^2)/(p^2 + q^2 + r^2) = 1$. Therefore the stated identity, and hence the Reason, is true.

Step 3: Test the explanatory link Simultaneous perpendicularity would make $\alpha = \beta = \gamma = 90^\circ$, so all three cosines would be zero. Their squared sum would then be 0, contradicting the Reason. The identity therefore directly explains why the proposed line cannot exist; option (B) misses this link, while (C) and (D) incorrectly declare one statement false.

Step 4: Conclusion Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: To test perpendicularity to all axes, put all three direction cosines equal to zero and check their squared sum.

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20. Assertion (A): For matrix $A = \begin{pmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{pmatrix}$**, where** $\theta \in [0, 2\pi]$**,** $|A| \in [2, 4]$**.**

Reason (R): $\cos\theta \in [-1, 1],\ \forall\, \theta \in [0, 2\pi]$**.** [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- Expand a 3×3 determinant along its first row using alternating signs $+, -, +$.
- If $-1 \le c \le 1$, then $0 \le c^2 \le 1$; apply this to $c = \cos\theta$.

Step 1: Judge the Assertion by expanding the determinant Write $c = \cos\theta$. Expansion along the first row gives

$$\begin{aligned} |A| &= 1(1 + c^2) - c(-c + c) + 1(c^2 + 1) \\ &= 2 + 2c^2. \end{aligned}$$

Since $0 \le \cos^2\theta \le 1$, this gives $2 \le |A| \le 4$. The endpoints occur at $\theta = \pi/2$ and $\theta = 0$, respectively, both in the specified interval. Thus the Assertion is true.

Step 2: Judge the Reason The cosine function always has values between -1 and 1, inclusive. In particular, this holds for every $\theta \in [0, 2\pi]$, so the Reason is true.

Step 3: Test the explanatory link The bound in the Reason gives $0 \le \cos^2\theta \le 1$. Substitution in the computed determinant $2 + 2\cos^2\theta$ supplies exactly the asserted interval. Hence the Reason explains the Assertion. Option (B) wrongly denies this explanation, and options (C) and (D) wrongly classify a true statement as false.

Step 4: Conclusion Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Simplify the determinant to $2 + 2\cos^2\theta$ before applying the cosine bound; square the bound carefully.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Find: $\int x\sqrt{1+2x}\,dx$ [2 marks]

Solution:

Concept:

- For a linear expression under a square root, substitute that expression as a new variable.
- The power rule is $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$, for $n \neq -1$.

Step 1: Remove the linear expression from the radical Let $u = 1+2x$. Then $du = 2\,dx$, $dx = du/2$, and $x = (u-1)/2$. Substituting both the factor x and the differential is essential:

$$I = \int x\sqrt{1+2x}\,dx = \frac{1}{4}\int (u-1)u^{1/2}\,du = \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)\,du.$$

The real-valued integrand is defined for $x \geq -1/2$.

Step 2: Integrate the powers and substitute back Apply the power rule to the two terms separately:

$$\begin{aligned} I &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \frac{u^{5/2}}{10} - \frac{u^{3/2}}{6} + C. \end{aligned}$$

Replacing u by $1+2x$, the expression may also be factored as $(3x-1)(1+2x)^{3/2}/15 + C$. To check, differentiating the unfactored expression gives

$$\frac{1}{2}(1+2x)^{3/2} - \frac{1}{2}(1+2x)^{1/2} = x\sqrt{1+2x},$$

which is the original integrand on the interior of its real domain.

Step 3: Conclusion

$$\boxed{\int x\sqrt{1+2x}\,dx = \frac{(1+2x)^{5/2}}{10} - \frac{(1+2x)^{3/2}}{6} + C.}$$

Quick Tip to Solve: After setting $u = 1+2x$, replace x by $(u-1)/2$ as well as dx by $du/2$.

OR

21(b). Evaluate: $\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ [2 marks]

Solution:

Concept:

- Substitution converts a composite expression into a simpler integrand: take $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t\,dt$.
- In a definite integral, transform both limits into the new variable; the elementary antiderivative is $\int \sin t\,dt = -\cos t$.

Step 1: Substitute and transform the limits Let I denote the required integral. Put $t = \sqrt{x}$; since $x \geq 0$, we use $t \geq 0$. The limits become

$$x = 0 \Rightarrow t = 0, \qquad x = \frac{\pi^2}{4} \Rightarrow t = \frac{\pi}{2}.$$

For positive t, the factor t in $dx = 2t\,dt$ cancels the denominator. Hence

$$I = 2\int_0^{\pi/2} \sin t\,dt.$$

The original integrand has limiting value 1 at zero, since $\sin t/t \to 1$; its endpoint singularity is removable.

Step 2: Evaluate at the transformed endpoints Integrate first, and then subtract the value at the lower endpoint from the value at the upper endpoint:

$$I = 2[-\cos t]_0^{\pi/2} = 2\left(-\cos\frac{\pi}{2} + \cos 0\right).$$

Use $\cos(\pi/2) = 0$ and $\cos 0 = 1$. No constant of integration is needed for a definite integral.

Step 3: Conclusion The required value is $\boxed{2}$.

Quick Tip to Solve: Set $t = \sqrt{x}$ immediately: the differential $2t\,dt$ cancels the square root in the denominator. Change both limits before integrating.

22. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}\,|\vec{a}|$. [2 marks]

Solution:

Concept:

- Perpendicular vectors have zero dot product. Thus each given perpendicularity condition gives a scalar equation.
- For vectors $\vec{u}, \vec{v}$, the dot product is distributive and symmetric, and $\vec{u} \cdot \vec{u} = |\vec{u}|^2$, the square of its magnitude.

Quick Tip to Solve: Convert each perpendicularity into a zero dot product; eliminate $\vec{a} \cdot \vec{b}$ before taking square roots of the magnitudes.

Step 1: Use the first perpendicularity condition Since $(\vec{a}+\vec{b}) \perp \vec{a}$, take their dot product and expand:

$$(\vec{a}+\vec{b}) \cdot \vec{a} = 0,$$
$$|\vec{a}|^2 + \vec{b} \cdot \vec{a} = 0.$$

By symmetry, this expresses the unknown mixed product as

$$\vec{a} \cdot \vec{b} = -|\vec{a}|^2.$$

Step 2: Substitute into the second condition The other given condition is $(2\vec{a}+\vec{b}) \perp \vec{b}$. Therefore

$$(2\vec{a}+\vec{b}) \cdot \vec{b} = 0,$$
$$2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 0,$$
$$-2|\vec{a}|^2 + |\vec{b}|^2 = 0.$$

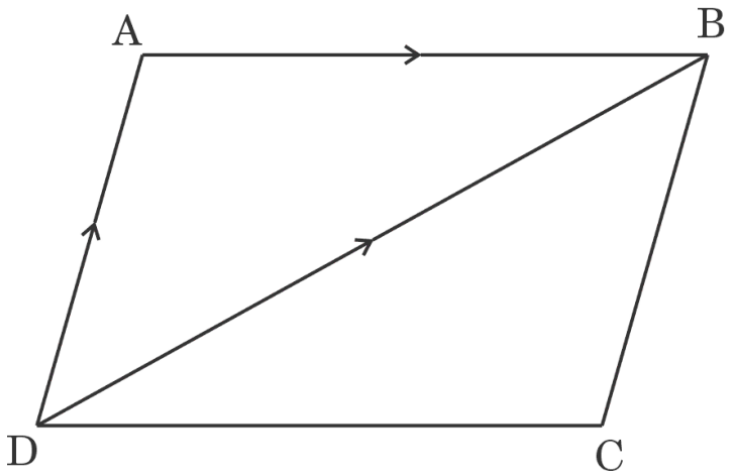
This eliminates the mixed product without needing to find the angle between the vectors. Rearrange to get $|\vec{b}|^2 = 2|\vec{a}|^2$, and take the positive square root because the vectors are non-zero and their magnitudes are positive.

Step 3: Conclusion Hence $\boxed{|\vec{b}| = \sqrt{2}\,|\vec{a}|}$, as required.

Quick Tip to Solve: Convert each perpendicularity into a zero dot product; eliminate $\vec{a} \cdot \vec{b}$ before taking square roots of the magnitudes.

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23. In the given figure, $ABCD$ is a parallelogram. If $\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\overrightarrow{AD}$ and hence find the area of parallelogram $ABCD$. [2 marks]



Solution:

Concept:

- Vector addition along the path $A \to D \to B$ gives $\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$.

Quick Tip to Solve: Track the diagonal's direction: $\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$. Use a cross-product magnitude for area, without dividing by two.

- The area of a parallelogram with adjacent side vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$; the magnitude is independent of the order of the two vectors.

Step 1: Find the adjacent side vector The given diagonal points from D towards B. Applying the triangle law with this direction,

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} - \overrightarrow{DB} \\ &= (2\hat{i} - 4\hat{j} + 5\hat{k}) - (3\hat{i} - 6\hat{j} + 2\hat{k}) \\ &= -\hat{i} + 2\hat{j} + 3\hat{k}.\end{aligned}$$

Step 2: Calculate the cross product and its magnitude Expand the determinant, remembering the negative sign in the middle cofactor:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ -1 & 2 & 3 \end{vmatrix} \\ &= (-12 - 10)\hat{i} - (6 + 5)\hat{j} + (4 - 4)\hat{k} \\ &= -22\hat{i} - 11\hat{j}.\end{aligned}$$

Take its magnitude for the area:

$$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(-22)^2 + (-11)^2 + 0^2} = \sqrt{605}.$$

Step 3: Conclusion $\boxed{\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 3\hat{k}}$ and the area of $ABCD$ is $\boxed{11\sqrt{5} \text{ square units}}$.

Quick Tip to Solve: Track the diagonal's direction: $\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD}$. Use a cross-product magnitude for area, without dividing by two.

24(a). **If $y = \sqrt{\cos x + y}$, prove that $\dfrac{dy}{dx} = \dfrac{\sin x}{1 - 2y}$.** [2 marks]

Solution:

Concept:

- In implicit differentiation, y is a function of x; the chain rule gives $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$.
- Squaring the radical equation removes the square root. The original equation also requires $y \geq 0$, and division by $2y - 1$ is valid wherever this factor is non-zero.

Step 1: Remove the square root and differentiate Starting with the given relation and squaring both sides,

$$y = \sqrt{\cos x + y} \quad \Longrightarrow \quad y^2 = \cos x + y.$$

Differentiate each term with respect to x. Because y depends on x, its derivative must be retained on

Quick Tip to Solve: Square the relation before differentiating, then collect every dy/dx term on one side. This avoids differentiating a nested radical.

both sides:

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}.$$

The derivative of $\cos x$ contributes a minus sign; the derivative of y is not zero.

Step 2: Collect the derivative terms Move the derivative term on the right to the left and factor it out:

$$2y\frac{dy}{dx} - \frac{dy}{dx} = -\sin x,$$
$$(2y-1)\frac{dy}{dx} = -\sin x.$$

For $2y - 1 \neq 0$, divide by this factor. Multiplying both numerator and denominator by -1 puts the expression in the requested form.

Step 3: Conclusion Hence $\boxed{\frac{dy}{dx} = \frac{\sin x}{1-2y}}$, wherever the derivative exists and $y \neq \frac{1}{2}$.

Quick Tip to Solve: Square the relation before differentiating, then collect every dy/dx term on one side. This avoids differentiating a nested radical.

OR

24(b). Show that the function $f(x) = |x|^3$ is differentiable at all points of its domain. [2 marks]

Solution:

Concept:

- The absolute value is defined for every real number, and $|x| = x$ for $x \geq 0$, whereas $|x| = -x$ for $x < 0$.
- Differentiability at a joining point is checked from the difference quotient $\lim_{h\to 0}[f(a+h)-f(a)]/h$; its left and right limits must agree and be finite.

Step 1: Differentiate away from zero The domain of f is $\mathbb{R}$. Write its formula on either side of zero:

$$f(x) = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

On each open half-line this is a polynomial, so it is differentiable there, with

$$f'(x) = \begin{cases} 3x^2, & x > 0, \\ -3x^2, & x < 0. \end{cases}$$

Only the joining point $x = 0$ remains to be checked directly.

Step 2: Use first principles at zero Here $f(0) = 0$. Compute the difference quotient separately for

Quick Tip to Solve: For a power of $|x|$, use polynomial formulas when $x \neq 0$, then test zero with the difference quotient; $|h|^3/h$ tends to zero from both sides.

positive and negative increments:

$$\frac{f(h)-f(0)}{h}=\frac{|h|^3}{h}=\begin{cases}h^2, & h>0,\\ -h^2, & h<0.\end{cases}$$

Therefore the right-hand derivative is $\lim_{h\to 0^+} h^2=0$, and the left-hand derivative is $\lim_{h\to 0^-}(-h^2)=0$. Both limits agree, establishing differentiability at the only possible joining point.

Step 3: Conclusion The function is differentiable at every $x\in\mathbb{R}$, with $\boxed{f'(x)=3x|x|}$; in particular, $f'(0)=0$.

Quick Tip to Solve: For a power of $|x|$, use polynomial formulas when $x\neq 0$, then test zero with the difference quotient; $|h|^3/h$ tends to zero from both sides.

25. Find the absolute maximum and minimum values of the function $f(x)=12\,x^{4/3}-6\,x^{1/3},\ x\in[0,1]$.

[2 marks]

Solution:

Concept:

- A continuous function on a closed interval attains its absolute maximum and minimum. Compare its values at the endpoints and all interior critical points.
- For $x>0$, the power rule gives $\frac{d}{dx}x^r=rx^{r-1}$. A derivative need not exist at an endpoint for the closed-interval method to apply.

Step 1: Find the interior critical point The given function is continuous throughout $[0,1]$. For $0<x<1$, differentiate and combine the powers of x:

$$f'(x)=16x^{1/3}-2x^{-2/3}=\frac{2(8x-1)}{x^{2/3}}.$$

Since the denominator is positive in the open interval, $f'(x)=0$ gives $x=\frac{1}{8}$. This is the only interior critical point. At $x=0$, the derivative is undefined, but the function value must still be tested because this point is an endpoint.

Step 2: Compare the candidate values Evaluate the function at both endpoints and at the critical point:

$$\begin{aligned}f(0)&=0,\\ f(1)&=12-6=6,\\ f\left(\frac{1}{8}\right)&=12\left(\frac{1}{16}\right)-6\left(\frac{1}{2}\right)=\frac{3}{4}-3=-\frac{9}{4}.\end{aligned}$$

Thus the smallest candidate value is $-\frac{9}{4}$, whereas the largest is 6. The change of f' from negative to positive at $\frac{1}{8}$ also confirms the interior minimum.

Step 3: Conclusion The absolute maximum is $\boxed{6}$, attained at $x=1$. The absolute minimum is $\boxed{-\frac{9}{4}}$, attained at $x=\frac{1}{8}$.

Quick Tip to Solve: For this fractional-power function, factor the derivative as $2(8x-1)/x^{2/3}$; then compare $x=0,\frac{1}{8},1$, retaining the endpoint where the derivative is undefined.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). Find: $\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$ [3 marks]

Solution:

Concept:

- Resolve a rational function with distinct quadratic denominator factors into simpler fractions, and determine the constants by equating coefficients.
- For a positive constant a, $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$, where C is an arbitrary constant.

Step 1: Set up the partial fractions The integrand is an even rational function, and both denominator factors have degree two. We can seek constants A, B such that

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{A}{x^2+4} + \frac{B}{x^2+9}.$$

Multiplication by the common denominator gives the polynomial identity

$$x^2 = A(x^2+9) + B(x^2+4) = (A+B)x^2 + 9A + 4B.$$

Equating the coefficients of x^2 and the constant terms yields $A + B = 1$ and $9A + 4B = 0$.

Step 2: Find the constants and integrate Substitute $B = 1 - A$ into the second equation to obtain $5A + 4 = 0$. Hence $A = -\frac{4}{5}$ and $B = \frac{9}{5}$. Therefore, writing the required integral as I,

$$\begin{aligned} I &= -\frac{4}{5}\int \frac{dx}{x^2+2^2} + \frac{9}{5}\int \frac{dx}{x^2+3^2} \\ &= -\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C. \end{aligned}$$

The scale factors $\frac{1}{2}$ and $\frac{1}{3}$ are essential: differentiating $\tan^{-1}(x/a)$ gives $a/(x^2+a^2)$, not $1/(x^2+a^2)$. Both quadratic denominators are positive for all real x, so no real point is excluded.

Step 3: Conclusion

$$\boxed{I = \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C.}$$

Quick Tip to Solve: Equate the x^2 and constant coefficients first; after resolving the fractions, include the $1/a$ factor for each $x^2 + a^2$ denominator.

OR

26(b). Evaluate: $\int_1^3 (|x-1|+|x-2|+|x-3|)\,dx$ [3 marks]

Solution:

Concept:

- The absolute value satisfies $|u| = u$ for $u \geq 0$ and $|u| = -u$ for $u < 0$. Remove each modulus only after checking its sign on the integration interval.
- A definite integral can be split at an interior sign-change point: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Step 1: Identify where the signs change On the entire interval $[1,3]$, $x-1 \geq 0$ and $x-3 \leq 0$. Hence $|x-1| = x-1$ and $|x-3| = 3-x$, whose sum is the constant 2. Only $|x-2|$ changes its formula inside the interval, at $x = 2$. Thus

$$|x-1|+|x-2|+|x-3| = \begin{cases} (x-1)+(2-x)+(3-x) = 4-x, & 1 \leq x \leq 2, \\ (x-1)+(x-2)+(3-x) = x, & 2 \leq x \leq 3. \end{cases}$$

Both expressions have value 2 at the joining point, so the integrand is continuous there.

Step 2: Integrate separately on the two intervals Let the required integral be I. Using the appropriate expression on each interval,

$$\begin{aligned} I &= \int_1^2 (4-x)\,dx + \int_2^3 x\,dx \\ &= \left[4x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 \\ &= \left[(8-2) - \left(4 - \frac{1}{2}\right)\right] + \left(\frac{9}{2} - 2\right) \\ &= \frac{5}{2} + \frac{5}{2}. \end{aligned}$$

As a check, the constant part 2 contributes $2(3-1) = 4$, and the remaining integral of $|x-2|$ contributes $\frac{1}{2} + \frac{1}{2} = 1$. This confirms the same total.

Step 3: Conclusion The value of the definite integral is $\boxed{5}$.

Quick Tip to Solve: Combine the outer terms first: $|x-1|+|x-3| = 2$ throughout $[1,3]$. Then split only at $x = 2$ for the remaining modulus.

27. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$. [3 marks]

Solution:

Concept:

- An equation $dy/dx = F(y/x)$ is homogeneous. Substitute $y = vx$, where v depends on x, and use $dy/dx = v + x\,dv/dx$.
- Separate the variables, integrate, and replace v by y/x. Any constant values excluded while dividing must be checked in the original equation.

Step 1: Apply the homogeneous substitution The given equation is defined where $xy \neq 0$. Its right-hand side can be written as

$$\frac{x^2 + y^2}{2xy} = \frac{1 + (y/x)^2}{2(y/x)}.$$

Put $y = vx$. Differentiating with the product rule and substituting gives

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}, \qquad x\frac{dv}{dx} = \frac{1 - v^2}{2v}.$$

This converts the homogeneous equation into a separable equation for v and x.

Step 2: Separate and integrate For $v^2 \neq 1$, bring the factors involving v to one side:

$$\frac{2v}{1 - v^2}\,dv = \frac{dx}{x}.$$

Since the derivative of $1 - v^2$ is $-2v$, integration gives

$$-\ln|1 - v^2| = \ln|x| + C_1, \qquad \ln|x(1 - v^2)| = C_2.$$

Absorbing the sign and exponential of the constant into a nonzero constant K, obtain $x(1 - v^2) = K$. Substitution of $v = y/x$ then yields

$$x - \frac{y^2}{x} = K, \qquad y^2 = x^2 - Kx.$$

Step 3: Include the excluded constant ratios and verify The values $v = 1$ and $v = -1$, excluded during separation, give $y = x$ and $y = -x$; both satisfy the original equation for $x \neq 0$. They are included by allowing the arbitrary constant to be zero. Rename $-K$ as C. Differentiating $y^2 = x^2 + Cx$ gives $2y\,dy/dx = 2x + C$. Since $C = (y^2 - x^2)/x$, this recovers $dy/dx = (x^2 + y^2)/(2xy)$, as required.

Step 4: Conclusion The general solution is $\boxed{y^2 = x^2 + Cx}$, where C is an arbitrary real constant, on solution intervals where $x \neq 0$ and $y \neq 0$.

Quick Tip to Solve: Use $y = vx$ because numerator and denominator both have degree two. After dividing by $1 - v^2$, remember to recover the solutions $y = \pm x$.

28. Solve the following linear programming problem graphically:

Maximise $z = 5x + 4y$ subject to the constraints

$x + 2y \geq 4$

$3x + y \leq 6$

$x + y \leq 4$

$x, y \geq 0$ [3 marks]

Solution:

Concept:

- Each linear inequality determines a half-plane; their common part in the first quadrant is the feasible region.
- The corner-point method states that a linear objective function on a nonempty bounded polygon attains its maximum at a vertex. Here the objective is $z = 5x + 4y$.

Step 1: Draw the boundary lines and select the half-planes The line $x + 2y = 4$ passes through $(4, 0)$ and $(0, 2)$; take its side away from the origin since $x + 2y \geq 4$. The line $3x + y = 6$ passes through $(2, 0)$ and $(0, 6)$; take its side containing the origin. The line $x + y = 4$ passes through $(4, 0)$ and $(0, 4)$; again take the side containing the origin. Restrict all three half-planes to $x \geq 0,\ y \geq 0$.

Step 2: Find the vertices of the common feasible region On the y-axis, the feasible segment runs from $A = (0, 2)$ to $B = (0, 4)$. The intersection of $3x + y = 6$ and $x + y = 4$ gives $2x = 2$, hence $C = (1, 3)$. For the remaining vertex, solve

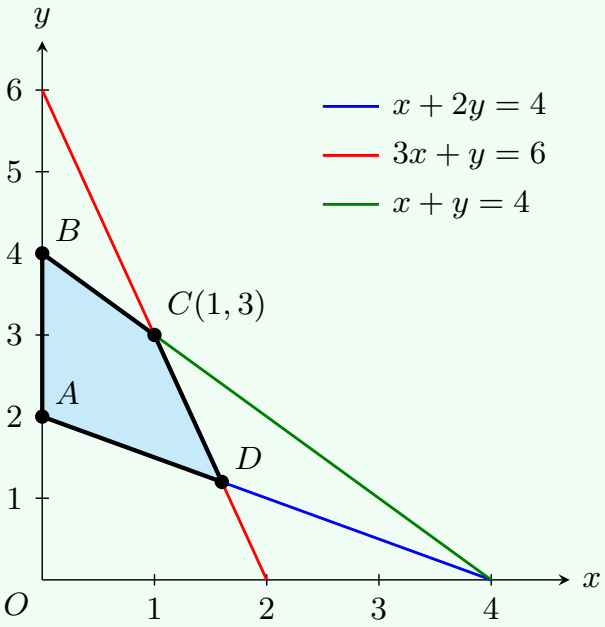
$$x + 2y = 4, \qquad 3x + y = 6,$$

$$y = 6 - 3x,$$

$$x + 2(6 - 3x) = 4 \quad \Rightarrow \quad x = \frac{8}{5}, \quad y = \frac{6}{5}.$$

Thus $D = (8/5, 6/5)$. Both intersections satisfy the third inequality. The common region is the closed, bounded quadrilateral $ABCD$, shaded in the graph. No point on the x-axis is feasible, since there we would require both $x \geq 4$ and $x \leq 2$.

Quick Tip to Solve: Find the common shaded region before listing vertices: the intercept $(4, 0)$ is excluded by $3x + y \leq 6$.



The shaded quadrilateral is the common feasible region.

Step 3: Evaluate the objective function at every vertex Evaluate $z = 5x + 4y$ at the four corners:

Vertex	$z = 5x + 4y$
$A(0, 2)$	8
$B(0, 4)$	16
$C(1, 3)$	17
$D(8/5, 6/5)$	$8 + 24/5 = 64/5$

Since the feasible region is bounded, comparison of these values determines the required maximum. The largest entry occurs at C.

Step 4: Conclusion The maximum value is $\boxed{z_{\max} = 17}$, attained at $\boxed{x = 1,\ y = 3}$.

Quick Tip to Solve: Find the common shaded region before listing vertices: the intercept $(4, 0)$ is excluded by $3x + y \leq 6$.

29. E **and** F **are two independent events such that** $P(\overline{E}) = 0.6$ **and** $P(E \cup F) = 0.6$**. Find** $P(F)$ **and** $P(\overline{E} \cup \overline{F})$**.** [3 marks]

Solution:

Concept:

- Complement rule: $P(\overline{E}) = 1 - P(E)$. For independent events, $P(E \cap F) = P(E)P(F)$.
- The addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. De Morgan's law gives $\overline{E} \cup \overline{F} = \overline{E \cap F}$.

Step 1: Express the probability of the union using independence The given complement probability

Quick Tip to Solve: First use independence in the union formula; then change a union of complements into the complement of an intersection.

yields

$$P(E) = 1 - P(\overline{E}) = 1 - 0.6 = 0.4 = \frac{2}{5}.$$

Let $P(F) = p$. Since E and F are independent, their intersection has probability $P(E \cap F) = 0.4p$. Independence permits this product; it does not mean that the two events are mutually exclusive.

Step 2: Determine the probability of the second event Substitute the given union probability into the addition rule:

$$0.6 = 0.4 + p - 0.4p,$$
$$0.2 = 0.6p,$$
$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

This is a valid probability between zero and one. Consequently,

$$P(E \cap F) = \frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}.$$

As a check, $2/5 + 1/3 - 2/15 = 9/15 = 0.6$, agreeing with the given union probability.

Step 3: Apply De Morgan's law to the union of complements The event $\overline{E} \cup \overline{F}$ means that at least one of E, F does not occur. Its complement is the event that both occur, namely $E \cap F$. Therefore,

$$P(\overline{E} \cup \overline{F}) = 1 - P(E \cap F) = 1 - \frac{2}{15} = \frac{13}{15}.$$

We must complement the intersection here; $1 - P(E \cup F)$ would instead be the probability that neither event occurs.

Step 4: Conclusion $\boxed{P(F) = \frac{1}{3}}$ and $\boxed{P(\overline{E} \cup \overline{F}) = \frac{13}{15}}$.

Quick Tip to Solve: First use independence in the union formula; then change a union of complements into the complement of an intersection.

30(a). A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as $R = \{(x, y) : |x^2 - y^2| < 8\}$. Check whether the relation R is reflexive, symmetric and transitive. [3 marks]

Solution:

Concept:

- A relation on A is reflexive if $(x, x) \in R$ for every $x \in A$, and symmetric if $(x, y) \in R$ implies $(y, x) \in R$.
- Transitivity requires that $(x, y) \in R$ and $(y, z) \in R$ together imply $(x, z) \in R$ for all $x, y, z \in A$. A single counterexample disproves transitivity.

Step 1: Check reflexivity for every element Take any $x \in A = \{1, 2, 3, 4, 5\}$. Then

$$|x^2 - x^2| = 0 < 8.$$

Quick Tip to Solve: For transitivity, try consecutive elements $1, 2, 3$: their squared differences are $3, 5, 8$, and equality to 8 is excluded.

Thus $(x, x) \in R$ for every element of A. Thus every diagonal pair belongs to the relation, including the pairs corresponding to the smallest and largest elements of the set. Therefore R is reflexive.

Step 2: Check symmetry by reversing an arbitrary pair Suppose $(x, y) \in R$. The defining condition gives $|x^2 - y^2| < 8$. Reversing the order changes only the sign inside the absolute value:

$$|y^2 - x^2| = |-(x^2 - y^2)| = |x^2 - y^2| < 8.$$

Hence $(y, x) \in R$. This argument applies to every pair in R, so the relation is symmetric.

Step 3: Test transitivity with three elements of the set Choose $x = 1,\ y = 2,\ z = 3$, all of which lie in A. We have

$$|1^2 - 2^2| = 3 < 8, \qquad |2^2 - 3^2| = 5 < 8.$$

Thus $(1, 2) \in R$ and $(2, 3) \in R$. However,

$$|1^2 - 3^2| = 8,$$

which does not satisfy the strict inequality < 8. Therefore $(1, 3) \notin R$. This provides the two required linked pairs but excludes their composed pair, so the universal condition for transitivity fails. The relation is not transitive.

Step 4: Conclusion R is reflexive and symmetric, but not transitive. Consequently, it is not an equivalence relation.

Quick Tip to Solve: For transitivity, try consecutive elements $1, 2, 3$: their squared differences are $3, 5, 8$, and equality to 8 is excluded.

OR

30(b). A function f is defined from $\mathbb{R} \to \mathbb{R}$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find function $f(x)$. Hence, check whether function $f(x)$ is one-one and onto or not. [3 marks]

Solution:

Concept:

- For an affine function $f(x) = ax + b$, two given input-output pairs determine the constants a and b.
- A function is one-one if $f(u) = f(v)$ implies $u = v$. It is onto if every element of its codomain is the image of some element of its domain.

Step 1: Use the two given values Substituting $x = 1$ and $x = 2$ into the defining formula gives

$$a + b = 1, \qquad 2a + b = 3.$$

Subtracting the first equation from the second gives $a = 2$, and then $b = 1 - 2 = -1$. Thus the candidate formula is $f(x) = 2x - 1$. It satisfies both data: $f(1) = 2 - 1 = 1$ and $f(2) = 4 - 1 = 3$.

Step 2: Test whether distinct inputs can share an image Let $u, v \in \mathbb{R}$ and suppose $f(u) = f(v)$. With

Quick Tip to Solve: After finding the nonzero slope, prove onto by explicitly solving $f(x) = t$ for an arbitrary real t.

the formula obtained above,

$$2u - 1 = 2v - 1 \implies 2u = 2v \implies u = v.$$

Therefore two inputs with the same image must be equal, which proves the one-one property over the entire domain, not merely at the two given inputs.

Step 3: Check every real number in the codomain Take an arbitrary $t \in \mathbb{R}$. Solving $2x - 1 = t$ for its possible preimage gives $x = (t+1)/2$, which is a real number and hence belongs to the domain. Direct substitution verifies

$$f\left(\frac{t+1}{2}\right) = 2\left(\frac{t+1}{2}\right) - 1 = t.$$

Since this construction works for every real t, the range equals the stated codomain $\mathbb{R}$.

Step 4: Conclusion $\boxed{f(x) = 2x - 1}$. The function is both one-one and onto; hence it is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Quick Tip to Solve: After finding the nonzero slope, prove onto by explicitly solving $f(x) = t$ for an arbitrary real t.

31(a). **If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.** [3 marks]

Solution:

Concept:

- In implicit differentiation, y is a function of x; the chain rule gives $\dfrac{d}{dx}\sqrt{1-y^2} = -\dfrac{y}{\sqrt{1-y^2}}\dfrac{dy}{dx}$.
- The constant a can be eliminated using the original relation. The derivative formula is proved on a differentiable interior branch, where $|x| < 1$ and $|y| < 1$.

Step 1: Introduce notation for the radicals Put $U = \sqrt{1-x^2}$, $V = \sqrt{1-y^2}$, and $p = dy/dx$. On the interior of the real domain, $U, V > 0$. The given relation becomes $U + V = a(x-y)$. Since $U + V > 0$, we have $x \neq y$, so

$$a = \frac{U+V}{x-y}.$$

We will use this expression after differentiating, keeping a constant during differentiation.

Step 2: Differentiate the implicit equation Applying the chain rule separately to the two radicals gives

$$-\frac{x}{U} - \frac{y}{V}p = a(1-p).$$

Collecting the terms containing p on one side, we obtain

$$\left(a - \frac{y}{V}\right)p = a + \frac{x}{U}.$$

The factor p in the second radical's derivative is essential because y varies with x.

Step 3: Eliminate the constant and simplify Substitute the expression for a, using $U^2 = 1 - x^2$ and

Quick Tip to Solve: Replace a using the original equation after implicit differentiation; both numerator and denominator then contain $1 + UV - xy$.

$V^2 = 1 - y^2$:

$$a + \frac{x}{U} = \frac{U(U+V) + x(x-y)}{U(x-y)} = \frac{1 + UV - xy}{U(x-y)},$$
$$a - \frac{y}{V} = \frac{V(U+V) - y(x-y)}{V(x-y)} = \frac{1 + UV - xy}{V(x-y)}.$$

Here $1 + UV - xy > 0$, since $|xy| < 1$ and $UV > 0$. Thus the common numerator is nonzero and the derivative equation can be divided by its coefficient. The ratio of the two expressions reduces to V/U.

Step 4: Conclusion Therefore, on the differentiable interior branch,

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}}.$$

Quick Tip to Solve: Replace a using the original equation after implicit differentiation; both numerator and denominator then contain $1 + UV - xy$.

OR

31(b). If $y = (\tan x)^x$, then find $\frac{dy}{dx}$. [3 marks]

Solution:

Concept:

- For a variable base raised to a variable exponent, logarithmic differentiation converts the power into a product: $\ln(u^v) = v \ln u$ for $u > 0$.
- The chain and product rules give $\frac{d}{dx} \ln y = \frac{1}{y}\frac{dy}{dx}$ and $\frac{d}{dx} \tan x = \sec^2 x$.

Step 1: Take natural logarithms Both the base $\tan x$ and the exponent x vary with x, so an ordinary fixed-exponent power rule is insufficient. Work on any open interval where $\tan x$ is finite and positive; the real-valued power $(\tan x)^x$ is then positive and differentiable. Taking logarithms of the given relation yields

$$\ln y = \ln\big((\tan x)^x\big) = x \ln(\tan x).$$

This converts the variable exponent into a factor to which the product rule applies.

Step 2: Differentiate both sides Use the chain rule on the left and the product rule on the right:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\{x \ln(\tan x)\}$$
$$= \ln(\tan x) + x\frac{\sec^2 x}{\tan x}.$$

The logarithm term comes from differentiating the exponent factor x; the second term accounts for the changing base. Retaining both contributions is necessary.

Step 3: Restore the original value of the function Multiply the differentiated equation by y and

Quick Tip to Solve: When both base and exponent contain x, take logarithms first and keep both terms produced by the product rule.

substitute $y = (\tan x)^x$. If desired, simplify the trigonometric quotient using

$$\frac{\sec^2 x}{\tan x} = \frac{1/\cos^2 x}{\sin x/\cos x} = \frac{1}{\sin x \cos x}.$$

All divisions are valid on the positive-tangent intervals specified above, where neither sine nor cosine is zero.

Step 4: Conclusion

$$\boxed{\frac{dy}{dx} = (\tan x)^x \left[\ln(\tan x) + \frac{x \sec^2 x}{\tan x}\right]}.$$

Equivalently, the bracket is $\ln(\tan x) + x/(\sin x \cos x)$.

Quick Tip to Solve: When both base and exponent contain x, take logarithms first and keep both terms produced by the product rule.

Section D (20 marks)

This section comprises long answer type questions (LA) of 5 marks each.

32(a). Evaluate: $\displaystyle\int_0^{\pi/2} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$ [5 marks]

Solution:

Concept:

- The product rule gives $\dfrac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$.
- The identity $\sin^2 x + \cos^2 x = 1$ helps simplify the trigonometric factor.

Step 1: Split the trigonometric factor Denote the required integral by I. On $[0, \pi/2]$, we have $1 + \cos x \geq 1$, so the denominator never vanishes and the integrand is continuous. Write

$$I = \int_0^{\pi/2} e^x \left(\frac{\sin x}{1+\cos x} + \frac{1}{1+\cos x}\right) dx.$$

Step 2: Identify the function and its derivative Put $f(x) = \dfrac{\sin x}{1+\cos x}$. By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}. \end{aligned}$$

Hence the entire factor multiplying e^x is exactly $f(x) + f'(x)$; neither term is omitted. In particular, the constant term in the numerator arises from differentiating the trigonometric quotient, while the

Quick Tip to Solve: When e^x multiplies this ratio, split its numerator into $\sin x$ and 1: the derivative of $\sin x/(1+\cos x)$ is exactly $1/(1+\cos x)$.

sine term arises from differentiating the exponential.

Step 3: Apply the product rule in reverse The fundamental theorem evaluates a definite integral by subtracting the antiderivative at the lower limit from its value at the upper limit. Here the product rule has already supplied the antiderivative. Therefore, substituting the identity gives

$$I = \int_0^{\pi/2} e^x \{f(x) + f'(x)\}\, dx$$
$$= [e^x f(x)]_0^{\pi/2} = \left[\frac{e^x \sin x}{1+\cos x}\right]_0^{\pi/2}.$$

At the upper endpoint, $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$. At the lower endpoint, $\sin 0 = 0$ and $\cos 0 = 1$. Therefore the boundary expression is

$$e^{\pi/2}\frac{1}{1+0} - e^0\frac{0}{1+1}.$$

Step 4: Conclusion The value of the definite integral is

$$\boxed{I = e^{\pi/2}}.$$

Quick Tip to Solve: When e^x multiplies this ratio, split its numerator into $\sin x$ and 1: the derivative of $\sin x/(1+\cos x)$ is exactly $1/(1+\cos x)$.

OR

32(b). Find: $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$ [5 marks]

Solution:

Concept:

- The identity $(\sin x - \cos x)^2 = 1 - \sin 2x$ changes the radical to the standard form $\sqrt{1-t^2}$, where $t = \sin x - \cos x$.
- For $-1 < t < 1$, $\displaystyle\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + C$, with $\sin^{-1}$ denoting the principal inverse sine in radians.

Step 1: Choose a substitution suited to the numerator Let the given integral be I, and set

$$t = \sin x - \cos x, \qquad dt = (\cos x + \sin x)\, dx.$$

This differential accounts for the entire numerator. The difference is the useful choice here, since differentiating the sum would instead produce a difference and would not match the numerator.

Step 2: Express the radical in terms of the new variable Square the chosen expression and use the

Quick Tip to Solve: With numerator $\sin x + \cos x$, try $t = \sin x - \cos x$. Its square converts $\sin 2x$ to $1 - t^2$, and these limits become equal in magnitude with opposite signs.

double-angle identity:

$$\begin{aligned} t^2 &= (\sin x - \cos x)^2 \\ &= \sin^2 x + \cos^2 x - 2\sin x \cos x \\ &= 1 - \sin 2x. \end{aligned}$$

Hence $\sin 2x = 1 - t^2$ and $\sqrt{\sin 2x} = \sqrt{1 - t^2}$. Since $2x \in [\pi/3, 2\pi/3]$, the quantity $\sin 2x$ is positive throughout the interval.

Step 3: Transform both limits At the lower and upper limits respectively,

$$\begin{aligned} x = \frac{\pi}{6}: \quad t &= \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1 - \sqrt{3}}{2}, \\ x = \frac{\pi}{3}: \quad t &= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3} - 1}{2}. \end{aligned}$$

Write $a = (\sqrt{3} - 1)/2$, so $0 < a < 1$ and the transformed limits are $-a$ and a.

Step 4: Integrate in the new variable The integral becomes the standard inverse-sine integral:

$$I = \int_{-a}^{a} \frac{dt}{\sqrt{1 - t^2}} = \left[\sin^{-1} t\right]_{-a}^{a} = \sin^{-1} a - \sin^{-1}(-a).$$

Use $\sin^{-1}(-a) = -\sin^{-1} a$, since inverse sine is odd. The subtraction therefore adds two equal positive contributions.

Step 5: Conclusion Therefore,

$$\boxed{I = 2\sin^{-1}\left(\frac{\sqrt{3} - 1}{2}\right).}$$

Quick Tip to Solve: With numerator $\sin x + \cos x$, try $t = \sin x - \cos x$. Its square converts $\sin 2x$ to $1 - t^2$, and these limits become equal in magnitude with opposite signs.

33. Using integration, find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$, included between the lines $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- For a region between two curves, its area is $A = \int_a^b (y_{\text{upper}} - y_{\text{lower}})\, dx$, where a and b are the bounding abscissae.
- The ellipse $x^2/16 + y^2/4 = 1$ is symmetric about both coordinate axes. Since the bounding lines are $x = -2$ and $x = 2$, the required region consists of four equal parts.
- The substitution $x = 4\sin\theta$ simplifies $\sqrt{16 - x^2}$. Use $\cos^2\theta = (1 + \cos 2\theta)/2$ to integrate the resulting expression.

Quick Tip to Solve: Use four times the first-quadrant area from 0 to 2; the upper boundary is $\frac{1}{2}\sqrt{16 - x^2}$, so retain the factor $\frac{1}{2}$ before substituting $x = 4\sin\theta$.

Step 1: Express the upper and lower boundaries Rearranging the equation of the ellipse gives

$$y^2 = 4\left(1 - \frac{x^2}{16}\right) = \frac{16 - x^2}{4}.$$

Thus, the upper half is $y = \frac{1}{2}\sqrt{16 - x^2}$, and the lower half is $y = -\frac{1}{2}\sqrt{16 - x^2}$.

Step 2: Set up the area integral using symmetry The first-quadrant portion of the required region lies under the upper half of the ellipse from $x = 0$ to $x = 2$. Multiplying its area by four accounts for both sides of each axis:

$$A = 4\int_0^2 \frac{1}{2}\sqrt{16 - x^2}\,dx = 2\int_0^2 \sqrt{16 - x^2}\,dx.$$

Step 3: Transform the integral Put $x = 4\sin\theta$, so $dx = 4\cos\theta\,d\theta$. The limits $x = 0$ and $x = 2$ become $\theta = 0$ and $\theta = \pi/6$, respectively. Since $\cos\theta$ is positive throughout this interval,

$$\sqrt{16 - x^2} = 4\cos\theta, \qquad A = 32\int_0^{\pi/6} \cos^2\theta\,d\theta.$$

Step 4: Evaluate the transformed integral Apply the double-angle identity and then substitute the limits:

$$\begin{aligned} A &= 16\int_0^{\pi/6} (1 + \cos 2\theta)\,d\theta \\ &= 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} \\ &= 16\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right). \end{aligned}$$

Step 5: Conclusion The area of the ellipse included between $x = -2$ and $x = 2$ is

$$\boxed{\frac{8\pi}{3} + 4\sqrt{3} \text{ square units}}.$$

Quick Tip to Solve: Use four times the first-quadrant area from 0 to 2; the upper boundary is $\frac{1}{2}\sqrt{16 - x^2}$, so retain the factor $\frac{1}{2}$ before substituting $x = 4\sin\theta$.

34. Equations of sides of a parallelogram $ABCD$ are as follows:

$$AB: \frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-1}{2}$$
$$BC: \frac{x-1}{3} = \frac{y+2}{-5} = \frac{z-5}{3}$$
$$CD: \frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$$
$$DA: \frac{x-2}{3} = \frac{y+3}{-5} = \frac{z-4}{3}$$

Find the equation of diagonal BD. [5 marks]

Solution:

Concept:

- A line $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c = t$ has parametric coordinates $(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$, where (a, b, c) are direction ratios.
- Adjacent sides meet at a vertex: $B = AB \cap BC$ and $D = CD \cap DA$. Find each intersection using separate parameters for the two lines and verify all three coordinates.
- A line through two distinct points B and D has direction vector $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB}$. Its equation is $\vec{r} = \overrightarrow{OB} + \lambda\overrightarrow{BD}$.

Step 1: Find vertex B from the first two sides Give the common ratios on AB and BC the values t and s, respectively. Their parametric forms are

$$AB: \quad (x, y, z) = (-1 + t,\ 2 - 2t,\ 1 + 2t),$$
$$BC: \quad (x, y, z) = (1 + 3s,\ -2 - 5s,\ 5 + 3s).$$

Equating the x- and y-coordinates at their intersection gives

$$t - 3s = 2, \qquad 2t - 5s = 4.$$

Substitute $t = 2 + 3s$ into the second equation to obtain

$$4 + 6s - 5s = 4, \qquad s = 0, \quad t = 2.$$

The z-coordinates also agree: $1 + 2(2) = 5 + 3(0) = 5$. Hence the intersection is $B = (1, -2, 5)$.

Step 2: Find vertex D from the last two sides Use parameters u on CD and v on DA:

$$CD: \quad (x, y, z) = (4 + u,\ -7 - 2u,\ 8 + 2u),$$
$$DA: \quad (x, y, z) = (2 + 3v,\ -3 - 5v,\ 4 + 3v).$$

Equating their first two coordinates gives

$$u - 3v = -2, \qquad 2u - 5v = -4.$$

As $u = -2 + 3v$, the second equation yields

$$-4 + 6v - 5v = -4, \qquad v = 0, \quad u = -2.$$

Quick Tip to Solve: Find only $AB \cap BC$ and $CD \cap DA$ for diagonal BD. Check the third coordinate after solving the first two; the point displayed in a side's equation need not be a named vertex.

Check the remaining coordinate: $8 + 2(-2) = 4 + 3(0) = 4$. Therefore $D = (2, -3, 4)$.

Step 3: Obtain the direction ratios of the diagonal Subtract the coordinates of B from those of D:

$$\overrightarrow{BD} = (2 - 1,\ -3 - (-2),\ 4 - 5) = (1, -1, -1).$$

Thus, using B as the point on the required line, its parametric representation is

$$x = 1 + \lambda, \qquad y = -2 - \lambda, \qquad z = 5 - \lambda.$$

At $\lambda = 0$ this gives B, and at $\lambda = 1$ it gives D, confirming that the line passes through both opposite vertices.

Step 4: Conclusion The equation of diagonal BD is

$$\boxed{\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z-5}{-1}}.$$

Equivalently, $\vec{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + \lambda(\hat{i} - \hat{j} - \hat{k})$, where $\lambda \in \mathbb{R}$.

Quick Tip to Solve: Find only $AB \cap BC$ and $CD \cap DA$ for diagonal BD. Check the third coordinate after solving the first two; the point displayed in a side's equation need not be a named vertex.

35(a). If $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix}$, find A^{-1} and use it to solve the following system of equations: $x - 2y = 10,\ 2x - y - z = 8,\ -2y + z = 7$. [5 marks]

Solution:

Concept:

- For a square matrix A, the inverse exists when $|A| \neq 0$, and $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j. The adjoint is the transpose of the cofactor matrix.
- A linear system $AX = B$ has the unique solution $X = A^{-1}B$ if A is invertible; here $X = (x, y, z)^T$.

Step 1: Check that the coefficient matrix is invertible Expand along the first row of the given matrix.

Quick Tip to Solve: Here the determinant is 1, so the inverse equals the adjoint. Remember to transpose the cofactor matrix before multiplying by $(10, 8, 7)^T$.

Include the negative sign of the second cofactor:

$$|A| = 1\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$
$$= 1(-1-2) + 2(2-0) = -3 + 4 = 1.$$

Since this determinant is nonzero, the inverse exists and the system has exactly one solution.

Step 2: Compute the cofactors and transpose their matrix Evaluate each minor and apply the alternating cofactor signs:

$$\begin{aligned} C_{11} &= -1-2 = -3, & C_{12} &= -(2-0) = -2, \\ C_{13} &= -4-0 = -4, & C_{21} &= -(-2-0) = 2, \\ C_{22} &= 1-0 = 1, & C_{23} &= -(-2-0) = 2, \\ C_{31} &= 2-0 = 2, & C_{32} &= -(-1-0) = 1, \\ C_{33} &= -1+4 = 3. \end{aligned}$$

Thus the cofactor matrix and its transpose are

$$C = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \text{adj } A = C^T = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

Dividing the adjoint by $|A| = 1$ leaves these entries unchanged, giving the required inverse.

Step 3: Multiply the constant vector by the inverse In the printed order of equations, the coefficient matrix is precisely A. Therefore

$$AX = B, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad B = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}.$$

Premultiplying by A^{-1}, rather than changing the equation order, gives

$$X = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}$$
$$= \begin{pmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}.$$

Quick Tip to Solve: Here the determinant is 1, so the inverse equals the adjoint. Remember to transpose the cofactor matrix before multiplying by $(10, 8, 7)^T$.

Step 4: Verify the values in the original system Substituting the obtained values checks all three equations independently:

$$x - 2y = 0 - 2(-5) = 10,$$
$$2x - y - z = 0 - (-5) - (-3) = 8,$$
$$-2y + z = -2(-5) - 3 = 7.$$

Each left-hand side equals its given constant, confirming the matrix multiplication and the signs.

Step 5: Conclusion The required inverse and solution are

$$\boxed{A^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}}, \qquad \boxed{x = 0,\ y = -5,\ z = -3}.$$

Quick Tip to Solve: Here the determinant is 1, so the inverse equals the adjoint. Remember to transpose the cofactor matrix before multiplying by $(10, 8, 7)^T$.

OR

35(b). **If** $A = \begin{pmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{pmatrix}$ **and** $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{pmatrix}$**, find the value of** $(a + x) - (b + y)$**.** [5 marks]

Solution:

Concept:

- A matrix and its inverse satisfy $AA^{-1} = I$, where I is the identity matrix: its diagonal entries are 1 and its off-diagonal entries are 0.
- The entry in row i, column j of a product is the dot product of row i of the first matrix with column j of the second. Select entries containing only one unknown to avoid simultaneous nonlinear equations.

Step 1: Identify useful entries of the identity product Let B denote the matrix given as A^{-1}. Then

$$\begin{pmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{pmatrix} = I.$$

The third row of A is fully known, so its products with the first and second columns of B determine b and y. Similarly, the third column of B is fully known and determines a and x from the first two rows of A. All four selected entries are off the main diagonal and must be zero.

Step 2: Find the unknown entries in the inverse From the $(3, 1)$ entry of $AB = I$,

$$3(1) + 1(-8) + 1(b) = 0 \quad \Longrightarrow \quad b = 5.$$

Quick Tip to Solve: Multiply the known third row of A into columns 1 and 2 of A^{-1}, then use the known third column of A^{-1}. These four zero entries isolate the unknowns immediately.

From the $(3,2)$ entry,

$$3(-1)+1(7)+1(y)=0 \quad \Longrightarrow \quad y=-4.$$

These equations use the third row of the original matrix, not the third row of its inverse; the order of matrix multiplication matters.

Step 3: Find the unknown entries in the original matrix Use the $(1,3)$ entry of $AB=I$:

$$(-1)(1)+a(-5)+2(3)=0 \quad \Longrightarrow \quad 5-5a=0 \quad \Longrightarrow \quad a=1.$$

Next, use the $(2,3)$ entry:

$$1(1)+2(-5)+x(3)=0 \quad \Longrightarrow \quad -9+3x=0 \quad \Longrightarrow \quad x=3.$$

Thus all four unknowns have been obtained using linear equations only.

Step 4: Check the complete identity and evaluate the expression Substituting these values into the entire product verifies that the unused entries also agree:

$$AB=\begin{pmatrix} -1-8+10 & 1+7-8 & -1-5+6 \\ 1-16+15 & -1+14-12 & 1-10+9 \\ 3-8+5 & -3+7-4 & 3-5+3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now form the two sums before subtracting. Since y is negative, retain its sign inside the parentheses:

$$a+x=1+3=4, \qquad b+y=5+(-4)=1.$$

The requested difference is therefore $4-1$.

Step 5: Conclusion The values are $a=1,\ x=3,\ b=5,\ y=-4$, and hence

$$\boxed{(a+x)-(b+y)=3}.$$

Quick Tip to Solve: Multiply the known third row of A into columns 1 and 2 of A^{-1}, then use the known third column of A^{-1}. These four zero entries isolate the unknowns immediately.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that, an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.

On the basis of the above information, answer the following questions: [4 marks]

36(i). Find the probability that an airplane reached its destination late. [2 marks]

Solution:

Concept:

- The total probability theorem combines conditional probabilities over mutually exclusive and exhaustive events.
- If S, M, L denote severe, moderate and light turbulence and D denotes late arrival, then $P(D) = P(S)P(D \mid S) + P(M)P(D \mid M) + P(L)P(D \mid L)$.

Step 1: Identify the three possible conditions The three turbulence categories are equally likely, so

$$P(S) = P(M) = P(L) = \frac{1}{3}.$$

Convert the given percentages to probabilities:

$$P(D \mid S) = \frac{55}{100}, \quad P(D \mid M) = \frac{37}{100}, \quad P(D \mid L) = \frac{17}{100}.$$

Every late flight belongs to exactly one of these three categories, so their joint probabilities can be added.

Step 2: Apply total probability Weight each late-arrival probability by the probability of its turbulence category:

$$\begin{aligned} P(D) &= \frac{1}{3} \cdot \frac{55}{100} + \frac{1}{3} \cdot \frac{37}{100} + \frac{1}{3} \cdot \frac{17}{100} \\ &= \frac{55 + 37 + 17}{300} = \frac{109}{300}. \end{aligned}$$

Equal category probabilities make this the arithmetic mean of the three conditional late-arrival probabilities.

Step 3: Conclusion The probability that the airplane reaches its destination late is $\boxed{\frac{109}{300}}$.

Quick Tip to Solve: With three equally likely turbulence categories, average 55%, 37% and 17%; adding them without dividing by three is incorrect.

36(ii). If the airplane reached its destination late, find the probability that it was due to moderate turbulence. [2 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: $P(M \mid D) = \dfrac{P(M)P(D \mid M)}{P(D)}$, where M denotes moderate turbulence and D late arrival.
- The denominator $P(D)$ includes late arrivals under all mutually exclusive turbulence conditions, not just moderate turbulence.

Step 1: Find the probabilities of late arrival Write S and L for severe and light turbulence. All three categories have probability $1/3$. Thus the total probability of a late flight is

$$\begin{aligned} P(D) &= P(S)P(D \mid S) + P(M)P(D \mid M) + P(L)P(D \mid L) \\ &= \frac{1}{3} \cdot \frac{55}{100} + \frac{1}{3} \cdot \frac{37}{100} + \frac{1}{3} \cdot \frac{17}{100} = \frac{109}{300}. \end{aligned}$$

The joint probability of moderate turbulence and late arrival is

$$P(M \cap D) = \frac{1}{3} \cdot \frac{37}{100} = \frac{37}{300}.$$

Step 2: Condition on the known late arrival Restrict the sample space to late flights and take the fraction of those flights that encountered moderate turbulence:

$$P(M \mid D) = \frac{P(M \cap D)}{P(D)} = \frac{37/300}{109/300} = \frac{37}{109}.$$

The given 37% describes late arrival when moderate turbulence is already known; it is not the reversed conditional probability asked here.

Step 3: Conclusion Given that the airplane arrived late, the probability that this was due to moderate turbulence is $\boxed{\dfrac{37}{109}}$.

Quick Tip to Solve: Since the three prior probabilities are equal, the required Bayes ratio simplifies directly to $37/(55 + 37 + 17)$.

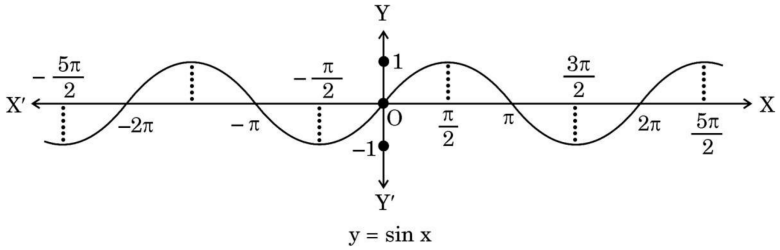
37. Case Study - 2

If a function $f : X \to Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x),\ y \in Y$. Function g is called the inverse of function f.

The domain of sine function is $\mathbb{R}$ and function sine: $\mathbb{R} \to \mathbb{R}$ is neither one-one nor onto. The following graph shows the sine function.

Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A.

On the basis of the above information, answer the following questions: [4 marks]



y = sin x

37(i). If A is the interval other than principal value branch, give an example of one such interval. [1 mark]

Solution:

Concept:

- An inverse exists when sine restricted to A is one-one and onto $[-1, 1]$.
- The principal branch uses $[-\pi/2, \pi/2]$; a different full monotonic interval can define another branch.

Step 1: Choose a different monotonic interval Consider $A = [\pi/2, 3\pi/2]$. Sine decreases strictly on this interval, since $\cos x < 0$ in its interior. Hence no two distinct inputs in this interval have the same sine value.

Step 2: Verify that every required value occurs At the endpoints, $\sin(\pi/2) = 1$ and $\sin(3\pi/2) = -1$. By continuity, the sine values cover all of $[-1, 1]$. Thus this restriction is both one-one and onto, so it has an inverse.

Step 3: Conclusion One such interval other than the principal value branch is $\boxed{A = [\pi/2, 3\pi/2]}$.

Quick Tip to Solve: Choose one complete decreasing half-wave of sine, from a maximum to the next minimum, to obtain a nonprincipal inverse branch.

37(ii). **If $\sin^{-1}(x)$ is defined from $[-1,1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$.** [1 mark]

Solution:

Concept:

- The principal value of $\sin^{-1} x$ lies in $[-\pi/2, \pi/2]$.
- To evaluate inverse sine, select the unique angle in this interval whose sine equals the given input.

Step 1: Identify the principal angles Since $\sin(-\pi/6) = -1/2$ and $-\pi/6$ belongs to the principal interval,

$$\sin^{-1}(-1/2) = -\frac{\pi}{6}.$$

Similarly, $\sin(\pi/2) = 1$, and the endpoint $\pi/2$ is included, giving $\sin^{-1}(1) = \pi/2$.

Step 2: Subtract using a common denominator Substitute these angles in the given order:

$$\sin^{-1}(-1/2) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{6} - \frac{3\pi}{6}.$$

The difference itself need not lie in the principal interval; only the individual inverse-sine values must do so.

Step 3: Conclusion The required value is $\boxed{-\frac{2\pi}{3}}$.

Quick Tip to Solve: Use the negative principal angle for $-1/2$, then keep the subtraction sign before $\pi/2$ to avoid a sign error.

37(iii)(a). **Draw the graph of $\sin^{-1} x$ from $[-1,1]$ to its principal value branch.** [2 marks]

Solution:

Concept:

- The graph of an inverse is obtained by interchanging the coordinates on the graph of the original one-one function, equivalently reflecting in $y = x$.
- For principal inverse sine, $y = \sin^{-1} x$ means $x = \sin y$ with $-\pi/2 \le y \le \pi/2$; its domain is $[-1,1]$.

Step 1: Find points on the inverse graph Restrict the original sine curve to its increasing principal interval. Interchanging the coordinates of its endpoint and origin values gives

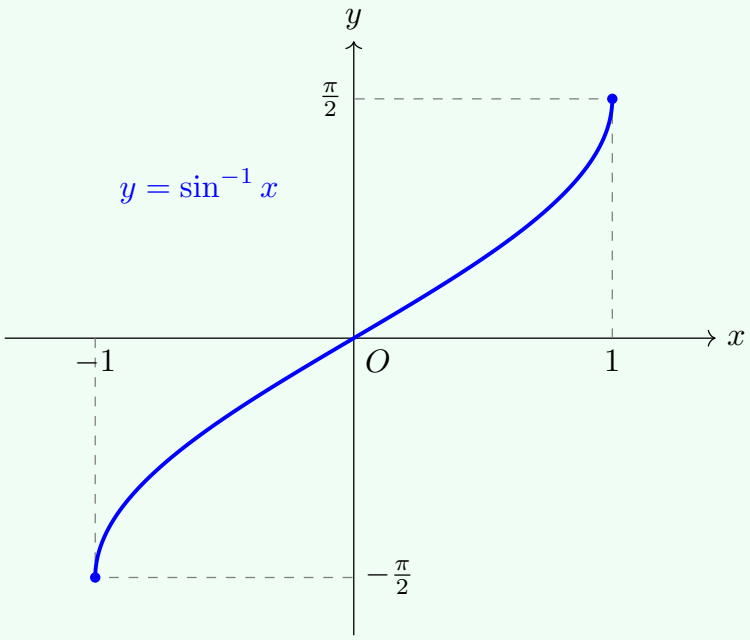
$$(-\pi/2, -1) \longmapsto (-1, -\pi/2),$$
$$(0,0) \longmapsto (0,0),$$
$$(\pi/2, 1) \longmapsto (1, \pi/2).$$

Both endpoints are included. For further guidance, the curve also passes through $(-1/2, -\pi/6)$ and $(1/2, \pi/6)$.

Step 2: Sketch the increasing principal branch Mark the three main points and join them by a smooth

Quick Tip to Solve: Swap the sine graph's endpoint coordinates: $(\pm\pi/2, \pm 1)$ become $(\pm 1, \pm\pi/2)$. Include both endpoints.

increasing curve. It is symmetric about the origin and becomes vertical at its two endpoints. The horizontal coordinate runs only from -1 to 1, while the vertical coordinate runs from $-\pi/2$ to $\pi/2$, in radians. The plotted branch therefore gives exactly one angle for every allowed input.



Principal branch of $y = \sin^{-1} x$; solid dots indicate included endpoints.

Step 3: Conclusion The graph shown is $y = \sin^{-1} x$, with domain $\boxed{[-1, 1]}$ and principal range $\boxed{[-\pi/2, \pi/2]}$.

Quick Tip to Solve: Swap the sine graph's endpoint coordinates: $(\pm\pi/2, \pm 1)$ become $(\pm 1, \pm\pi/2)$. Include both endpoints.

OR

37(iii)(b). Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$. [2 marks]

Solution:

Concept:

- The real-valued principal inverse sine $\sin^{-1} u$ requires $-1 \le u \le 1$, and its values lie in $[-\pi/2, \pi/2]$.
- The domain is the set of allowed inputs x; the range is the set of attained outputs $f(x)$. Multiplying all outputs by a positive constant multiplies both range endpoints by that constant.

Step 1: Restrict the input of inverse sine Here the input of inverse sine is $1 - x$. Therefore impose its real-domain restriction:

$$-1 \le 1 - x \le 1.$$

The left inequality gives $x \le 2$, while the right inequality gives $x \ge 0$. Both must hold simultaneously. The endpoints are allowed because inverse sine is defined at both -1 and 1.

Quick Tip to Solve: Apply the input restriction to $1 - x$ first; the outside factor 2 changes the range, not the domain.

Step 2: Determine all attained output values As x increases continuously from 0 to 2, the quantity $1-x$ runs through the entire interval $[-1,1]$, in reverse order. Thus $\sin^{-1}(1-x)$ attains every value from $-\pi/2$ to $\pi/2$. Multiplying these values by 2 gives

$$-\pi \le 2\sin^{-1}(1-x) \le \pi.$$

Indeed, $f(0)=\pi$ and $f(2)=-\pi$; continuity ensures that all values between these extremes occur.

Step 3: Conclusion The domain is $\boxed{[0,2]}$, and the range is $\boxed{[-\pi,\pi]}$.

Quick Tip to Solve: Apply the input restriction to $1-x$ first; the outside factor 2 changes the range, not the domain.

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38. Case Study - 3

The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.

A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

On the basis of the above information, answer the following questions: [4 marks]

38(i). Express θ in terms of height of the camera installed on the pole and x. [1 mark]

Solution:

Concept:

- For an acute angle in a right triangle, tangent equals the opposite side divided by the adjacent side.
- The principal inverse tangent converts a positive tangent ratio into the unique angle in $(0,\pi/2)$.

Step 1: Identify the perpendicular and base The pole is vertical and the car is on the horizontal road. Relative to the angle of elevation θ at the car, the opposite side is the camera height $5\,\text{m}$, and the adjacent side is the horizontal distance $x\,\text{m}$, where $x>0$.

Step 2: Apply the tangent relation Taking the ratio of the two sides gives

$$\tan\theta = \frac{5}{x}.$$

The elevation angle is acute, so applying the principal inverse tangent yields the required angle without any additional multiple of π.

Step 3: Conclusion $\boxed{\theta = \tan^{-1}\left(\frac{5}{x}\right)}$, for $x>0$, with θ measured in radians.

Quick Tip to Solve: For an angle of elevation at the car, put the vertical height over the horizontal distance inside inverse tangent.

38(ii). Find $\frac{d\theta}{dx}$. [1 mark]

Solution:

Concept:

- The geometry gives $\theta = \tan^{-1}(5/x)$, where $x > 0$ is the horizontal distance in metres.
- The chain rule gives $\frac{d}{dx}(\tan^{-1} u) = \frac{u'}{1+u^2}$, with the angle measured in radians.

Step 1: Differentiate the inner function Set $u = 5/x = 5x^{-1}$. The height is constant, so differentiation with respect to distance gives

$$\frac{du}{dx} = -5x^{-2} = -\frac{5}{x^2}.$$

Step 2: Use the inverse tangent derivative Apply the chain rule and multiply numerator and denominator by x^2:

$$\frac{d\theta}{dx} = \frac{-5/x^2}{1+25/x^2} = -\frac{5}{x^2+25}.$$

The negative sign agrees with the geometry: increasing the distance lowers the angle of elevation.

Step 3: Conclusion $\boxed{\frac{d\theta}{dx} = -\frac{5}{x^2+25}}$, in radians per metre.

Quick Tip to Solve: Differentiate $5/x$ as well as inverse tangent; omitting the inner derivative loses the essential negative sign.

38(iii)(a). Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole. [2 marks]

Solution:

Concept:

- The angle of elevation satisfies $\tan\theta = 5/x$, where x is the horizontal distance from the pole in metres and θ is measured in radians.
- For related rates, $\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt}$. The car moving away from the pole makes dx/dt positive.

Step 1: Relate the angular rate to the car's velocity Using $\theta = \tan^{-1}(5/x)$, differentiation with respect to x gives

$$\frac{d\theta}{dx} = \frac{-5/x^2}{1+25/x^2} = -\frac{5}{x^2+25}.$$

Since the distance changes with time, multiply by the car's velocity:

$$\frac{d\theta}{dt} = -\frac{5}{x^2+25}\frac{dx}{dt}.$$

The negative coefficient shows that the angle decreases when the car moves away.

Step 2: Substitute the instantaneous distance and speed At the required instant, $x = 50\,\text{m}$ and

Quick Tip to Solve: Multiply the derivative with respect to distance by 20; a car travelling away must produce a negative angular rate.

$dx/dt = 20\,\text{m s}^{-1}$. Therefore

$$\frac{d\theta}{dt} = -\frac{5(20)}{50^2+25}$$
$$= -\frac{100}{2525} = -\frac{4}{101}\,\text{rad s}^{-1}.$$

Substituting $x = 50$ after differentiation retains the fact that x is a changing distance, not a constant throughout the motion.

Step 3: Conclusion $\boxed{\frac{d\theta}{dt} = -\frac{4}{101}\,\text{rad s}^{-1}}$. The angle of elevation is decreasing at $\frac{4}{101}\,\text{rad s}^{-1}$.

Quick Tip to Solve: Multiply the derivative with respect to distance by 20; a car travelling away must produce a negative angular rate.

OR

38(iii)(b). If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car. [2 marks]

Solution:

Concept:

- The fixed height gives $\tan\theta = 5/x$, where x is the car's horizontal distance in metres and the elevation angle θ is in radians.
- The chain rule relates the rates by $\frac{d\theta}{dt} = -\frac{5}{x^2+25}\frac{dx}{dt}$. Speed is the non-negative magnitude $v = |dx/dt|$.

Step 1: Find the angular sensitivity at the given distance Differentiate $\theta = \tan^{-1}(5/x)$ with respect to the changing distance:

$$\frac{d\theta}{dx} = \frac{-5/x^2}{1+25/x^2} = -\frac{5}{x^2+25}.$$

At $x = 50\,\text{m}$, this becomes

$$\frac{d\theta}{dx} = -\frac{5}{2525} = -\frac{1}{505}.$$

Thus each metre of increasing distance produces a decrease of the angle at this instant.

Step 2: Recover the speed from the angular rate The printed angular rate is positive: $d\theta/dt = 3/101\,\text{rad s}^{-1}$. Using the chain rule gives

$$\frac{3}{101} = -\frac{1}{505}\frac{dx}{dt}, \qquad \frac{dx}{dt} = -\frac{505(3)}{101} = -15\,\text{m s}^{-1}.$$

The negative velocity means this other car approaches the pole. Its speed is the magnitude of this velocity. If the stated rate is read as an angular-rate magnitude instead, the speed obtained is the same, since $|d\theta/dt| = v/505$.

Step 3: Conclusion The speed of the other car is $\boxed{15\,\text{m s}^{-1}}$.

Quick Tip to Solve: Solve for dx/dt using the new angular rate; then take its magnitude because the question asks for speed.