

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-2-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If the sum of all the elements of a 3×3 scalar matrix is 9, then the product of all its elements is: [1 mark]

- (A) 0
- (B) 9
- (C) 27
- (D) 729

Correct Answer: (A) 0

Solution:

Concept:

- A scalar matrix has the form $A = \lambda I_n$: every diagonal entry is λ , and every off-diagonal entry is zero.
- The product of all entries includes the off-diagonal entries; a single zero factor makes this product zero.

Step 1: Determine the scalar entry Write $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$. Its nine entries have sum $3\lambda = 9$, giving $\lambda = 3$.

Step 2: Include all nine entries in the product There are three entries equal to 3 and six entries equal to 0. Thus their product is $3^3 \cdot 0^6 = 0$. Option (B) repeats the sum, while (C) multiplies only the diagonal entries. Option (D) also ignores the zero entries; neither is the product requested.

Step 3: Conclusion The product of all elements is $\boxed{0}$. Hence, the correct option is (A).

Quick Tip to Solve: For a scalar matrix of order greater than one, notice its off-diagonal zeros before multiplying any entries.

2. Let $f : R_+ \rightarrow [-5, \infty)$ be defined as $f(x) = 9x^2 + 6x - 5$, where R_+ is the set of all non-negative real numbers. Then, f is: [1 mark]

- (A) one-one
- (B) onto
- (C) bijective
- (D) neither one-one nor onto

Correct Answer: (C) bijective

Solution:

Concept:

- A function is one-one if equal outputs force equal inputs, and onto if every element of the stated codomain has a preimage.
- A function that is both one-one and onto is bijective. Here the domain includes zero: $R_+ = [0, \infty)$.

Step 1: Establish that the function is one-one For $a, b \geq 0$, suppose $f(a) = f(b)$. Factoring gives $(a - b)[9(a + b) + 6] = 0$. Since $9(a + b) + 6 > 0$, we obtain $a = b$. Thus distinct non-negative inputs cannot have the same output.

Step 2: Establish that the function is onto Completing the square gives $f(x) = (3x + 1)^2 - 6$. For any $y \geq -5$, choose $x = (\sqrt{y + 6} - 1)/3 \geq 0$. Substitution gives $f(x) = y$, so every codomain value is attained, including -5 at $x = 0$. Options (A) and (B) each state only one of the two properties; (C) specifies both. Option (D) contradicts both checks.

Step 3: Conclusion The function is bijective. Hence, the correct option is (C).

Quick Tip to Solve: After proving injectivity, check the exact codomain by solving $f(x)=y$ for a non-negative preimage.

3. If $\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = k abc$, then the value of k is: [1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) 4

Correct Answer: (D) 4

Solution:

Concept:

- Taking a common factor out of each column multiplies a determinant by the product of those factors.
- In expansion along the first row, the cofactor signs are $+, -, +$; each minor is a 2×2 determinant.

Quick Tip to Solve: Factor each column first so that the remaining determinant contains only 1 and -1, reducing sign errors.

Step 1: Factor the columns Factoring a, b, c from the first, second and third columns respectively gives

$$D = abc \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

This determinant identity holds even if a column factor is zero, since no division is needed.

Step 2: Expand the numerical determinant Expanding along the first row gives

$$\begin{aligned} D &= abc\{(-1)[1 - 1] - [(-1) - 1] + [1 - (-1)]\} \\ &= abc(0 + 2 + 2) = 4abc. \end{aligned}$$

In particular, the two nonzero cofactor contributions add rather than cancel. The coefficients 0, 1, 2 in options (A), (B), (C) do not match this expansion.

Step 3: Conclusion Comparing with $D = kabc$, the coefficient is $k = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Factor each column first so that the remaining determinant contains only 1 and -1, reducing sign errors.

4. The number of points of discontinuity of $f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$ is: [1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) infinite

Correct Answer: (B) 1

Solution:

Concept:

- At a joining point a , continuity requires $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$.
- The functions $|x| + 3$, $-2x$, and $6x + 2$ are continuous on their own intervals, so only the joins -3 and 3 need checking.

Step 1: Test the join at -3 The first branch gives $f(-3) = |-3| + 3 = 6$ and $\lim_{x \rightarrow -3^-} f(x) = 6$. The middle branch gives $\lim_{x \rightarrow -3^+} f(x) = -2(-3) = 6$. Both limits equal the function value, so this join is continuous.

Step 2: Test the join at 3 From the middle branch, $\lim_{x \rightarrow 3^-} f(x) = -6$. From the last branch, $\lim_{x \rightarrow 3^+} f(x) = 20 = f(3)$. The unequal one-sided limits give a discontinuity at 3. There are no other candidate points. Thus (A) misses this jump, (C) incorrectly counts the continuous join at -3 , and (D) ignores continuity within each branch.

Quick Tip to Solve: For piecewise continuous formulas, compare both one-sided limits only at the branch boundaries.

Step 3: Conclusion There is exactly $\boxed{1}$ point of discontinuity. Hence, the correct option is (B).

Quick Tip to Solve: For piecewise continuous formulas, compare both one-sided limits only at the branch boundaries.

5. The function $f(x) = x^3 - 3x^2 + 12x - 18$ is:

[1 mark]

- (A) strictly decreasing on R
- (B) strictly increasing on R
- (C) neither strictly increasing nor strictly decreasing on R
- (D) strictly decreasing on $(-\infty, 0)$

Correct Answer: (B) strictly increasing on R

Solution:

Concept:

- A differentiable function with positive derivative throughout an interval is strictly increasing there.
- Completing the square reveals the sign of a quadratic derivative for every real input, without estimating its roots.

Step 1: Differentiate and complete the square For $f(x) = x^3 - 3x^2 + 12x - 18$,

$$f'(x) = 3x^2 - 6x + 12 = 3[(x - 1)^2 + 3].$$

Since $(x - 1)^2 \geq 0$, the derivative satisfies $f'(x) \geq 9 > 0$ for every $x \in R$.

Step 2: Apply the monotonicity criterion For any $a < b$, the mean value theorem gives $f(b) - f(a) = f'(c)(b - a) > 0$ for some $c \in (a, b)$. Thus the function increases strictly over the whole real line. Options (A) and (D) demand decreasing behaviour, inconsistent with its positive derivative even on $(-\infty, 0)$. Option (C) fails because the derivative never changes sign.

Step 3: Conclusion The function is strictly increasing on R . Hence, the correct option is (B).

Quick Tip to Solve: Write the quadratic derivative as a positive multiple of a square plus a positive constant to decide monotonicity globally.

6. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ is equal to:

[1 mark]

(A) π

(B) Zero (0)

(C) $\int_0^{\pi/2} \frac{2 \sin x}{1 + \sin x \cos x} dx$

(D) $\frac{\pi^2}{4}$

Correct Answer: (B) Zero (0)

Solution:

Concept:

- Reflection of a definite integral gives $\int_0^a g(x) dx = \int_0^a g(a-x) dx$.
- At $a = \pi/2$, reflection interchanges sine and cosine. The denominator $1 + \sin x \cos x$ stays positive throughout this interval.

Step 1: Reflect the integration variable Let the given integral be I . Substituting $x = \pi/2 - t$ reverses the limits and introduces $dx = -dt$. Hence

$$I = \int_0^{\pi/2} \frac{\cos t - \sin t}{1 + \cos t \sin t} dt = -I.$$

The numerator changes sign while the denominator remains unchanged.

Step 2: Use cancellation and check the options The identity $I = -I$ yields $2I = 0$. No singularity obstructs this argument, since the denominator is at least 1. Option (C) is strictly positive: its printed integrand $2 \sin x / (1 + \sin x \cos x)$ is positive in the interior. Options (A) and (D) are also positive, so none of these equals the cancellation value.

Step 3: Conclusion The integral equals $\boxed{0}$. Hence, the correct option is (B).

Quick Tip to Solve: Before finding an antiderivative, replace x by $\pi/2$ minus x and check whether the integrand changes sign.

7. The differential equation $\frac{dy}{dx} = F(x, y)$ will not be a homogeneous differential equation, if $F(x, y)$ is: [1 mark]

(A) $\cos x - \sin\left(\frac{y}{x}\right)$

(B) $\frac{y}{x}$

(C) $\frac{x^2 + y^2}{xy}$

(D) $\cos^2\left(\frac{x}{y}\right)$

Correct Answer: (A) $\cos x - \sin\left(\frac{y}{x}\right)$

Solution:

Concept:

- The first-order equation $dy/dx = F(x, y)$ is homogeneous when F has degree zero, that is, $F(tx, ty) = F(x, y)$ for admissible nonzero scaling factors.
- Expressions depending only on y/x , or equivalently on x/y where defined, satisfy this degree-zero test.

Step 1: Test the separate dependence on x In option (A), $F(tx, ty) = \cos(tx) - \sin(y/x)$. The ratio in the sine term remains fixed, but $\cos(tx)$ generally differs from $\cos x$. Consequently F is not invariant under scaling. The presence of an unscaled argument x prevents homogeneity.

Step 2: Verify that the remaining expressions are homogeneous Option (B) is y/x , a ratio. In (C),

$$\frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x},$$

which also depends only on ratios, on its domain $xy \neq 0$. Option (D), $\cos^2(x/y)$, retains the same argument under scaling. Thus (B), (C), and (D) all have degree zero wherever their respective formulas are defined.

Step 3: Conclusion The expression in (A) does not give a homogeneous differential equation. Hence, the correct option is (A).

Quick Tip to Solve: Scale x and y together; ratios stay fixed, but a lone x inside a trigonometric function generally changes.

8. For any two vectors $\vec{a}$ and $\vec{b}$, which of the following statements is always true?

[1 mark]

- (A) $\vec{a} \cdot \vec{b} \geq |\vec{a}| |\vec{b}|$
- (B) $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$
- (C) $\vec{a} \cdot \vec{b} \leq |\vec{a}| |\vec{b}|$
- (D) $\vec{a} \cdot \vec{b} < |\vec{a}| |\vec{b}|$

Correct Answer: (C) $\vec{a} \cdot \vec{b} \leq |\vec{a}| |\vec{b}|$

Solution:

Concept:

- For nonzero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is their angle.
- The cosine always satisfies $-1 \leq \cos \theta \leq 1$; magnitudes are nonnegative.

Step 1: Bound the dot product Multiplying $\cos \theta \leq 1$ by the nonnegative product of the magnitudes gives $\vec{a} \cdot \vec{b} \leq |\vec{a}| |\vec{b}|$. If either vector is zero, both sides are zero, so this bound also holds then.

Step 2: Test the alternatives Equality holds for nonzero vectors in the same direction, so the strict inequality (D) is not always true. For perpendicular nonzero vectors the dot product is zero while the product of magnitudes is positive; this rules out both (A) and (B).

Step 3: Conclusion The always-valid statement is $\vec{a} \cdot \vec{b} \leq |\vec{a}| |\vec{b}|$. Hence, the correct option is (C).

Quick Tip to Solve: Use $\cos \theta \leq 1$, and test parallel vectors before accepting a strict dot-product inequality.

9. The coordinates of the foot of the perpendicular drawn from the point $(0, 1, 2)$ on the x -axis are given by:

[1 mark]

- (A) $(1, 0, 0)$
- (B) $(2, 0, 0)$
- (C) $(\sqrt{5}, 0, 0)$
- (D) $(0, 0, 0)$

Correct Answer: (D) $(0, 0, 0)$

Solution:

Concept:

- Every point on the x -axis has the form $F = (t, 0, 0)$.
- A perpendicular to the x -axis has zero scalar product with its direction vector $(1, 0, 0)$.

Step 1: Express the perpendicular condition Let $P = (0, 1, 2)$ and $F = (t, 0, 0)$. The vector from the foot to the point is $\vec{FP} = (-t, 1, 2)$. Hence perpendicularity requires $(-t, 1, 2) \cdot (1, 0, 0) = -t = 0$.

Step 2: Check the possible feet The perpendicular condition gives $t = 0$. The other listed points have nonzero first coordinates 1, 2 and $\sqrt{5}$, respectively, so their joining vectors to P have a nonzero component along the axis and cannot be perpendicular to it.

Step 3: Conclusion The foot is the origin $(0, 0, 0)$. Hence, the correct option is (D).

Quick Tip to Solve: For projection onto the x -axis, retain the point's first coordinate and replace the other two by zero.

10. The common region determined by all the constraints of a linear programming problem is called:

[1 mark]

- (A) an unbounded region
- (B) an optimal region
- (C) a bounded region
- (D) a feasible region

Correct Answer: (D) a feasible region

Solution:

Concept:

- A feasible point satisfies every constraint of a linear programming problem, including nonnegativity conditions.
- The feasible region is the intersection of the solution regions of all the constraints.

Step 1: Apply the definition Each linear inequality determines a half-plane. A point in their common region obeys all the inequalities simultaneously and is therefore feasible. The entire common region is consequently termed the feasible region.

Step 2: Distinguish the other terms The feasible region may be bounded or unbounded, depending on the constraints, so neither (A) nor (C) is its general name. Optimality refers to maximizing or minimizing the objective function over feasible points; constraints alone do not designate an optimal region. Thus (B) is also inappropriate.

Step 3: Conclusion The common region is called a feasible region. Hence, the correct option is (D).

Quick Tip to Solve: Associate constraints with feasibility; use the objective function only when deciding optimality.

11. Let E be an event of a sample space S of an experiment, then $P(S | E) =$

[1 mark]

- (A) $P(S \cap E)$
- (B) $P(E)$
- (C) 1
- (D) 0

Correct Answer: (C) 1

Solution:

Concept:

- For $P(E) > 0$, conditional probability is $P(A | E) = \frac{P(A \cap E)}{P(E)}$.
- Since an event E is a subset of its sample space S , $S \cap E = E$.

Step 1: Simplify the intersection Conditioning on E means considering outcomes in E . Every such outcome is already in S , so the intersection of S with E is exactly E . Its probability is therefore $P(S \cap E) = P(E)$.

Step 2: Substitute in the conditional probability For the conditional probability to be defined,

Quick Tip to Solve: The sample space remains certain after conditioning on any event of positive probability.

$P(E) > 0$. Thus

$$P(S | E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1.$$

Options (A) and (B) are not universally equal to this value, because $P(E)$ need not be one; (D) incorrectly makes the sample space impossible.

Step 3: Conclusion $P(S | E) = 1$, whenever the stated conditional probability is defined. Hence, the correct option is (C).

Quick Tip to Solve: The sample space remains certain after conditioning on any event of positive probability.

12. If $A = [a_{ij}]$ be a 3×3 matrix, where $a_{ij} = i - 3j$, then which of the following is false? [1 mark]

- (A) $a_{11} < 0$
- (B) $a_{12} + a_{21} = -6$
- (C) $a_{13} > a_{31}$
- (D) $a_{31} = 0$

Correct Answer: (C) $a_{13} > a_{31}$

Solution:

Concept:

- In $a_{ij} = i - 3j$, the first index i denotes the row and the second index j denotes the column.
- To identify the false statement, calculate each relevant entry directly rather than interchanging its indices.

Step 1: Calculate the required entries Substituting the indicated indices gives

$$a_{11} = 1 - 3 = -2, \quad a_{12} = 1 - 6 = -5,$$

$$a_{21} = 2 - 3 = -1, \quad a_{13} = 1 - 9 = -8,$$

$$a_{31} = 3 - 3 = 0.$$

These entries are sufficient to check all four statements.

Step 2: Check every statement Statement (A) holds because $-2 < 0$. Statement (B) holds because $a_{12} + a_{21} = -5 - 1 = -6$. Statement (D) holds because $a_{31} = 0$. In contrast, (C) requires $-8 > 0$, which is false: actually $a_{13} < a_{31}$.

Step 3: Conclusion The false statement is $a_{13} > a_{31}$. Hence, the correct option is (C).

Quick Tip to Solve: Calculate a_{13} and a_{31} separately: swapping row and column indices changes their values.

13. The derivative of $\tan^{-1}(x^2)$ with respect to x is:

[1 mark]

- (A) $\frac{x}{1+x^4}$
(B) $\frac{2x}{1+x^4}$
(C) $-\frac{2x}{1+x^4}$
(D) $\frac{1}{1+x^4}$

Correct Answer: (B) $\frac{2x}{1+x^4}$

Solution:

Concept:

- The inverse tangent derivative is $\frac{d}{du} \tan^{-1} u = \frac{1}{1+u^2}$, where u is real.
- The chain rule gives $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ for a composite function.

Step 1: Differentiate the inner function Put $u = x^2$. Then $y = \tan^{-1} u$ and $\frac{du}{dx} = 2x$. The factor $2x$ must multiply the inverse tangent derivative, because the argument is x^2 , rather than simply x .

Step 2: Apply the chain rule and compare We obtain

$$\frac{dy}{dx} = \frac{1}{1+u^2} (2x) = \frac{2x}{1+x^4}.$$

The numerator in (A) omits the factor two; (C) introduces an incorrect minus sign; and (D) omits the inner derivative altogether. Also, $1+x^4 > 0$ for every real x , so the expression is defined throughout the real line.

Step 3: Conclusion The derivative is $\frac{2x}{1+x^4}$. Hence, the correct option is (B).

Quick Tip to Solve: In $\tan^{-1}(x^2)$, square the argument for the denominator and differentiate it for the numerator.

14. The degree of the differential equation $(y'')^2 + (y')^3 = x \sin(y')$ is:

[1 mark]

- (A) 1
(B) 2
(C) 3
(D) not defined

Correct Answer: (D) not defined

Solution:

Concept:

- The degree of a differential equation is defined only when it is polynomial in its derivatives, after removing radicals and fractions involving derivatives when possible.

Quick Tip to Solve: Check that every derivative appears polynomially before using the exponent of the highest-order derivative.

- When this polynomial form exists, the degree is the exponent of the highest-order derivative in that form.

Step 1: Inspect how the derivatives occur The equation is $(y'')^2 + (y')^3 = x \sin(y')$. Although the left side is polynomial in the derivatives, the right side contains the first derivative inside the sine function. Thus the full equation is not polynomial in its derivatives.

Step 2: Separate degree from order The highest derivative appearing is y'' , so the order is two. However, its exponent two cannot be declared the degree without first meeting the polynomial requirement. The term $\sin(y')$ prevents this, and the cube of y' does not make the degree three. Therefore none of the numerical degree options applies.

Step 3: Conclusion The degree is not defined. Hence, the correct option is (D).

Quick Tip to Solve: Check that every derivative appears polynomially before using the exponent of the highest-order derivative.

15. The unit vector perpendicular to both vectors $\hat{i} + \hat{k}$ and $\hat{i} - \hat{k}$ is:

[1 mark]

- (A) $2\hat{j}$
 (B) $\hat{j}$
 (C) $\frac{\hat{i} - \hat{k}}{\sqrt{2}}$
 (D) $\frac{\hat{i} + \hat{k}}{\sqrt{2}}$

Correct Answer: (B) $\hat{j}$

Solution:

Concept:

- A vector is perpendicular to another vector when their dot product is zero. A unit vector has magnitude 1.
- The vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular unit vectors along the coordinate axes.

Step 1: Determine the common perpendicular Both given vectors, $\vec{a} = \hat{i} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{k}$, lie in the xz -plane. Their cross product is $\vec{a} \times \vec{b} = 2\hat{j}$. Thus a perpendicular unit direction is obtained by dividing by its magnitude 2.

Step 2: Check all listed candidates For $\hat{j}$, both dot products are zero and its magnitude is 1. Option (A), $2\hat{j}$, is perpendicular but has magnitude 2, so is not a unit vector. Option (C) is parallel to $\vec{b}$, and its dot product with $\vec{b}$ is $\sqrt{2}$, not zero. Option (D) is parallel to $\vec{a}$, with dot product $\sqrt{2}$ against $\vec{a}$.

Step 3: Conclusion The listed unit vector perpendicular to both is $\hat{j}$. Hence, the correct option is (B).

Quick Tip to Solve: Check perpendicularity and unit length separately: a multiple of the correct direction may still have the wrong magnitude.

16. Direction ratios of a vector parallel to the line $\frac{x-1}{2} = -y = \frac{2z+1}{6}$ are:

[1 mark]

- (A) 2, -1, 6
- (B) 2, 1, 6
- (C) 2, 1, 3
- (D) 2, -1, 3

Correct Answer: (D) 2, -1, 3

Solution:

Concept:

- In $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, the direction ratios are a, b, c .
- Direction ratios are proportional to the coefficients of a common parameter in the coordinates.

Step 1: Write the printed line using a parameter Let $\frac{x-1}{2} = -y = \frac{2z+1}{6} = t$. Then

$$x = 1 + 2t, \quad y = -t, \quad z = -\frac{1}{2} + 3t.$$

The factor 2 multiplying z must be divided out before its direction ratio is read.

Step 2: Identify and compare the direction ratios The coefficients of t give $(2, -1, 3)$. Option (A) incorrectly uses 6 for the third ratio without dividing by 2. Option (B) has both the incorrect third ratio and the wrong sign for the second ratio. Option (C) uses the correct first and third ratios but the wrong sign for the second ratio. None of these triples is proportional to $(2, -1, 3)$.

Step 3: Conclusion The required direction ratios are $[2, -1, 3]$. Hence, the correct option is (D).

Quick Tip to Solve: First make each coordinate have coefficient one in the numerator; $(2z+1)/6$ becomes $(z+1/2)/3$.

17. If $F(x) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $[F(x)]^2 = F(kx)$, then the value of k is:

[1 mark]

- (A) 1
- (B) 2
- (C) 0
- (D) -2

Correct Answer: (B) 2

Solution:

Concept:

- Matrix multiplication uses the row-by-column rule. The square of a matrix is its product with itself.
- The identities $\cos^2 x - \sin^2 x = \cos 2x$ and $2 \sin x \cos x = \sin 2x$ identify the resulting entries.

Quick Tip to Solve: Squaring this rotation matrix adds its angles; inspect the off-diagonal sine entries to distinguish $2x$ from $-2x$.

Step 1: Multiply the matrix by itself Put $c = \cos x$ and $s = \sin x$. Direct multiplication gives

$$F(x)^2 = \begin{pmatrix} c^2 - s^2 & -2cs & 0 \\ 2cs & c^2 - s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The third row and column remain unchanged because their only non-zero entry is the bottom-right 1.

Step 2: Recognise the double-angle matrix Using the double-angle identities,

$$F(x)^2 = \begin{pmatrix} \cos 2x & -\sin 2x & 0 \\ \sin 2x & \cos 2x & 0 \\ 0 & 0 & 1 \end{pmatrix} = F(2x).$$

Option (A) would leave the angle unchanged, and option (C) would give the identity matrix for every x . Option (D) reverses the signs of the sine entries. These do not give the stated matrix identity for general x .

Step 3: Conclusion The angle multiplier is $k = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Squaring this rotation matrix adds its angles; inspect the off-diagonal sine entries to distinguish $2x$ from $-2x$.

18. If a line makes an angle of 30° with the positive direction of x -axis, 120° with the positive direction of y -axis, then the angle which it makes with the positive direction of z -axis is: [1 mark]

- (A) 90°
- (B) 120°
- (C) 60°
- (D) 0°

Correct Answer: (A) 90°

Solution:

Concept:

- If a directed line makes angles α, β, γ with the positive axes, its direction cosines are $\cos \alpha, \cos \beta, \cos \gamma$.
- Direction cosines satisfy $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, with each direction angle between 0° and 180° .

Step 1: Substitute the two known angles Here $\alpha = 30^\circ$ and $\beta = 120^\circ$, so $\cos \alpha = \sqrt{3}/2$ and $\cos \beta = -1/2$. Therefore

$$\cos^2 \gamma = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 = 1 - \frac{3}{4} - \frac{1}{4} = 0.$$

Step 2: Select the admissible angle We have $\cos \gamma = 0$. In the range of direction angles, this gives

Quick Tip to Solve: When the two known squared direction cosines add to one, the third cosine is zero and the remaining angle is 90° degrees.

$\gamma = 90^\circ$. Options (B) and (C) each have squared cosine $1/4$, while option (D) has squared cosine 1. Each would violate the direction-cosine identity because the first two squared cosines already add to 1.

Step 3: Conclusion The line makes 90° with the positive z -axis. Hence, the correct option is (A).

Quick Tip to Solve: When the two known squared direction cosines add to one, the third cosine is zero and the remaining angle is 90 degrees.

19. Assertion (A): For any symmetric matrix A , $B'AB$ is a skew-symmetric matrix.

Reason (R): A square matrix P is skew-symmetric if $P' = -P$.

Select the correct answer from the codes (A), (B), (C) and (D):

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A real square matrix A is symmetric if $A^T = A$, and skew-symmetric if $A^T = -A$; the prime denotes transpose.
- Transposition reverses the order of a product: $(XYZ)^T = Z^T Y^T X^T$.

Step 1: Judge the assertion For a compatible matrix B , let $M = B^T AB$. Since $A^T = A$,

$$M^T = (B^T AB)^T = B^T A^T B = B^T AB = M.$$

Thus the product is symmetric. It is not always skew-symmetric: choosing $A = I$ and $B = I$ gives $M = I$, for which $I^T = I \neq -I$. The assertion is false.

Step 2: Judge the reason The condition $P^T = -P$ is exactly the definition of a skew-symmetric square matrix. Hence the reason is true.

Step 3: Check the explanation and answer codes The true definition in the reason cannot establish the false assertion. Codes (A) and (B) require both statements to be true, and code (C) requires a true assertion and false reason. None has the truth values obtained here; code (D) does.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Transpose B-transpose A B in reverse order; a symmetric middle matrix makes the whole product symmetric.

20. Assertion (A): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Reason (R): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.

Select the correct answer from the codes (A), (B), (C) and (D):

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- The dot product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$, where θ is the angle between the vectors.
- The cross product is anti-commutative: $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$.

Step 1: Judge the assertion Interchanging the vectors leaves their magnitudes and the angle between them unchanged. Thus $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = \vec{b} \cdot \vec{a}$, so the assertion is true.

Step 2: Judge the reason The direction of a cross product follows the right-hand rule and reverses when the order is reversed. For example, $\hat{i} \times \hat{j} = \hat{k}$, whereas $\hat{j} \times \hat{i} = -\hat{k}$. These are non-zero vectors satisfying the question's hypothesis, and they disprove the claimed universal equality. Hence the reason is false. Parallel vectors give zero cross products, but that special case does not make the general claim true.

Step 3: Check the explanation and answer codes A false cross-product claim cannot explain the true dot-product assertion. Codes (A) and (B) require the reason to be true, and code (D) requires the assertion to be false. The obtained truth values therefore match code (C).

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Remember that reversing a dot product changes nothing, whereas reversing a non-zero cross product changes its sign.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

[2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$, so negative arguments yield negative principal values.

Quick Tip to Solve: Write the three principal angles first, then combine their fractions using denominator 12 to avoid sign mistakes.

- The principal range of $\cot^{-1} x$ is $(0, \pi)$. Choose the angle in this range whose cotangent equals the argument.
- Evaluate an inner ordinary trigonometric function before applying the outer inverse function.

Step 1: Evaluate the first two principal values Because $\tan(-\pi/6) = -1/\sqrt{3}$ and $-\pi/6$ lies in the principal tangent range,

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$

Also, $\cot(\pi/3) = 1/\sqrt{3}$ and $\pi/3 \in (0, \pi)$, giving

$$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

Step 2: Evaluate the nested term and add The inner value is $\sin(-\pi/2) = -1$. Its inverse tangent is $\tan^{-1}(-1) = -\pi/4$, since this angle lies in the principal range. Hence the requested sum is

$$\begin{aligned} S &= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{-2\pi + 4\pi - 3\pi}{12} = -\frac{\pi}{12}. \end{aligned}$$

Taking a positive value for either inverse tangent of a negative argument would violate its principal range.

Step 3: Conclusion The required value is $\boxed{-\frac{\pi}{12}}$.

Quick Tip to Solve: Write the three principal angles first, then combine their fractions using denominator 12 to avoid sign mistakes.

OR

21(b). Find the domain of the function $f(x) = \sin^{-1}(x^2 - 4)$. Also, find its range.

[2 marks]

Solution:

Concept:

- The real function $\sin^{-1} t$ is defined precisely for $-1 \leq t \leq 1$.
- Its principal values lie in $[-\pi/2, \pi/2]$. The range of a composite function depends on which input values its inner expression actually attains.

Step 1: Impose the real-domain condition For $f(x) = \sin^{-1}(x^2 - 4)$ to be real, its argument must satisfy

$$-1 \leq x^2 - 4 \leq 1 \iff 3 \leq x^2 \leq 5.$$

Taking account of both positive and negative values of x , this is equivalent to $\sqrt{3} \leq |x| \leq \sqrt{5}$. Thus the allowed intervals are $[-\sqrt{5}, -\sqrt{3}]$ and $[\sqrt{3}, \sqrt{5}]$; all four endpoints are included.

Step 2: Determine all possible output values As x runs through $[\sqrt{3}, \sqrt{5}]$, the continuous expression

Quick Tip to Solve: First bound the argument between -1 and 1; when taking square roots, retain both signs of x .

$x^2 - 4$ takes every value from -1 to 1 . Therefore the outer inverse sine attains its full principal interval. In particular, $f(\pm\sqrt{3}) = -\pi/2$ and $f(\pm\sqrt{5}) = \pi/2$, so neither endpoint is excluded from the range.

Step 3: Conclusion The domain is $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$, and the range is $[-\pi/2, \pi/2]$.

Quick Tip to Solve: First bound the argument between -1 and 1 ; when taking square roots, retain both signs of x .

22(a). If $f(x) = |\tan 2x|$, then find the value of $f'(x)$ at $x = \frac{\pi}{3}$.

[2 marks]

Solution:

Concept:

- For a nonzero real expression u , $|u| = u$ when $u > 0$ and $|u| = -u$ when $u < 0$. Determine the sign locally before differentiating a modulus.
- By the chain rule, $\frac{d}{dx} \tan(2x) = 2 \sec^2(2x)$, wherever the tangent is defined.

Step 1: Determine the sign near the given point At $x = \pi/3$, the inner argument is $2x = 2\pi/3$, which lies in the second quadrant. Hence $\tan(2\pi/3) = -\sqrt{3} < 0$. Tangent is continuous and nonzero near this argument, so it remains negative in a sufficiently small neighbourhood of the given point. In that neighbourhood, $f(x) = |\tan 2x| = -\tan 2x$.

Step 2: Differentiate the appropriate branch and substitute Using the negative branch and the chain rule gives

$$f'(x) = -2 \sec^2(2x),$$

$$f'(\pi/3) = -2 \sec^2(2\pi/3) = -2 \left(\frac{1}{-1/2} \right)^2 = -8.$$

The squared secant is positive; the negative sign comes from the modulus branch. There is no cusp at this point because the inner tangent does not vanish here.

Step 3: Conclusion Therefore, $f'(\pi/3) = -8$.

Quick Tip to Solve: Evaluate the sign of $\tan(2x)$ first; differentiating the modulus as if it were absent reverses this answer.

OR

22(b). If $y = \operatorname{cosec}(\cot^{-1} x)$, then prove that $\sqrt{1+x^2} \frac{dy}{dx} - x = 0$.

[2 marks]

Solution:

Concept:

- The principal angle $\theta = \cot^{-1} x$ lies in $(0, \pi)$, so $\sin \theta > 0$ and $\operatorname{cosec} \theta > 0$ for every real x .
- The identity $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$ reduces the composite function to an algebraic expression, which

Quick Tip to Solve: Use the cosecant-squared identity first, and select the positive square root from the principal cotangent angle.

can then be differentiated using the chain rule.

Step 1: Simplify the composite function with its correct sign Put $\theta = \cot^{-1} x$. Then $\cot \theta = x$, and the trigonometric identity gives

$$y^2 = \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta = 1 + x^2.$$

Because $\theta \in (0, \pi)$, its sine is positive. Thus $y = \operatorname{cosec} \theta$ is positive, and the appropriate square root is $y = \sqrt{1 + x^2}$, even when x itself is negative.

Step 2: Differentiate and verify the required expression Applying the chain rule to the positive square root yields

$$\frac{dy}{dx} = \frac{1}{2}(1 + x^2)^{-1/2}(2x) = \frac{x}{\sqrt{1 + x^2}}.$$

Consequently,

$$\sqrt{1 + x^2} \frac{dy}{dx} - x = \sqrt{1 + x^2} \frac{x}{\sqrt{1 + x^2}} - x = x - x = 0.$$

The denominator is always positive for real x , so the calculation applies throughout the function's real domain.

Step 3: Conclusion Hence $\boxed{\sqrt{1 + x^2} \frac{dy}{dx} - x = 0}$, as required.

Quick Tip to Solve: Use the cosecant-squared identity first, and select the positive square root from the principal cotangent angle.

23. If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}$ ($x \neq 0$) respectively, find the value of $(M - m)$. [2 marks]

Solution:

Concept:

- At an interior local extremum of a differentiable function, $f'(x) = 0$. A point outside the domain cannot be a critical point of the function.
- At a stationary point a , $f''(a) < 0$ indicates a local maximum and $f''(a) > 0$ indicates a local minimum. The extreme values are $f(a)$, rather than the coordinates a .

Step 1: Locate and classify the stationary points For $f(x) = x + 1/x$, with $x \neq 0$, differentiation gives

$$f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

Solving $f'(x) = 0$ gives $x^2 = 1$, hence $x = -1$ and $x = 1$. Both points belong to the domain. Since $f''(-1) = -2 < 0$, there is a local maximum at $x = -1$; since $f''(1) = 2 > 0$, there is a local minimum at $x = 1$. The excluded point $x = 0$ is not an extremum.

Step 2: Evaluate the function at these points The local maximum value and local minimum value are

Quick Tip to Solve: Substitute each stationary point into f before subtracting; local maximum and minimum values need not be globally ordered.

respectively

$$M = f(-1) = -1 - 1 = -2, \quad m = f(1) = 1 + 1 = 2.$$

Therefore their requested difference is $M - m = (-2) - 2$. A local maximum on the negative branch can be numerically smaller than a local minimum on the positive branch, because these extrema describe separate neighbourhoods in a disconnected domain.

Step 3: Conclusion Thus $M - m = -4$.

Quick Tip to Solve: Substitute each stationary point into f before subtracting; local maximum and minimum values need not be globally ordered.

24. Find: $\int \frac{e^{4x} - 1}{e^{4x} + 1} dx.$

[2 marks]

Solution:

Concept:

- Rewrite a numerator relative to the denominator to separate a constant term from a logarithmic-derivative term.
- If $g(x) > 0$, then $\int \frac{g'(x)}{g(x)} dx = \ln g(x) + C$, where C is an arbitrary constant of integration.

Step 1: Rewrite the integrand Let I denote the required integral. Because $e^{4x} - 1 = 2e^{4x} - (e^{4x} + 1)$, division by the positive denominator gives

$$\frac{e^{4x} - 1}{e^{4x} + 1} = \frac{2e^{4x}}{e^{4x} + 1} - 1.$$

This form is useful because the derivative of $e^{4x} + 1$ is $4e^{4x}$, which is a constant multiple of the numerator of the first term.

Step 2: Integrate by substitution Set $u = e^{4x} + 1$, so $du = 4e^{4x} dx$. Then

$$\begin{aligned} I &= 2 \int \frac{e^{4x}}{e^{4x} + 1} dx - \int 1 dx \\ &= \frac{1}{2} \int \frac{du}{u} - x \\ &= \frac{1}{2} \ln(e^{4x} + 1) - x + C. \end{aligned}$$

Since $e^{4x} + 1 > 0$ for all real x , absolute-value signs in the logarithm are unnecessary. Differentiating this expression gives $2e^{4x}/(e^{4x} + 1) - 1$, which is exactly the original integrand and verifies the coefficient of the logarithm.

Step 3: Conclusion Therefore, $\int \frac{e^{4x} - 1}{e^{4x} + 1} dx = \frac{1}{2} \ln(e^{4x} + 1) - x + C$.

Quick Tip to Solve: Express the numerator as twice the exponential minus the denominator to expose a logarithmic derivative directly.

25. Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain.

[2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive at every point of that interval.
- The standard derivatives are $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}e^{-x} = -e^{-x}$, and $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$.

Step 1: Determine the domain and differentiate Both exponential terms, the polynomial term and $\tan^{-1} x$ are defined and differentiable for every real x . Thus the domain is $\mathbb{R}$. Differentiating each term, taking care of the negative sign before e^{-x} , gives

$$f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2} = e^x + e^{-x} + \frac{x^2}{1+x^2}.$$

Step 2: Establish the sign on the whole domain For every real x , both e^x and e^{-x} are strictly positive, while $x^2/(1+x^2) \geq 0$. Hence $f'(x) > 0$ throughout $\mathbb{R}$, including $x = 0$, where $f'(0) = 2$. More explicitly, if $a < b$, the mean value theorem gives $f(b) - f(a) = f'(c)(b-a) > 0$ for some $c \in (a, b)$. This verifies strict increase, rather than merely non-decrease.

Step 3: Conclusion Therefore $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing on its entire domain $\mathbb{R}$.

Quick Tip to Solve: Combine $1 - 1/(1+x^2)$ into $x^2/(1+x^2)$; its non-negative sign makes the derivative's positivity immediate.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). If $x = e^{\cos 3t}$ and $y = e^{\sin 3t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

[3 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$ and $y = y(t)$, the chain rule gives $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ wherever $dx/dt \neq 0$.
- For the natural logarithm, $\log(e^u) = u$. Also $\frac{d}{dt}e^{u(t)} = e^{u(t)}u'(t)$, which requires differentiating the exponent as well.

Step 1: Express the trigonometric factors through logarithms The given equations are $x = e^{\cos 3t}$ and $y = e^{\sin 3t}$. Exponentials are positive, so both logarithms are defined for all real t . Taking natural logarithms yields

$$\log x = \cos 3t, \quad \log y = \sin 3t.$$

These identities will allow the final derivative to be written entirely in terms of x and y .

Quick Tip to Solve: Take logarithms first to identify $\cos 3t = \log x$ and $\sin 3t = \log y$, then cancel the common factor 3 in the derivative ratio.

Step 2: Differentiate both coordinates with respect to the parameter Applying the chain rule separately to each exponential gives

$$\begin{aligned}\frac{dx}{dt} &= e^{\cos 3t}(-3 \sin 3t) = -3x \sin 3t, \\ \frac{dy}{dt} &= e^{\sin 3t}(3 \cos 3t) = 3y \cos 3t.\end{aligned}$$

The factor 3 comes from the inner angle $3t$. The signs differ because the derivative of cosine is negative sine, whereas the derivative of sine is cosine.

Step 3: Form the parametric derivative and substitute Where $\sin 3t \neq 0$, we have $dx/dt \neq 0$, and hence

$$\frac{dy}{dx} = \frac{3y \cos 3t}{-3x \sin 3t} = -\frac{y \cos 3t}{x \sin 3t} = -\frac{y \log x}{x \log y}.$$

The same restriction makes $\log y \neq 0$, so the displayed denominator is non-zero. At $\sin 3t = 0$, neither the ratio formula nor the requested expression defines a finite derivative; thus the identity is understood at points where it is defined.

Step 4: Conclusion Therefore $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$, wherever $\log y \neq 0$, as required.

Quick Tip to Solve: Take logarithms first to identify $\cos 3t = \log x$ and $\sin 3t = \log y$, then cancel the common factor 3 in the derivative ratio.

OR

26(b). Show that: $\frac{d}{dx}(|x|) = \frac{x}{|x|}$, $x \neq 0$.

[3 marks]

Solution:

Concept:

- The absolute value function is piecewise defined: $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$.
- On each open interval where the defining expression is differentiable, differentiate that expression. A formula involving division by $|x|$ requires $x \neq 0$.

Step 1: Examine the positive half-line Suppose $x > 0$. In a sufficiently small neighbourhood of this point, all inputs are positive and $|x| = x$. Consequently,

$$\frac{d}{dx}|x| = \frac{d}{dx}x = 1.$$

Also, on this interval, $x/|x| = x/x = 1$. Thus the derivative and the proposed right-hand side agree at every positive input.

Step 2: Examine the negative half-line Suppose $x < 0$. In a sufficiently small neighbourhood of this point, the absolute value expression is $|x| = -x$, so

$$\frac{d}{dx}|x| = \frac{d}{dx}(-x) = -1.$$

Quick Tip to Solve: Split absolute-value differentiation at zero: use $|x| = x$ on the positive side and $|x| = -x$ on the negative side.

Meanwhile, $x/|x| = x/(-x) = -1$. Hence the two sides agree at every negative input as well. The denominator never vanishes on either half-line.

Step 3: Explain the excluded point and combine the cases At $x = 0$, the difference quotient is $|h|/h$. It equals 1 for $h > 0$ and -1 for $h < 0$, so its two one-sided limits differ and the derivative does not exist. The expression $x/|x|$ is also undefined there. All non-zero real inputs have already been covered by the two cases above; the exclusion in the question is therefore necessary and sufficient.

Step 4: Conclusion Combining the positive and negative cases, $\frac{d}{dx}|x| = \frac{x}{|x|}$ for every $x \neq 0$, as required.

Quick Tip to Solve: Split absolute-value differentiation at zero: use $|x| = x$ on the positive side and $|x| = -x$ on the negative side.

27(a). Evaluate: $\int_{-2}^2 \sqrt{\frac{2-x}{2+x}} dx$.

[3 marks]

Solution:

Concept:

- A trigonometric substitution $x = 2 \cos \theta$, with $0 \leq \theta \leq \pi$, transforms expressions containing $2 - x$ and $2 + x$ using half-angle identities.
- The identities $1 - \cos \theta = 2 \sin^2(\theta/2)$, $1 + \cos \theta = 2 \cos^2(\theta/2)$, and $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$ remove the radical.

Step 1: Substitute and determine the limits Write $I = \int_{-2}^2 \sqrt{\frac{2-x}{2+x}} dx$. Put $x = 2 \cos \theta$, so $dx = -2 \sin \theta d\theta$. At $x = -2$, $\theta = \pi$; at $x = 2$, $\theta = 0$. The integrand is unbounded at the lower endpoint, so the integral is interpreted as a one-sided improper integral there. The transformed integrand below extends continuously to that endpoint, establishing convergence.

Step 2: Simplify the square root with the half-angle identities For $0 < \theta < \pi$, both half-angle sine and cosine are positive. Therefore,

$$\sqrt{\frac{2-2\cos\theta}{2+2\cos\theta}} = \sqrt{\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}} = \tan \frac{\theta}{2}.$$

Reversing the limits absorbs the negative sign in dx , giving

$$I = 2 \int_0^\pi \sin \theta \tan \frac{\theta}{2} d\theta = 2 \int_0^\pi (1 - \cos \theta) d\theta.$$

Here $\sin \theta \tan(\theta/2) = 2 \sin^2(\theta/2) = 1 - \cos \theta$. This last expression has a finite limit at both endpoints.

Step 3: Evaluate the elementary definite integral The antiderivative of $1 - \cos \theta$ is $\theta - \sin \theta$. Consequently,

$$I = 2[\theta - \sin \theta]_0^\pi = 2\{(\pi - \sin \pi) - (0 - \sin 0)\} = 2\pi.$$

The changed limits and the factor 2 from dx have both been retained throughout the evaluation.

Quick Tip to Solve: With $x = 2 \cos \theta$, combine $\sin \theta \tan(\theta/2)$ into $1 - \cos \theta$ before integrating, and reverse the limits.

Step 4: Conclusion Thus $\int_{-2}^2 \sqrt{\frac{2-x}{2+x}} dx = 2\pi$.

Quick Tip to Solve: With $x = 2 \cos \theta$, combine $\sin \theta \tan(\theta/2)$ into $1 - \cos \theta$ before integrating, and reverse the limits.

OR

27(b). Find: $\int \frac{1}{x[(\log x)^2 - 3 \log x - 4]} dx$.

[3 marks]

Solution:

Concept:

- A logarithmic substitution uses $t = \log x$, so that $dt = dx/x$, where $\log$ denotes the natural logarithm and $x > 0$.
- Factor a quadratic before using partial fractions. For distinct factors, write the rational function as a sum of terms of the form $A/(t - a)$.
- The standard formula is $\int dt/(t - a) = \log |t - a| + C$, on any interval not containing $t = a$.

Step 1: Substitute for the logarithm Let $t = \log x$. Then $dt = dx/x$, and the required integral becomes

$$I = \int \frac{dt}{t^2 - 3t - 4} = \int \frac{dt}{(t - 4)(t + 1)}.$$

The substitution removes both the logarithm in the quadratic and the factor x outside it. The factorisation follows because -4 and 1 have product -4 and sum -3 .

Step 2: Resolve the rational expression into partial fractions Put

$$\frac{1}{(t - 4)(t + 1)} = \frac{A}{t - 4} + \frac{B}{t + 1}.$$

Multiplication by the denominator gives $1 = A(t + 1) + B(t - 4)$. Setting $t = 4$ yields $A = 1/5$; setting $t = -1$ yields $B = -1/5$. Therefore

$$I = \frac{1}{5} \int \frac{dt}{t - 4} - \frac{1}{5} \int \frac{dt}{t + 1}.$$

In particular, the two logarithmic terms have opposite signs; replacing the difference by a sum would not reproduce the numerator 1.

Step 3: Integrate and check the substitution Applying the logarithmic integral formula gives

$$I = \frac{1}{5} \log |t - 4| - \frac{1}{5} \log |t + 1| + C.$$

Restore $t = \log x$. As a check, differentiation of this expression gives

$$\frac{1}{5x} \left(\frac{1}{\log x - 4} - \frac{1}{\log x + 1} \right) = \frac{1}{x[(\log x)^2 - 3 \log x - 4]},$$

Quick Tip to Solve: Whenever a quadratic in $\log x$ is accompanied by dx/x , substitute $t = \log x$ before attempting partial fractions.

exactly the given integrand. The antiderivative is used separately on intervals excluding $x = e^{-1}$ and $x = e^4$, where the original denominator vanishes.

Step 4: Conclusion

$$\int \frac{dx}{x[(\log x)^2 - 3 \log x - 4]} = \frac{1}{5} \log \left| \frac{\log x - 4}{\log x + 1} \right| + C.$$

Quick Tip to Solve: Whenever a quadratic in $\log x$ is accompanied by dx/x , substitute $t = \log x$ before attempting partial fractions.

28(a). Find the particular solution of the differential equation given by $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$; $y = 2$, when $x = 1$. [3 marks]

Solution:

Concept:

- A homogeneous equation $dy/dx = F(y/x)$ is reduced by putting $y = vx$, where v is a function of x . Then $dy/dx = v + x dv/dx$.
- A separable equation is integrated after collecting the dependent variable on one side and the independent variable on the other.

Step 1: Express the equation in homogeneous form Rearrange the printed differential equation as

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2} = \frac{y}{x} + \frac{1}{2} \left(\frac{y}{x} \right)^2.$$

Its right-hand side depends only on y/x . Put $y = vx$, giving $dy/dx = v + x dv/dx$. Substitution yields

$$v + x \frac{dv}{dx} = v + \frac{v^2}{2}, \quad x \frac{dv}{dx} = \frac{v^2}{2}.$$

Step 2: Separate variables and integrate The initial condition gives $v(1) = 2$, so the solution sought is not the zero solution. Divide by v^2 and separate:

$$\frac{dv}{v^2} = \frac{dx}{2x}.$$

Integrating gives

$$-\frac{1}{v} = \frac{1}{2} \log |x| + C.$$

Restore $v = y/x$ to obtain

$$\frac{2x}{y} + \log |x| = C_1,$$

where $C_1 = -2C$. The zero solution excluded by division does not satisfy the initial condition.

Quick Tip to Solve: After substituting $y = vx$, cancel the common v terms before separating variables, and use $v(1) = 2$ to check the constant.

Step 3: Apply the initial condition and verify Insert $x = 1, y = 2$ into the integrated relation:

$$C_1 = \frac{2(1)}{2} + \log 1 = 1.$$

Since the initial point has $x > 0$, use $\log |x| = \log x$. Hence $2x/y = 1 - \log x$. The resulting expression gives $y(1) = 2$, as required. Its denominator vanishes at $x = e$, so the interval containing the initial point is $0 < x < e$.

Step 4: Conclusion The required particular solution is

$$y = \frac{2x}{1 - \log x}, \quad 0 < x < e.$$

Quick Tip to Solve: After substituting $y = vx$, cancel the common v terms before separating variables, and use $v(1) = 2$ to check the constant.

OR

28(b). Find the general solution of the differential equation: $y dx = (x + 2y^2) dy$.

[3 marks]

Solution:

Concept:

- When a differential equation is linear in x as a function of y , use the form $dx/dy + P(y)x = Q(y)$, rather than treating y as the dependent variable.
- An integrating factor is I.F. = $e^{\int P(y) dy}$. Multiplication by it converts the left-hand side into a product derivative.

Step 1: Write the linear equation with dependent variable x The given equation is $y dx = (x + 2y^2) dy$. Regard x as a differentiable function of y . On an interval where $y \neq 0$, divide by $y dy$ and rearrange:

$$\frac{dx}{dy} - \frac{x}{y} = 2y.$$

This is linear in x , with $P(y) = -1/y$ and $Q(y) = 2y$.

Step 2: Find and apply an integrating factor The integrating factor is

$$e^{\int -dy/y} = e^{-\log |y|} = \frac{1}{|y|}.$$

On either interval $y > 0$ or $y < 0$, we may use the proportional integrating factor $1/y$, since a nonzero constant multiple is equally valid. Multiplying the equation by $1/y$ gives

$$\frac{1}{y} \frac{dx}{dy} - \frac{x}{y^2} = 2.$$

Quick Tip to Solve: In $y dx = (x + 2y^2) dy$, treat x as dependent on y ; the integrating factor $1/y$ immediately produces $d(x/y)/dy$.

The quotient rule identifies its left-hand side as

$$\frac{d}{dy} \left(\frac{x}{y} \right) = 2.$$

Step 3: Integrate and verify the family Integrate with respect to y :

$$\frac{x}{y} = 2y + C, \quad x = 2y^2 + Cy.$$

There is no initial condition, so C remains arbitrary. As a check, differentiation gives $dx/dy = 4y + C$. Thus $y(dx/dy) = 4y^2 + Cy = x + 2y^2$, which verifies the original equation. Although the derivation used $y \neq 0$, the polynomial family also extends through $y = 0$ and satisfies the original equation there.

Step 4: Conclusion The general solution, with x treated as a function of y , is

$$x = 2y^2 + Cy, \quad C \in \mathbb{R}.$$

Quick Tip to Solve: In $y dx = (x + 2y^2) dy$, treat x as dependent on y ; the integrating factor $1/y$ immediately produces $d(x/y)/dy$.

29. The position vectors of vertices of $\triangle ABC$ are $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$ and $C(3\hat{i} - 4\hat{j} - 4\hat{k})$. Find all the angles of $\triangle ABC$. [3 marks]

Solution:

Concept:

- The vector from a vertex P to a vertex Q is $\overrightarrow{PQ} = \vec{q} - \vec{p}$, the difference of their position vectors.
- For two nonzero vectors $\vec{u}, \vec{v}$, the included angle θ satisfies $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$. At a triangle vertex, both vectors must point away from that vertex.
- A zero dot product means the vectors are perpendicular; the three interior angles of a triangle sum to 180° .

Step 1: Find the side vectors and their lengths Subtracting the given position vectors gives

$$\begin{aligned}\overrightarrow{AB} &= -\hat{i} - 2\hat{j} - 6\hat{k}, \\ \overrightarrow{AC} &= \hat{i} - 3\hat{j} - 5\hat{k}, \\ \overrightarrow{BC} &= 2\hat{i} - \hat{j} + \hat{k}.\end{aligned}$$

Hence $|AB| = \sqrt{1 + 4 + 36} = \sqrt{41}$, $|AC| = \sqrt{1 + 9 + 25} = \sqrt{35}$, and $|BC| = \sqrt{4 + 1 + 1} = \sqrt{6}$. These are nonzero, so the angle formula applies at each vertex.

Step 2: Calculate the angle at A The two vectors beginning at A are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Their dot product

Quick Tip to Solve: At each vertex use two outgoing side vectors; reversing just one vector gives the supplementary angle instead of the interior angle.

is $(-1)(1) + (-2)(-3) + (-6)(-5) = 35$. Therefore

$$\cos A = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}}.$$

This positive cosine shows that the interior angle at A is acute.

Step 3: Calculate the angles at B and C At B , use $\overrightarrow{BA} = (1, 2, 6)$ and $\overrightarrow{BC} = (2, -1, 1)$. Thus $\overrightarrow{BA} \cdot \overrightarrow{BC} = 2 - 2 + 6 = 6$, giving

$$\cos B = \frac{6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}}.$$

At C , the outgoing vectors are $\overrightarrow{CA} = (-1, 3, 5)$ and $\overrightarrow{CB} = (-2, 1, -1)$. Their dot product is $2 + 3 - 5 = 0$, so they are perpendicular. As a check, $|AC|^2 + |BC|^2 = 35 + 6 = 41 = |AB|^2$, consistent with a right angle at C .

Step 4: Conclusion The angles are $A = \cos^{-1} \sqrt{35/41}$, $B = \cos^{-1} \sqrt{6/41}$, $C = 90^\circ$. Equivalently, $A = \tan^{-1} \sqrt{6/35}$ and $B = 90^\circ - A$.

Quick Tip to Solve: At each vertex use two outgoing side vectors; reversing just one vector gives the supplementary angle instead of the interior angle.

30. A pair of dice is thrown simultaneously. If X denotes the absolute difference of the numbers appearing on top of the dice, then find the probability distribution of X . [3 marks]

Solution:

Concept:

- Two fair dice give $6 \times 6 = 36$ equally likely ordered outcomes (a, b) , where $a, b \in \{1, 2, 3, 4, 5, 6\}$.
- The random variable is $X = |a - b|$. Its distribution assigns $P(X = k)$ to every possible value k , with the probabilities summing to one.
- For an event in this finite equally likely sample space, probability equals the number of favourable outcomes divided by 36.

Step 1: Identify the possible absolute differences The smallest difference is 0, obtained when the dice show the same number. The largest is 5, obtained from $(1, 6)$ or $(6, 1)$. Every intermediate integer difference is attainable, so the possible values are 0, 1, 2, 3, 4, 5. We retain ordered outcomes because (a, b) and (b, a) refer to different results of the two dice when $a \neq b$.

Step 2: Count the outcomes for each value For $X = 0$, the six outcomes are $(1, 1), (2, 2), \dots, (6, 6)$. For $k > 0$, outcomes with $b - a = k$ are $(1, 1+k), \dots, (6-k, 6)$, numbering $6-k$. Reversing each gives another $6-k$ outcomes with $a - b = k$. The two groups do not overlap for a positive difference. Thus the numbers of favourable outcomes for $k = 1, 2, 3, 4, 5$ are respectively 10, 8, 6, 4, 2. For example, a difference of 4 occurs for $(1, 5), (2, 6), (5, 1), (6, 2)$.

Step 3: Divide by the sample-space size and verify the total Therefore $P(X = 0) = 6/36$, and $P(X = k) = 2(6-k)/36$ for $k = 1, \dots, 5$. The favourable counts add to $6 + 10 + 8 + 6 + 4 + 2 = 36$, exhausting the sample space with no overlap. Equivalently, the probabilities add to $(3 + 5 + 4 + 3 + 2 + 1)/18 = 1$,

Quick Tip to Solve: Count equal dice separately: there are six zero-difference outcomes, while every positive difference has two possible orders.

as required for a probability distribution.

Step 4: Conclusion The probability distribution of X is

X	0	1	2	3	4	5
$P(X)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Quick Tip to Solve: Count equal dice separately: there are six zero-difference outcomes, while every positive difference has two possible orders.

31. Find: $\int x^2 \sin^{-1}(x^{3/2}) dx$.

[3 marks]

Solution:

Concept:

- A substitution should simplify the inverse-trigonometric argument: take $t = x^{3/2}$, so $x^3 = t^2$ and $x^2 dx = \frac{2}{3}t dt$.
- Integration by parts uses $\int u dv = uv - \int v du$, with $\frac{d}{dt} \sin^{-1} t = \frac{1}{\sqrt{1-t^2}}$ for $-1 < t < 1$.
- Use $\sin^2 \theta = (1 - \cos 2\theta)/2$. The real integrand has domain $0 \leq x \leq 1$; differentiate on $0 < x < 1$.

Step 1: Substitute and integrate by parts Write $I = \int x^2 \sin^{-1}(x^{3/2}) dx$. With $t = x^{3/2}$, differentiation of $x^3 = t^2$ gives $3x^2 dx = 2t dt$. Hence

$$I = \frac{2}{3} \int t \sin^{-1} t dt.$$

Choose $u = \sin^{-1} t$, $dv = t dt$, so $du = dt/\sqrt{1-t^2}$ and $v = t^2/2$. Integration by parts yields

$$I = \frac{t^2}{3} \sin^{-1} t - \frac{1}{3} J, \quad J = \int \frac{t^2}{\sqrt{1-t^2}} dt.$$

Step 2: Evaluate the remaining integral Put $t = \sin \theta$, choosing $\theta = \sin^{-1} t \in [0, \pi/2]$. On the interior of this interval $dt = \cos \theta d\theta$ and $\sqrt{1-t^2} = \cos \theta$, so

$$\begin{aligned} J &= \int \sin^2 \theta d\theta \\ &= \frac{\theta}{2} - \frac{\sin 2\theta}{4} \\ &= \frac{1}{2} \sin^{-1} t - \frac{1}{2} t \sqrt{1-t^2}. \end{aligned}$$

The integration constants from this calculation are absorbed into the final single constant C .

Quick Tip to Solve: Set the entire inverse-sine argument equal to t ; then use x cubed equals t squared to transform x squared dx directly.

Step 3: Substitute back and check by differentiation Substitution into the expression for I gives

$$I = \frac{2t^2 - 1}{6} \sin^{-1} t + \frac{t\sqrt{1-t^2}}{6} + C.$$

Differentiating this expression with respect to t , the terms involving $1/\sqrt{1-t^2}$ cancel, leaving $\frac{2}{3}t \sin^{-1} t$. Since $dt/dx = \frac{3}{2}\sqrt{x}$, its derivative with respect to x is $x^2 \sin^{-1}(x^{3/2})$, verifying the result.

Step 4: Conclusion Thus

$$\int x^2 \sin^{-1}(x^{3/2}) dx = \frac{(2x^3 - 1) \sin^{-1}(x^{3/2}) + x^{3/2} \sqrt{1-x^3}}{6} + C.$$

Quick Tip to Solve: Set the entire inverse-sine argument equal to t ; then use x cubed equals t squared to transform x squared dx directly.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). Show that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{2x}{1+x^2}$ is neither one-one nor onto. Further, find set A so that the given function $f : \mathbb{R} \rightarrow A$ becomes an onto function. [5 marks]

Solution:

Concept:

- A function is one-one if distinct inputs always have distinct images. Thus, a single pair $x_1 \neq x_2$ with $f(x_1) = f(x_2)$ disproves injectivity.
- A function is onto its codomain if every element of that codomain is attained. Its range is the set of values actually attained; restricting the codomain to this range makes the function onto.
- Since $1+x^2 > 0$ for every real x , the quotient is defined throughout $\mathbb{R}$. The inequalities $(x-1)^2 \geq 0$ and $(x+1)^2 \geq 0$ give sharp bounds for the quotient.

Step 1: Exhibit two distinct inputs with the same image Choose the real inputs 2 and $\frac{1}{2}$. Direct substitution gives

$$f(2) = \frac{4}{5}, \quad f\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}.$$

Although $2 \neq \frac{1}{2}$, their images are equal. Thus the function is not one-one.

Step 2: Bound the range and test the given codomain For any real input x , the nonnegative squares give

$$\begin{aligned} (x-1)^2 \geq 0 &\implies 2x \leq 1+x^2, \\ (x+1)^2 \geq 0 &\implies -2x \leq 1+x^2. \end{aligned}$$

Quick Tip to Solve: Use reciprocal inputs to disprove injectivity, then bound the quotient with $(x \pm 1)^2 \geq 0$; check that all intermediate values occur before declaring the range.

Dividing each inequality by the positive number $1 + x^2$ yields

$$-1 \leq \frac{2x}{1+x^2} \leq 1.$$

Thus no number outside $[-1, 1]$ is an image. For example, 2 belongs to the given codomain $\mathbb{R}$, but $f(x) = 2$ has no real solution. It is therefore not onto. The endpoints themselves are attained: $f(1) = 1$ and $f(-1) = -1$.

Step 3: Prove that every value between the bounds is attained Let $y \in [-1, 1]$ be arbitrary. If $y = 0$, choose $x = 0$, so $f(x) = y$. If $y \neq 0$, solve the equation

$$y = \frac{2x}{1+x^2} \iff yx^2 - 2x + y = 0.$$

Its discriminant is $4 - 4y^2 \geq 0$, because $|y| \leq 1$. Hence it has the real solutions

$$x = \frac{1 \pm \sqrt{1-y^2}}{y}.$$

Either root satisfies $y(1+x^2) = 2x$. Since the denominator is positive, division recovers $f(x) = y$. Hence every $y \in [-1, 1]$ is attained and the range is exactly $[-1, 1]$.

Step 4: Conclusion The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is neither one-one nor onto. Taking $A = [-1, 1]$ makes $f: \mathbb{R} \rightarrow A$ onto, since every element of this codomain has a real preimage.

Quick Tip to Solve: Use reciprocal inputs to disprove injectivity, then bound the quotient with $(x \pm 1)^2 \geq 0$; check that all intermediate values occur before declaring the range.

OR

32(b). A relation R is defined on $N \times N$ (where N is the set of natural numbers) as: $(a, b) R (c, d) \iff a - c = b - d$. Show that R is an equivalence relation. [5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation precisely when it is reflexive, symmetric and transitive. All three conditions must hold for arbitrary elements of the underlying set.
- Here the elements of the underlying set $\mathbb{N} \times \mathbb{N}$ are ordered pairs. The rule $(a, b) R (c, d)$ means $a - c = b - d$; equivalently, it means $a - b = c - d$.
- Reflexivity compares an element with itself, symmetry reverses a related pair of elements, and transitivity links two successive relations through the same middle element.

Step 1: Establish reflexivity Take any ordered pair $(a, b) \in \mathbb{N} \times \mathbb{N}$. To test its relation with itself, put $(c, d) = (a, b)$ in the defining rule. Then

$$a - a = 0 = b - b.$$

Quick Tip to Solve: Rewrite the rule as $a - b = c - d$: related ordered pairs have the same coordinate difference. For transitivity, add the two given equalities so that the middle coordinates cancel.

Therefore $(a, b)R(a, b)$. As the pair was arbitrary, R is reflexive.

Step 2: Establish symmetry Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, and suppose that $(a, b)R(c, d)$. By definition,

$$a - c = b - d.$$

Multiplying both sides by -1 gives

$$c - a = d - b.$$

This is exactly the equality required by the rule for $(c, d)R(a, b)$. The reverse relation therefore holds for arbitrary ordered pairs, proving symmetry.

Step 3: Establish transitivity Take any three ordered pairs $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Suppose that

$$(a, b)R(c, d) \quad \text{and} \quad (c, d)R(e, f).$$

The two hypotheses give, respectively,

$$a - c = b - d, \quad c - e = d - f.$$

Add these equalities side by side. The coordinates of the middle ordered pair cancel:

$$\begin{aligned}(a - c) + (c - e) &= (b - d) + (d - f), \\ a - e &= b - f.\end{aligned}$$

The last equality is precisely the defining condition for $(a, b)R(e, f)$. Hence R is transitive.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive on $\mathbb{N} \times \mathbb{N}$. Therefore R is an equivalence relation.

Quick Tip to Solve: Rewrite the rule as $a - b = c - d$: related ordered pairs have the same coordinate difference. For transitivity, add the two given equalities so that the middle coordinates cancel.

33. Find the equation of the line which bisects the line segment joining points $A(2, 3, 4)$ and $B(4, 5, 8)$ and is perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. [5 marks]

Solution:

Concept:

- A line bisecting a segment passes through its midpoint. For endpoints $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, this is $M((x_1 + x_2)/2, (y_1 + y_2)/2, (z_1 + z_2)/2)$.
- In a symmetric line equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the direction ratios are a, b, c . Two directions are perpendicular when their scalar product is zero.
- A vector perpendicular to two nonparallel direction vectors $\mathbf{u}$ and $\mathbf{v}$ is their cross product $\mathbf{u} \times \mathbf{v}$. Any nonzero scalar multiple gives the same line direction.

Step 1: Find the point through which the required line passes Since the line bisects the segment

Quick Tip to Solve: Use the denominators for direction ratios, take their cross product, and pass the resulting direction through the midpoint of the segment.

joining $A(2, 3, 4)$ and $B(4, 5, 8)$, it passes through

$$M = \left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2} \right) = (3, 4, 6).$$

Here bisecting the segment specifies its midpoint; the two perpendicularity conditions determine the required direction.

Step 2: Determine a direction perpendicular to both given lines Reading the denominators of the printed symmetric equations gives

$$\mathbf{u} = 3\hat{\mathbf{i}} - 16\hat{\mathbf{j}} + 7\hat{\mathbf{k}}, \quad \mathbf{v} = 3\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 5\hat{\mathbf{k}}.$$

Their cross product is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= (80 - 56)\hat{\mathbf{i}} - (-15 - 21)\hat{\mathbf{j}} + (24 + 48)\hat{\mathbf{k}} \\ &= 24\hat{\mathbf{i}} + 36\hat{\mathbf{j}} + 72\hat{\mathbf{k}}. \end{aligned}$$

Dividing by 12, we may use the direction ratios 2, 3, 6. The cross product is nonzero, so these conditions determine a unique direction up to a nonzero scalar factor.

Step 3: Form the line and check its defining conditions The line through M with this direction has parametric coordinates

$$x = 3 + 2t, \quad y = 4 + 3t, \quad z = 6 + 6t, \quad t \in \mathbb{R}.$$

Setting $t = 0$ gives the midpoint $(3, 4, 6)$. Also,

$$\begin{aligned} (2, 3, 6) \cdot (3, -16, 7) &= 6 - 48 + 42 = 0, \\ (2, 3, 6) \cdot (3, 8, -5) &= 6 + 24 - 30 = 0. \end{aligned}$$

Thus the line meets both required perpendicularity conditions. Eliminating the common parameter gives its symmetric form.

Step 4: Conclusion The required line is $\boxed{\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-6}{6}}$, or equivalently $\mathbf{r} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + t(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$, where $t \in \mathbb{R}$.

Quick Tip to Solve: Use the denominators for direction ratios, take their cross product, and pass the resulting direction through the midpoint of the segment.

34(a). Solve the following system of equations, using matrices: $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$, $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$, $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$, where $x, y, z \neq 0$. [5 marks]

Solution:

Concept:

- Reciprocal substitution converts equations in $1/x, 1/y, 1/z$ into linear equations. Put $u = 1/x$, $v = 1/y$, and $w = 1/z$; the given restriction $x, y, z \neq 0$ makes these substitutions valid.
- A linear system is written $AX = B$, where A is the coefficient matrix, X the column of unknowns and B the constant column. If $|A| \neq 0$, its unique solution is $X = A^{-1}B$.
- The inverse is $A^{-1} = \text{adj } A / |A|$, where $\text{adj } A$ is the transpose of the cofactor matrix. Recover the original variables by taking reciprocals of the solved unknowns.

Step 1: Rewrite the equations as a matrix system Set $u = 1/x$, $v = 1/y$, and $w = 1/z$. The equations become

$$2u + 3v + 10w = 4, \quad 4u - 6v + 5w = 1, \quad 6u + 9v - 20w = 2.$$

Hence $AX = B$, with

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Step 2: Calculate the determinant and inverse Expanding along the first row,

$$\begin{aligned} |A| &= 2[(-6)(-20) - 5(9)] - 3[4(-20) - 5(6)] + 10[4(9) - (-6)(6)] \\ &= 2(75) - 3(-110) + 10(72) = 1200. \end{aligned}$$

Since $|A| \neq 0$, the inverse exists. Computing signed minors gives the cofactor matrix

$$C = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}.$$

For example, $C_{12} = -[4(-20) - 5(6)] = 110$ and $C_{23} = -[2(9) - 3(6)] = 0$. Transposing C therefore gives

$$A^{-1} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

Quick Tip to Solve: Solve the matrix system for the reciprocals first; remember to invert each answer at the end and check the signs in the second and third equations.

Step 3: Solve for the reciprocal variables Multiply the inverse by the constant column:

$$\begin{aligned} X = A^{-1}B &= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}. \end{aligned}$$

Thus $1/x = 1/2$, $1/y = 1/3$, and $1/z = 1/5$. All three reciprocal variables are nonzero, so taking their reciprocals yields valid original variables.

Step 4: Verify the original equations For $x = 2$, $y = 3$, and $z = 5$, the left sides of the original equations are

$$\begin{aligned} \frac{2}{2} + \frac{3}{3} + \frac{10}{5} &= 1 + 1 + 2 = 4, \\ \frac{4}{2} - \frac{6}{3} + \frac{5}{5} &= 2 - 2 + 1 = 1, \\ \frac{6}{2} + \frac{9}{3} - \frac{20}{5} &= 3 + 3 - 4 = 2. \end{aligned}$$

These agree with the stated right sides, and x, y, z satisfy the nonzero restriction. The invertible matrix also establishes uniqueness of the reciprocal solution and hence of this solution.

Step 5: Conclusion The solution of the given system is $\boxed{x = 2, \quad y = 3, \quad z = 5}$.

Quick Tip to Solve: Solve the matrix system for the reciprocals first; remember to invert each answer at the end and check the signs in the second and third equations.

OR

34(b). If $A = \begin{pmatrix} 1 & \cot x \\ -\cot x & 1 \end{pmatrix}$, then show that $A'A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}$.

[5 marks]

Solution:

Concept:

- The transpose A' is obtained by interchanging rows and columns. In particular, the two off-diagonal entries exchange their positions.
- For $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, $B^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- The identities $1 + \cot^2 x = \csc^2 x$, $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$ convert the resulting matrix into the required form.

Step 1: Find the transpose and establish invertibility Since $\cot x$ occurs in the question, $\sin x \neq 0$. Put

Quick Tip to Solve: Set $c = \cot x$ before multiplying; afterwards multiply each entry's numerator and denominator by $\sin^2 x$ to expose the double-angle identities.

$c = \cot x$ to keep the matrix multiplication clear. Then

$$A = \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix}.$$

Its determinant is $|A| = 1 \cdot 1 - c(-c) = 1 + c^2$. This is strictly positive for every real value of c , so the inverse exists throughout the domain of the given matrix.

Step 2: Calculate the inverse and multiply in the stated order Applying the inverse formula gives

$$A^{-1} = \frac{1}{1 + c^2} \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix}.$$

Therefore, using row-by-column multiplication,

$$\begin{aligned} A'A^{-1} &= \frac{1}{1 + c^2} \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix} \\ &= \frac{1}{1 + c^2} \begin{pmatrix} 1 - c^2 & -2c \\ 2c & 1 - c^2 \end{pmatrix}. \end{aligned}$$

Both diagonal entries are equal, and the lower off-diagonal entry has the opposite sign to the upper one. These signs follow directly from the transpose and must be retained.

Step 3: Simplify each distinct entry by double-angle identities Restoring $c = \cot x$, multiply numerator and denominator by $\sin^2 x$. This is permitted because $\sin x \neq 0$. For either diagonal entry,

$$\frac{1 - \cot^2 x}{1 + \cot^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = -\cos 2x.$$

For the lower off-diagonal entry,

$$\frac{2 \cot x}{1 + \cot^2 x} = \frac{2 \sin x \cos x}{\sin^2 x + \cos^2 x} = \sin 2x.$$

The upper off-diagonal entry is the negative of this quantity, hence it equals $-\sin 2x$. Thus all four entries match the corresponding entries of the claimed matrix.

Step 4: Conclusion Hence, for every x for which $\cot x$ is defined,

$$A'A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}.$$

Quick Tip to Solve: Set $c = \cot x$ before multiplying; afterwards multiply each entry's numerator and denominator by $\sin^2 x$ to expose the double-angle identities.

35. If A_1 denotes the area of region bounded by $y^2 = 4x$, $x = 1$ and x -axis in the first quadrant and A_2 denotes the area of region bounded by $y^2 = 4x$, $x = 4$, find $A_1 : A_2$. [5 marks]

Solution:

Concept:

- For a region between $y = f(x) \geq 0$ and the x -axis from $x = a$ to $x = b$, its area is $\int_a^b f(x) dx$.
- The parabola $y^2 = 4x$ has upper branch $y = 2\sqrt{x}$ and lower branch $y = -2\sqrt{x}$. It is symmetric about the x -axis.
- The area between two branches is the integral of upper ordinate minus lower ordinate. A region confined to the first quadrant includes only the upper half.

Step 1: Identify the boundaries of the first region The upper branch of the parabola meets the x -axis at the vertex $(0, 0)$. At $x = 1$, the first-quadrant intersection is $(1, 2)$. Thus the region specified for A_1 extends from $x = 0$ to $x = 1$, and a vertical strip has height $2\sqrt{x} - 0$. The explicit first-quadrant restriction means that the lower branch contributes nothing to A_1 .

Step 2: Evaluate the first-quadrant area Integrating the strip heights,

$$\begin{aligned} A_1 &= \int_0^1 2\sqrt{x} dx = 2 \left[\frac{x^{3/2}}{3/2} \right]_0^1 \\ &= \frac{4}{3} \left[x^{3/2} \right]_0^1 = \frac{4}{3} \text{ square units.} \end{aligned}$$

This is the area under the upper branch alone; it is already positive, so no absolute-value adjustment is needed.

Step 3: Identify the entire region bounded by the parabola and the second line At $x = 4$, the equation $y^2 = 16$ gives two intersections, $(4, 4)$ and $(4, -4)$. The parabola and the vertical line together enclose a region on both sides of the x -axis. Unlike A_1 , the definition of A_2 has no first-quadrant restriction and does not include the x -axis as a boundary. Therefore its vertical strip height is

$$y_{\text{upper}} - y_{\text{lower}} = 2\sqrt{x} - (-2\sqrt{x}) = 4\sqrt{x}, \quad 0 \leq x \leq 4.$$

Equivalently, symmetry makes this entire area twice the area under the upper branch from 0 to 4.

Step 4: Evaluate the second area and reduce the ratio Consequently,

$$\begin{aligned} A_2 &= \int_0^4 4\sqrt{x} dx = \frac{8}{3} \left[x^{3/2} \right]_0^4 \\ &= \frac{8}{3} (4^{3/2} - 0) = \frac{8}{3} \cdot 8 = \frac{64}{3} \text{ square units.} \end{aligned}$$

The area ratio is therefore

$$A_1 : A_2 = \frac{4}{3} : \frac{64}{3} = 4 : 64 = 1 : 16.$$

As an independent boundary check, horizontal strips for the second region extend from $x = y^2/4$ to $x = 4$, with $-4 \leq y \leq 4$. They give $\int_{-4}^4 (4 - y^2/4) dy = 64/3$, confirming that both halves have been included.

Quick Tip to Solve: Read the region restrictions separately: only A_1 is in the first quadrant; for A_2 , double the upper-branch area because the whole parabola and line enclose both halves.

Step 5: Conclusion The areas are $A_1 = \frac{4}{3}$ and $A_2 = \frac{64}{3}$ square units. Hence $A_1 : A_2 = 1 : 16$.

Quick Tip to Solve: Read the region restrictions separately: only A_1 is in the first quadrant; for A_2 , double the upper-branch area because the whole parabola and line enclose both halves.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Overspeeding increases fuel consumption and decreases fuel economy as a result of tyre rolling friction and air resistance. While vehicles reach optimal fuel economy at different speeds, fuel mileage usually decreases rapidly at speeds above 80 km/h.

The relation between fuel consumption F (l/100 km) and speed V (km/h) under some constraints is given as $F = \frac{V^2}{500} - \frac{V}{4} + 14$.

On the basis of the above information, answer the following questions:

[4 marks]

36(i). Find F , when $V = 40$ km/h.

[1 mark]

Solution:

Concept:

- The fuel-consumption function is $F(V) = \frac{V^2}{500} - \frac{V}{4} + 14$, where V is the speed in km/h and F is measured in litres per 100 km.
- To evaluate a function at a given speed, substitute that speed into every occurrence of V .

Step 1: Substitute the given speed At $V = 40$, the quadratic term uses 40^2 , while the linear term uses 40. Thus

$$F(40) = \frac{40^2}{500} - \frac{40}{4} + 14.$$

Step 2: Simplify the numerical terms Since $40^2 = 1600$, we have $1600/500 = 3.2$ and $40/4 = 10$. Therefore

$$F(40) = 3.2 - 10 + 14 = 7.2.$$

This value describes the amount of fuel consumed per 100 km at the stated speed.

Step 3: Conclusion $F = 7.2$ litres per 100 km when $V = 40$ km/h.

Quick Tip to Solve: Square the speed before dividing by 500; retain the negative sign in the linear term.

36(ii). Find $\frac{dF}{dV}$.

[1 mark]

Solution:

Concept:

- For a power of V , the power rule gives $\frac{d}{dV} V^n = nV^{n-1}$.
- Differentiate each term separately: a constant multiple stays outside the derivative, and the derivative of a constant is zero.

Step 1: Differentiate the quadratic term For $F(V) = \frac{V^2}{500} - \frac{V}{4} + 14$, the first term gives

$$\frac{d}{dV} \left(\frac{V^2}{500} \right) = \frac{2V}{500} = \frac{V}{250}.$$

The denominator remains constant; only the power of V is differentiated.

Step 2: Differentiate the remaining terms The derivative of $-V/4$ is $-1/4$, while 14 contributes zero. Adding the termwise derivatives gives

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4} + 0.$$

This derivative is the rate of change of consumption per unit change in speed.

Step 3: Conclusion $\boxed{\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}}$.

Quick Tip to Solve: The constant 14 disappears on differentiation; the negative linear term contributes $-1/4$.

36(iii)(a). Find the speed V for which fuel consumption F is minimum.

[2 marks]

Solution:

Concept:

- An interior minimum of a differentiable function occurs at a stationary point, so first solve $F'(V) = 0$.
- The second-derivative test confirms a strict minimum when $F''(V) > 0$; here V denotes speed in km/h.

Step 1: Find the stationary speed From the given consumption model,

$$F'(V) = \frac{V}{250} - \frac{1}{4}.$$

Quick Tip to Solve: After setting the first derivative to zero, use the positive constant second derivative to confirm the minimum.

Equating this to zero gives

$$\begin{aligned}\frac{V}{250} - \frac{1}{4} &= 0, \\ \frac{V}{250} &= \frac{1}{4}, \\ V &= \frac{250}{4} = 62.5.\end{aligned}$$

This positive value is admissible as a vehicle speed.

Step 2: Verify that consumption is minimized Differentiating once more,

$$F''(V) = \frac{1}{250} > 0.$$

Hence the stationary point is a strict minimum. Moreover, the second derivative is positive for every speed, so the quadratic opens upward and this is its global minimum. Equivalently, $F'(V) < 0$ for $V < 62.5$ and $F'(V) > 0$ for $V > 62.5$: consumption decreases up to this speed and increases afterwards.

Step 3: Conclusion Fuel consumption is minimum at $V = 62.5 \text{ km/h}$.

Quick Tip to Solve: After setting the first derivative to zero, use the positive constant second derivative to confirm the minimum.

OR

36(iii)(b). Find the quantity of fuel required to travel 600 km at the speed V at which $\frac{dF}{dV} = -0.01$.
[2 marks]

Solution:

Concept:

- The rate condition concerns $F'(V) = \frac{V}{250} - \frac{1}{4}$, rather than the value of $F(V)$ itself.
- If F litres are consumed per 100 km at a fixed speed, the fuel for distance d km is $\frac{d}{100}F$ litres.

Step 1: Find the speed from the rate condition Use the stated value $dF/dV = -0.01$:

$$\begin{aligned}\frac{V}{250} - \frac{1}{4} &= -0.01, \\ \frac{V}{250} &= 0.25 - 0.01 = 0.24, \\ V &= 250(0.24) = 60.\end{aligned}$$

Thus the vehicle travels at 60 km/h. The negative derivative describes decreasing consumption at this speed; it does not mean the consumption itself is negative.

Quick Tip to Solve: Solve the derivative condition for speed first, then evaluate F and multiply by 600/100.

Step 2: Evaluate consumption and scale to the journey Substitute this speed into the original function:

$$\begin{aligned} F(60) &= \frac{60^2}{500} - \frac{60}{4} + 14 \\ &= 7.2 - 15 + 14 = 6.2. \end{aligned}$$

Therefore the consumption rate is 6.2 litres per 100 km. The journey of 600 km comprises $600/100 = 6$ such distance intervals, so the fuel required is

$$6 \times 6.2 = 37.2 \text{ litres.}$$

Step 3: Conclusion At $V = 60$ km/h, the quantity of fuel required for 600 km is 37.2 litres.

Quick Tip to Solve: Solve the derivative condition for speed first, then evaluate F and multiply by $600/100$.

37. The month of September is celebrated as the Rashtriya Poshan Maah across the country. Following a healthy and well-balanced diet is crucial in order to supply the body with the proper nutrients it needs. A balanced diet also keeps us mentally fit and promotes improved level of energy.

A dietician wishes to minimize the cost of a diet involving two types of foods, food X (x kg) and food Y (y kg) which are available at the rate of ₹ 16/kg and ₹ 20/kg respectively. The feasible region satisfying the constraints is shown in Figure-2.

On the basis of the above information, answer the following questions:

[4 marks]

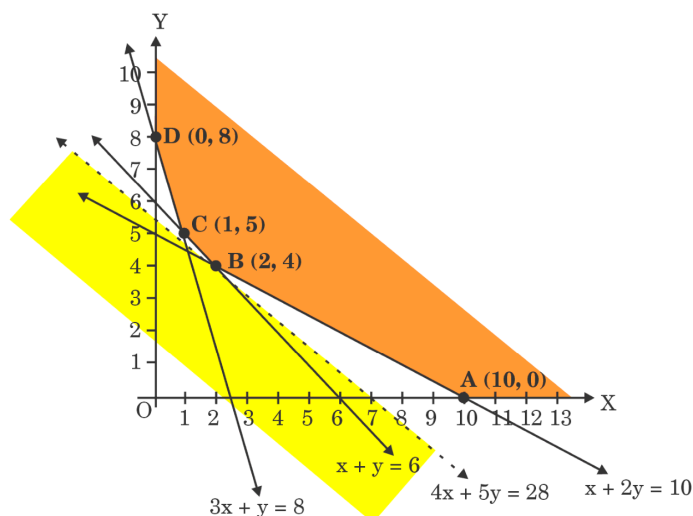


Figure-2

37(i). Identify and write all the constraints which determine the given feasible region in Figure-2.

[2 marks]

Solution:

Concept:

- A boundary line $ax + by = c$ divides the plane into two half-planes. Substitute a point on the shaded feasible side to choose the correct inequality.
- Food quantities cannot be negative, so $x \geq 0$ and $y \geq 0$. Solid constraint boundaries are included in the feasible region.

Step 1: Read the constraint boundary lines The lower boundary of the orange feasible region follows the three solid lines shown in Figure-2:

$$3x + y = 8, \quad x + y = 6, \quad x + 2y = 10.$$

Their consecutive intersections produce the displayed vertices $D(0, 8)$, $C(1, 5)$, $B(2, 4)$ and $A(10, 0)$. Each line therefore contributes a constraint, together with the coordinate axes.

Step 2: Choose the feasible half-planes The orange feasible region lies above these lower boundaries. For example, a point such as $(3, 6)$ lies on its shaded side:

$$3(3) + 6 = 15 \geq 8, \quad 3 + 6 = 9 \geq 6, \quad 3 + 2(6) = 15 \geq 10.$$

Hence all three inequalities have the greater-than-or-equal direction. The dashed line $4x + 5y = 28$ is a cost level line, so it is not an additional constraint; the yellow shading does not replace the orange feasible region.

Step 3: Conclusion The constraints are $3x + y \geq 8, \quad x + y \geq 6, \quad x + 2y \geq 10, \quad x \geq 0, \quad y \geq 0$.

Quick Tip to Solve: Use a point inside the orange region to fix inequality directions; do not include the dashed objective line.

37(ii). If the objective is to minimize cost $Z = 16x + 20y$, find the values of x and y at which cost is minimum. Also, find minimum cost assuming that minimum cost is possible for the given unbounded region. [2 marks]

Solution:

Concept:

- The objective function is $Z = 16x + 20y$, where x and y are food quantities in kg and Z is the cost in rupees.
- When a minimum exists for a linear objective over the feasible region, a corner point attains it. In an unbounded region, also check that a lower cost level does not meet the region.

Quick Tip to Solve: Compare all four corner costs, then express Z as $12(x+y) + 4(x+2y)$ to certify the global lower bound.

Step 1: Evaluate the objective at all vertices At the four labelled vertices, the costs are

$$Z(D) = 16(0) + 20(8) = 160,$$

$$Z(C) = 16(1) + 20(5) = 116,$$

$$Z(B) = 16(2) + 20(4) = 112,$$

$$Z(A) = 16(10) + 20(0) = 160.$$

The least cost is at B .

Step 2: Check the lower bound on the unbounded region Every feasible point satisfies $x + y \geq 6$ and $x + 2y \geq 10$. Combining these inequalities with positive multipliers gives

$$\begin{aligned} Z &= 16x + 20y \\ &= 12(x + y) + 4(x + 2y) \\ &\geq 12(6) + 4(10) = 112. \end{aligned}$$

Thus no feasible point has cost below 112. Equality holds only at $(2, 4)$, the intersection of $x + y = 6$ and $x + 2y = 10$. This point also satisfies $3x + y \geq 8$.

Step 3: Conclusion The minimum occurs at $x = 2$ kg, $y = 4$ kg, with $Z_{\min} = ₹112$.

Quick Tip to Solve: Compare all four corner costs, then express Z as $12(x+y) + 4(x+2y)$ to certify the global lower bound.

38. Airplanes are by far the safest mode of transportation when the number of transported passengers are measured against personal injuries and fatality totals.

Previous records state that the probability of an airplane crash is 0.00001%. Further, there are 95% chances that there will be survivors after a plane crash. Assume that in case of no crash, all travellers survive.

Let E_1 be the event that there is a plane crash and E_2 be the event that there is no crash. Let A be the event that passengers survive after the journey.

On the basis of the above information, answer the following questions:

[4 marks]

38(i). Find the probability that the airplane will not crash.

[1 mark]

Solution:

Concept:

- If E_1 is a crash and E_2 is no crash, the events are complementary: $P(E_2) = 1 - P(E_1)$.
- Convert a percentage to a probability by dividing its numerical value by 100.

Step 1: Convert the given percentage The printed crash probability is 0.00001%, so

$$P(E_1) = \frac{0.00001}{100} = 0.0000001 = 10^{-7}.$$

Keeping the percentage symbol during conversion is essential; 0.00001 itself is not the given probability.

Step 2: Use the complementary event A flight either crashes or does not crash. These possibilities are

Quick Tip to Solve: First divide the crash percentage by 100, then subtract that probability from 1.

mutually exclusive and exhaustive, giving

$$P(E_2) = 1 - 10^{-7} = 1 - 0.0000001 = 0.9999999.$$

Step 3: Conclusion The probability that the airplane will not crash is $\boxed{0.9999999}$.

Quick Tip to Solve: First divide the crash percentage by 100, then subtract that probability from 1.

38(ii). Find $P(A | E_1) + P(A | E_2)$.

[1 mark]

Solution:

Concept:

- $P(A | E_1)$ describes survival when a crash is known to have occurred.
- $P(A | E_2)$ describes survival when there is no crash; read both conditional probabilities directly from the stem.

Step 1: Read survival given a crash The passage gives a 95% chance of survivors after a crash. Hence the stated survival event satisfies

$$P(A | E_1) = \frac{95}{100} = 0.95.$$

This is conditional on a crash, so it is not multiplied by the crash probability for this part.

Step 2: Read survival given no crash and add All travellers survive when there is no crash, so $P(A | E_2) = 1$. Therefore

$$P(A | E_1) + P(A | E_2) = 0.95 + 1 = 1.95.$$

This sum concerns two different conditioning events; it is not itself the probability of a single event and can exceed 1.

Step 3: Conclusion The required sum is $\boxed{1.95}$.

Quick Tip to Solve: Use 0.95 for survival after a crash and 1 for survival without a crash; simply add them here.

38(iii)(a). Find $P(A)$.

[2 marks]

Solution:

Concept:

- Crash E_1 and no crash E_2 form a mutually exclusive, exhaustive partition of the outcomes.
- For survival event A , total probability gives $P(A) = P(E_1)P(A | E_1) + P(E_2)P(A | E_2)$.

Step 1: List the event and conditional probabilities Convert the printed crash percentage before

Quick Tip to Solve: Check the total by subtracting the chance of a crash with no survival, $10^{-7} \times 0.05$, from 1.

applying the formula:

$$\begin{aligned}P(E_1) &= \frac{0.00001}{100} = 10^{-7}, \\P(E_2) &= 1 - 10^{-7} = 0.9999999, \\P(A | E_1) &= \frac{95}{100} = 0.95, \\P(A | E_2) &= 1.\end{aligned}$$

The last equality follows from the assumption that all travellers survive if no crash occurs.

Step 2: Apply the theorem of total probability A surviving journey can occur through either a crash with survivors or a journey without a crash. Weight each conditional probability by the probability of that case:

$$\begin{aligned}P(A) &= (10^{-7})(0.95) + (1 - 10^{-7})(1) \\&= 0.000000095 + 0.9999999 \\&= 0.999999995.\end{aligned}$$

As a check, non-survival has probability $(10^{-7})(0.05) = 5 \times 10^{-9}$, whose complement gives the same value.

Step 3: Conclusion The probability of survival is $P(A) = 0.999999995$.

Quick Tip to Solve: Check the total by subtracting the chance of a crash with no survival, $10^{-7} \times 0.05$, from 1.

OR

38(iii)(b). Find $P(E_2 | A)$.

[2 marks]

Solution:

Concept:

- Bayes' theorem reverses conditioning: $P(E_2 | A) = P(E_2)P(A | E_2)/P(A)$, where A is survival and E_2 is no crash.
- Since crash E_1 and no crash E_2 partition the outcomes, $P(A) = P(E_1)P(A | E_1) + P(E_2)P(A | E_2)$.

Step 1: Calculate the probability of survival The given percentage gives $P(E_1) = 0.00001/100 = 10^{-7}$, so $P(E_2) = 1 - 10^{-7}$. Also, $P(A | E_1) = 0.95$ and $P(A | E_2) = 1$. Thus

$$\begin{aligned}P(A) &= (10^{-7})(0.95) + (1 - 10^{-7})(1) \\&= 0.999999995.\end{aligned}$$

This denominator includes survival both with a crash and without a crash.

Step 2: Apply Bayes' theorem to no crash given survival Given that passengers survived, restrict the sample space to survival outcomes. The no-crash contribution to those outcomes is $P(E_2)P(A | E_2) =$

Quick Tip to Solve: Put the no-crash survival contribution in the numerator and all survival contributions in the denominator.

0.9999999. Hence

$$\begin{aligned}P(E_2 | A) &= \frac{0.9999999 \times 1}{0.999999995} \\&= \frac{199999980}{199999999}.\end{aligned}$$

The ratio is less than 1, because survival can also occur after a crash; knowing survival does not guarantee no crash.

Step 3: Conclusion The required probability is $P(E_2 | A) = \frac{199999980}{199999999}$.

Quick Tip to Solve: Put the no-crash survival contribution in the numerator and all survival contributions in the denominator.