CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-2-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The number of solutions of the differential equation $\frac{dy}{dx} - y = 1$, given that $y(0) = 1$, is: [1 mark]

(A) 0

(B) 1

(C) 2

(D) infinitely many

Correct Answer: (B) 1

Solution:

Concept:

- A first-order linear equation $y' + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$.
- An initial condition determines the constant in the general solution; distinct constants otherwise give different solution curves.

Step 1: Find the general solution Here $P(x) = -1$, so the integrating factor is e^{-x}. Multiplying the equation by it gives

$$\frac{d}{dx}(ye^{-x}) = e^{-x}.$$

Integrating yields $ye^{-x} = -e^{-x} + C$, or $y = -1 + Ce^{x}$.

Step 2: Apply the initial condition At $x = 0$, the condition becomes $1 = -1 + C$, fixing $C = 2$. Thus the family has exactly one member satisfying the condition. Zero solutions is impossible because this member exists; two or infinitely many would require more than one admissible value of C.

Step 3: Conclusion The unique solution is $y = 2e^{x} - 1$, so the number of solutions is 1. Hence, the correct option is (B).

Quick Tip to Solve: Count solutions only after imposing the initial condition: here it fixes the single constant uniquely.

2. For any two vectors $\vec{a}$ and $\vec{b}$, which of the following statements is always true? [1 mark]

(A) $\vec{a} \cdot \vec{b} \geq |\vec{a}|\,|\vec{b}|$

(B) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$

(C) $\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|$

(D) $\vec{a} \cdot \vec{b} < |\vec{a}|\,|\vec{b}|$

Correct Answer: (C) $\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|$

Solution:

Concept:

- For nonzero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where θ is their angle.

Quick Tip to Solve: Test an alleged universal dot-product inequality with perpendicular vectors and with vectors in the same direction.

- The bound $-1 \leq \cos\theta \leq 1$ gives upper and lower bounds for the dot product.

Step 1: Apply the upper bound Since the product of magnitudes is nonnegative, multiplication preserves $\cos\theta \leq 1$. Consequently,

$$\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|.$$

If either vector is zero, both sides are zero, so the same inequality holds.

Step 2: Check the other claims For nonzero perpendicular vectors the dot product is zero while the product of magnitudes is positive. Thus (A) and (B) are not always true. For parallel vectors pointing in the same direction, equality holds, ruling out the strict inequality in (D).

Step 3: Conclusion The universally valid statement is $\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|$. Hence, the correct option is (C).

Quick Tip to Solve: Test an alleged universal dot-product inequality with perpendicular vectors and with vectors in the same direction.

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3. The coordinates of the foot of the perpendicular drawn from the point $(0, 1, 2)$ on the x-axis are given by: [1 mark]

(A) $(1, 0, 0)$

(B) $(2, 0, 0)$

(C) $(\sqrt{5}, 0, 0)$

(D) $(0, 0, 0)$

Correct Answer: (D) $(0, 0, 0)$

Solution:

Concept:

- Every point on the x-axis has coordinates $(t, 0, 0)$.
- A line perpendicular to the x-axis has a direction vector whose dot product with $(1, 0, 0)$ is zero.

Step 1: Write the perpendicularity condition Let the required foot be $H = (t, 0, 0)$ and the given point be $P = (0, 1, 2)$. Then $\overrightarrow{PH} = (t, -1, -2)$. Since PH is perpendicular to the axis,

$$(t, -1, -2) \cdot (1, 0, 0) = t = 0.$$

Step 2: Check the proposed feet The perpendicular foot must retain the original x-coordinate and set the other two coordinates to zero. Although the points in (A), (B) and (C) lie on the axis, their nonzero x-coordinates make the corresponding joining lines nonperpendicular. In particular, $\sqrt{5}$ is the distance to the axis, not the coordinate of the foot.

Step 3: Conclusion The foot of the perpendicular is $(0, 0, 0)$. Hence, the correct option is (D).

Quick Tip to Solve: For projection on the x-axis, retain the x-coordinate and replace the y- and z-coordinates by zero.

4. The common region determined by all the constraints of a linear programming problem is called: [1 mark]

(A) an unbounded region

(B) an optimal region

(C) a bounded region

(D) a feasible region

Correct Answer: (D) a feasible region

Solution:

Concept:

- A feasible solution is a point satisfying every constraint of a linear programming problem, including nonnegativity constraints when imposed.
- The feasible region is the set of all feasible solutions, obtained by intersecting the regions determined by the constraints.

Step 1: Identify the common set Each linear inequality restricts the permissible points to a half-plane. A point belongs to their common region precisely when it satisfies every restriction simultaneously. Therefore this common region is the set of feasible solutions.

Step 2: Distinguish the other descriptions A feasible region can be bounded or unbounded depending on the constraints, so neither (A) nor (C) names the common region in general. Optimality depends additionally on maximising or minimising the objective function; merely satisfying the constraints does not make every point optimal. Hence (B) is not the required definition.

Step 3: Conclusion The common region determined by all the constraints is a feasible region. Hence, the correct option is (D).

Quick Tip to Solve: Use constraints to identify feasibility; use the objective function only when identifying an optimal solution.

5. Let E be an event of a sample space S of an experiment, then $P(S \mid E) =$ [1 mark]

(A) $P(S \cap E)$

(B) $P(E)$

(C) 1

(D) 0

Correct Answer: (C) 1

Solution:

Concept:

- For an event E with $P(E) > 0$, conditional probability is $P(A \mid E) = \dfrac{P(A \cap E)}{P(E)}$.
- Every event E is a subset of the sample space S, so $S \cap E = E$.

Step 1: Simplify the intersection Conditioning on E means restricting possible outcomes to E. All such outcomes still lie in the sample space, and therefore $P(S \cap E) = P(E)$.

Quick Tip to Solve: Before using conditional probability with the sample space, simplify $S \cap E$ to E.

Step 2: Apply the conditional probability formula Under the condition $P(E) > 0$ required for the displayed conditional probability to be defined,

$$P(S \mid E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1.$$

Options (A) and (B) both equal $P(E)$, which need not be one. Option (D) cannot apply because membership in the sample space is certain.

Step 3: Conclusion Whenever the stated conditional probability is defined, $P(S \mid E) = 1$. Hence, the correct option is (C).

Quick Tip to Solve: Before using conditional probability with the sample space, simplify $S \cap E$ to E.

6. The number of all scalar matrices of order 3, with each entry -1, 0 or 1, is: [1 mark]

(A) 1

(B) 3

(C) 2

(D) 3^9

Correct Answer: (B) 3

Solution:

Concept:

- A scalar matrix of order 3 has the form kI_3: all diagonal entries equal k, and all off-diagonal entries are zero.
- The zero matrix is also a scalar matrix, obtained by taking $k = 0$.

Step 1: Apply the scalar-matrix restrictions Only the common diagonal value is free. The six off-diagonal entries are forced to zero, which is among the permitted entries. Since the diagonal entries must come from $\{-1, 0, 1\}$, the common value k has exactly these three choices.

Step 2: Count and test the alternatives The resulting matrices are $-I_3$, $0_{3\times3}$ and I_3, and they are distinct. Option (A) omits two admissible values; (C) omits the zero scalar matrix. The count 3^9 in (D) applies when all nine entries can be chosen independently, which violates the scalar-matrix restrictions.

Step 3: Conclusion There are 3 scalar matrices satisfying the conditions. Hence, the correct option is (B).

Quick Tip to Solve: Count choices for the single scalar k, not choices for nine entries; remember that $k = 0$ is allowed.

7. $\frac{d}{dx}\left[\cos(\log x + e^x)\right]$ **at** $x = 1$ **is:** [1 mark]

(A) $-\sin e$

(B) $\sin e$

(C) $-(1+e)\sin e$

(D) $(1+e)\sin e$

Correct Answer: (C) $-(1+e)\sin e$

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$.
- For $x > 0$, $\frac{d}{dx}\log x = 1/x$, while $\frac{d}{dx}e^x = e^x$.

Step 1: Differentiate the outer and inner functions Take $u = \log x + e^x$. Then $u\prime = 1/x + e^x$, and hence

$$\frac{d}{dx}\cos(\log x + e^x) = -\sin(\log x + e^x)\left(\frac{1}{x} + e^x\right).$$

Step 2: Substitute and check the factors At $x = 1$, the angle becomes $\log 1 + e^1 = e$, and the inner derivative becomes $1 + e$. Option (A) drops this inner derivative, (B) also loses the minus sign from differentiating cosine, and (D) has the wrong sign.

Step 3: Conclusion The derivative at $x = 1$ is $-(1+e)\sin e$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate before substituting $x = 1$; retain both the minus sign of cosine and the inner factor $1 + e$.

8. The degree of the differential equation $(y'')^2 + (y')^3 = x\sin(y')$ **is:** [1 mark]

(A) 1

(B) 2

(C) 3

(D) not defined

Correct Answer: (D) not defined

Solution:

Concept:

- The degree of a differential equation is defined only when it can be expressed as a polynomial in its derivatives.
- For such a polynomial equation, the degree is the power of the highest-order derivative; order and degree are different concepts.

Step 1: Check the polynomial requirement The expression contains $\sin(y\prime)$, a trigonometric function of a derivative. Thus

$$(y\prime\prime)^2 + (y\prime)^3 - x\sin(y\prime) = 0$$

Quick Tip to Solve: Before reading off any derivative exponent, check that no derivative occurs inside sine, cosine or another nonpolynomial function.

is not a polynomial in the derivatives. There is no radical or fractional power that can be cleared algebraically to remove this trigonometric dependence.

Step 2: Avoid confusing order with degree The highest derivative is y'', so the order is two. However, its exponent two is not the degree because the polynomial requirement has failed. Similarly, the exponent three of y' and the first derivative inside sine do not supply a degree; options (A), (B) and (C) all assign an unjustified numerical degree.

Step 3: Conclusion The degree of the given differential equation is not defined. Hence, the correct option is (D).

Quick Tip to Solve: Before reading off any derivative exponent, check that no derivative occurs inside sine, cosine or another nonpolynomial function.

9. The unit vector perpendicular to both vectors $\hat{i}+\hat{k}$ and $\hat{i}-\hat{k}$ is: [1 mark]

(A) $2\hat{j}$

(B) $\hat{j}$

(C) $\dfrac{\hat{i}-\hat{k}}{\sqrt{2}}$

(D) $\dfrac{\hat{i}+\hat{k}}{\sqrt{2}}$

Correct Answer: (B) $\hat{j}$

Solution:

Concept:

- Two nonzero vectors are perpendicular when their dot product is zero: $\vec{a}\cdot\vec{b}=0$.
- A unit vector has magnitude one; $\hat{i},\hat{j},\hat{k}$ are mutually perpendicular unit vectors.

Step 1: Check perpendicularity Both given vectors lie in the xz-plane. Thus $\hat{j}$ is perpendicular to both, as verified by

$$\hat{j}\cdot(\hat{i}+\hat{k})=0, \qquad \hat{j}\cdot(\hat{i}-\hat{k})=0.$$

Also, $|\hat{j}|=1$, so it satisfies the unit-length condition.

Step 2: Test the other options Option (A), $2\hat{j}$, is perpendicular to both vectors but has magnitude 2, so it is not a unit vector. Option (C) is parallel to the second given vector, while option (D) is parallel to the first; each therefore fails the required perpendicularity to both. Although $-\hat{j}$ also works, it is not a listed option.

Step 3: Conclusion The required listed unit vector is $\hat{j}$. Hence, the correct option is (B).

Quick Tip to Solve: Check both perpendicularity and magnitude: a multiple of $\hat{j}$ need not be a unit vector.

10. Direction ratios of a vector parallel to the line $\frac{x-1}{2} = -y = \frac{2z+1}{6}$ are: [1 mark]

(A) 2, -1, 6

(B) 2, 1, 6

(C) 2, 1, 3

(D) 2, -1, 3

Correct Answer: (D) 2, -1, 3

Solution:

Concept:

- A line in symmetric form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ has direction ratios a, b, c.
- Equivalently, the coefficients of a common parameter in x, y, z give direction ratios.

Step 1: Introduce a common parameter Put

$$\frac{x-1}{2} = -y = \frac{2z+1}{6} = t.$$

Solving each equality for its coordinate gives

$$x = 1 + 2t, \qquad y = -t, \qquad z = -\frac{1}{2} + 3t.$$

The coefficient of t in z is 3, because the numerator contains $2z$.

Step 2: Compare the direction vectors The coefficients of t form the direction vector $2\hat{i} - \hat{j} + 3\hat{k}$. Option (A) incorrectly uses 6 as the last ratio without dividing the numerator by 2. Options (B) and (C) give a positive second ratio, whereas $-y = t$ implies $y = -t$. Only option (D) has all three correct coefficients.

Step 3: Conclusion Direction ratios are $2, -1, 3$. Hence, the correct option is (D).

Quick Tip to Solve: First make each coordinate coefficient in the numerator equal to one; rewrite $(2z+1)/6$ as $(z+1/2)/3$.

11. If $F(x) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **and** $[F(x)]^2 = F(kx)$**, then the value of** k **is:** [1 mark]

(A) 1

(B) 2

(C) 0

(D) -2

Correct Answer: (B) 2

Solution:

Concept:

- Matrix squaring means multiplication $[F(x)]^2 = F(x)F(x)$, using row-by-column products.
- The double-angle identities are $\cos^2 x - \sin^2 x = \cos 2x$ and $2\sin x \cos x = \sin 2x$.

Step 1: Multiply the matrix by itself The last row and column leave the third diagonal entry equal to 1. Multiplying the upper-left block yields

$$[F(x)]^2 = \begin{pmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x & 0 \\ 2\sin x \cos x & \cos^2 x - \sin^2 x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 2: Apply the double-angle identities Substituting the identities gives

$$[F(x)]^2 = \begin{pmatrix} \cos 2x & -\sin 2x & 0 \\ \sin 2x & \cos 2x & 0 \\ 0 & 0 & 1 \end{pmatrix} = F(2x).$$

This is an identity in x. The choices 1 and 0 do not double the angle, while -2 reverses the signs of the sine entries and therefore does not agree for general x.

Step 3: Conclusion Comparing with $F(kx)$ gives $k = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Recognize the upper-left block as a rotation matrix: multiplying two equal rotations adds their angles.

12. If a line makes an angle of 30° with the positive direction of x-axis, 120° with the positive direction of y-axis, then the angle which it makes with the positive direction of z-axis is: [1 mark]

(A) 90°

(B) 120°

(C) 60°

(D) 0°

Correct Answer: (A) 90°

Solution:

Concept:

- For direction angles α, β, γ, the direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.
- They satisfy $l^2 + m^2 + n^2 = 1$, and each direction angle lies between 0° and 180°.

Step 1: Substitute the given angles Using $\alpha = 30^\circ$ and $\beta = 120^\circ$,

$$\cos^2\gamma = 1 - \cos^2 30^\circ - \cos^2 120^\circ = 1 - \frac{3}{4} - \frac{1}{4} = 0.$$

The negative value $\cos 120^\circ = -1/2$ contributes a positive square.

Step 2: Identify the angle Since $\cos\gamma = 0$, its unique value in the direction-angle range is 90°. The other listed angles 120°, 60° and 0° have cosine squares $1/4, 1/4, 1$, respectively. None can be added to the existing total $3/4 + 1/4$ while preserving a sum of one.

Step 3: Conclusion The angle with the positive z-axis is 90°. Hence, the correct option is (A).

Quick Tip to Solve: Square the two known direction cosines first; if their sum is one, the third direction cosine must be zero.

13. If the sum of all the elements of a 3×3 scalar matrix is 9, then the product of all its elements is: [1 mark]

(A) 0

(B) 9

(C) 27

(D) 729

Correct Answer: (A) 0

Solution:

Concept:

- A scalar matrix has the form λI_n: all diagonal entries equal λ, and all off-diagonal entries are zero.
- The product of all entries is different from the determinant; a single zero entry makes that product zero.

Quick Tip to Solve: For the product of all entries of a scalar matrix of order greater than one, look for off-diagonal zeros immediately.

Step 1: Find the scalar diagonal entry Write the matrix as

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The sum of its nine entries is $3\lambda = 9$, so each diagonal entry is $\lambda = 3$. The remaining six entries are zero.

Step 2: Use every entry in the product The requested product includes all nine elements, hence contains six zero factors. In particular,

$$\prod_{i=1}^{3}\prod_{j=1}^{3} a_{ij} = 3 \cdot 0 \cdot 0 \cdot 0 \cdot 3 \cdot 0 \cdot 0 \cdot 0 \cdot 3 = 0.$$

The tempting value 27 is the product of only the diagonal entries, namely the determinant, and does not answer this question. The values 9 and 729 also ignore the zero entries.

Step 3: Conclusion The product of all the elements is 0. Hence, the correct option is (A).

Quick Tip to Solve: For the product of all entries of a scalar matrix of order greater than one, look for off-diagonal zeros immediately.

14. Which of the following statements is not true about equivalence classes A_i $(i = 1, 2, \ldots, n)$ formed by an equivalence relation R defined on a set A? [1 mark]

(A) $\bigcup_{i=1}^{n} A_i = A$

(B) $A_i \cap A_j \neq \varnothing,\ i \neq j$

(C) $x \in A_i$ and $x \in A_j \Rightarrow A_i = A_j$

(D) All elements of A_i are related to each other, for all i

Correct Answer: (B) $A_i \cap A_j \neq \varnothing,\ i \neq j$

Solution:

Concept:

- An equivalence relation is reflexive, symmetric and transitive; its equivalence classes partition the underlying set.
- Any two equivalence classes are either identical or disjoint, and the union of all classes is the entire set.

Step 1: Check coverage and intersection Every element belongs to its own equivalence class, so $\bigcup_{i=1}^{n} A_i = A$; option (A) is true. Distinct classes cannot share an element. Therefore the assertion in option (B), $A_i \cap A_j \neq \varnothing$ for distinct classes, is not a property of an equivalence relation.

Step 2: Check the remaining statements If $x \in A_i$ and $x \in A_j$, the common element links every member of the two classes by symmetry and transitivity, forcing $A_i = A_j$. Thus option (C) is true. Elements belonging to one class are related to a common representative and hence to one another, so

Quick Tip to Solve: Remember that equivalence classes form a partition: they cover the set, and two distinct classes never overlap.

option (D) is also true. These facts establish precisely why nonempty intersection cannot occur for distinct classes.

Step 3: Conclusion The statement that is not true is $A_i \cap A_j \neq \varnothing$ for distinct classes. Hence, the correct option is (B).

Quick Tip to Solve: Remember that equivalence classes form a partition: they cover the set, and two distinct classes never overlap.

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15. If $\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = k\,abc$**, then the value of** k **is:** [1 mark]

(A) 0

(B) 1

(C) 2

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- A common factor in a column can be taken outside a determinant; extracting one factor from each column multiplies the determinant by their product.
- Expansion along the first row uses the alternating signs $+, -, +$.

Step 1: Extract the column factors Taking a, b, c from the first, second and third columns respectively gives the polynomial identity

$$\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = abc \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

This identity also holds if any of a, b, c is zero; no division by them is needed.

Step 2: Expand the numerical determinant Expanding along its first row,

$$\begin{aligned} D &= (-1)\big[(-1)(-1) - 1\big] - 1\big[1(-1) - 1\big] \\ &\quad + 1\big[1 - (-1)\big] \\ &= 0 - (-2) + 2 = 4. \end{aligned}$$

Thus the original determinant is $4abc$. The values $0, 1, 2$ do not match the coefficient obtained from the complete cofactor expansion; in particular, the two nonzero terms must both be retained.

Step 3: Conclusion The coefficient is $k = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Factor out a, b, c columnwise before expanding; the remaining determinant contains only 1 and -1.

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16. The number of points of discontinuity of $f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$ **is:** [1 mark]

(A) 0

(B) 1

(C) 2

(D) infinite

Correct Answer: (B) 1

Solution:

Concept:

- A function is continuous at $x = a$ when its left-hand limit, right-hand limit and value $f(a)$ all coincide.
- Each given expression is continuous on its own interval, so only the joining points -3 and 3 need checking.

Step 1: Check the join at minus three For $x \leq -3$, $|x| = -x$, so the left-hand expression tends to 6. The middle expression also gives $-2(-3) = 6$. Thus

$$\lim_{x \to -3^-} f(x) = 6 = \lim_{x \to -3^+} f(x) = f(-3).$$

There is no discontinuity at -3.

Step 2: Check the join at three At the other join,

$$\lim_{x \to 3^-} f(x) = -6, \qquad \lim_{x \to 3^+} f(x) = 20 = f(3).$$

These one-sided limits differ, so there is a jump at 3. There are no further candidates: $|x|$ is used only for $x \leq -3$, and around zero the function is simply $-2x$. Consequently the choices zero, two and infinitely many discontinuities are excluded.

Step 3: Conclusion There is exactly one point of discontinuity, $x = 3$. Hence, the correct option is (B).

Quick Tip to Solve: Check only the piecewise boundaries, and use the branch containing equality to determine the function value at each join.

17. The function $f(x) = x^3 - 3x^2 + 12x - 18$ is: [1 mark]

(A) strictly decreasing on R

(B) strictly increasing on R

(C) neither strictly increasing nor strictly decreasing on R

(D) strictly decreasing on $(-\infty, 0)$

Correct Answer: (B) strictly increasing on R

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- Completing the square determines the sign of a quadratic for every real input, since a real square is non-negative.

Step 1: Differentiate the polynomial The function is differentiable on all of $\mathbb{R}$. Differentiate each term:

$$f'(x) = 3x^2 - 6x + 12 = 3\big((x-1)^2 + 3\big).$$

Step 2: Check the sign and the options Since $(x-1)^2 \geq 0$, we have $f'(x) \geq 9 > 0$ for every real x. Thus the derivative never vanishes or becomes negative. Options (A) and (D), which claim decrease, are false, including on the negative half-line. Option (C) is also false because the function has a definite increasing behaviour on the whole real line.

Step 3: Conclusion The function is strictly increasing on $\mathbb{R}$. Hence, the correct option is (B).

Quick Tip to Solve: Complete the square in $f'(x)$: the positive lower bound 9 settles monotonicity on the entire real line at once.

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18. If $\int_0^2 2e^{2x}\,dx = \int_0^a e^x\,dx$, then the value of 'a' is: [1 mark]

(A) 1

(B) 2

(C) 4

(D) 1/2

Correct Answer: (C) 4

Solution:

Concept:

- The derivative of e^{kx} is ke^{kx}, so $\int ke^{kx}\,dx = e^{kx} + C$.
- A definite integral equals the antiderivative at the upper limit minus its value at the lower limit. The exponential function is one-to-one on $\mathbb{R}$.

Step 1: Evaluate both definite integrals The factor 2 in the left integrand is precisely the derivative of

Quick Tip to Solve: With the substitution $u = 2x$, change the left upper limit from 2 to 4; this directly matches the right integral.

its exponent:

$$\int_0^2 2e^{2x}\,dx = [e^{2x}]_0^2 = e^4 - 1.$$

Similarly,

$$\int_0^a e^x\,dx = [e^x]_0^a = e^a - 1.$$

Step 2: Equate the values The given equality implies $e^4 - 1 = e^a - 1$, hence $e^a = e^4$. Injectivity of the exponential gives a unique real value of a. The choices 1, 2, and $1/2$ give different exponential values and therefore fail the equality.

Step 3: Conclusion $\boxed{a = 4}$. Hence, the correct option is (C).

Quick Tip to Solve: With the substitution $u = 2x$, change the left upper limit from 2 to 4; this directly matches the right integral.

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19. Assertion (A): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
Reason (R): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$.
Select the correct answer from the codes (A), (B), (C) and (D): [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- The scalar product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- The vector product is anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$; interchanging the order reverses its direction.

Step 1: Evaluate the assertion If θ is the angle between the two vectors, both scalar products equal $|\vec{a}||\vec{b}| \cos\theta$. Interchanging the vectors changes neither their magnitudes nor this angle. Assertion (A) is therefore true.

Step 2: Evaluate the reason For example, $\hat{i} \times \hat{j} = \hat{k}$, whereas $\hat{j} \times \hat{i} = -\hat{k}$. This disproves the claimed equality for general non-zero vectors. Parallel vectors give zero products in both orders, but that special case cannot establish the universal claim. Reason (R) is false.

Step 3: Check the explanation and codes A false reason cannot explain the true assertion. Codes (A) and (B) both require a true reason, while (D) requires a false assertion; each conflicts with the results above.

Quick Tip to Solve: Test a proposed cross-product identity with $\hat{i}$ and $\hat{j}$; reversing their order immediately reveals the sign change.

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Test a proposed cross-product identity with $\hat{i}$ and $\hat{j}$; reversing their order immediately reveals the sign change.

20. Assertion (A): For any symmetric matrix A, B'AB is a skew-symmetric matrix.

Reason (R): A square matrix P is skew-symmetric if $P' = -P$.

Select the correct answer from the codes (A), (B), (C) and (D): [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A square matrix A is symmetric when $A' = A$, where the prime denotes transpose; it is skew-symmetric when $A' = -A$.
- Transposing a product reverses the order of its factors: $(XYZ)' = Z'Y'X'$, and $(B')' = B$.

Step 1: Evaluate the assertion For conformable matrices, put $P = B'AB$. Since $A' = A$,

$$P' = (B'AB)' = B'A'(B')' = B'AB = P.$$

Thus P is symmetric. It need not be skew-symmetric: taking $A = B = I$ gives $P = I$, whose transpose is not $-I$. The assertion is false.

Step 2: Evaluate the reason The equation $P' = -P$ is exactly the definition of a skew-symmetric square matrix. Therefore the reason is true.

Step 3: Check the explanation and codes The true definition does not establish the assertion: the transpose calculation gives $P' = P$, not $P' = -P$ in general. Codes (A), (B), and (C) all require a true assertion and therefore cannot apply here.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Reverse factor order when transposing $B'AB$, then use $A' = A$; an identity-matrix example disproves the skew-symmetry claim.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}$ $(x \neq 0)$ respectively, find the value of $(M - m)$. [2 marks]

Solution:

Concept:

- Interior local extrema of a differentiable function occur at stationary points, where $f'(x) = 0$.
- At a stationary point c, $f''(c) < 0$ implies a local maximum and $f''(c) > 0$ implies a local minimum. The extremum value is $f(c)$, not c.

Step 1: Find the stationary points The domain excludes $x = 0$, and the function is differentiable at every other real number. Differentiate:

$$f'(x) = 1 - \frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}.$$

Solving $1 - 1/x^2 = 0$ gives $x^2 = 1$, so the stationary points are $x = -1$ and $x = 1$. The excluded point 0 is not a candidate.

Step 2: Classify the points and evaluate the function At $x = -1$, $f''(-1) = -2 < 0$, so the local maximum value is

$$M = f(-1) = -1 + \frac{1}{-1} = -2.$$

At $x = 1$, $f''(1) = 2 > 0$, so the local minimum value is

$$m = f(1) = 1 + 1 = 2.$$

These extrema lie on different branches of the domain. A local maximum on one branch can be smaller than a local minimum on the other, so their signs should not be exchanged.

Step 3: Conclusion $\boxed{M - m = -2 - 2 = -4}$.

Quick Tip to Solve: Classify $x = -1$ and $x = 1$ first, then substitute into f; the question asks for extremum values, not stationary coordinates.

22. Evaluate: $\displaystyle\int_0^{a^3} \frac{x^2}{x^6 + a^6}\, dx$. [2 marks]

Solution:

Concept:

- With $t = x^3$, the differential is $dt = 3x^2\, dx$, which converts the sixth power in the denominator into a square.
- For a non-zero constant b, $\int dt/(t^2 + b^2) = b^{-1} \tan^{-1}(t/b) + C$. A definite substitution must also transform both limits.

Quick Tip to Solve: Match $x^2\, dx$ with $d(x^3)/3$, and cube the original upper limit: $(a^3)^3 = a^9$.

Step 1: Substitute and change the limits Let I denote the given integral and take $a \neq 0$, so the denominator stays positive for real x. Put $t = x^3$. When $x = 0$, $t = 0$, and when $x = a^3$, $t = (a^3)^3 = a^9$. Therefore,

$$I = \frac{1}{3}\int_0^{a^9} \frac{dt}{t^2 + (a^3)^2}.$$

The upper limit changes to a^9; leaving it as a^3 after this substitution would change the integral.

Step 2: Apply the inverse tangent formula Using $b = a^3$, evaluate the antiderivative at both transformed limits:

$$\begin{aligned} I &= \frac{1}{3a^3}\left[\tan^{-1}\left(\frac{t}{a^3}\right)\right]_0^{a^9} \\ &= \frac{1}{3a^3}\left(\tan^{-1}(a^6) - \tan^{-1}(0)\right). \end{aligned}$$

This formula also works for negative non-zero a; the factor a^3 retains the sign appropriate to the reversed integration interval.

Step 3: Conclusion $\boxed{\int_0^{a^3} \frac{x^2}{x^6 + a^6}\,dx = \frac{\tan^{-1}(a^6)}{3a^3}}$, for $a \neq 0$.

Quick Tip to Solve: Match $x^2\,dx$ with $d(x^3)/3$, and cube the original upper limit: $(a^3)^3 = a^9$.

23. Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- The derivative formulae needed here are $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$; apply the chain rule to e^{-x}.

Step 1: Identify the domain and differentiate Both exponential terms, the linear term and the inverse tangent are defined and differentiable for every real x. Thus the domain is $\mathbb{R}$, a single interval. Differentiating term by term gives

$$f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2}.$$

The sign before e^{-x} is positive because differentiating its exponent contributes another negative sign.

Step 2: Establish the sign of the derivative Combine the last two terms to obtain

$$f'(x) = e^x + e^{-x} + \frac{x^2}{1+x^2}.$$

For every real x, both exponential terms are strictly positive. Also, $x^2 \geq 0$ and $1 + x^2 > 0$, so the fraction is non-negative. Their sum is therefore strictly positive, including at $x = 0$, where the fraction

Quick Tip to Solve: Rewrite $1 - 1/(1+x^2)$ as $x^2/(1+x^2)$; every term in the derivative then has an immediate sign.

alone vanishes. The increasing-function test consequently applies on the entire domain.

Step 3: Conclusion Since $f'(x) > 0$ for every $x \in \mathbb{R}$, the given function is strictly increasing throughout its domain $\mathbb{R}$.

Quick Tip to Solve: Rewrite $1 - 1/(1+x^2)$ as $x^2/(1+x^2)$; every term in the derivative then has an immediate sign.

24(a). **Find the value of** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$. [2 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$, so a negative argument gives a negative angle.
- The principal value of $\cot^{-1} t$ lies in $(0, \pi)$. For a positive argument it lies in the first quadrant.

Step 1: Evaluate the first two principal values Since $\tan(-\pi/6) = -1/\sqrt{3}$ and $-\pi/6$ lies in the principal interval of inverse tangent,

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$

Also, $\cot(\pi/3) = 1/\sqrt{3}$, and $\pi/3 \in (0, \pi)$. Hence

$$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

Choosing the principal intervals prevents adding an extraneous multiple of π to either value.

Step 2: Evaluate the nested term and add First evaluate the sine inside the final inverse tangent. Since $\sin(-\pi/2) = -1$, we get

$$\tan^{-1}[\sin(-\pi/2)] = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

The angle $-\pi/4$ is in the principal range of inverse tangent. Using a common denominator of 12, the required sum becomes

$$-\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{(-2+4-3)\pi}{12}.$$

Step 3: Conclusion The value of the given expression is $\boxed{-\frac{\pi}{12}}$.

Quick Tip to Solve: Evaluate the inner sine before applying inverse tangent, and use $\pi/3$, not $\pi/6$, for $\cot^{-1}(1/\sqrt{3})$.

OR

24(b). Find the domain of the function $f(x) = \sin^{-1}(x^2 - 4)$. Also, find its range. [2 marks]

Solution:

Concept:

- For real values, $\sin^{-1} u$ is defined exactly when $-1 \le u \le 1$.
- The principal range of inverse sine is $[-\pi/2, \pi/2]$. The range of a composite function depends on which inner values are attained on its domain.

Step 1: Restrict the input of inverse sine Put $u = x^2 - 4$. For the function to be real, its inner expression must belong to $[-1, 1]$. Therefore

$$\begin{aligned} -1 \le x^2 - 4 \le 1 &\iff 3 \le x^2 \le 5 \\ &\iff \sqrt{3} \le |x| \le \sqrt{5}. \end{aligned}$$

The absolute-value inequality allows both negative and positive inputs. Thus x lies either between $-\sqrt{5}$ and $-\sqrt{3}$, or between $\sqrt{3}$ and $\sqrt{5}$, with all endpoints included.

Step 2: Determine all possible output values On the positive interval $[\sqrt{3}, \sqrt{5}]$, the expression $x^2 - 4$ increases continuously from -1 to 1. Thus every allowed input of inverse sine is attained. In particular,

$$f(\sqrt{3}) = \sin^{-1}(-1) = -\frac{\pi}{2}, \qquad f(\sqrt{5}) = \sin^{-1}(1) = \frac{\pi}{2}.$$

Continuity and strict increase of inverse sine show that every value between these endpoint outputs occurs. Its principal range rules out any output beyond them.

Step 3: Conclusion The domain and range are respectively

$$\boxed{[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}, \qquad \boxed{[-\pi/2, \pi/2]}.$$

Quick Tip to Solve: Solve $3 \le x^2 \le 5$ using $|x|$ so the negative interval is retained, then check that the inner expression covers all of $[-1, 1]$.

25(a). Check the differentiability of $f(x) = |\cos x|$ at $x = \dfrac{\pi}{2}$. [2 marks]

Solution:

Concept:

- A function is differentiable at $x = a$ only if its left-hand and right-hand derivatives exist, are finite and equal.
- The modulus satisfies $|u| = u$ when $u \ge 0$ and $|u| = -u$ when $u < 0$. Apply this using the sign of $\cos x$ near $\pi/2$.

Step 1: Write the local expressions on each side At $a = \pi/2$, we have $f(a) = 0$. Just to its left,

Quick Tip to Solve: At a zero inside a modulus, first check the sign on each side; differentiating $\cos x$ unchanged on both sides would miss the corner.

$\cos x > 0$; just to its right, $\cos x < 0$. Thus, in a neighbourhood of a,

$$f(x) = \begin{cases} \cos x, & x \le \pi/2, \\ -\cos x, & x > \pi/2. \end{cases}$$

Both expressions tend to zero at the joining point, so the function is continuous there. Continuity alone, however, does not settle differentiability.

Step 2: Compare the one-sided derivatives Using $\cos(\pi/2 + h) = -\sin h$, the derivative from the left is

$$f'_-(a) = \lim_{h \to 0^-} \frac{-\sin h}{h} = -1.$$

From the right the modulus changes the sign of the cosine, giving

$$f'_+(a) = \lim_{h \to 0^+} \frac{\sin h}{h} = 1.$$

Here $\lim_{h \to 0} \sin h/h = 1$ is used with angles measured in radians. The two finite one-sided limits disagree, so the two-sided derivative does not exist.

Step 3: Conclusion $\boxed{f(x) = |\cos x| \text{ is not differentiable at } x = \pi/2}$, because its left and right derivatives are -1 and 1, respectively.

Quick Tip to Solve: At a zero inside a modulus, first check the sign on each side; differentiating $\cos x$ unchanged on both sides would miss the corner.

OR

25(b). **If $y = A \sin 2x + B \cos 2x$ and $\dfrac{d^2y}{dx^2} - ky = 0$, find the value of k.** [2 marks]

Solution:

Concept:

- The chain rule gives $\dfrac{d}{dx}\sin(2x) = 2\cos(2x)$ and $\dfrac{d}{dx}\cos(2x) = -2\sin(2x)$.
- The constants A and B remain unchanged during differentiation. To find a coefficient in a differential equation, express the second derivative in terms of the original function.

Step 1: Differentiate twice using the chain rule Starting with $y = A \sin 2x + B \cos 2x$, differentiate each term, including the factor 2 from the inner angle:

$$\frac{dy}{dx} = 2A \cos 2x - 2B \sin 2x.$$

Differentiate once more. The derivative of cosine contributes a negative sign, whereas the already negative sine term stays negative after differentiation:

$$\frac{d^2y}{dx^2} = -4A \sin 2x - 4B \cos 2x.$$

Quick Tip to Solve: For a combination of $\sin 2x$ and $\cos 2x$, the second derivative is $-2^2 y$; match the minus sign in $y'' - ky = 0$ carefully.

Each second derivative contains the factor $2^2 = 4$ from the two applications of the chain rule.

Step 2: Compare with the given differential equation Factor the expression for the second derivative and recognise the original y:

$$\frac{d^2y}{dx^2} = -4(A\sin 2x + B\cos 2x) = -4y.$$

Thus the given family satisfies $y'' + 4y = 0$. Comparing this with the printed form $y'' - ky = 0$ gives $-k = 4$. This comparison is an identity for the family with arbitrary constants A, B, so no division by a possibly zero value of y is needed.

Step 3: Conclusion The required coefficient is $\boxed{k = -4}$, for which $y'' - ky = y'' + 4y = 0$.

Quick Tip to Solve: For a combination of $\sin 2x$ and $\cos 2x$, the second derivative is -2^2y; match the minus sign in $y'' - ky = 0$ carefully.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). Find the particular solution of the differential equation given by $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$; $y = 2$, when $x = 1$. [3 marks]

Solution:

Concept:

- A first-order equation is homogeneous when its right-hand side is a function of y/x. Put $y = vx$, where v is a function of x, so that $dy/dx = v + x\,dv/dx$.
- Separate the variables after substitution and integrate. The initial condition fixes the constant of integration and selects the interval containing the initial point.

Step 1: Express the equation in homogeneous form Near the initial point $x = 1$, we have $x \neq 0$. Rearranging the given equation gives

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{2x^2}.$$

Both terms depend only on the ratio y/x, so use $y = vx$. On differentiating and substituting,

$$v + x\frac{dv}{dx} = v + \frac{v^2}{2}, \qquad x\frac{dv}{dx} = \frac{v^2}{2}.$$

Step 2: Separate and integrate The given condition gives $v = 2$ at $x = 1$, so we seek the nonzero branch of v. Separating its variables yields

$$\frac{dv}{v^2} = \frac{dx}{2x}.$$

Quick Tip to Solve: After setting $y = vx$, cancel the v terms before separating; the initial ratio is $v(1) = 2$.

Integrating both sides and replacing v by y/x,

$$-\frac{1}{v} = \frac{1}{2}\ln|x| + C, \qquad -\frac{x}{y} = \frac{1}{2}\ln|x| + C.$$

The constant solution $y = 0$, omitted when dividing by v^2, cannot satisfy the specified initial value.

Step 3: Apply the initial condition Substituting $x = 1$ and $y = 2$ into the integrated relation gives $C = -1/2$. Thus the required branch obeys

$$-\frac{x}{y} = \frac{1}{2}\ln x - \frac{1}{2}.$$

Here $x > 0$ because the interval contains 1. Its denominator must remain nonzero, and the nearest positive singular point is $x = e$.

Step 4: Conclusion The particular solution is

$$\boxed{y = \frac{2x}{1 - \ln x}}, \qquad 0 < x < e.$$

At $x = 1$, this gives $y = 2$, as required.

Quick Tip to Solve: After setting $y = vx$, cancel the v terms before separating; the initial ratio is $v(1) = 2$.

OR

26(b). Find the general solution of the differential equation: $y\,dx = (x + 2y^2)\,dy$. [3 marks]

Solution:

Concept:

- Treat x as the dependent variable and y as the independent variable. A linear equation then has the form $dx/dy + P(y)x = Q(y)$.
- Its integrating factor is $I = e^{\int P(y)\,dy}$. Multiplication by I makes the left-hand side the derivative $d(Ix)/dy$.

Step 1: Write the equation as a linear equation in x From $y\,dx = (x + 2y^2)\,dy$, divide by dy and then by y, on an interval where $y \neq 0$:

$$\frac{dx}{dy} - \frac{x}{y} = 2y.$$

Comparing with the linear form gives $P(y) = -1/y$ and $Q(y) = 2y$. This choice avoids treating the equation as a nonlinear equation for y.

Step 2: Use the integrating factor The exponential formula gives $e^{-\ln|y|} = 1/|y|$. A nonzero constant multiple is equally valid on each fixed-sign interval, so take $I = 1/y$. Multiplying the equation by it gives

$$\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2.$$

Quick Tip to Solve: Read $y\,dx = (x + 2y^2)\,dy$ as a linear equation for $x(y)$; multiplying by $1/y$ produces $d(x/y)/dy$.

By the quotient rule, the complete left-hand side is $d(x/y)/dy$. Hence integration with respect to y yields

$$\frac{x}{y} = 2y + C,$$

where C is an arbitrary real constant.

Step 3: Check the integrated family Multiplying the integrated relation by y gives a polynomial expression for x. Its derivative is $dx/dy = 4y + C$. Substitution in the original differential equation gives

$$y\frac{dx}{dy} = 4y^2 + Cy, \qquad x + 2y^2 = 4y^2 + Cy.$$

Thus both sides agree. The polynomial family also extends through $y = 0$, although the integrating-factor calculation used $y \neq 0$.

Step 4: Conclusion The general solution, with x expressed as a function of y, is

$$\boxed{x = 2y^2 + Cy}, \qquad C \in \mathbb{R}.$$

Quick Tip to Solve: Read $y\,dx = (x + 2y^2)\,dy$ as a linear equation for $x(y)$; multiplying by $1/y$ produces $d(x/y)/dy$.

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27. If vectors $\vec{a}$, $\vec{b}$ and $2\vec{a} + 3\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$. [3 marks]

Solution:

Concept:

- A unit vector has magnitude 1. For any vector $\vec{u}$, its squared magnitude equals $\vec{u} \cdot \vec{u}$.
- The dot product satisfies $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where $\theta \in [0, \pi]$ is the angle between the two nonzero vectors.

Step 1: Use the unit-vector conditions Since all three given vectors are unit vectors, we have

$$|\vec{a}|^2 = 1, \qquad |\vec{b}|^2 = 1, \qquad |2\vec{a} + 3\vec{b}|^2 = 1.$$

Square the magnitude of the linear combination by taking its dot product with itself. This retains the information about the angle between the vectors.

Step 2: Expand the dot product Using distributivity and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,

$$\begin{aligned}(2\vec{a} + 3\vec{b}) \cdot (2\vec{a} + 3\vec{b}) &= 4|\vec{a}|^2 + 12\vec{a} \cdot \vec{b} + 9|\vec{b}|^2, \\ 1 &= 4 + 12\vec{a} \cdot \vec{b} + 9, \\ 12\vec{a} \cdot \vec{b} &= -12.\end{aligned}$$

The cross term is $12\vec{a} \cdot \vec{b}$, because the two mixed terms each contribute $6\vec{a} \cdot \vec{b}$.

Step 3: Relate the dot product to the angle The calculation gives $\vec{a} \cdot \vec{b} = -1$. As both magnitudes are

Quick Tip to Solve: Square $|2\vec{a} + 3\vec{b}| = 1$ and keep the full mixed term $2(2)(3)\vec{a} \cdot \vec{b}$; it immediately determines $\cos\theta$.

1,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = -1.$$

This is consistent with oppositely directed unit vectors: if $\vec{b} = -\vec{a}$, then $2\vec{a} + 3\vec{b} = -\vec{a}$, which indeed has unit magnitude.

Step 4: Conclusion The angle between $\vec{a}$ and $\vec{b}$ is

$$\boxed{\theta = \pi \text{ radians} = 180^\circ}.$$

Quick Tip to Solve: Square $|2\vec{a} + 3\vec{b}| = 1$ and keep the full mixed term $2(2)(3)\vec{a}\cdot\vec{b}$; it immediately determines $\cos\theta$.

28. A pair of dice is thrown simultaneously. If X denotes the absolute difference of the numbers appearing on top of the dice, then find the probability distribution of X. [3 marks]

Solution:

Concept:

- For two fair dice, an outcome is an ordered pair (i, j), with $1 \le i, j \le 6$. The 36 ordered pairs are equally likely.
- A probability distribution lists each possible value k of a random variable and its probability. Here $X = |i - j|$, so $P(X = k)$ is the number of ordered pairs with absolute difference k, divided by 36.

Step 1: Identify the possible differences The smallest absolute difference is 0, obtained when both dice show the same number. The largest is 5, obtained from $(1, 6)$ or $(6, 1)$. Every integer between these extremes occurs, so

$$X \in \{0, 1, 2, 3, 4, 5\}.$$

Negative values cannot occur because the difference is taken in absolute value.

Step 2: Count the outcomes for each difference For $X = 0$, the outcomes are $(1, 1), (2, 2), \ldots, (6, 6)$, giving six outcomes. For $k \ge 1$, the pairs with the second number larger are

$$(1, 1 + k), (2, 2 + k), \ldots, (6 - k, 6).$$

There are $6 - k$ such pairs. Reversing their order gives another $6 - k$ distinct outcomes, so the count for a positive difference is $2(6 - k)$. Therefore the counts for $k = 1, 2, 3, 4, 5$ are respectively $10, 8, 6, 4, 2$.

Step 3: Convert the counts into probabilities Divide each count by the total of 36 equally likely outcomes. The counts cover the sample space because

$$6 + 10 + 8 + 6 + 4 + 2 = 36.$$

Accordingly, the resulting probabilities are nonnegative and their sum is $36/36 = 1$. Note that the six possible differences themselves are not equally likely: for example, adjacent numbers occur more

Quick Tip to Solve: Count zero difference separately; for a positive difference k, use $2(6 - k)$ ordered pairs and divide by 36.

often than numbers differing by five.

Step 4: Conclusion The required probability distribution is

k	0	1	2	3	4	5
$P(X=k)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

For all other values of k, $P(X=k)=0$.

Quick Tip to Solve: Count zero difference separately; for a positive difference k, use $2(6-k)$ ordered pairs and divide by 36.

29. Find:

$$\int x^2 \cdot \sin^{-1}\left(x^{3/2}\right)\, dx$$

[3 marks]

Solution:

Concept:

- Substitution converts the fractional power inside the inverse sine to a single variable. For $t = x^{3/2}$, we have $x^3 = t^2$, so $3x^2\,dx = 2t\,dt$.
- Integration by parts uses $\int u\,dv = uv - \int v\,du$. Choose the inverse trigonometric function as u, since $\frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-t^2}}$.
- The identity $t^2 = 1-(1-t^2)$ separates the remaining integral into standard forms.

Step 1: Substitute for the fractional power Let $I = \int x^2 \sin^{-1}(x^{3/2})\,dx$ and set $t = x^{3/2}$. Then

$$x^2\,dx = \frac{2t}{3}\,dt, \qquad I = \frac{2}{3}\int t\sin^{-1} t\,dt.$$

For real values, the original integrand has $0 \le x \le 1$, and hence $0 \le t \le 1$.

Step 2: Apply integration by parts Take $u = \sin^{-1} t$ and $dv = t\,dt$, so that $du = dt/\sqrt{1-t^2}$ and $v = t^2/2$. Therefore,

$$I = \frac{t^2}{3}\sin^{-1} t - \frac{1}{3}J, \qquad J = \int \frac{t^2}{\sqrt{1-t^2}}\,dt.$$

Step 3: Evaluate the remaining standard integral Using $t^2 = 1-(1-t^2)$, write

$$J = \int \frac{dt}{\sqrt{1-t^2}} - \int \sqrt{1-t^2}\,dt.$$

To evaluate the second integral, put $t = \sin\theta$, with $0 < \theta < \pi/2$. Then $dt = \cos\theta\,d\theta$ and the square

Quick Tip to Solve: Start with $t = x^{3/2}$: differentiating $x^3 = t^2$ immediately turns $x^2\,dx$ into $\frac{2}{3}t\,dt$, making integration by parts simple.

root is $\cos\theta$. Hence

$$\begin{aligned}\int \sqrt{1-t^2}\,dt &= \int \cos^2\theta\,d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} \\ &= \frac{1}{2}\sin^{-1}t + \frac{t}{2}\sqrt{1-t^2}.\end{aligned}$$

Thus $J = \frac{1}{2}\sin^{-1}t - \frac{t}{2}\sqrt{1-t^2}$, with its integration constant absorbed into the final constant.

Step 4: Conclusion Substituting J and then $t = x^{3/2}$, we obtain

$$\boxed{\int x^2\sin^{-1}(x^{3/2})\,dx = \frac{2x^3-1}{6}\sin^{-1}(x^{3/2}) + \frac{x^{3/2}\sqrt{1-x^3}}{6} + C.}$$

Here C is an arbitrary constant.

Quick Tip to Solve: Start with $t = x^{3/2}$: differentiating $x^3 = t^2$ immediately turns $x^2\,dx$ into $\frac{2}{3}t\,dt$, making integration by parts simple.

30(a). **If $x^{30}y^{20} = (x+y)^{50}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.** [3 marks]

Solution:

Concept:

- Logarithmic differentiation converts powers into multiples: $\ln|u^n| = n\ln|u|$, wherever $u \neq 0$.
- For a differentiable function $y = y(x)$, the chain rule gives $\dfrac{d}{dx}\ln|y| = \dfrac{y'}{y}$, where $y' = \dfrac{dy}{dx}$.

Step 1: Take logarithms of the given relation Work at a point with $x \neq 0$, as required for the expression y/x. The original equation then forces both $y \neq 0$ and $x + y \neq 0$. Taking logarithms of absolute values therefore gives

$$30\ln|x| + 20\ln|y| = 50\ln|x+y|.$$

Absolute values allow the same differentiation argument for negative as well as positive nonzero quantities.

Step 2: Differentiate and collect the derivative terms Differentiating both sides with respect to x, and applying the chain rule to y and $x + y$, gives

$$\frac{30}{x} + \frac{20y'}{y} = \frac{50(1+y')}{x+y}.$$

Quick Tip to Solve: Take logarithms first to reduce the powers 30, 20 and 50, then factor the collected terms as $x(2x-3y)$ and $y(2x-3y)$.

Multiply by $xy(x+y)$, divide by 10, and rearrange:

$$3y(x+y)+2x(x+y)y' = 5xy(1+y'),$$
$$(2x^2-3xy)y' = 2xy-3y^2,$$
$$x(2x-3y)y' = y(2x-3y).$$

Step 3: Check the factor before cancellation The factor $2x-3y$ cannot vanish at a nonzero point of the curve. Otherwise $y=2x/3$, and substitution into the original equation, followed by division by x^{50}, would require

$$\left(\frac{2}{3}\right)^{20} = \left(\frac{5}{3}\right)^{50}.$$

This is impossible because the left side is less than 1, whereas the right side is greater than 1. Thus the common factor may be cancelled.

Step 4: Conclusion Cancelling $2x-3y$ and dividing by x yields $\boxed{\frac{dy}{dx}=\frac{y}{x}}$, at every differentiable point of the curve with $x \neq 0$, as required.

Quick Tip to Solve: Take logarithms first to reduce the powers 30, 20 and 50, then factor the collected terms as $x(2x-3y)$ and $y(2x-3y)$.

OR

30(b). Find $\frac{dy}{dx}$, if $5^x+5^y=5^{x+y}$. [3 marks]

Solution:

Concept:

- For a positive constant $a \neq 1$, $\frac{d}{dx}a^{u(x)} = a^{u(x)}\ln a\, u'(x)$. The derivative of the exponent is required by the chain rule.
- In implicit differentiation, treat y as a function of x. Hence $\frac{d}{dx}(x+y) = 1+y'$, with $y' = \frac{dy}{dx}$.

Step 1: Differentiate the exponential relation Starting from $5^x+5^y=5^{x+y}$, differentiate both sides with respect to x:

$$5^x\ln 5 + 5^y \ln 5\, y' = 5^{x+y}\ln 5\,(1+y').$$

The exponent on the right depends on both x and y; its derivative is therefore $1+y'$. Since $\ln 5 \neq 0$, divide through by this common factor.

Step 2: Collect the terms containing the derivative Expand the right side, then move all terms containing y' to the left:

$$5^x + 5^y y' = 5^{x+y} + 5^{x+y}y',$$
$$(5^y - 5^{x+y})y' = 5^{x+y} - 5^x.$$

Use the given relation itself to simplify both coefficients:

$$5^y - 5^{x+y} = -5^x, \qquad 5^{x+y} - 5^x = 5^y.$$

Quick Tip to Solve: After differentiating, replace 5^{x+y} by 5^x+5^y; both coefficients then simplify immediately.

Thus the derivative equation becomes $-5^x y' = 5^y$. The coefficient 5^x is always positive, so division is valid.

Step 3: Express the result using only x if desired Factoring the original relation gives

$$5^y(5^x - 1) = 5^x, \qquad \frac{5^y}{5^x} = \frac{1}{5^x - 1}.$$

Its positive left factor 5^y shows that $5^x - 1 > 0$, so $x > 0$. The simplified denominator cannot vanish on the real curve. Both forms of the derivative are therefore equivalent.

Step 4: Conclusion

$$\boxed{\frac{dy}{dx} = -5^{y-x} = -\frac{1}{5^x - 1}.}$$

Quick Tip to Solve: After differentiating, replace 5^{x+y} by $5^x + 5^y$; both coefficients then simplify immediately.

31(a). Evaluate: $\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\, dx$. [3 marks]

Solution:

Concept:

- On symmetric limits, substituting $x = -t$ gives $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{a} f(-x)\, dx$, whenever the integrals converge.
- For $-2 < x < 2$, both $2 - x$ and $2 + x$ are positive, so radicals can be combined using the positive denominator $\sqrt{4 - x^2}$.
- The standard primitive is $\int dx/\sqrt{a^2 - x^2} = \sin^{-1}(x/a) + C$ for $a > 0$ and $|x| < a$.

Step 1: Reflect the integrand about the origin Put $I = \int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\, dx$. The lower endpoint is singular, but the integrand behaves like $2/\sqrt{x+2}$ there, whose integral is finite. Thus the given improper integral converges. With $x = -t$, we have $dx = -dt$; reversing the transformed limits gives

$$I = \int_{-2}^{2} \sqrt{\frac{2+t}{2-t}}\, dt = \int_{-2}^{2} \sqrt{\frac{2+x}{2-x}}\, dx.$$

The letter of integration is a dummy variable, so it may be renamed as x in the last expression.

Step 2: Add the two representations and integrate For interior points of the interval,

$$\sqrt{\frac{2-x}{2+x}} + \sqrt{\frac{2+x}{2-x}} = \frac{(2-x) + (2+x)}{\sqrt{(2-x)(2+x)}} = \frac{4}{\sqrt{4-x^2}}.$$

Both reflected endpoint singularities are integrable, so adding the integrals is valid. Therefore

$$2I = 4\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}} = 4\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\pi.$$

Quick Tip to Solve: For a square-root ratio on symmetric limits, add its reflected form first: the numerators here add to the constant 4.

The inverse-sine values at the endpoints are understood as their one-sided limits.

Step 3: Conclusion Dividing by 2, the required integral is $\boxed{2\pi}$.

Quick Tip to Solve: For a square-root ratio on symmetric limits, add its reflected form first: the numerators here add to the constant 4.

OR

31(b). Find: $\int \dfrac{1}{x\left[(\log x)^2 - 3\log x - 4\right]}\,dx$. [3 marks]

Solution:

Concept:

- A logarithm accompanied by dx/x suggests $t = \log x$, since $dt = dx/x$. Here $\log$ denotes the natural logarithm and $x > 0$.
- Factor a quadratic denominator into linear factors before resolving the rational function into partial fractions.
- The primitive $\int dt/(t-a) = \log|t-a| + C$ uses an absolute value and applies on any interval avoiding $t = a$.

Step 1: Replace the logarithm by a single variable Let

$$I = \int \frac{dx}{x[(\log x)^2 - 3\log x - 4]}.$$

Set $t = \log x$, so that $dt = dx/x$. The entire factor $1/x$ is absorbed into the differential, and the remaining denominator is a polynomial in t. Hence

$$I = \int \frac{dt}{t^2 - 3t - 4} = \int \frac{dt}{(t-4)(t+1)}.$$

The factors follow because -4 and 1 have product -4 and sum -3.

Step 2: Resolve into partial fractions Write

$$\frac{1}{(t-4)(t+1)} = \frac{A}{t-4} + \frac{B}{t+1}.$$

Multiplying by the common denominator gives $1 = A(t+1) + B(t-4)$. Substituting $t = 4$ yields $A = 1/5$, and substituting $t = -1$ yields $B = -1/5$. Therefore

$$I = \frac{1}{5}\int \left(\frac{1}{t-4} - \frac{1}{t+1}\right) dt = \frac{1}{5}\log|t-4| - \frac{1}{5}\log|t+1| + C.$$

Step 3: Conclusion Restoring $t = \log x$, the required indefinite integral is

$$\boxed{\frac{1}{5}\log\left|\frac{\log x - 4}{\log x + 1}\right| + C}.$$

Quick Tip to Solve: Absorb dx/x with $t = \log x$ before factoring; the two roots 4 and -1 immediately give the coefficient $1/5$.

This holds on intervals where $x > 0$, excluding $x = e^{-1}$ and $x = e^4$.

Quick Tip to Solve: Absorb dx/x with $t = \log x$ before factoring; the two roots 4 and -1 immediately give the coefficient $1/5$.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32. Find the value of p for which the lines $\vec{r} = \lambda\hat{i} + (2\lambda + 1)\hat{j} + (3\lambda + 2)\hat{k}$ and $\vec{r} = \hat{i} - 3\mu\hat{j} + (\mu p + 7)\hat{k}$ are perpendicular to each other and also intersect. Also, find the point of intersection of the given lines. [5 marks]

Solution:

Concept:

- A vector line $\vec{r} = \vec{a} + t\vec{b}$ passes through the point with position vector $\vec{a}$ and has direction vector $\vec{b}$, whose components are the coefficients of its parameter.
- Two nonzero direction vectors are perpendicular exactly when their scalar product is zero: $\vec{b}_1 \cdot \vec{b}_2 = 0$.
- Lines in three-dimensional space intersect only if the same pair of parameter values makes all three Cartesian coordinates equal. Perpendicular directions alone do not establish intersection.

Step 1: Identify the direction vectors Separate the constant terms from the parameter terms in each given line:

$$L_1: \quad \vec{r} = (\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}),$$
$$L_2: \quad \vec{r} = (\hat{i} + 7\hat{k}) + \mu(-3\hat{j} + p\hat{k}).$$

Thus direction vectors are $\vec{b}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b}_2 = -3\hat{j} + p\hat{k}$. The coefficient of $\hat{i}$ in the second direction vector is zero: the constant $\hat{i}$ specifies its fixed x-coordinate, not a direction component. Both direction vectors are nonzero for every value of p.

Step 2: Impose the perpendicularity condition Taking the scalar product gives

$$\vec{b}_1 \cdot \vec{b}_2 = (1)(0) + (2)(-3) + (3)p = -6 + 3p.$$

Perpendicular directions require $-6 + 3p = 0$, so the only candidate is $p = 2$. This computation determines the angle between the directions. We must still check that the two lines actually meet for this value; otherwise the directions could be orthogonal while the lines remain skew.

Step 3: Solve the coordinate equations for intersection At a common point, the Cartesian forms of the lines must satisfy

$$\lambda = 1, \qquad 2\lambda + 1 = -3\mu, \qquad 3\lambda + 2 = p\mu + 7.$$

The first equation gives $\lambda = 1$. Substituting it into the second yields $3 = -3\mu$, hence $\mu = -1$. For $p = 2$, the third equation becomes

$$3(1) + 2 = 2(-1) + 7, \qquad 5 = 5.$$

Quick Tip to Solve: Read directions from parameter coefficients, then use the second line's fixed $x = 1$ to obtain λ immediately and verify intersection.

Thus the same parameter values satisfy every coordinate equation, establishing intersection.

Step 4: Calculate and verify the common point On L_1, putting $\lambda = 1$ gives

$$\vec{r} = \hat{i} + 3\hat{j} + 5\hat{k}.$$

On L_2, putting $p = 2$ and $\mu = -1$ gives the same vector, $\vec{r} = \hat{i} - 3(-1)\hat{j} + [2(-1) + 7]\hat{k} = \hat{i} + 3\hat{j} + 5\hat{k}$. This confirms the common point on both given lines.

Step 5: Conclusion The required value is $\boxed{p = 2}$, and the lines intersect at $\boxed{(1, 3, 5)}$.

Quick Tip to Solve: Read directions from parameter coefficients, then use the second line's fixed $x = 1$ to obtain λ immediately and verify intersection.

33(a). **If** $A = \begin{pmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ **and** $B^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$**, then find** $(AB)^{-1}$**. Also, find** $|(AB)^{-1}|$**.** [5 marks]

Solution:

Concept:

- For a square matrix A with nonzero determinant, $A^{-1} = \text{adj}\, A/|A|$. The adjugate is the transpose of the matrix of cofactors.
- For invertible matrices, $(AB)^{-1} = B^{-1}A^{-1}$: the order reverses. Also, $|UV| = |U||V|$ and $|A^{-1}| = 1/|A|$.

Step 1: Check that the matrix is invertible Expanding the determinant of A along the first row gives

$$|A| = 5(3 \cdot 1 - 2 \cdot 2) + 4(2 \cdot 2 - 3 \cdot 1) = -5 + 4 = -1.$$

Since this determinant is nonzero, the inverse exists and we can use the adjugate formula.

Step 2: Calculate the cofactors and inverse A cofactor is the corresponding minor multiplied by $(-1)^{i+j}$. For example,

$$C_{12} = -(2 \cdot 1 - 2 \cdot 1) = 0, \qquad C_{21} = -(0 \cdot 1 - 4 \cdot 2) = 8.$$

Evaluating all nine cofactors yields the cofactor matrix

$$C = \begin{pmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{pmatrix}.$$

Quick Tip to Solve: Since B^{-1} is supplied, compute only A^{-1}, then multiply $B^{-1}A^{-1}$. Use determinant multiplication to avoid expanding the final matrix.

Transpose this matrix and divide by $|A| = -1$:

$$\operatorname{adj} A = \begin{pmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{pmatrix}.$$

Step 3: Multiply in the reversed order The given matrix is already B^{-1}, so we do not need to find B. Applying the product-inverse rule,

$$(AB)^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{pmatrix}.$$

Each entry is a row-column scalar product. For instance, the first row entries are

$$1 + 0 - 3 = -2, \quad -8 - 3 + 30 = 19, \quad 12 + 6 - 45 = -27.$$

Continuing for the other two rows gives

$$(AB)^{-1} = \begin{pmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{pmatrix}.$$

Step 4: Evaluate the determinant of the inverse product Subtract the first row from each of the second and third rows of B^{-1}. These operations do not change its determinant:

$$|B^{-1}| = \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Therefore the multiplicative property gives

$$|(AB)^{-1}| = |B^{-1}|\,|A^{-1}| = 1\left(\frac{1}{-1}\right) = -1.$$

Step 5: Conclusion The required inverse and its determinant are

$$\boxed{(AB)^{-1} = \begin{pmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{pmatrix}}, \qquad \boxed{|(AB)^{-1}| = -1}.$$

Quick Tip to Solve: Since B^{-1} is supplied, compute only A^{-1}, then multiply $B^{-1}A^{-1}$. Use determinant multiplication to avoid expanding the final matrix.

OR

33(b). Given $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, find A^{-1}. Use it to solve the following system of equations: $x + y + z = 1,\ 2x + 3y + 2z = 2,\ x + y + 2z = 4$. [5 marks]

Solution:

Concept:

- A square matrix A has an inverse when $|A| \neq 0$, and then $A^{-1} = \operatorname{adj} A/|A|$. The adjugate is the transpose of the cofactor matrix.
- A system with coefficient matrix A, unknown column X, and constant column D is $AX = D$. Multiplying on the left by A^{-1} gives the unique solution $X = A^{-1}D$.

Step 1: Evaluate the determinant Expand along the first row, retaining the alternating signs:

$$\begin{aligned} |A| &= 1(3 \cdot 2 - 2 \cdot 1) - 1(2 \cdot 2 - 2 \cdot 1) + 1(2 \cdot 1 - 3 \cdot 1) \\ &= 4 - 2 - 1 = 1. \end{aligned}$$

Thus A is nonsingular and the inverse-matrix method applies to the given equations.

Step 2: Form the cofactor matrix and its transpose The cofactor C_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j. Here the first-row cofactors are $4, -2, -1$. The remaining cofactors are

$$\begin{aligned} &C_{21} = -(1 \cdot 2 - 1 \cdot 1) = -1, \quad C_{22} = 1 \cdot 2 - 1 \cdot 1 = 1, \quad C_{23} = -(1 \cdot 1 - 1 \cdot 1) = 0, \\ &C_{31} = 1 \cdot 2 - 1 \cdot 3 = -1, \quad C_{32} = -(1 \cdot 2 - 1 \cdot 2) = 0, \quad C_{33} = 1 \cdot 3 - 1 \cdot 2 = 1. \end{aligned}$$

Consequently,

$$C = \begin{pmatrix} 4 & -2 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \operatorname{adj} A = C^T = \begin{pmatrix} 4 & -1 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

As $|A| = 1$, division by the determinant leaves the adjugate unchanged:

$$A^{-1} = \begin{pmatrix} 4 & -1 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Step 3: Express and solve the system in matrix form The coefficients in the three equations occur in exactly the row order of A. Therefore

$$AX = D, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad D = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before using it as the adjugate. Keep the constants in the same order as the equation rows when computing $A^{-1}(1, 2, 4)^T$.

Multiply both sides on the left by the inverse just found:

$$X = A^{-1}D = \begin{pmatrix} 4 & -1 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4-2-4 \\ -2+2+0 \\ -1+0+4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

Step 4: Verify the values in all equations Substitution provides a direct check on both the matrix multiplication and the ordering of constants:

$$\begin{aligned} x + y + z &= -2 + 0 + 3 = 1, \\ 2x + 3y + 2z &= -4 + 0 + 6 = 2, \\ x + y + 2z &= -2 + 0 + 6 = 4. \end{aligned}$$

All three right-hand sides agree with the given system.

Step 5: Conclusion The inverse and the unique solution are

$$\boxed{A^{-1} = \begin{pmatrix} 4 & -1 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}, \qquad \boxed{x = -2,\ y = 0,\ z = 3}.$$

Quick Tip to Solve: Transpose the cofactor matrix before using it as the adjugate. Keep the constants in the same order as the equation rows when computing $A^{-1}(1, 2, 4)^T$.

34. If A_1 denotes the area of region bounded by $y^2 = 4x$, $x = 1$ and x-axis in the first quadrant and A_2 denotes the area of region bounded by $y^2 = 4x$, $x = 4$, find $A_1 : A_2$. [5 marks]

Solution:

Concept:

- The area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$, from $x = a$ to $x = b$, is $\int_a^b [u(x) - l(x)]\, dx$.
- The parabola $y^2 = 4x$ has branches $y = 2\sqrt{x}$ and $y = -2\sqrt{x}$, for $x \geq 0$. Its enclosed regions are symmetric about the x-axis when both branches form the boundary.
- The power rule gives $\int x^{1/2}\, dx = \frac{2}{3}x^{3/2} + C$. Bounds must be chosen from the actual boundaries of each region.

Step 1: Identify the first region and its limits For A_1, the question explicitly restricts the region to the first quadrant and includes the x-axis as a boundary. Thus its upper boundary is $y = 2\sqrt{x}$ and its lower boundary is $y = 0$. The parabola meets the x-axis at the origin, while the vertical boundary is $x = 1$. Therefore the relevant interval is $0 \leq x \leq 1$; at its right endpoint the upper curve has height 2.

Quick Tip to Solve: Apply the first-quadrant restriction only to A_1. The region defining A_2 includes both branches, so double its upper-half area before taking the ratio.

Step 2: Integrate to find the first area A vertical strip has height $2\sqrt{x}$ and width dx. Hence

$$A_1 = \int_0^1 2\sqrt{x}\,dx$$
$$= \frac{4}{3}\left[x^{3/2}\right]_0^1 = \frac{4}{3} \quad \text{square units.}$$

Step 3: Identify the complete second region For A_2, only the parabola and $x = 4$ are specified as boundaries; no first-quadrant restriction or x-axis boundary is given for this region. Substituting $x = 4$ gives $y^2 = 16$, so the line meets the parabola at $(4, 4)$ and $(4, -4)$. The enclosed region therefore includes the portions above and below the x-axis. For $0 \le x \le 4$, the height of a vertical strip is

$$2\sqrt{x} - (-2\sqrt{x}) = 4\sqrt{x}.$$

Step 4: Evaluate the second area and compare Integrating this full height gives

$$A_2 = \int_0^4 4\sqrt{x}\,dx$$
$$= \frac{8}{3}\left[x^{3/2}\right]_0^4 = \frac{8}{3}\cdot 8 = \frac{64}{3} \quad \text{square units.}$$

As a check, horizontal strips give the same area: $\int_{-4}^{4}(4 - y^2/4)\,dy = 64/3$. Both areas use the same square units, so their ratio is dimensionless.

Step 5: Conclusion The areas are $A_1 = \frac{4}{3}$ and $A_2 = \frac{64}{3}$ square units. Therefore

$$\boxed{A_1 : A_2 = \frac{4}{3} : \frac{64}{3} = 1 : 16}.$$

Quick Tip to Solve: Apply the first-quadrant restriction only to A_1. The region defining A_2 includes both branches, so double its upper-half area before taking the ratio.

35(a). Show that a function $f : R \to R$ defined by $f(x) = \frac{2x}{1+x^2}$ is neither one-one nor onto. Further, find set A so that the given function $f : R \to A$ becomes an onto function. [5 marks]

Solution:

Concept:

- A function is one-one when equal outputs imply equal inputs. Two distinct inputs with the same image disprove this property.
- A function is onto its codomain when every element of that codomain has a preimage. Thus an onto codomain must equal the range.
- Since $1 + x^2 > 0$ for real x, nonnegative squares $(x-1)^2$ and $(x+1)^2$ give bounds for $2x/(1+x^2)$.

Step 1: Disprove the one-one property The denominator never vanishes, so the function is defined on

Quick Tip to Solve: Use reciprocal inputs 2 and $1/2$ to disprove one-one behavior. For the onto codomain, establish the bounds and prove that every value between them has a preimage.

all of $\mathbb{R}$. Choose the distinct inputs 2 and $\frac{1}{2}$:

$$f(2) = \frac{4}{1+4} = \frac{4}{5}, \qquad f\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}.$$

Since $2 \neq \frac{1}{2}$ but their images agree, the function is not one-one.

Step 2: Bound the outputs and disprove surjectivity onto the reals For every real x,

$$1 - f(x) = \frac{1+x^2-2x}{1+x^2} = \frac{(x-1)^2}{1+x^2} \geq 0,$$
$$1 + f(x) = \frac{1+x^2+2x}{1+x^2} = \frac{(x+1)^2}{1+x^2} \geq 0.$$

Thus $-1 \leq f(x) \leq 1$. In particular, $2 \in \mathbb{R}$ has no preimage, so the given function with codomain $\mathbb{R}$ is not onto. We still need to prove that every value between the bounds is attained.

Step 3: Establish the exact range and choose the codomain Let $y \in [-1, 1]$. Solving for a preimage gives

$$\frac{2x}{1+x^2} = y \quad \Longleftrightarrow \quad yx^2 - 2x + y = 0.$$

If $y = 0$, take $x = 0$. If $y \neq 0$, this quadratic has discriminant

$$\Delta = 4 - 4y^2 = 4(1-y^2) \geq 0,$$

so it has a real root $x = (1 \pm \sqrt{1-y^2})/y$. Since the denominator $1 + x^2$ is always positive, this root also satisfies the original equation; no extraneous root was introduced. The endpoints are attained explicitly by $f(-1) = -1$ and $f(1) = 1$.

Consequently the range is exactly $[-1, 1]$. A codomain must contain every output, and surjectivity requires every codomain element to be an output. Therefore choosing $A = [-1, 1]$ makes $f : \mathbb{R} \to A$ onto. This codomain restriction does not change the repeated images found above.

Step 4: Conclusion The given function $f : \mathbb{R} \to \mathbb{R}$ is neither one-one nor onto. The required set for $f : \mathbb{R} \to A$ to be onto is $\boxed{A = [-1, 1]}$.

Quick Tip to Solve: Use reciprocal inputs 2 and $1/2$ to disprove one-one behavior. For the onto codomain, establish the bounds and prove that every value between them has a preimage.

OR

35(b). A relation R is defined on $N \times N$ (where N is the set of natural numbers) as: $(a, b)\, R\, (c, d) \iff a - c = b - d$. Show that R is an equivalence relation. [5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation if it is reflexive, symmetric and transitive. Here the elements of the underlying set $N \times N$ are ordered pairs, so each property must be checked for ordered pairs, not for individual natural numbers.
- The defining condition is $(a, b)R(c, d) \iff a - c = b - d$. Equivalently, $a - b = c - d$: two ordered pairs are related precisely when their first coordinate minus their second coordinate has the same value.

Step 1: Verify reflexivity Let (a, b) be any element of $N \times N$. To show that it is related to itself, substitute $(c, d) = (a, b)$ into the defining condition. Then

$$a - a = 0 = b - b.$$

Hence $(a, b)R(a, b)$ for every ordered pair $(a, b) \in N \times N$. Thus R is reflexive. This does not require $a = b$, since each coordinate is subtracted from itself.

Step 2: Verify symmetry Let $(a, b), (c, d) \in N \times N$ and suppose $(a, b)R(c, d)$. By definition,

$$a - c = b - d.$$

Multiplying both sides by -1 gives

$$c - a = d - b.$$

This is exactly the condition for $(c, d)R(a, b)$. Thus whenever the first pair is related to the second, the second is related to the first, proving symmetry.

Step 3: Verify transitivity Take arbitrary $(a, b), (c, d), (e, f) \in N \times N$. Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. The two hypotheses give

$$a - c = b - d, \qquad c - e = d - f.$$

Add the equalities so that the coordinates of the middle pair cancel:

$$(a - c) + (c - e) = (b - d) + (d - f),$$
$$a - e = b - f.$$

Therefore $(a, b)R(e, f)$, as required for transitivity. Although coordinates belong to N, their differences may be integers; this causes no difficulty because the relation only tests equality of those differences.

Step 4: Conclusion All three properties hold on $N \times N$. Therefore R is an equivalence relation.

Quick Tip to Solve: Rewrite the condition as $a - b = c - d$: equality of coordinate differences suggests all three equivalence properties immediately.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. The month of September is celebrated as the Rashtriya Poshan Maah across the country. Following a healthy and well-balanced diet is crucial in order to supply the body with the proper nutrients it needs. A balanced diet also keeps us mentally fit and promotes improved level of energy.

A dietician wishes to minimize the cost of a diet involving two types of foods, food X (x kg) and food Y (y kg) which are available at the rate of Rs. 16/kg and Rs. 20/kg respectively. The feasible region satisfying the constraints is shown in Figure-2.

On the basis of the above information, answer the following questions: [4 marks]

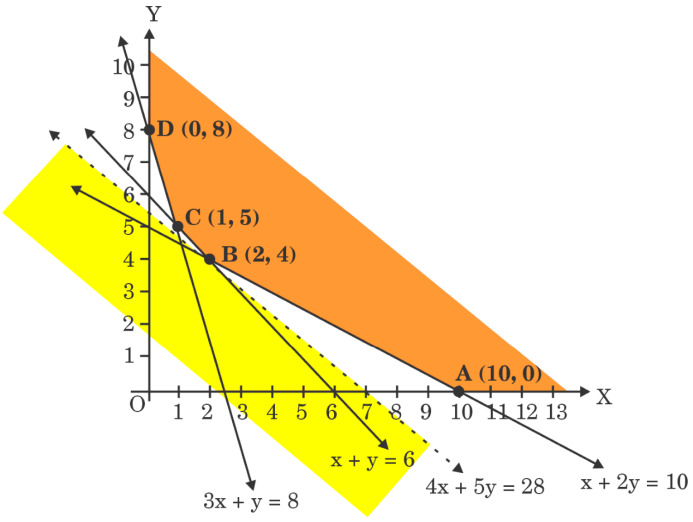


Figure-2

36(i). Identify and write all the constraints which determine the given feasible region in Figure-2. [2 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes of its constraints; solid boundaries are included.
- The masses of either food cannot be negative, so $x \geq 0$ and $y \geq 0$, where x, y are masses in kilograms.

Step 1: Read the boundary lines The three boundary segments DC, CB and BA lie respectively on $3x + y = 8$, $x + y = 6$ and $x + 2y = 10$. The dashed line $4x + 5y = 28$ is an objective level line, not a boundary edge.

Step 2: Choose the feasible half-planes The feasible region is above and to the right of $DCBA$, opposite the origin for each line. The origin makes all three left sides zero, so the feasible inequalities use $\geq$. Non-negativity confines the region to the first quadrant.

Quick Tip to Solve: Test the origin against each boundary line: it is outside each feasible half-plane, so all three inequalities use $\geq$.

Step 3: Conclusion The constraints are

$$\boxed{3x + y \geq 8, \quad x + y \geq 6, \quad x + 2y \geq 10, \quad x \geq 0, \quad y \geq 0.}$$

Quick Tip to Solve: Test the origin against each boundary line: it is outside each feasible half-plane, so all three inequalities use $\geq$.

36(ii). If the objective is to minimize cost Z = 16x + 20y, find the values of x and y at which cost is minimum. Also, find minimum cost assuming that minimum cost is possible for the given unbounded region. [2 marks]

Solution:

Concept:

- The total cost for x kg of food X and y kg of food Y is $Z = 16x + 20y$, measured in rupees.
- For this linear programme, if a finite minimum exists, it occurs at a corner point of the feasible region. In an unbounded region, a candidate must also be consistent with the absence of feasible points of lower cost.

Step 1: Compare the corner-point costs The graph gives the corners $D(0, 8)$, $C(1, 5)$, $B(2, 4)$ and $A(10, 0)$. Substituting each into the objective gives

$$\begin{aligned} Z(D) &= 16(0) + 20(8) = 160, \\ Z(C) &= 16(1) + 20(5) = 116, \\ Z(B) &= 16(2) + 20(4) = 112, \\ Z(A) &= 16(10) + 20(0) = 160. \end{aligned}$$

Thus B has the least cost among the vertices.

Step 2: Verify the minimum over the unbounded region Every feasible point satisfies $x + y \geq 6$ and $x + 2y \geq 10$. Multiplying these inequalities by 12 and 4, respectively, and adding gives

$$16x + 20y = 12(x + y) + 4(x + 2y) \geq 72 + 40 = 112.$$

The point $(2, 4)$ satisfies both equalities and also $3x + y = 10 \geq 8$, with $x, y \geq 0$. Therefore it is feasible and attains the lower bound; unboundedness does not permit a lower cost.

Step 3: Conclusion The cost is minimum at $\boxed{x = 2\,\text{kg},\ y = 4\,\text{kg}}$. The minimum cost is $\boxed{\text{Rs. } 112}$.

Quick Tip to Solve: The identity $Z = 12(x + y) + 4(x + 2y)$ proves the lower bound directly and checks that the best corner really minimizes cost in the unbounded region.

37. Airplanes are by far the safest mode of transportation when the number of transported passengers are measured against personal injuries and fatality totals.

Previous records state that the probability of an airplane crash is 0.00001%. Further, there are 95% chances that there will be survivors after a plane crash. Assume that in case of no crash, all travellers survive.

Let E_1 be the event that there is a plane crash and E_2 be the event that there is no crash. Let A be the event that passengers survive after the journey.

On the basis of the above information, answer the following questions: [4 marks]

37(i). Find the probability that the airplane will not crash. [1 mark]

Solution:

Concept:

- Crash and no crash are complementary events: $E_2 = E_1'$, so $P(E_2) = 1 - P(E_1)$.
- Convert a percentage to a probability by dividing by 100; the percentage value must not be used directly as a decimal probability.

Step 1: Convert the crash percentage The stated crash probability is 0.00001%. Thus

$$P(E_1) = \frac{0.00001}{100} = 0.0000001 = 10^{-7}.$$

Step 2: Apply the complement rule Exactly one of crash or no crash occurs on the journey. Therefore

$$P(E_2) = 1 - 10^{-7} = 0.9999999.$$

The survival probability after a crash is irrelevant to this complement calculation.

Step 3: Conclusion The probability that the airplane will not crash is $\boxed{0.9999999}$.

Quick Tip to Solve: Divide 0.00001% by 100 before subtracting from 1; this gives 10^{-7}, not 10^{-5}.

37(ii). Find $P(A \mid E_1) + P(A \mid E_2)$. [1 mark]

Solution:

Concept:

- The conditional probability $P(A \mid E_i)$ describes survival A when the specified event E_i is known to have occurred.
- The sum requested here adds probabilities under two different conditions. It is not the unconditional probability $P(A)$, which would require weighting by $P(E_1)$ and $P(E_2)$.

Step 1: Read survival probabilities under each condition After a crash, the stated survival chance is 95%, so $P(A \mid E_1) = 0.95$. With no crash, all travellers survive, hence $P(A \mid E_2) = 1$.

Step 2: Add the conditional probabilities Substituting the two values gives

$$P(A \mid E_1) + P(A \mid E_2) = 0.95 + 1 = 1.95.$$

Quick Tip to Solve: Read 95% and certainty as 0.95 and 1; do not weight them by crash probabilities when only their sum is asked.

A sum exceeding 1 is valid here: these probabilities refer to different conditions and are not complementary events in one conditional sample space.

Step 3: Conclusion The required sum is $\boxed{1.95}$.

Quick Tip to Solve: Read 95% and certainty as 0.95 and 1; do not weight them by crash probabilities when only their sum is asked.

37(iii)(a). Find $P(A)$. [2 marks]

Solution:

Concept:

- The crash event E_1 and the no-crash event E_2 partition the sample space, so $P(E_2) = 1 - P(E_1)$.
- The law of total probability gives $P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$, where A denotes survival after the journey.

Step 1: Convert the percentage and identify the conditional probabilities A percentage must be divided by 100 to become a probability. Hence

$$P(E_1) = \frac{0.00001}{100} = 10^{-7}, \qquad P(E_2) = 1 - 10^{-7}.$$

From the passage, survival after a crash has probability $P(A \mid E_1) = 0.95$, whereas survival without a crash is certain: $P(A \mid E_2) = 1$.

Step 2: Add the probabilities of the two survival routes Survival can occur either after a crash or when there is no crash. These cases are disjoint, so

$$\begin{aligned} P(A) &= 10^{-7}(0.95) + (1 - 10^{-7})(1) \\ &= 1 - 0.05 \times 10^{-7} \\ &= 1 - 5 \times 10^{-9}. \end{aligned}$$

Equivalently, the only nonsurvival route has probability $P(E_1)P(A^c \mid E_1) = 10^{-7}(0.05)$.

Step 3: Conclusion The required probability is $\boxed{P(A) = 0.999999995}$.

Quick Tip to Solve: Convert the crash percentage to 10^{-7} first; then subtract the crash-and-no-survival probability from 1.

OR

37(iii)(b). Find $P(E_2 \mid A)$. [2 marks]

Solution:

Concept:

- Bayes' theorem gives $P(E_2 \mid A) = P(E_2)P(A \mid E_2)/P(A)$, provided $P(A) > 0$.
- Since crash and no crash are complementary, the total survival probability is $P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$.

Step 1: Obtain the probabilities from the passage The crash probability is 0.00001%, which is 10^{-7} after dividing by 100. Therefore

$$P(E_1) = 10^{-7}, \quad P(E_2) = 1 - 10^{-7}, \quad P(A \mid E_1) = 0.95, \quad P(A \mid E_2) = 1.$$

We condition on survival A, exactly as printed. Survival can also occur following a crash, so knowing that passengers survive does not make no crash certain.

Step 2: Apply total probability followed by Bayes' theorem First evaluate the denominator over both possible flight outcomes:

$$P(A) = 0.95 \times 10^{-7} + 1 - 10^{-7} = 0.999999995.$$

Hence the conditional probability is

$$P(E_2 \mid A) = \frac{(1 - 10^{-7}) \times 1}{0.999999995} = \frac{0.9999999}{0.999999995} = \frac{199999980}{199999999}.$$

Step 3: Conclusion $\boxed{P(E_2 \mid A) = \frac{199999980}{199999999}}$, approximately 0.999999905.

Quick Tip to Solve: For no crash given survival, use the no-crash survival contribution as numerator and total survival probability as denominator.

38. Overspeeding increases fuel consumption and decreases fuel economy as a result of tyre rolling friction and air resistance. While vehicles reach optimal fuel economy at different speeds, fuel mileage usually decreases rapidly at speeds above 80 km/h.

The relation between fuel consumption F (l/100 km) and speed V (km/h) under some constraints is given as $F = \frac{V^2}{500} - \frac{V}{4} + 14$.

On the basis of the above information, answer the following questions: [4 marks]

38(i). Find F, when V = 40 km/h. [1 mark]

Solution:

Concept:

- The model $F(V) = V^2/500 - V/4 + 14$ expresses fuel consumption in litres per 100 km when speed V is in km/h.
- To evaluate the consumption at a stated speed, substitute that same speed in every occurrence of V.

Step 1: Substitute the given speed Insert $V = 40$ into the fuel-consumption model:

$$F(40) = \frac{40^2}{500} - \frac{40}{4} + 14.$$

The squared term requires $40^2 = 1600$; the linear term retains its negative sign.

Step 2: Simplify the numerical expression Evaluate the divisions before adding:

$$F(40) = \frac{1600}{500} - 10 + 14 = 3.2 - 10 + 14 = 7.2.$$

This is consumption per 100 km, rather than a total quantity for an unspecified journey.

Step 3: Conclusion At 40 km/h, $\boxed{F = 7.2 \text{ litres per } 100 \text{ km}}$.

Quick Tip to Solve: Square 40 before dividing by 500, and retain the model's unit of litres per 100 km.

38(ii). Find $\frac{dF}{dV}$. [1 mark]

Solution:

Concept:

- The power rule states $\frac{d}{dV}V^n = nV^{n-1}$, and a constant has derivative zero.
- Differentiation is linear: differentiate each term of $F(V) = V^2/500 - V/4 + 14$ separately, keeping its constant multiplier.

Step 1: Differentiate the polynomial term by term Here the independent variable is speed V. Thus

$$\frac{dF}{dV} = \frac{1}{500}(2V) - \frac{1}{4}(1) + 0.$$

Quick Tip to Solve: The quadratic term gives $V/250$, the linear term gives $-1/4$, and the constant 14 disappears.

The term 14 contributes zero because it does not change with speed, whereas the derivative of $-V/4$ is the constant $-1/4$.

Step 2: Reduce the coefficient of the speed Simplify $2/500$ to $1/250$, obtaining

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}.$$

This derivative describes the instantaneous change in the consumption rate per unit increase in speed.

Step 3: Conclusion $\boxed{\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}}$.

Quick Tip to Solve: The quadratic term gives $V/250$, the linear term gives $-1/4$, and the constant 14 disappears.

38(iii)(a). Find the speed V for which fuel consumption F is minimum. [2 marks]

Solution:

Concept:

- At an interior stationary point of a differentiable function, its first derivative is zero.
- The second-derivative test gives a strict local minimum when $F'(V) = 0$ and $F''(V) > 0$. For an upward-opening quadratic, this minimum is also global.

Step 1: Find the stationary speed Differentiate the model with respect to V:

$$F'(V) = \frac{2V}{500} - \frac{1}{4} = \frac{V}{250} - \frac{1}{4}.$$

For a stationary value of fuel consumption, set this derivative equal to zero:

$$\frac{V}{250} - \frac{1}{4} = 0 \quad \Longrightarrow \quad V = \frac{250}{4} = 62.5.$$

This speed is positive and lies below the 80 km/h value discussed in the passage.

Step 2: Verify that the stationary point is a minimum Differentiate once more:

$$F''(V) = \frac{1}{250} > 0.$$

Thus the stationary point gives a minimum, not a maximum. In fact the positive coefficient of V^2 makes the quadratic open upward everywhere, so this is the unique minimum over nonnegative speeds. Alternatively, $F'(V)$ changes from negative to positive as V passes through 62.5.

Step 3: Conclusion Fuel consumption is minimum at $\boxed{V = 62.5\,\text{km/h}}$.

Quick Tip to Solve: After solving $F'(V) = 0$, check $F''(V) > 0$; a stationary speed alone does not establish a minimum.

OR

38(iii)(b). Find the quantity of fuel required to travel 600 km at the speed V at which $\frac{dF}{dV} = -0.01$. [2 marks]

Solution:

Concept:

- The derivative $F'(V) = V/250 - 1/4$ specifies how the fuel-consumption rate changes with speed. A stated derivative value determines the required speed.
- If F litres are consumed per 100 km, the quantity needed over distance d km at the same speed is $Q = (d/100)F$ litres.

Step 1: Determine the speed from the derivative condition Differentiate the given model and use the printed condition:

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4} = -0.01,$$
$$\frac{V}{250} = 0.25 - 0.01 = 0.24,$$
$$V = 60\,\text{km/h}.$$

The negative derivative describes decreasing consumption at this speed; it is not itself the amount of fuel consumed.

Step 2: Evaluate consumption and scale to the journey distance Substitute $V = 60$ into the original model, not into its derivative:

$$F(60) = \frac{60^2}{500} - \frac{60}{4} + 14 = 7.2 - 15 + 14 = 6.2.$$

The rate is therefore 6.2 litres per 100 km. A journey of 600 km contains six such intervals, so

$$Q = \frac{600}{100} \times 6.2 = 37.2\,\text{litres}.$$

Step 3: Conclusion At the required speed 60 km/h, the fuel needed for 600 km is $\boxed{37.2\,\text{litres}}$.

Quick Tip to Solve: Use the derivative to find speed, the original function to find consumption, and multiply by 6 for a 600 km journey.