

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. All questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators **is not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $x = at$, $y = \frac{a}{t}$, then $\frac{dy}{dx}$ is:

[1 mark]

- (A) t^2
- (B) $-t^2$
- (C) $\frac{1}{t^2}$
- (D) $-\frac{1}{t^2}$

Correct Answer: (D) $-\frac{1}{t^2}$

Solution:

Concept:

- For parametric equations, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- The derivative of t^{-1} is $-t^{-2}$.

Step 1: Differentiate both coordinates With a constant, $x = at$ gives $dx/dt = a$, while $y = at^{-1}$ gives $dy/dt = -a/t^2$. The parameter must satisfy $t \neq 0$, and the usual parametric derivative requires $a \neq 0$.

Step 2: Form the ratio and check the options Consequently,

$$\frac{dy}{dx} = \frac{-a/t^2}{a} = -\frac{1}{t^2}.$$

Options (A) and (B) use the reciprocal incorrectly. Option (C) has the correct magnitude but loses the negative sign from differentiating t^{-1} .

Step 3: Conclusion $\frac{dy}{dx} = -\frac{1}{t^2}$. Hence, the correct option is (D).

Quick Tip to Solve: Differentiate with respect to the same parameter first, then divide; keep the minus sign from the reciprocal.

2. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{\log y}$ is:

[1 mark]

- (A) $\log y = x + c$
- (B) $y \log y - y = x + c$
- (C) $\log y - y = x + c$
- (D) $y \log y + y = x + c$

Correct Answer: (B) $y \log y - y = x + c$

Solution:

Concept:

- A separable differential equation allows the variables to be placed on opposite sides: $\log y \, dy = dx$.
- Integration by parts gives $\int u \, dv = uv - \int v \, du$; here $u = \log y$ and $dv = dy$.

Step 1: Separate and integrate On intervals where $y > 0$ and $y \neq 1$, multiply by $\log y$ to obtain $\log y \, dy = dx$. Therefore

$$\int \log y \, dy = x + c.$$

Taking $u = \log y$, $du = dy/y$, and $v = y$ yields $\int \log y \, dy = y \log y - y$.

Step 2: Verify and reject the alternatives Thus $y \log y - y = x + c$. Differentiating its left side with respect to x gives $\log y (dy/dx) = 1$, exactly the given equation. The derivatives with respect to y of the other left sides are $1/y$, $1/y - 1$, and $\log y + 2$, respectively, so none gives the required factor $\log y$.

Step 3: Conclusion The solution is $y \log y - y = x + c$. Hence, the correct option is (B).

Quick Tip to Solve: For the integral of a logarithm, choose the logarithm as the first function in integration by parts.

3. The vector with terminal point $A(2, -3, 5)$ and initial point $B(3, -4, 7)$ is:

[1 mark]

- (A) $\hat{i} - \hat{j} + 2\hat{k}$
- (B) $\hat{i} + \hat{j} + 2\hat{k}$
- (C) $-\hat{i} - \hat{j} - 2\hat{k}$
- (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Solution:

Concept:

- A directed vector equals terminal position vector minus initial position vector: $\overrightarrow{BA} = \vec{A} - \vec{B}$.
- Coordinate differences are taken in the same order in all three components.

Step 1: Identify the direction The vector starts at $B(3, -4, 7)$ and ends at $A(2, -3, 5)$. Hence it is $\overrightarrow{BA}$, so each coordinate of B must be subtracted from the corresponding coordinate of A .

Step 2: Subtract and compare every option

$$\overrightarrow{BA} = (2 - 3)\hat{i} + [-3 - (-4)]\hat{j} + (5 - 7)\hat{k} = -\hat{i} + \hat{j} - 2\hat{k}.$$

Quick Tip to Solve: Write terminal minus initial above your working before subtracting signed coordinates.

Option (A) reverses the direction. Option (B) has incorrect first and third signs, while option (C) has an incorrect second sign. Only (D) gives all three required differences.

Step 3: Conclusion $\vec{BA} = -\hat{i} + \hat{j} - 2\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: Write terminal minus initial above your working before subtracting signed coordinates.

4. The distance of point $P(a, b, c)$ from y -axis is:

[1 mark]

(A) b

(B) b^2

(C) $\sqrt{a^2 + c^2}$

(D) $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$

Solution:

Concept:

- Every point on the y -axis has the form $(0, t, 0)$, where t is real.
- The perpendicular distance is the minimum Euclidean distance from the point to this axis.

Step 1: Locate the perpendicular foot For $P(a, b, c)$, the perpendicular foot on the y -axis is $F(0, b, 0)$. Keeping the second coordinate unchanged ensures that PF has no component along the axis.

Step 2: Calculate the distance and check choices

$$PF = \sqrt{(a-0)^2 + (b-b)^2 + (c-0)^2} = \sqrt{a^2 + c^2}.$$

More generally, distance squared to $(0, t, 0)$ is $a^2 + c^2 + (b-t)^2$, minimized at $t = b$. Options (A) and (B) wrongly depend on position along the axis; (D) is the squared distance rather than distance.

Step 3: Conclusion The distance is $\sqrt{a^2 + c^2}$. Hence, the correct option is (C).

Quick Tip to Solve: For distance from a coordinate axis, omit the coordinate along that axis and square the other two.

5. The number of corner points of the feasible region determined by constraints $x \geq 0$, $y \geq 0$, $x + y \geq 4$ is: [1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (C) 2

Solution:

Concept:

- A corner point is a vertex where two boundary portions of the feasible region meet.
- An intersection of boundary lines counts only if it satisfies all the constraints.

Step 1: Find possible boundary intersections The boundary lines are $x = 0$, $y = 0$, and $x + y = 4$. Their pairwise intersections are $(0, 0)$, $(0, 4)$, and $(4, 0)$. These exhaust the possible finite vertices of the region.

Step 2: Check feasibility and count vertices The origin violates $x + y \geq 4$, so it is excluded. Both $(0, 4)$ and $(4, 0)$ satisfy all three inequalities and are corners. The feasible region extends indefinitely above the segment joining them, but unboundedness does not remove its finite vertices. Thus (A) and (B) undercount, while (D) incorrectly includes the origin.

Step 3: Conclusion There are 2 corner points. Hence, the correct option is (C).

Quick Tip to Solve: List boundary intersections and test every inequality; an unbounded region can still have corner points.

6. If matrices A and B are of order 1×3 and 3×1 respectively, then the order of $A'B'$ is: [1 mark]

- (A) 1×1
- (B) 3×1
- (C) 1×3
- (D) 3×3

Correct Answer: (D) 3×3

Solution:

Concept:

- Transposition interchanges rows and columns: an $m \times n$ matrix becomes $n \times m$.
- The product of an $m \times n$ matrix and an $n \times p$ matrix has order $m \times p$.

Step 1: Transpose the given orders Since A has order 1×3 , its transpose A' has order 3×1 . Since B has order 3×1 , its transpose B' has order 1×3 .

Step 2: Use the multiplication rule The adjacent dimensions in $(3 \times 1)(1 \times 3)$ agree, so $A'B'$ exists. The outer dimensions give three rows and three columns. Option (A) is the order of AB , not the requested transposed product. Options (B) and (C) merely reproduce the individual transpose orders

Quick Tip to Solve: Transpose each order first, then match the inner dimensions and retain the outer dimensions.

and do not describe their product.

Step 3: Conclusion The order of A/B is 3×3 . Hence, the correct option is (D).

Quick Tip to Solve: Transpose each order first, then match the inner dimensions and retain the outer dimensions.

7. A relation R defined on a set of human beings as $R = \{(x, y) : x \text{ is 5 cm shorter than } y\}$ is: [1 mark]

- (A) reflexive only
- (B) reflexive and transitive
- (C) symmetric and transitive
- (D) neither transitive, nor symmetric, nor reflexive

Correct Answer: (D) neither transitive, nor symmetric, nor reflexive

Solution:

Concept:

- Reflexivity requires xRx ; symmetry requires $xRy \Rightarrow yRx$.
- Transitivity requires xRy and yRz to imply xRz . Here xRy means $h(y) - h(x) = 5$ cm, with h denoting height.

Step 1: Test reflexivity and symmetry No person is 5 cm shorter than themselves, since $h(x) - h(x) = 0$. Thus the relation is not reflexive. If x is 5 cm shorter than y , then y is taller than x , so the reversed pair does not satisfy the relation. It is not symmetric either.

Step 2: Test transitivity and compare the alternatives If x is 5 cm shorter than y , and y is 5 cm shorter than z , then $h(z) - h(x) = 10$ cm, not 5 cm. For example, heights 160, 165, 170 cm give two related consecutive pairs but not the first-to-last pair. Thus transitivity fails. Options (A) and (B) require reflexivity, and (C) requires symmetry and transitivity, so all three fail.

Step 3: Conclusion The relation is neither transitive, nor symmetric, nor reflexive. Hence, the correct option is (D).

Quick Tip to Solve: For an exact height difference, add consecutive differences when testing transitivity; the sum must still equal the stated difference.

8. If a matrix has 36 elements, the number of possible orders it can have, is:

[1 mark]

- (A) 13
- (B) 3
- (C) 5
- (D) 9

Correct Answer: (D) 9

Solution:

Concept:

- A matrix of order $m \times n$ has mn elements, where both dimensions are positive integers.
- Ordered factor pairs give distinct matrix orders; interchanging rows and columns usually changes the order.

Step 1: Factor the element count The condition is $mn = 36$. Since $36 = 2^2 \cdot 3^2$, each positive divisor chosen as the row count determines exactly one column count, namely $n = 36/m$.

Step 2: List all admissible orders The possible orders are $1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6, 9 \times 4, 12 \times 3, 18 \times 2, 36 \times 1$. There are nine orders. Options (A), (B) and (C) give incorrect counts; counting only unordered factor pairs would incorrectly give five.

Step 3: Conclusion The number of possible orders is 9. Hence, the correct option is (D).

Quick Tip to Solve: Count positive divisors of the element count, retaining both orders of each unequal factor pair.

9. Which of the following statements is true for the function

$$f(x) = \begin{cases} x^2 + 3, & x \neq 0 \\ 1, & x = 0 \end{cases} ?$$

[1 mark]

- (A) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R}$
- (B) $f(x)$ is continuous $\forall x \in \mathbb{R}$
- (C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$
- (D) $f(x)$ is discontinuous at infinitely many points

Correct Answer: (C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$

Solution:

Concept:

- Continuity at c requires $\lim_{x \rightarrow c} f(x) = f(c)$.
- A polynomial is continuous and differentiable everywhere; differentiability implies continuity.

Step 1: Check the exceptional point For nonzero x , $f(x) = x^2 + 3$, so $\lim_{x \rightarrow 0} f(x) = 3$. But $f(0) = 1$. The limit differs from the function value, hence the function is discontinuous, and therefore not differentiable, at zero.

Step 2: Check every other point and the options For each $c \neq 0$, a sufficiently small neighbourhood

Quick Tip to Solve: For a piecewise polynomial, compare the limiting polynomial value with the separately assigned value first.

excludes zero, and the function equals the polynomial $x^2 + 3$ throughout that neighbourhood. It is continuous and differentiable there, with derivative $2x$. Thus (A) and (B) fail at zero, while (D) fails because zero is the only discontinuity.

Step 3: Conclusion The function is continuous and differentiable on $\mathbb{R} \setminus \{0\}$. Hence, the correct option is (C).

Quick Tip to Solve: For a piecewise polynomial, compare the limiting polynomial value with the separately assigned value first.

10. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) . Then, this function $f(x)$ is strictly increasing in (a, b) if: [1 mark]

(A) $f'(x) < 0, \forall x \in (a, b)$

(B) $f'(x) > 0, \forall x \in (a, b)$

(C) $f'(x) = 0, \forall x \in (a, b)$

(D) $f(x) > 0, \forall x \in (a, b)$

Correct Answer: (B) $f'(x) > 0, \forall x \in (a, b)$

Solution:

Concept:

- If f is continuous on $[a, b]$ and differentiable on (a, b) , the mean value theorem applies to any smaller closed subinterval.
- A positive first derivative throughout an interval is sufficient for strict increase.

Step 1: Apply the mean value theorem Take any $a < x_1 < x_2 < b$. There exists $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. If $f'(x) > 0$ throughout (a, b) , both factors on the right are positive, giving $f(x_2) > f(x_1)$.

Step 2: Eliminate the other printed conditions Option (A), a negative derivative everywhere, instead gives strict decrease. Option (C), zero derivative everywhere, gives a constant function by the same theorem. Option (D) states only that function values are positive: a positive function such as $f(x) = 2 - x$ on $[0, 1]$ can decrease, so positivity alone does not ensure increase.

Step 3: Conclusion The required sufficient condition is $f'(x) > 0$ for all $x \in (a, b)$. Hence, the correct option is (B).

Quick Tip to Solve: Use the sign of the first derivative to determine increase; positive function values alone say nothing about direction.

11. If $\begin{pmatrix} x+y & 2 \\ 5 & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$, then the value of $\left(\frac{24}{x} + \frac{24}{y}\right)$ is:

[1 mark]

- (A) 7
- (B) 6
- (C) 8
- (D) 18

Correct Answer: (D) 18

Solution:

Concept:

- Equal matrices have equal corresponding entries.
- For nonzero x, y , $1/x + 1/y = (x+y)/(xy)$.

Step 1: Read the useful entries Equality of the upper-left entries gives $x+y=6$, and equality of the lower-right entries gives $xy=8$. The other corresponding entries are already equal. Since the product is nonzero, neither x nor y is zero, so both requested fractions are defined.

Step 2: Combine the fractions Using the sum and product directly,

$$\frac{24}{x} + \frac{24}{y} = 24 \left(\frac{x+y}{xy} \right) = 24 \left(\frac{6}{8} \right) = 18.$$

Solving individually would give $x, y = 2, 4$ in either order, which yields the same result. Options (A), (B) and (C) disagree with this value.

Step 3: Conclusion The value is 18. Hence, the correct option is (D).

Quick Tip to Solve: When equal matrices give the sum and product, evaluate reciprocal sums directly without solving for each variable.

12. If $f(x)$ is an odd function, then $\int_{-\pi/2}^{\pi/2} f(x) \cos^3 x \, dx$ equals:

[1 mark]

- (A) $2 \int_0^{\pi/2} f(x) \cos^3 x \, dx$
- (B) 0
- (C) $2 \int_0^{\pi/2} f(x) \, dx$
- (D) $2 \int_0^{\pi/2} \cos^3 x \, dx$

Correct Answer: (B) 0

Solution:

Concept:

- An odd function satisfies $f(-x) = -f(x)$; an even function satisfies $g(-x) = g(x)$.
- The integral of an integrable odd function over $[-a, a]$ is zero.

Quick Tip to Solve: Check the parity of the complete integrand before applying the symmetric-limit integration rule.

Step 1: Determine the parity of the integrand Cosine is even, so $\cos^3(-x) = \cos^3 x$. Writing $h(x) = f(x) \cos^3 x$, we obtain $h(-x) = f(-x) \cos^3(-x) = -f(x) \cos^3 x = -h(x)$. Thus the complete integrand is odd.

Step 2: Use the symmetric bounds The bounds $-\pi/2$ and $\pi/2$ are symmetric about zero. The contributions on the negative and positive halves cancel, so the integral is zero. Option (A) is the doubling rule for an even integrand, which does not apply here. Options (C) and (D) discard a factor of the integrand and do not express the required integral.

Step 3: Conclusion The integral equals 0. Hence, the correct option is (B).

Quick Tip to Solve: Check the parity of the complete integrand before applying the symmetric-limit integration rule.

13. Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin \theta = \frac{3}{5}$. Then, $\hat{a} \cdot \hat{b}$ is equal to: [1 mark]

- (A) $\pm \frac{3}{5}$
(B) $\pm \frac{3}{4}$
(C) $\pm \frac{4}{5}$
(D) $\pm \frac{4}{3}$

Correct Answer: (C) $\pm \frac{4}{5}$

Solution:

Concept:

- For two vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$. Unit vectors have magnitude one.
- The identity $\sin^2 \theta + \cos^2 \theta = 1$ determines the magnitude of the cosine.

Step 1: Find the possible cosine values Given $\sin \theta = 3/5$, we have $\cos^2 \theta = 1 - 9/25 = 16/25$. Therefore $\cos \theta = \pm 4/5$. The angle between vectors lies in $[0, \pi]$, where the stated positive sine permits both an acute and an obtuse angle; the sign is not fixed.

Step 2: Use the unit-vector condition and test the choices Because both vectors have magnitude one, $\hat{a} \cdot \hat{b} = 1 \cdot 1 \cdot \cos \theta$. Option (A) uses the sine instead of the cosine; (B) uses an incorrect ratio. Option (D) has magnitude greater than one, impossible for the dot product of two unit vectors.

Step 3: Conclusion We obtain $\hat{a} \cdot \hat{b} = \pm 4/5$. Hence, the correct option is (C).

Quick Tip to Solve: For unit vectors the dot product is the cosine, and a sine value alone may leave the cosine sign undetermined.

14. The integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, $-1 < x < 1$, is: [1 mark]

- (A) $\frac{1}{x^2 - 1}$
(B) $\frac{1}{\sqrt{x^2 - 1}}$
(C) $\frac{1}{1 - x^2}$
(D) $\frac{1}{\sqrt{1 - x^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1 - x^2}}$

Solution:

Concept:

- For $dy/dx + P(x)y = Q(x)$, an integrating factor is $e^{\int P(x) dx}$.
- On $-1 < x < 1$, $1 - x^2 > 0$, so its logarithm and positive square root are real.

Step 1: Put the equation in linear form Dividing the equation by the nonzero quantity $1 - x^2$ gives

$$\frac{dy}{dx} + \frac{x}{1 - x^2}y = \frac{ax}{1 - x^2}.$$

Hence $P(x) = x/(1 - x^2)$. The constant a belongs to the right-hand side and does not affect the integrating factor.

Step 2: Integrate the coefficient and compare options With $u = 1 - x^2$, $du = -2x dx$, so

$$\int \frac{x}{1 - x^2} dx = -\frac{1}{2} \log(1 - x^2).$$

Exponentiating gives $(1 - x^2)^{-1/2}$. Options (A) and (C) have the wrong exponent; option (B) contains the square root of a negative quantity on the specified real interval.

Step 3: Conclusion The integrating factor is $\frac{1}{\sqrt{1 - x^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: Always divide by the coefficient of the derivative before identifying the linear-equation coefficient $P(x)$.

15. If the direction cosines of a line are $\sqrt{3}k, \sqrt{3}k, \sqrt{3}k$, then the value of k is:

[1 mark]

- (A) ± 1
- (B) $\pm\sqrt{3}$
- (C) ± 3
- (D) $\pm\frac{1}{3}$

Correct Answer: (D) $\pm\frac{1}{3}$

Solution:

Concept:

- Direction cosines l, m, n of a line obey $l^2 + m^2 + n^2 = 1$.
- The sign of all direction cosines may reverse when the direction along the same line is reversed.

Step 1: Apply the unit-vector identity Here $l = m = n = \sqrt{3}k$. Substitution into the direction-cosine identity gives

$$(\sqrt{3}k)^2 + (\sqrt{3}k)^2 + (\sqrt{3}k)^2 = 1, \quad 9k^2 = 1.$$

Thus $k^2 = 1/9$; taking both square roots is necessary because no orientation of the line is specified.

Step 2: Check the alternatives For $k = \pm 1/3$, the sum of squares is exactly 1. Option (A), $k = \pm 1$, instead gives 9; option (B), $k = \pm\sqrt{3}$, gives 27; option (C), $k = \pm 3$, gives 81. These violate the normalization required of direction cosines.

Step 3: Conclusion $k = \pm\frac{1}{3}$. Hence, the correct option is (D).

Quick Tip to Solve: Square each direction cosine first; three equal entries must have squared value $1/3$ each.

16. A linear programming problem deals with the optimization of a/an:

[1 mark]

- (A) logarithmic function
- (B) linear function
- (C) quadratic function
- (D) exponential function

Correct Answer: (B) linear function

Solution:

Concept:

- Linear programming maximizes or minimizes a linear objective function subject to linear constraints.
- A two-variable objective has the form $Z = ax + by$, where a, b are constants and x, y are decision variables.

Step 1: Identify the objective Optimization means finding the greatest or least permitted value of the objective over the feasible region. In a linear programming problem, this objective is linear in the

Quick Tip to Solve: Check the objective as well as the constraints: both must be linear for a linear programming problem.

decision variables: each variable occurs only to the first power, and no product of decision variables appears.

Step 2: Compare all listed function types Option (B) supplies precisely this linear objective. A logarithmic objective in (A), a quadratic objective in (C), or an exponential objective in (D) introduces nonlinear dependence on the variables. Such objectives do not satisfy the definition of linear programming, even if some constraints happen to be linear.

Step 3: Conclusion The optimized function is a linear function. Hence, the correct option is (B).

Quick Tip to Solve: Check the objective as well as the constraints: both must be linear for a linear programming problem.

17. If $P(A | B) = P(A' | B)$, then which of the following statements is true?

[1 mark]

- (A) $P(A) = P(A')$
- (B) $P(A) = 2P(B)$
- (C) $P(A \cap B) = \frac{1}{2}P(B)$
- (D) $P(A \cap B) = 2P(B)$

Correct Answer: (C) $P(A \cap B) = \frac{1}{2}P(B)$

Solution:

Concept:

- For $P(B) > 0$, conditional probability is $P(A | B) = P(A \cap B)/P(B)$.
- Complementary events satisfy $P(A | B) + P(A' | B) = 1$.

Step 1: Use the complementary conditional probabilities The given conditional probabilities are equal and together sum to 1, because $A \cap B$ and $A' \cap B$ partition B . Therefore each conditional probability equals $1/2$. The condition concerns probabilities within B , rather than probabilities in the entire sample space.

Step 2: Convert to an intersection and test the options Using the definition,

$$\frac{P(A \cap B)}{P(B)} = \frac{1}{2} \implies P(A \cap B) = \frac{1}{2}P(B).$$

This is option (C). Option (A) need not hold outside B , and nothing implies the unconditional relation in (B). Option (D) is impossible here, since $A \cap B \subseteq B$ and $P(B) > 0$.

Step 3: Conclusion The intersection probability is half the probability of the conditioning event. Hence, the correct option is (C).

Quick Tip to Solve: Equal conditional probabilities of complementary events are each one half; multiply by the conditioning probability.

18. $\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}$ is equal to:

[1 mark]

- (A) $2x^3$
- (B) 2
- (C) 0
- (D) $2x^3 - 2$

Correct Answer: (B) 2

Solution:

Concept:

- For a 2×2 determinant, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- The factorization $(u - v)(u^2 + uv + v^2) = u^3 - v^3$ can simplify cubic products.

Step 1: Expand along the first row Let the given determinant be D . Keeping the subtraction sign before the second product gives

$$D = (x + 1)(x^2 - x + 1) - (x - 1)(x^2 + x + 1).$$

The first product is $x^3 + 1$, while the second is $x^3 - 1$, by the sum and difference of cubes identities.

Step 2: Simplify and compare the choices Thus $D = (x^3 + 1) - (x^3 - 1) = 2$. All cubic terms cancel, leaving a constant independent of x . Option (A) retains an incorrect cubic term, option (C) loses the two constant contributions, and option (D) has both an incorrect cubic term and an incorrect constant. Only (B) gives this determinant for every x .

Step 3: Conclusion $D = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Recognize the sum and difference of cubes before expanding; carefully subtract the entire second product.

19. Assertion (A): For matrix $A = \begin{pmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{pmatrix}$, where $\theta \in [0, 2\pi]$, $|A| \in [2, 4]$.

Reason (R): $\cos \theta \in [-1, 1]$, $\forall \theta \in [0, 2\pi]$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The determinant can be evaluated by expansion along the first row.
- For $c = \cos \theta$, $-1 \leq c \leq 1$ implies $0 \leq c^2 \leq 1$.

Step 1: Judge the assertion Put $c = \cos \theta$. Expanding the determinant along the first row,

$$\begin{aligned} |A| &= 1(1 + c^2) - c(-c + c) + 1(c^2 + 1) \\ &= 2 + 2c^2. \end{aligned}$$

Since $0 \leq c^2 \leq 1$, this gives $2 \leq |A| \leq 4$. Both endpoints are attained on the stated interval: $c = 0$ gives 2, and $c = \pm 1$ gives 4. The assertion is true.

Step 2: Judge the reason The cosine function takes values in $[-1, 1]$ for every real angle, in particular for $\theta \in [0, 2\pi]$. Thus the reason is true.

Step 3: Check the explanatory link The cosine bound stated in the reason supplies precisely the bound on c^2 used in the determinant calculation. It therefore explains the asserted interval. Option (B) incorrectly denies this connection; options (C) and (D) incorrectly reject one of the true statements.

Step 4: Conclusion Both statements are true, and the reason correctly explains the assertion. Hence, the correct option is (A).

Quick Tip to Solve: Evaluate the determinant as a polynomial in $\cos \theta$, then apply the cosine range to that polynomial.

20. Assertion (A): A line in space cannot be drawn perpendicular to x , y and z axes simultaneously.

Reason (R): For any line making angles α, β, γ with the positive directions of x , y and z axes respectively, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- A line making angles α, β, γ with the coordinate axes has direction cosines $\cos \alpha, \cos \beta, \cos \gamma$.
- The direction cosines obey $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Step 1: Judge the assertion If a line were perpendicular to all three coordinate axes, its angle with each positive axis direction would be 90° . All three direction cosines would then be zero. A nonzero direction vector cannot have all three components zero, so no such line exists. The assertion is true.

Step 2: Judge the reason A unit vector along the line has components equal to its direction cosines. Its squared length is 1, so the sum of the squares of these components is 1. This proves the stated identity and makes the reason true.

Step 3: Check whether the reason explains the assertion Putting $\alpha = \beta = \gamma = 90^\circ$ in the reason would give $0 + 0 + 0 = 1$, a contradiction. Hence the identity directly rules out the line described in the assertion. Option (B) misses this direct explanation, while (C) and (D) incorrectly declare a true statement false.

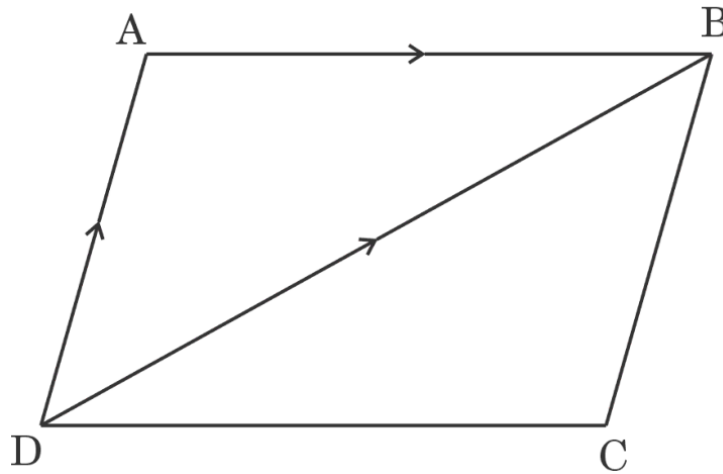
Step 4: Conclusion Both statements are true, and the reason correctly explains the assertion. Hence, the correct option is (A).

Quick Tip to Solve: For perpendicularity to an axis set its direction cosine to zero, then check the sum-of-squares identity.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. In the given figure, $ABCD$ is a parallelogram. If $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\vec{AD}$ and hence find the area of parallelogram $ABCD$. [2 marks]



Solution:

Concept:

- Along a connected path, vector displacements add: $\vec{DA} + \vec{AB} = \vec{DB}$.
- The area of a parallelogram with adjacent side vectors $\vec{u}, \vec{v}$ is $|\vec{u} \times \vec{v}|$.

Step 1: Find the adjacent side vector The path from D to B through A gives $\vec{DB} = -\vec{AD} + \vec{AB}$. Therefore

$$\begin{aligned}\vec{AD} &= \vec{AB} - \vec{DB} \\ &= (2\hat{i} - 4\hat{j} + 5\hat{k}) - (3\hat{i} - 6\hat{j} + 2\hat{k}) \\ &= -\hat{i} + 2\hat{j} + 3\hat{k}.\end{aligned}$$

The minus sign on $\vec{AD}$ in the path equation is essential: the path uses D to A , whereas the requested vector runs from A to D .

Step 2: Compute the area from the cross product The adjacent sides issuing from A are $\vec{AB}$ and $\vec{AD}$. Their cross product is

$$\begin{aligned}\vec{AB} \times \vec{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ -1 & 2 & 3 \end{vmatrix} \\ &= \hat{i}(-12 - 10) - \hat{j}(6 + 5) + \hat{k}(4 - 4) \\ &= -22\hat{i} - 11\hat{j}.\end{aligned}$$

Its magnitude is $\sqrt{(-22)^2 + (-11)^2} = \sqrt{605} = 11\sqrt{5}$. This magnitude, rather than the vector itself, gives the nonnegative area of the parallelogram.

Step 3: Conclusion $\vec{AD} = -\hat{i} + 2\hat{j} + 3\hat{k}$, and the area of $ABCD$ is $11\sqrt{5}$ square units.

Quick Tip to Solve: Follow the arrows around triangle DAB before subtracting vectors; use adjacent sides in the cross product.

22(a). Check the differentiability of function $f(x) = [x]$ at $x = -3$, where $[\cdot]$ denotes greatest integer function. [2 marks]

Solution:

Concept:

- The greatest integer function $[x]$ is the largest integer less than or equal to x . It is constant between successive integers.
- Differentiability at a requires both one-sided difference quotients $\frac{f(a+h) - f(a)}{h}$ to have the same finite limit. Differentiability also implies continuity.

Step 1: Check the values on the two sides At the given point, $f(-3) = [-3] = -3$. For $-1 < h < 0$, the number $-3 + h$ lies between -4 and -3 , so $f(-3 + h) = -4$. For $0 < h < 1$, it lies between -3 and -2 , so $f(-3 + h) = -3$. Thus the left-hand limit is -4 , whereas the right-hand limit and the function value are -3 . The function has a jump at the given integer.

Step 2: Examine the derivative limits The right-hand difference quotient is $\frac{-3 - (-3)}{h} = 0$, giving a right derivative of 0. On the left it is $\frac{-4 - (-3)}{h} = -\frac{1}{h}$, which tends to $+\infty$ as $h \rightarrow 0^-$. Therefore the left derivative does not exist as a finite real number. The two derivative limits cannot agree; equivalently, the failure of continuity already rules out differentiability.

Step 3: Conclusion The function $f(x) = [x]$ is not differentiable at $x = -3$.

Quick Tip to Solve: For a greatest integer function at an integer, check the value immediately to the left: it drops by one, producing a jump.

OR

22(b). If $x^{1/3} + y^{1/3} = 1$, find $\frac{dy}{dx}$ at the point $\left(\frac{1}{8}, \frac{1}{8}\right)$. [2 marks]

Solution:

Concept:

- For implicit differentiation, treat y as a function of x , so $\frac{d}{dx}(y^{1/3}) = \frac{1}{3}y^{-2/3}\frac{dy}{dx}$ when $y \neq 0$.
- At a specified point, first check that its coordinates satisfy the original relation, and substitute into the derivative only after differentiation.

Step 1: Verify the point and differentiate At $x = y = \frac{1}{8}$, both cube roots equal $\frac{1}{2}$; hence their sum is 1, so the point lies on the curve. Differentiate the given relation with respect to x :

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0.$$

The derivative of the constant on the right is zero, and the chain-rule factor dy/dx must accompany the term involving y .

Step 2: Substitute the coordinates Multiplying by $3y^{2/3}$, which is permitted near the point because

Quick Tip to Solve: When the two coordinates are equal and nonzero in this symmetric cube-root relation, the implicit derivative reduces immediately to minus one.

$y \neq 0$, gives

$$\frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}}.$$

Now $\left(\frac{1}{8}\right)^{2/3} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Thus the numerator and denominator are equal and nonzero at the given point, and the required slope is $-\frac{1/4}{1/4}$.

Step 3: Conclusion At $\left(\frac{1}{8}, \frac{1}{8}\right)$, $\boxed{\frac{dy}{dx} = -1}$.

Quick Tip to Solve: When the two coordinates are equal and nonzero in this symmetric cube-root relation, the implicit derivative reduces immediately to minus one.

23. Find local maximum value and local minimum value (whichever exists) for the function $f(x) = 4x^2 + \frac{1}{x}$ ($x \neq 0$). [2 marks]

Solution:

Concept:

- A local extremum at an interior point of a differentiable domain can occur only where $f'(x) = 0$. The excluded point $x = 0$ is not eligible.
- At a stationary point a , the second derivative test gives a local minimum if $f''(a) > 0$, and a local maximum if $f''(a) < 0$.

Step 1: Find all stationary points Write $f(x) = 4x^2 + x^{-1}$, for $x \neq 0$. Differentiating gives

$$f'(x) = 8x - \frac{1}{x^2} = \frac{8x^3 - 1}{x^2}.$$

Since $x^2 \neq 0$ throughout the domain, $f'(x) = 0$ means $8x^3 = 1$, giving the only real stationary point $x = \frac{1}{2}$. In particular, there is no stationary point in the negative half of the domain.

Step 2: Classify the point and calculate its value Differentiate once more:

$$f''(x) = 8 + \frac{2}{x^3}, \quad f''\left(\frac{1}{2}\right) = 8 + 16 = 24 > 0.$$

Hence the stationary point is a strict local minimum. Its function value is

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + \frac{1}{1/2} = 1 + 2 = 3.$$

There are no other stationary points and the function is differentiable everywhere in its domain. Since 0 is excluded, it cannot furnish an extremum; consequently there is no local maximum.

Step 3: Conclusion The local minimum value is $\boxed{3}$, attained at $x = \frac{1}{2}$. No local maximum exists.

Quick Tip to Solve: Clear the positive denominator in the first derivative before finding critical points, and never count an excluded point as an extremum.

24(a). Find: $\int x \sqrt{1+2x} dx$

[2 marks]

Solution:

Concept:

- Use substitution $u = 1 + 2x$ to convert the square root into a power of a single variable; then $x = (u - 1)/2$ and $dx = du/2$.
- The power rule is $\int u^n du = u^{n+1}/(n+1) + C$ for $n \neq -1$, where C is an arbitrary constant.

Step 1: Transform the integral Let $I = \int x \sqrt{1+2x} dx$. With $u = 1 + 2x$, account for both the factor x and the differential:

$$I = \int \frac{u-1}{2} u^{1/2} \frac{du}{2} = \frac{1}{4} \int (u^{3/2} - u^{1/2}) du.$$

The product has now become a difference of powers, each of which can be integrated directly.

Step 2: Integrate and return to the original variable Applying the power rule term by term gives

$$\begin{aligned} I &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{10} (1+2x)^{5/2} - \frac{1}{6} (1+2x)^{3/2} + C. \end{aligned}$$

Factoring $(1+2x)^{3/2}$, the remaining coefficient is $(1+2x)/10 - 1/6 = (3x-1)/15$. As a check, differentiating the unfactored result gives $\frac{1}{2}(1+2x)^{3/2} - \frac{1}{2}(1+2x)^{1/2} = x\sqrt{1+2x}$, the required integrand.

Step 3: Conclusion

$$\int x \sqrt{1+2x} dx = \frac{3x-1}{15} (1+2x)^{3/2} + C.$$

Quick Tip to Solve: After setting u equal to the expression under the root, replace x as well as dx ; the two factors of one-half produce the coefficient one-quarter.

OR

24(b). Evaluate: $\int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

[2 marks]

Solution:

Concept:

- With $t = \sqrt{x}$, we have $x = t^2$ and $dx = 2t dt$. This cancels the square root in the denominator.
- For a definite integral, change both limits when changing variables. The apparent singularity at zero is removable, since $\sin t/t \rightarrow 1$ as $t \rightarrow 0$.

Step 1: Substitute and change the limits Let $I = \int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$. Put $t = \sqrt{x}$, taking the nonnegative

Quick Tip to Solve: For a square root inside a trigonometric function, set t equal to that root; transform the limits before evaluating the antiderivative.

square root. When $x = 0$, $t = 0$; when $x = \pi^2/4$, $t = \pi/2$. For $t > 0$, substitution gives

$$\frac{\sin \sqrt{x}}{\sqrt{x}} dx = \frac{\sin t}{t} 2t dt = 2 \sin t dt.$$

The cancellation is performed away from zero; the continuous limiting value at the endpoint justifies taking the lower limit as zero.

Step 2: Evaluate the trigonometric integral The transformed integral is elementary:

$$\begin{aligned} I &= 2 \int_0^{\pi/2} \sin t dt \\ &= 2[-\cos t]_0^{\pi/2} \\ &= 2\left(-\cos \frac{\pi}{2} + \cos 0\right) = 2(0 + 1). \end{aligned}$$

The upper limit is $\pi/2$, not $\pi^2/4$, because the new variable is the square root of the old one. No arbitrary integration constant is required for a definite integral.

Step 3: Conclusion

$$\boxed{\int_0^{\pi^2/4} \frac{\sin \sqrt{x}}{\sqrt{x}} dx = 2.}$$

Quick Tip to Solve: For a square root inside a trigonometric function, set t equal to that root; transform the limits before evaluating the antiderivative.

25. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}|\vec{a}|$. [2 marks]

Solution:

Concept:

- Perpendicular vectors have zero scalar product: $\vec{u} \perp \vec{v}$ implies $\vec{u} \cdot \vec{v} = 0$.
- The scalar product is distributive and symmetric, and $\vec{a} \cdot \vec{a} = |\vec{a}|^2$. Magnitudes $|\vec{a}|$ and $|\vec{b}|$ are non-negative.

Step 1: Use the first perpendicularity condition Since $(\vec{a} + \vec{b}) \perp \vec{a}$, taking the scalar product with $\vec{a}$ gives

$$(\vec{a} + \vec{b}) \cdot \vec{a} = 0 \implies |\vec{a}|^2 + \vec{a} \cdot \vec{b} = 0.$$

Thus $\vec{a} \cdot \vec{b} = -|\vec{a}|^2$. This expresses the unknown mixed scalar product in terms of the magnitude of $\vec{a}$.

Step 2: Substitute in the second condition From $(2\vec{a} + \vec{b}) \perp \vec{b}$,

$$2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 0.$$

Quick Tip to Solve: Convert each perpendicularity statement into a zero dot product, then eliminate the common term $\vec{a} \cdot \vec{b}$.

Substituting the mixed scalar product obtained above gives

$$-2|\vec{a}|^2 + |\vec{b}|^2 = 0, \quad |\vec{b}|^2 = 2|\vec{a}|^2.$$

Both magnitudes are positive because the vectors are non-zero. Therefore taking the positive square root yields the required relation; a negative square root cannot represent a vector's magnitude.

Step 3: Conclusion Hence $|\vec{b}| = \sqrt{2}|\vec{a}|$, as required.

Quick Tip to Solve: Convert each perpendicularity statement into a zero dot product, then eliminate the common term $\vec{a} \cdot \vec{b}$.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Solve the following linear programming problem graphically:

Minimise $z = 5x - 2y$ **subject to the constraints**

$$x + 2y \leq 120$$

$$x + y \geq 60$$

$$x - 2y \geq 0$$

$$x, y \geq 0$$

[3 marks]

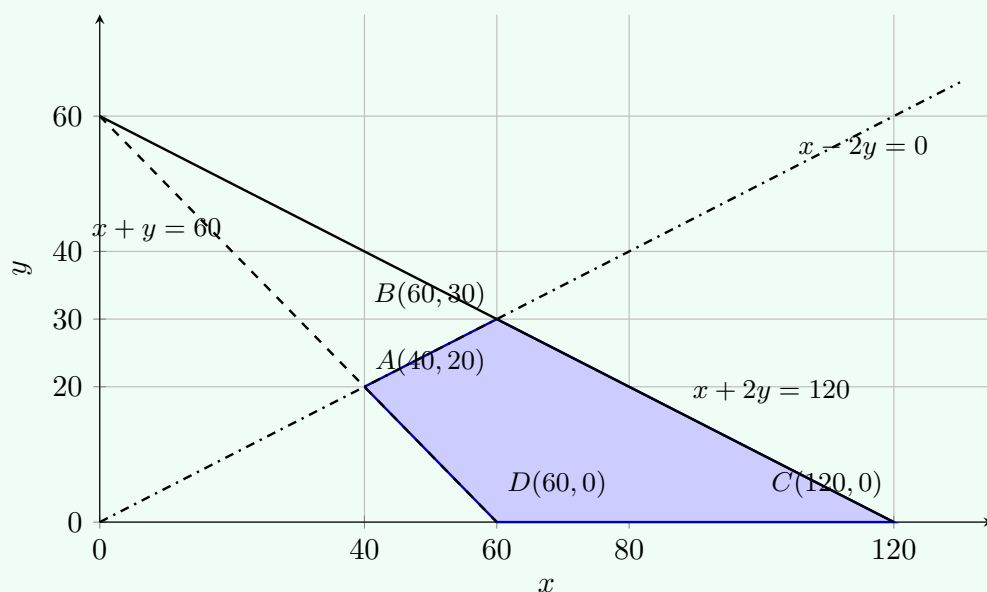
Solution:

Concept:

- The feasible region is the common part of the half-planes determined by all constraints, including $x, y \geq 0$.
- A linear objective function on a non-empty bounded polygon attains its minimum at a corner point. Here the objective is $z = 5x - 2y$.

Step 1: Plot the boundary lines and select the half-planes The line $x + 2y = 120$ passes through (120, 0) and (0, 60); retain the side on or below it. The line $x + y = 60$ passes through (60, 0) and (0, 60); retain the side on or above it. Finally, $x - 2y = 0$, or $y = x/2$, passes through (0, 0) and (80, 40); retain the side $y \leq x/2$. Work only in the first quadrant. The shaded polygon below is their common intersection, with its boundary included.

Quick Tip to Solve: Find the feasible vertices before evaluating the objective; the intersection (0, 60) is not feasible for $x \geq 2y$.



The shaded region is the feasible polygon $ABCD$.

Step 2: Find the vertices of the feasible polygon Intersecting $x + y = 60$ with $x = 2y$ gives $3y = 60$, hence $A = (40, 20)$. Intersecting $x + 2y = 120$ with $x = 2y$ gives $4y = 120$, hence $B = (60, 30)$. On the x -axis, the constraints give $60 \leq x \leq 120$, so the other vertices are $C = (120, 0)$ and $D = (60, 0)$. Each of these four points satisfies every inequality. The intersection $(0, 60)$ of the first two boundary lines is excluded because it violates $x - 2y \geq 0$.

Step 3: Evaluate the objective function at the vertices The region $ABCD$ is bounded, so the corner-point method applies. Substitution gives

Vertex (x, y)	$z = 5x - 2y$
$A(40, 20)$	$200 - 40 = 160$
$B(60, 30)$	$300 - 60 = 240$
$C(120, 0)$	600
$D(60, 0)$	300

The smallest value in this comparison is 160, occurring at A . Since adjacent vertices give larger objective values, no whole edge shares this minimum.

Step 4: Conclusion The minimum value is $z_{\min} = 160$, attained at $x = 40, y = 20$.

Quick Tip to Solve: Find the feasible vertices before evaluating the objective; the intersection $(0, 60)$ is not feasible for $x \geq 2y$.

27. E and F are two independent events such that $P(\overline{E}) = 0.6$ and $P(E \cup F) = 0.6$. Find $P(F)$ and $P(\overline{E} \cup \overline{F})$. [3 marks]

Solution:

Concept:

- For an event E , the complement rule is $P(E) = 1 - P(\overline{E})$.
- Independence gives $P(E \cap F) = P(E)P(F)$, while the addition rule gives $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- De Morgan's law gives $\overline{E} \cup \overline{F} = \overline{E \cap F}$, so its probability is $1 - P(E \cap F)$.

Step 1: Find the probability of E The supplied probability is that of the complement of E , not E itself. Therefore

$$P(E) = 1 - P(\overline{E}) = 1 - 0.6 = 0.4.$$

Write $P(F) = p$, with $0 \leq p \leq 1$. Because E and F are independent, their intersection has probability $P(E \cap F) = 0.4p$. Independence does not mean that the events are mutually exclusive, so this intersection must be retained in the union formula.

Step 2: Use the probability of the union to determine F Applying the addition rule to the given union,

$$0.6 = 0.4 + p - 0.4p,$$

$$0.2 = 0.6p,$$

$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

This value is a valid probability. Consequently $P(E \cap F) = \frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}$. As a check, $\frac{2}{5} + \frac{1}{3} - \frac{2}{15} = \frac{3}{5} = 0.6$, reproducing the given union probability.

Step 3: Apply De Morgan's law to the union of complements The union $\overline{E} \cup \overline{F}$ contains every outcome except those belonging to both E and F . Hence

$$P(\overline{E} \cup \overline{F}) = P(\overline{E \cap F}) = 1 - \frac{2}{15} = \frac{13}{15}.$$

This is a union of complements, so subtracting $P(E \cup F)$ from one would instead give the probability of their intersection and would answer a different expression.

Step 4: Conclusion Thus $P(F) = \frac{1}{3}$ and $P(\overline{E} \cup \overline{F}) = \frac{13}{15}$.

Quick Tip to Solve: For the union of two complements, use De Morgan's law first: subtract the original intersection probability from one.

28(a). A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as $R = \{(x, y) : |x^2 - y^2| < 8\}$. Check whether the relation R is reflexive, symmetric and transitive. [3 marks]

Solution:

Concept:

- A relation on A is reflexive if $(x, x) \in R$ for every $x \in A$.
- Symmetry requires $(y, x) \in R$ whenever $(x, y) \in R$. Transitivity requires $(x, z) \in R$ whenever both (x, y) and (y, z) belong to R .

Step 1: Check reflexivity For any $x \in A = \{1, 2, 3, 4, 5\}$, we have $|x^2 - x^2| = 0 < 8$. Thus all five diagonal pairs $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ satisfy the defining condition. Since this holds for every element of the given set, R is reflexive.

Step 2: Check symmetry Suppose $(x, y) \in R$. Then $|x^2 - y^2| < 8$. Reversing the order changes the sign inside the absolute value but leaves its value unchanged:

$$|y^2 - x^2| = |-(x^2 - y^2)| = |x^2 - y^2| < 8.$$

Hence $(y, x) \in R$. This implication applies to every pair in the relation, so R is symmetric.

Step 3: Test transitivity using three elements Choose $x = 1, y = 2, z = 3$, all of which lie in A . Then

$$|1^2 - 2^2| = 3 < 8, \quad |2^2 - 3^2| = 5 < 8.$$

Consequently $(1, 2) \in R$ and $(2, 3) \in R$. However,

$$|1^2 - 3^2| = 8,$$

which does not satisfy the strict inequality < 8 . Therefore $(1, 3) \notin R$. These two related pairs with a missing required third pair disprove transitivity.

Step 4: Conclusion The relation R is reflexive and symmetric, but it is not transitive. Therefore it is not an equivalence relation.

Quick Tip to Solve: To disprove transitivity here, use 1, 2 and 3: the endpoint square difference is exactly 8, excluded by the strict inequality.

OR

28(b). A function f is defined from $\mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find function $f(x)$. Hence, check whether function $f(x)$ is one-one and onto or not. [3 marks]

Solution:

Concept:

- A function is one-one if equality of its outputs implies equality of its inputs.
- A function from $\mathbb{R}$ to $\mathbb{R}$ is onto if every real number in its codomain has a real preimage.

Quick Tip to Solve: For onto, start with an arbitrary real output y and solve for x ; showing this input is real completes the proof.

Step 1: Determine the constants Substituting the given inputs into $f(x) = ax + b$, the condition $f(1) = 1$ gives $a + b = 1$, while $f(2) = 3$ gives $2a + b = 3$. Subtracting the first equation from the second yields $a = 2$. Substituting this into $a + b = 1$ yields $b = -1$. Therefore the function determined by both conditions is $f(x) = 2x - 1$. Indeed, substituting $x = 1$ and $x = 2$ gives the required outputs 1 and 3.

Step 2: Prove that the function is one-one Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) = f(x_2)$. Using the formula obtained above,

$$2x_1 - 1 = 2x_2 - 1 \implies 2x_1 = 2x_2 \implies x_1 = x_2.$$

Thus equal outputs cannot come from distinct inputs. Since x_1 and x_2 were arbitrary, f is one-one on its entire domain.

Step 3: Prove that the function is onto Take any $y \in \mathbb{R}$, the stated codomain. Solve $f(x) = y$:

$$2x - 1 = y \implies x = \frac{y + 1}{2}.$$

This input is real for every real y , so it belongs to the domain. Substitution verifies

$$f\left(\frac{y + 1}{2}\right) = 2\left(\frac{y + 1}{2}\right) - 1 = y.$$

Every element of the codomain therefore has a preimage, proving that f is onto.

Step 4: Conclusion $f(x) = 2x - 1$. The function is both one-one and onto, and hence is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Quick Tip to Solve: For onto, start with an arbitrary real output y and solve for x ; showing this input is real completes the proof.

29(a). If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, prove that

$$\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}.$$

[3 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x , while the parameter a is constant.
- For positive radicals, $\frac{d}{dx} \sqrt{1 - y^2} = -\frac{y}{\sqrt{1 - y^2}} \frac{dy}{dx}$. The difference-of-squares identity will eliminate a .

Step 1: Obtain an auxiliary identity from the given relation Put $U = \sqrt{1 - x^2}$ and $V = \sqrt{1 - y^2}$. At

Quick Tip to Solve: Multiply the sum of the two radicals by their difference first; the resulting identity cancels the parameter after implicit differentiation.

an interior point where the required derivative is defined, $U, V > 0$. The given equation is

$$U + V = a(x - y).$$

Since $U + V > 0$, both $a \neq 0$ and $x - y \neq 0$. Also,

$$(U + V)(U - V) = U^2 - V^2 = y^2 - x^2 = -(x - y)(x + y).$$

Substitute $U + V = a(x - y)$ and cancel $x - y$ to obtain

$$a(U - V) = -(x + y), \quad aU + x = aV - y.$$

This second identity relates exactly the two factors that arise upon differentiation.

Step 2: Differentiate implicitly and simplify the ratio Write $p = \frac{dy}{dx}$. Differentiating the original relation with respect to x gives

$$-\frac{x}{U} - \frac{y}{V}p = a(1 - p).$$

Collect all terms containing p on one side:

$$\left(a - \frac{y}{V}\right)p = a + \frac{x}{U}, \quad p = \frac{V(aU + x)}{U(aV - y)}.$$

Use $aU + x = aV - y$ from the first step to cancel the common factor. This factor cannot vanish at such an interior point: if both $aU + x = 0$ and $aV - y = 0$, then $x = -aU$, $y = aV$, and substitution into $U + V = a(x - y)$ would give $1 = -a^2$, impossible for real a . Therefore the cancellation is valid.

Step 3: Conclusion Hence, wherever the displayed derivative is defined,

$$\boxed{\frac{dy}{dx} = \frac{V}{U} = \sqrt{\frac{1 - y^2}{1 - x^2}}}.$$

Quick Tip to Solve: Multiply the sum of the two radicals by their difference first; the resulting identity cancels the parameter after implicit differentiation.

OR

29(b). If $y = (\tan x)^x$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- Logarithmic differentiation is useful when both the base and the exponent depend on x : for positive $u(x)$, $\log(u^v) = v \log u$.
- The product rule is $(fg)' = f'g + fg'$, and $\frac{d}{dx} \log(\tan x) = \frac{\sec^2 x}{\tan x}$. Here $\log$ denotes the natural

Quick Tip to Solve: For a variable base raised to a variable exponent, take natural logarithms first and differentiate the resulting product.

logarithm.

Step 1: Take logarithms to bring down the variable exponent The function is $y = (\tan x)^x$. Work on any open interval where $\tan x > 0$ and $\tan x$ is defined, so the real logarithm and this variable power are differentiable. Taking natural logarithms on both sides,

$$\log y = x \log(\tan x).$$

The exponent x has become a multiplicative factor. Both factors on the right depend on x , so their derivative requires the product rule. Treating the exponent as constant would omit a necessary term.

Step 2: Apply the product and chain rules Differentiate the logarithmic identity:

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{dx}{dx} \log(\tan x) + x \frac{d}{dx} \log(\tan x) \\ &= \log(\tan x) + x \frac{1}{\tan x} \sec^2 x.\end{aligned}$$

The factor $1/y$ on the left comes from the chain rule, and $\sec^2 x$ on the right is the derivative of $\tan x$. Multiply by y , then substitute the original expression $y = (\tan x)^x$. The two terms account separately for the changing exponent and changing base.

Step 3: Conclusion Therefore,

$$\frac{dy}{dx} = (\tan x)^x \left[\log(\tan x) + \frac{x \sec^2 x}{\tan x} \right].$$

Quick Tip to Solve: For a variable base raised to a variable exponent, take natural logarithms first and differentiate the resulting product.

30(a). Find: $\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$

[3 marks]

Solution:

Concept:

- Partial fractions split a rational function with distinct quadratic factors into simpler fractions that can be integrated separately.
- The standard formula is $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$, where $a > 0$ and C is an arbitrary constant.

Step 1: Decompose the rational function Since both numerator and denominator are even functions of x , use constant numerators:

$$\frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{A}{x^2 + 4} + \frac{B}{x^2 + 9}.$$

Quick Tip to Solve: After finding the partial-fraction constants, divide each by the positive square root of its quadratic constant.

Multiplying by the denominator gives the identity

$$x^2 = A(x^2 + 9) + B(x^2 + 4) = (A + B)x^2 + 9A + 4B.$$

Equating coefficients yields $A + B = 1$ and $9A + 4B = 0$. Subtracting four times the first equation from the second gives $5A = -4$, so $A = -4/5$ and $B = 9/5$. These values give a zero constant term and the required coefficient of x^2 .

Step 2: Integrate the two quadratic fractions By linearity of integration,

$$\begin{aligned} I &= -\frac{4}{5} \int \frac{dx}{x^2 + 2^2} + \frac{9}{5} \int \frac{dx}{x^2 + 3^2} \\ &= -\frac{4}{5} \cdot \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \frac{9}{5} \cdot \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C. \end{aligned}$$

Both quadratic denominators are positive for every real x , so this antiderivative holds throughout the real line. As a check, differentiating the two inverse-tangent terms gives $-4/[5(x^2 + 4)] + 9/[5(x^2 + 9)]$, whose combined numerator is x^2 .

Step 3: Conclusion $\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = -\frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) + C.$

Quick Tip to Solve: After finding the partial-fraction constants, divide each by the positive square root of its quadratic constant.

OR

30(b). Evaluate: $\int_1^3 (|x - 1| + |x - 2| + |x - 3|) dx$

[3 marks]

Solution:

Concept:

- The absolute-value rule is $|u| = u$ for $u \geq 0$ and $|u| = -u$ for $u < 0$. Split a definite integral at interior points where these signs change.
- Definite integrals are additive over adjacent intervals: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

Step 1: Remove the absolute values on each interval On the whole interval $[1, 3]$, we have $x - 1 \geq 0$ and $x - 3 \leq 0$. Hence $|x - 1| = x - 1$ and $|x - 3| = 3 - x$; their sum is the constant 2. The remaining term $|x - 2|$ changes its expression at $x = 2$. For $1 \leq x \leq 2$, it equals $2 - x$, and therefore the complete integrand is $(x - 1) + (2 - x) + (3 - x) = 4 - x$. For $2 \leq x \leq 3$, it equals $x - 2$, giving the complete integrand $(x - 1) + (x - 2) + (3 - x) = x$. Both expressions equal 2 at the splitting point, so the integrand is continuous there.

Quick Tip to Solve: First combine the endpoint terms: on $[1, 3]$, their sum is 2; only the middle absolute value needs splitting.

Step 2: Evaluate the two definite integrals Using the expressions just obtained,

$$\begin{aligned} I &= \int_1^2 (4-x) dx + \int_2^3 x dx \\ &= \left[4x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} \right]_2^3 \\ &= \left(8 - 2 - 4 + \frac{1}{2} \right) + \left(\frac{9}{2} - 2 \right) \\ &= \frac{5}{2} + \frac{5}{2}. \end{aligned}$$

A separate check follows from the simplified form $2 + |x - 2|$. Its constant part contributes $2(3 - 1) = 4$. The absolute-value term contributes $\int_1^2 (2 - x) dx + \int_2^3 (x - 2) dx = 1/2 + 1/2 = 1$, agreeing with the direct calculation.

Step 3: Conclusion $\int_1^3 (|x - 1| + |x - 2| + |x - 3|) dx = 5$.

Quick Tip to Solve: First combine the endpoint terms: on $[1, 3]$, their sum is 2; only the middle absolute value needs splitting.

31. Solve the following differential equation: $(\tan^{-1} y - x) dy = (1 + y^2) dx$.

[3 marks]

Solution:

Concept:

- Treat x as a function of y . A first-order linear equation has the form $dx/dy + P(y)x = Q(y)$.
- Its integrating factor is $IF = e^{\int P(y) dy}$, and multiplication by it makes the left side the derivative of x IF.

Step 1: Write the equation in linear form The given relation is $(\tan^{-1} y - x) dy = (1 + y^2) dx$. Since $1 + y^2 > 0$ for real y , it becomes

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}.$$

Thus $P(y) = 1/(1 + y^2)$ and $Q(y) = \tan^{-1} y/(1 + y^2)$. The dependent variable in this linear form is x , not y ; this choice is what allows the standard integrating-factor method.

Step 2: Find the integrating factor and integrate The integrating factor is

$$IF = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}.$$

Multiplication by this positive factor gives

$$\frac{d}{dy} (xe^{\tan^{-1} y}) = \frac{(\tan^{-1} y)e^{\tan^{-1} y}}{1 + y^2}.$$

Quick Tip to Solve: When the equation is linear in x after division by dy , solve for x as a function of y and use $t = \text{inverse tangent of } y$.

Put $t = \tan^{-1} y$, so $dt = dy/(1 + y^2)$. Integration by parts, with $u = t$ and $dv = e^t dt$, gives

$$xe^{\tan^{-1} y} = \int te^t dt = te^t - e^t + C = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C.$$

Dividing by the integrating factor gives $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$. To verify, differentiate with respect to y : $dx/dy = [1 - Ce^{-\tan^{-1} y}]/(1 + y^2)$. The solution also gives $\tan^{-1} y - x = 1 - Ce^{-\tan^{-1} y}$, reproducing the original equation exactly.

Step 3: Conclusion The general solution is $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$, where C is an arbitrary real constant.

Quick Tip to Solve: When the equation is linear in x after division by dy , solve for x as a function of y and use $t = \text{inverse tangent of } y$.

Section D (20 marks)

This section comprises long answer type questions (LA) of 5 marks each.

32. Find the equation of a line l_2 which is the mirror image of the line l_1 with respect to line l : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$, given that line l_1 passes through the point $P(1, 6, 3)$ and parallel to line l . [5 marks]

Solution:

Concept:

- A line through a point with position vector $\mathbf{a}$ and direction vector $\mathbf{d}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$, where $t \in \mathbb{R}$.
- To reflect a point P in a line, find its perpendicular foot H on that line. The reflected point P' satisfies $\overrightarrow{OP'} = 2\overrightarrow{OH} - \overrightarrow{OP}$.
- Reflection about a line leaves the direction parallel to that line unchanged. Thus, a line parallel to the reflecting line is mapped to a parallel line through the reflected point.

Step 1: Identify the reflecting line and the given parallel line From the symmetric equation of l , take $A = (0, 1, 2)$ on l and direction vector $\mathbf{d} = (1, 2, 3)$. Therefore,

$$l : \mathbf{r} = (0, 1, 2) + t(1, 2, 3).$$

Since l_1 passes through $P = (1, 6, 3)$ and is parallel to l , its points are $P + s\mathbf{d}$. To determine its reflected line, we first reflect P and then retain the parallel direction $\mathbf{d}$.

Step 2: Locate the perpendicular foot of P on l Let the foot be $H = (t, 1 + 2t, 2 + 3t)$. The vector from P to H is $(t - 1, 2t - 5, 3t - 1)$. Its scalar product with the direction of l must vanish:

$$\begin{aligned}(t - 1) + 2(2t - 5) + 3(3t - 1) &= 0, \\ 14t - 14 &= 0, \\ t &= 1.\end{aligned}$$

Quick Tip to Solve: For reflection of a parallel line, project one point onto the mirror line, use $P' = 2H - P$, and retain the original direction ratios.

Hence $H = (1, 3, 5)$. Indeed, $\overrightarrow{PH} = (0, -3, 2)$ has scalar product $0 - 6 + 6 = 0$ with $(1, 2, 3)$, confirming perpendicularity.

Step 3: Reflect the point and determine the image direction The foot H is the midpoint of PP' , so

$$P' = 2H - P = (2, 6, 10) - (1, 6, 3) = (1, 0, 7).$$

More generally, the perpendicular foot of a point $P + s\mathbf{d}$ on l_1 is $H + s\mathbf{d}$: translating both endpoints parallel to l preserves their perpendicular separation. Its reflection is

$$2(H + s\mathbf{d}) - (P + s\mathbf{d}) = P' + s\mathbf{d}.$$

Thus the image is a line through $(1, 0, 7)$ with direction ratios $1, 2, 3$.

Step 4: Conclusion The required mirror image is

$$l_2 : \frac{x-1}{1} = \frac{y}{2} = \frac{z-7}{3}.$$

Quick Tip to Solve: For reflection of a parallel line, project one point onto the mirror line, use $P' = 2H - P$, and retain the original direction ratios.

33(a). If $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix}$, find A^{-1} and use it to solve the following system of equations: $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$. [5 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$, and $A^{-1} = \frac{1}{|A|} \text{adj } A$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.
- A linear system with coefficient matrix A can be written $AX = B$. If A is invertible, its unique solution is $X = A^{-1}B$.

Step 1: Evaluate the determinant Expand along the first row of the given matrix:

$$\begin{aligned} |A| &= 1 \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \\ &= 1(-1 - 2) + 2(2 - 0) \\ &= -3 + 4 = 1. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the system has exactly one solution. The zero in the first row makes the third term of this expansion vanish.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the right-hand-side column; check the resulting three values in every original equation.

Step 2: Compute the cofactors and the inverse Apply the alternating cofactor signs to the nine two-by-two minors:

$$C_{11} = (-1)(1) - (-1)(-2) = -3,$$

$$C_{12} = -[2(1) - (-1)(0)] = -2,$$

$$C_{13} = 2(-2) - (-1)(0) = -4,$$

$$C_{21} = -[(-2)(1) - 0(-2)] = 2,$$

$$C_{22} = 1(1) - 0(0) = 1,$$

$$C_{23} = -[1(-2) - (-2)(0)] = 2,$$

$$C_{31} = (-2)(-1) - 0(-1) = 2,$$

$$C_{32} = -[1(-1) - 0(2)] = 1,$$

$$C_{33} = 1(-1) - (-2)(2) = 3.$$

Therefore, the cofactor matrix and its transpose are

$$C = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

Dividing by $|A| = 1$ gives

$$A^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

Step 3: Use the inverse to solve the system The order of the unknowns is x, y, z ; hence the given equations are

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}.$$

Premultiplying by A^{-1} , we obtain

$$\begin{aligned} X &= \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}. \end{aligned}$$

Step 4: Verify in all three original equations Substitution gives $0 - 2(-5) = 10$, $2(0) - (-5) - (-3) = 8$, and $-2(-5) + (-3) = 7$. Each right-hand side is reproduced, confirming the solution and the order of the components of X .

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the right-hand-side column; check the resulting three values in every original equation.

Step 5: Conclusion

$$A^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}, \quad \boxed{x = 0, y = -5, z = -3}.$$

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the right-hand-side column; check the resulting three values in every original equation.

OR

33(b). If $A = \begin{pmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{pmatrix}$, find the value of $(a + x) - (b + y)$. [5 marks]

Solution:**Concept:**

- For an invertible square matrix, its given inverse satisfies $AA^{-1} = I_3$, where I_3 has diagonal entries 1 and all other entries 0.
- The entry in row i , column j of a product is the scalar product of row i of the first matrix and column j of the second matrix. Thus selected entries give simple linear equations for the unknowns.

Step 1: Form the inverse identity Write the supplied inverse as B . The condition $AB = I_3$ gives

$$\begin{pmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{pmatrix} = I_3.$$

We select entries whose equations contain only one unknown. This avoids computing a determinant or a complete inverse afresh.

Step 2: Determine the bottom-row unknowns The third row of A is completely known. Its products with the first and second columns of B must both be zero, because they are off-diagonal entries:

$$(AB)_{31} = 3(1) + 1(-8) + 1(b) = b - 5 = 0,$$

$$(AB)_{32} = 3(-1) + 1(7) + 1(y) = y + 4 = 0.$$

Therefore $b = 5$ and $y = -4$. Notice that the diagonal entry is also consistent: $(AB)_{33} = 3 - 5 + 3 = 1$.

Step 3: Determine the remaining unknowns The third column of B is completely known, so use it with the first two rows of A :

$$(AB)_{13} = (-1)(1) + a(-5) + 2(3) = 5 - 5a = 0,$$

$$(AB)_{23} = 1(1) + 2(-5) + x(3) = 3x - 9 = 0.$$

Quick Tip to Solve: Use the fully known third row and third column in $AA^{-1} = I$ first; each selected off-diagonal entry gives one unknown directly.

These yield $a = 1$ and $x = 3$, respectively. Both entries are off the main diagonal and therefore equal zero, not one.

Step 4: Verify the inverse and evaluate the expression With these values the matrices become

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{pmatrix}.$$

Their product is

$$AB = \begin{pmatrix} -1 - 8 + 10 & 1 + 7 - 8 & -1 - 5 + 6 \\ 1 - 16 + 15 & -1 + 14 - 12 & 1 - 10 + 9 \\ 3 - 8 + 5 & -3 + 7 - 4 & 3 - 5 + 3 \end{pmatrix} = I_3.$$

Hence all entries, including those not initially used, agree with the given inverse. Substituting in the requested expression gives $(1 + 3) - [5 + (-4)] = 4 - 1$.

Step 5: Conclusion Thus $a = 1$, $x = 3$, $b = 5$, $y = -4$, and $(a + x) - (b + y) = 3$.

Quick Tip to Solve: Use the fully known third row and third column in $AA^{-1} = I$ first; each selected off-diagonal entry gives one unknown directly.

34(a). Find: $\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx$

[5 marks]

Solution:

Concept:

- The identity $\sin^2 x + \cos^2 x = 1$ can convert a denominator involving both trigonometric functions into a polynomial in $\cos x$.
- For the substitution $u = \cos x - 2$, the differential is $du = -\sin x dx$. Integrate the resulting powers using $\int u^{-1} du = \ln |u|$ and $\int u^{-2} du = -u^{-1}$.

Step 1: Rewrite the denominator as a square Let the requested indefinite integral be I . Using $\sin^2 x = 1 - \cos^2 x$, simplify the denominator:

$$\begin{aligned} 5 - \sin^2 x - 4 \cos x &= 5 - (1 - \cos^2 x) - 4 \cos x \\ &= \cos^2 x - 4 \cos x + 4 \\ &= (\cos x - 2)^2. \end{aligned}$$

Consequently,

$$I = \int \frac{(3 \cos x - 2) \sin x}{(\cos x - 2)^2} dx.$$

The factor $\sin x dx$ now suggests substituting for $\cos x$. The denominator cannot vanish for real x , since $-1 \leq \cos x \leq 1$.

Step 2: Make the substitution and separate powers Put $u = \cos x - 2$. Then $\cos x = u + 2$,

Quick Tip to Solve: First replace $\sin^2 x$ by $1 - \cos^2 x$; the denominator becomes a perfect square and $u = \cos x - 2$ reduces the integral to two powers.

$du = -\sin x \, dx$, and the numerator factor becomes

$$3 \cos x - 2 = 3(u + 2) - 2 = 3u + 4.$$

Thus the transformed integral is

$$\begin{aligned} I &= - \int \frac{3u + 4}{u^2} du \\ &= -3 \int \frac{1}{u} du - 4 \int u^{-2} du. \end{aligned}$$

The leading minus sign comes from the differential and applies to both terms. Keeping this sign is essential when integrating the reciprocal-square term.

Step 3: Integrate and substitute back Apply the standard logarithmic and power rules:

$$\begin{aligned} I &= -3 \ln |u| - 4(-u^{-1}) + C \\ &= -3 \ln |u| + \frac{4}{u} + C \\ &= -3 \ln |\cos x - 2| + \frac{4}{\cos x - 2} + C. \end{aligned}$$

Since $2 - \cos x > 0$, we may equivalently write the logarithm as $\ln(2 - \cos x)$ and the rational term as $-4/(2 - \cos x)$. The single constant C combines the constants from the two integrations.

Step 4: Check by differentiation Differentiate the expression obtained before simplifying its signs:

$$\begin{aligned} \frac{d}{dx} \left[-3 \ln |\cos x - 2| + \frac{4}{\cos x - 2} \right] &= \frac{3 \sin x}{\cos x - 2} + \frac{4 \sin x}{(\cos x - 2)^2} \\ &= \frac{[3(\cos x - 2) + 4] \sin x}{(\cos x - 2)^2} \\ &= \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x}. \end{aligned}$$

This exactly reproduces the printed integrand and checks the sign of both terms.

Step 5: Conclusion

$$\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx = -3 \ln(2 - \cos x) - \frac{4}{2 - \cos x} + C.$$

Quick Tip to Solve: First replace $\sin^2 x$ by $1 - \cos^2 x$; the denominator becomes a perfect square and $u = \cos x - 2$ reduces the integral to two powers.

34(b). Evaluate: $\int_{-2}^2 \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4} dx$

[5 marks]

Solution:

Concept:

- An odd function satisfies $f(-x) = -f(x)$, so its integral over $[-a, a]$ is zero. An even function satisfies $g(-x) = g(x)$, and its integral is twice the integral over $[0, a]$.
- The modulus satisfies $|-x| = |x|$, and $|x| = x$ when $x \geq 0$. Here $x^2 + 4|x| + 4 = (|x| + 2)^2$, which is positive throughout the interval.
- For $u > 0$, $\int u^{-1} du = \ln u$ and $\int u^{-2} du = -u^{-1}$. A substitution must transform both limits of a definite integral.

Step 1: Separate the odd and even contributions Let the required integral be I . Split its numerator to obtain

$$I = \int_{-2}^2 \frac{x^3}{x^2 + 4|x| + 4} dx + \int_{-2}^2 \frac{|x| + 1}{x^2 + 4|x| + 4} dx.$$

In the first integrand the numerator changes sign under $x \mapsto -x$, whereas the denominator remains unchanged. Thus this integrand is odd, and its integral is zero. The second integrand is even because both its numerator and denominator remain unchanged under the same replacement. There is no singularity to obstruct these symmetry properties, since the denominator is at least 4.

Step 2: Use symmetry and remove the modulus Applying the even-function property to the remaining integral gives

$$I = 2 \int_0^2 \frac{x + 1}{x^2 + 4x + 4} dx = 2 \int_0^2 \frac{x + 1}{(x + 2)^2} dx.$$

The numerator can now be written as $x + 1 = (x + 2) - 1$. Hence

$$I = 2 \int_0^2 \left(\frac{1}{x + 2} - \frac{1}{(x + 2)^2} \right) dx.$$

This decomposition is useful because both terms have elementary antiderivatives, with the sign of the second term handled explicitly.

Step 3: Integrate and evaluate the endpoints Put $u = x + 2$, so $du = dx$. The limits $x = 0$ and $x = 2$ become $u = 2$ and $u = 4$, respectively. Therefore

$$\begin{aligned} I &= 2 \int_2^4 \left(\frac{1}{u} - \frac{1}{u^2} \right) du \\ &= 2 \left[\ln u + \frac{1}{u} \right]_2^4 \\ &= 2 \left(\ln 4 + \frac{1}{4} - \ln 2 - \frac{1}{2} \right). \end{aligned}$$

To check the antiderivative, differentiating $\ln u + 1/u$ gives $1/u - 1/u^2$, exactly the transformed integrand. Also, the surviving even integrand is positive, consistent with a positive final value.

Quick Tip to Solve: Split off the odd x^3 term before removing the modulus; then factor the denominator as $(x + 2)^2$ on the positive half-interval.

Step 4: Conclusion The required definite integral is $\boxed{2 \ln 2 - \frac{1}{2}}$.

Quick Tip to Solve: Split off the odd x^3 term before removing the modulus; then factor the denominator as $(x + 2)^2$ on the positive half-interval.

35. Using integration, find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$, included between the lines $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, from $x = a$ to $x = b$, is $\int_a^b [f(x) - g(x)] dx$.
- The ellipse $x^2/16 + y^2/4 = 1$ is symmetric about both coordinate axes. Its upper and lower branches are $y = \pm 2\sqrt{1 - x^2/16}$, for $-4 \leq x \leq 4$.
- The substitution $x = 4 \sin \theta$ simplifies $\sqrt{16 - x^2}$. The identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ then makes the resulting integral elementary.

Step 1: Express the vertical height of the region Rearranging the equation of the ellipse,

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}, \quad y = \pm \frac{1}{2} \sqrt{16 - x^2}.$$

For every x between -2 and 2 , the upper boundary is $\frac{1}{2}\sqrt{16 - x^2}$ and the lower boundary is $-\frac{1}{2}\sqrt{16 - x^2}$. Consequently, the height of a vertical strip is their difference, $\sqrt{16 - x^2}$. Both halves of the ellipse contribute to the requested area; using just the upper branch would count only half of it.

Step 2: Set up the area integral and exploit symmetry If A denotes the required area, the strip method gives

$$A = \int_{-2}^2 \sqrt{16 - x^2} dx = 2 \int_0^2 \sqrt{16 - x^2} dx.$$

The second equality uses the evenness of $\sqrt{16 - x^2}$. Equivalently, this is four times the area in the first quadrant between $x = 0$ and $x = 2$, since the ellipse and the bounding lines are symmetric about both axes. The integration limits come from the two stated vertical lines, rather than the endpoints $x = \pm 4$ of the full ellipse.

Step 3: Apply a trigonometric substitution Set $x = 4 \sin \theta$. On the interval $0 \leq x \leq 2$, take $0 \leq \theta \leq \pi/6$, so $\cos \theta \geq 0$. Thus

$$dx = 4 \cos \theta d\theta, \quad \sqrt{16 - x^2} = 4 \cos \theta.$$

The lower limit becomes $\theta = 0$, and the upper limit satisfies $\sin \theta = 1/2$, giving $\theta = \pi/6$. Therefore

$$A = 32 \int_0^{\pi/6} \cos^2 \theta d\theta.$$

Quick Tip to Solve: Subtract the lower branch from the upper branch first, then use even symmetry to halve the interval without losing a factor of two.

Step 4: Evaluate the transformed integral Using the double-angle identity,

$$\begin{aligned} A &= 16 \int_0^{\pi/6} (1 + \cos 2\theta) d\theta \\ &= 16 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/6} \\ &= 16 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right). \end{aligned}$$

As a size check, the strip has width 4, and its height lies between $2\sqrt{3}$ and 4. Its area must therefore lie between $8\sqrt{3}$ and 16; the evaluated result satisfies these bounds and is smaller than the full ellipse area 8π .

Step 5: Conclusion The area of the ellipse included between $x = -2$ and $x = 2$ is $\boxed{\frac{8\pi}{3} + 4\sqrt{3}}$ square units.

Quick Tip to Solve: Subtract the lower branch from the upper branch first, then use even symmetry to halve the interval without losing a factor of two.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

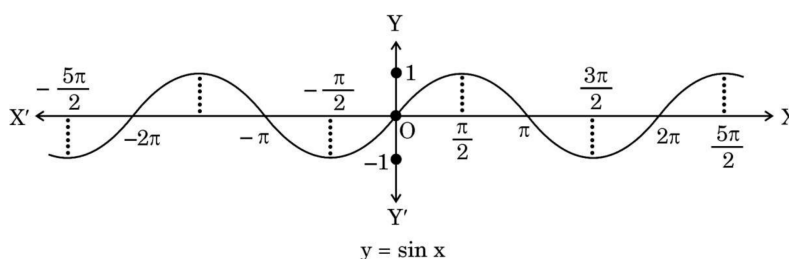
If a function $f : X \rightarrow Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \rightarrow X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Function g is called the inverse of function f .

The domain of sine function is $\mathbb{R}$ and function sine: $\mathbb{R} \rightarrow \mathbb{R}$ is neither one-one nor onto. The following graph shows the sine function.

Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A .

On the basis of the above information, answer the following questions:

[4 marks]



36(i). If A is the interval other than principal value branch, give an example of one such interval.

[1 mark]

Solution:

Concept:

- The restricted sine function has an inverse when it is one-one and maps its chosen interval onto $[-1, 1]$.
- The principal interval is $[-\pi/2, \pi/2]$; a different complete monotonic branch is also possible.

Step 1: Choose another branch Take $A = [\pi/2, 3\pi/2]$. On this interval sine decreases strictly from $\sin(\pi/2) = 1$ to $\sin(3\pi/2) = -1$.

Step 2: Check invertibility By continuity, every value in $[-1, 1]$ occurs, and strict decrease ensures that it occurs exactly once. Thus the restricted sine function is both one-one and onto. This interval differs from the principal branch.

Step 3: Conclusion One example is $A = [\pi/2, 3\pi/2]$.

Quick Tip to Solve: Choose an interval between consecutive extrema of sine, covering its full range once.

36(ii). If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$. [1 mark]

Solution:

Concept:

- Principal values of $\sin^{-1} x$ lie in $[-\pi/2, \pi/2]$.
- Choose the angle in that interval whose sine equals the given input.

Step 1: Evaluate both principal values Since $\sin(-\pi/6) = -1/2$ and $-\pi/6$ lies in the principal interval, $\sin^{-1}(-1/2) = -\pi/6$. Similarly, $\sin^{-1}(1) = \pi/2$, the upper endpoint of the principal interval.

Step 2: Subtract the angles Use a common denominator of 6:

$$\sin^{-1}(-1/2) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{3\pi}{6} = -\frac{4\pi}{6}.$$

The negative input selects a negative principal angle; no supplementary positive angle is admissible.

Step 3: Conclusion The required value is $\boxed{-2\pi/3}$.

Quick Tip to Solve: Write the two principal angles first, then subtract using the same denominator.

36(iii)(a). Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch. [2 marks]

Solution:

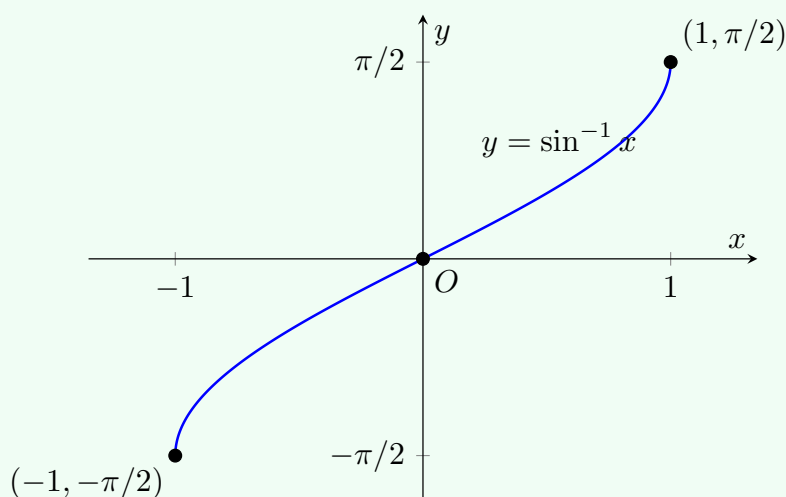
Concept:

- For the principal inverse sine, $y = \sin^{-1} x$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
- Interchanging coordinates on $x = \sin y$ gives the graph of the inverse function.

Step 1: Locate the defining points The endpoint values are $\sin^{-1}(-1) = -\pi/2$ and $\sin^{-1}(1) = \pi/2$, while $\sin^{-1}(0) = 0$. Thus the curve passes through $(-1, -\pi/2)$, $(0, 0)$, and $(1, \pi/2)$. Both endpoints belong to the graph because the domain is closed.

Step 2: Draw the principal branch The restricted sine function is strictly increasing, so its inverse is also strictly increasing. Draw a continuous curve through these three points. Its slope is $1/\sqrt{1-x^2}$ for $-1 < x < 1$: it is finite at the origin and becomes unbounded near the endpoints. Consequently, the curve has vertical endpoint tangents, lies below the horizontal axis for negative inputs and above it for positive inputs, and is symmetric about the origin.

Quick Tip to Solve: Mark both endpoints and the origin before drawing the increasing inverse-sine curve.



Principal graph of inverse sine

Step 3: Conclusion The displayed curve is $y = \sin^{-1} x, -1 \leq x \leq 1$, with principal range $[-\pi/2, \pi/2]$.

Quick Tip to Solve: Mark both endpoints and the origin before drawing the increasing inverse-sine curve.

OR

36(iii)(b). Find the domain and range of $f(x) = 2 \sin^{-1}(1 - x)$.

[2 marks]

Solution:

Concept:

- The real principal inverse sine requires its input to lie in $[-1, 1]$.
- Its output lies in $[-\pi/2, \pi/2]$. Multiplication by the positive constant 2 doubles both range endpoints.

Step 1: Determine the domain For $f(x) = 2 \sin^{-1}(1 - x)$ to be real,

$$-1 \leq 1 - x \leq 1.$$

Subtracting 1 gives $-2 \leq -x \leq 0$. Multiplication by -1 reverses the inequalities, giving $0 \leq x \leq 2$. The endpoints are included because inverse sine is defined at inputs 1 and -1 .

Step 2: Determine and verify the range As x runs through $[0, 2]$, the expression $1 - x$ covers all of $[-1, 1]$. Therefore $\sin^{-1}(1 - x)$ covers $[-\pi/2, \pi/2]$, and multiplying by 2 gives $[-\pi, \pi]$. In particular, $f(0) = \pi$ and $f(2) = -\pi$. Continuity ensures that every intermediate value is attained; these are actual range endpoints, not merely bounds.

Step 3: Conclusion $\text{Domain}(f) = [0, 2], \text{Range}(f) = [-\pi, \pi]$.

Quick Tip to Solve: Apply the input restriction to the entire expression $1 - x$; only then scale the output range.

37. Case Study - 2

The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.

A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ .

On the basis of the above information, answer the following questions:

[4 marks]

37(i). Express θ in terms of height of the camera installed on the pole and x .

[1 mark]

Solution:

Concept:

- For a right triangle, $\tan \theta$ equals opposite side divided by adjacent side.
- The pole is vertical, its height is 5 m, and the horizontal car-to-base distance is x m.

Step 1: Form the trigonometric relation The angle of elevation is measured at the car. The camera height is the opposite side and the distance from the base is the adjacent side. Hence $\tan \theta = 5/x$ for $x > 0$.

Step 2: Express the angle The angle of elevation is acute, so it is the principal inverse tangent of the positive ratio $5/x$. Applying inverse tangent therefore determines a unique angle in $(0, \pi/2)$. The detection distance does not change this geometric relation.

Step 3: Conclusion $\theta = \tan^{-1}(5/x), \quad x > 0$.

Quick Tip to Solve: Use the height as the opposite side and the horizontal distance as the adjacent side.

37(ii). Find $\frac{d\theta}{dx}$.

[1 mark]

Solution:

Concept:

- From the right-triangle geometry, $\theta = \tan^{-1}(5/x)$ for $x > 0$.
- The chain rule gives $d(\tan^{-1} u)/dx = u'/(1 + u^2)$; the angle is in radians.

Step 1: Differentiate the inner expression Let $u = 5/x = 5x^{-1}$. Since the camera height is constant, $du/dx = -5x^{-2} = -5/x^2$.

Step 2: Apply the chain rule and simplify

$$\frac{d\theta}{dx} = \frac{-5/x^2}{1 + 25/x^2} = \frac{-5/x^2}{(x^2 + 25)/x^2} = -\frac{5}{x^2 + 25}.$$

The cancellation is valid because $x > 0$. The negative sign agrees with the geometry: moving farther from a fixed-height pole reduces the angle of elevation.

Step 3: Conclusion $d\theta/dx = -5/(x^2 + 25)$, measured in radians per metre.

Quick Tip to Solve: Simplify the chain-rule fraction by multiplying numerator and denominator by x squared.

37(iii)(a). Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole. [2 marks]

Solution:

Concept:

- The right triangle formed by the pole, ground and line of sight gives $\tan \theta = 5/x$, where x is the horizontal distance in metres and θ is measured in radians.
- For related rates, apply the chain rule: $d\theta/dt = (d\theta/dx)(dx/dt)$. Moving away makes dx/dt positive.

Step 1: Differentiate the geometric relation For $x > 0$, write $\theta = \tan^{-1}(5/x)$. The derivative of the inverse tangent, followed by the chain rule, gives

$$\frac{d\theta}{dx} = \frac{1}{1 + 25/x^2} \left(-\frac{5}{x^2} \right) = -\frac{5}{x^2 + 25}.$$

The pole height remains constant, while the horizontal distance changes. Hence the angle decreases as the distance increases.

Step 2: Substitute the distance and speed The car travels away at $dx/dt = 20$ m/s. Thus, at $x = 50$ m,

$$\begin{aligned} \frac{d\theta}{dt} &= -\frac{5}{x^2 + 25} \frac{dx}{dt} \\ &= -\frac{5 \times 20}{50^2 + 25} = -\frac{100}{2525} = -\frac{4}{101} \text{ rad/s.} \end{aligned}$$

The negative sign is physically consistent: an observer farther from the pole sees a smaller angle of elevation.

Step 3: Conclusion The rate of change is $-\frac{4}{101}$ rad/s; the angle decreases at $\frac{4}{101}$ rad/s.

Quick Tip to Solve: Keep the sign of the distance rate: moving away gives positive dx/dt but a negative angle rate.

OR

37(iii)(b). If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car. [2 marks]

Solution:

Concept:

- For a camera at height 5 m, the angle of elevation satisfies $\theta = \tan^{-1}(5/x)$, with $x > 0$ the horizontal distance.
- The signed rates obey $d\theta/dt = -5(dx/dt)/(x^2 + 25)$. Speed is the magnitude $|dx/dt|$, irrespective of direction.

Quick Tip to Solve: First solve for the signed distance rate, then take its absolute value when the question asks for speed.

Step 1: Relate the two rates Differentiate $\theta = \tan^{-1}(5/x)$ with respect to time, using the chain rule:

$$\frac{d\theta}{dt} = \frac{1}{1 + (5/x)^2} \left(-\frac{5}{x^2} \right) \frac{dx}{dt} = -\frac{5}{x^2 + 25} \frac{dx}{dt}.$$

At the specified distance $x = 50$ m, this reduces to $d\theta/dt = -(1/505)(dx/dt)$. The fixed pole height is not differentiated as a variable.

Step 2: Use the printed angle rate The printed rate is $d\theta/dt = 3/101$ rad/s. Substituting it gives

$$\frac{3}{101} = -\frac{1}{505} \frac{dx}{dt} \implies \frac{dx}{dt} = -\frac{505 \times 3}{101} = -15 \text{ m/s}.$$

A positive angle rate means the car is approaching the pole, since its angle of elevation is increasing. The question asks for speed, so take the magnitude of this signed distance rate.

Step 3: Conclusion The speed of the car is 15 m/s. For the positive printed angle rate, its motion is towards the pole.

Quick Tip to Solve: First solve for the signed distance rate, then take its absolute value when the question asks for speed.

38. Case Study - 3

According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that, an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.

On the basis of the above information, answer the following questions:

[4 marks]

38(i). Find the probability that an airplane reached its destination late.

[2 marks]

Solution:

Concept:

- Let S, M, L denote severe, moderate and light turbulence, and D denote late arrival. These turbulence events form a mutually exclusive and exhaustive partition.
- The law of total probability is $P(D) = \sum_E P(E)P(D | E)$, where E ranges over S, M, L .

Step 1: Record the prior and conditional probabilities The three turbulence types have equal probabilities, so $P(S) = P(M) = P(L) = 1/3$. The given percentages describe late arrival conditional on each turbulence type:

$$P(D | S) = \frac{55}{100}, \quad P(D | M) = \frac{37}{100}, \quad P(D | L) = \frac{17}{100}.$$

These are different conditional chances, even though the turbulence types themselves are equally likely.

Step 2: Combine the disjoint ways of arriving late A flight can arrive late with severe, moderate or

Quick Tip to Solve: With equally likely turbulence types, average the three conditional late-arrival probabilities.

light turbulence. The events $D \cap S$, $D \cap M$ and $D \cap L$ are disjoint and cover all late arrivals. Therefore,

$$\begin{aligned} P(D) &= P(S)P(D | S) + P(M)P(D | M) + P(L)P(D | L) \\ &= \frac{1}{3} \left(\frac{55}{100} + \frac{37}{100} + \frac{17}{100} \right) = \frac{109}{300}. \end{aligned}$$

Each conditional probability is weighted by its turbulence probability; simply adding the three percentages would exceed one.

Step 3: Conclusion The probability of late arrival is $\boxed{\frac{109}{300}}$.

Quick Tip to Solve: With equally likely turbulence types, average the three conditional late-arrival probabilities.

38(ii). If the airplane reached its destination late, find the probability that it was due to moderate turbulence. [2 marks]

Solution:

Concept:

- Let S, M, L be the severe, moderate and light turbulence events, and D be late arrival. Each turbulence type has prior probability $1/3$.
- Bayes theorem gives $P(M | D) = P(M)P(D | M)/P(D)$. The denominator includes late arrivals from every turbulence type.

Step 1: Find the overall late-arrival probability The conditional late-arrival probabilities are $55/100$, $37/100$ and $17/100$ for severe, moderate and light turbulence, respectively. By total probability,

$$\begin{aligned} P(D) &= \frac{1}{3} \frac{55}{100} + \frac{1}{3} \frac{37}{100} + \frac{1}{3} \frac{17}{100} \\ &= \frac{109}{300}. \end{aligned}$$

The common factor $1/3$ reflects the equal likelihood of all three turbulence categories.

Step 2: Condition on the known late arrival The probability of both moderate turbulence and late arrival is $P(M \cap D) = P(M)P(D | M) = (1/3)(37/100) = 37/300$. Thus Bayes theorem yields

$$P(M | D) = \frac{P(M \cap D)}{P(D)} = \frac{37/300}{109/300} = \frac{37}{109}.$$

The given 37% is the chance of lateness when moderate turbulence occurs; the question reverses the conditioning by telling us that the flight was late.

Step 3: Conclusion The required probability is $\boxed{\frac{37}{109}}$.

Quick Tip to Solve: For equal prior probabilities, divide the moderate late percentage by the sum of all three late percentages.