

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-3-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $A = [a_{ij}]$ is an identity matrix, then which of the following is true?

[1 mark]

(A) $a_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}$

(B) $a_{ij} = 1, \forall i, j$

(C) $a_{ij} = 0, \forall i, j$

(D) $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

Correct Answer: (D) $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

Solution:

Concept:

- An identity matrix is a square matrix whose main diagonal entries are all 1 and whose other entries are all 0.
- The entry a_{ij} is on the main diagonal precisely when its row and column indices satisfy $i = j$.

Step 1: Identify diagonal entries For $i = j$, the definition of the identity matrix gives $a_{ij} = 1$. Thus option (A), which puts zero on the diagonal, and option (C), which makes every entry zero, do not describe an identity matrix.

Step 2: Identify off-diagonal entries For $i \neq j$, we have $a_{ij} = 0$. Option (B), which makes every entry one, does not give the defining rule for an identity matrix. Option (D) has both the required diagonal and off-diagonal entries.

Step 3: Conclusion Therefore $a_{ij} = 1$ if $i = j$, and $a_{ij} = 0$ if $i \neq j$. The correct option is (D).

Quick Tip to Solve: Check the diagonal first: an identity matrix has ones there and zeros everywhere else.

2. Let R_+ denote the set of all non-negative real numbers. Then the function $f : R_+ \rightarrow R_+$ defined as $f(x) = x^2 + 1$ is: [1 mark]

- (A) one-one but not onto
- (B) onto but not one-one
- (C) both one-one and onto
- (D) neither one-one nor onto

Correct Answer: (A) one-one but not onto

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ forces $x_1 = x_2$ for inputs in its domain.
- A function is onto when every member of its codomain is the image of some input.

Step 1: Test whether the function is one-one Let $x_1, x_2 \geq 0$ and suppose $f(x_1) = f(x_2)$. Then $x_1^2 + 1 = x_2^2 + 1$, so $x_1^2 = x_2^2$. Since both inputs are non-negative, taking non-negative square roots gives $x_1 = x_2$. Thus the function is one-one.

Step 2: Compare the range with the codomain For $x \geq 0$, $x^2 + 1 \geq 1$, and every $y \geq 1$ is obtained by taking $x = \sqrt{y - 1}$. The range is therefore $[1, \infty)$, whereas the codomain is $[0, \infty)$. In particular, zero has no preimage, so the function is not onto. Options (B) and (C) incorrectly assert onto behaviour; (D) incorrectly denies one-one behaviour.

Step 3: Conclusion The function is one-one but not onto. The correct option is (A).

Quick Tip to Solve: On non-negative inputs, squaring is injective; check whether the added constant leaves out codomain values.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix such that $\text{adj } A = A$. Then, $(a + b + c + d)$ is equal to: [1 mark]

- (A) $2a$
- (B) $2b$
- (C) $2c$
- (D) 0

Correct Answer: (A) $2a$

Solution:

Concept:

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjugate is $\text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- Equal matrices have equal corresponding entries.

Step 1: Compare corresponding entries Using $\text{adj } A = A$, we get $d = a$, $-b = b$, and $-c = c$. Consequently $b = 0$, $c = 0$, and the two diagonal entries are equal. No condition requires a itself to be zero.

Quick Tip to Solve: For a two by two adjugate, interchange the diagonal entries and negate both off-diagonal entries.

Step 2: Evaluate the requested sum Substituting the entry relations gives

$$a + b + c + d = a + 0 + 0 + a = 2a.$$

Options (B), (C), and (D) are zero after these substitutions, which need not equal the sum: for example, $A = I_2$ satisfies the hypothesis and has sum 2. Thus those expressions are not the general answer.

Step 3: Conclusion The required sum is $2a$. The correct option is (A).

Quick Tip to Solve: For a two by two adjugate, interchange the diagonal entries and negate both off-diagonal entries.

4. A function $f(x) = |1 - x + |x||$ is:

[1 mark]

- (A) discontinuous at $x = 1$ only
- (B) discontinuous at $x = 0$ only
- (C) discontinuous at $x = 0, 1$
- (D) continuous everywhere

Correct Answer: (D) continuous everywhere

Solution:

Concept:

- The absolute-value function is continuous on the real line. Sums and compositions of continuous functions remain continuous.
- To check a piecewise formula at a joining point, compare its one-sided limits and its value.

Step 1: Remove the nested absolute values For $x \geq 0$, $|x| = x$, so $f(x) = |1| = 1$. For $x < 0$, $|x| = -x$, giving $f(x) = |1 - 2x| = 1 - 2x$, because $1 - 2x > 0$ there. Each of these expressions is continuous on its own interval.

Step 2: Check the joining point and options At $x = 0$, $\lim_{x \rightarrow 0^-} f(x) = 1$, $\lim_{x \rightarrow 0^+} f(x) = 1$, and $f(0) = 1$. Hence there is no discontinuity at zero. Near $x = 1$, the function is the constant 1, so it is continuous there too. This rules out options (A), (B), and (C).

Step 3: Conclusion The function is continuous everywhere on $\mathbb{R}$. The correct option is (D).

Quick Tip to Solve: Nested absolute values preserve continuity; piecewise formulas can confirm the behaviour at zero.

5. If the sides of a square are decreasing at the rate of 1.5 cm/s, the rate of decrease of its perimeter is: [1 mark]

- (A) 1.5 cm/s
- (B) 6 cm/s
- (C) 3 cm/s
- (D) 2.25 cm/s

Correct Answer: (B) 6 cm/s

Solution:

Concept:

- If a square has side length s , its perimeter is $P = 4s$.
- A decreasing length has a negative time derivative; the rate of decrease is the positive magnitude of that derivative.

Step 1: Translate the given rate Let t denote time in seconds. Since each side decreases at 1.5 cm/s, its derivative is $\frac{ds}{dt} = -1.5$ cm/s. The perimeter depends linearly on the side length, so no current side length is needed.

Step 2: Differentiate the perimeter formula Differentiating $P = 4s$ with respect to time gives

$$\frac{dP}{dt} = 4 \frac{ds}{dt} = 4(-1.5) = -6 \text{ cm/s.}$$

Thus the perimeter decreases at the magnitude 6 cm/s. Option (A) counts only one side, option (C) counts only two sides, and option (D) squares the numerical rate without using the perimeter formula.

Step 3: Conclusion The rate of decrease of the perimeter is 6 cm/s. The correct option is (B).

Quick Tip to Solve: For a square, multiply the side-length rate by four to obtain the perimeter rate.

6. $\int_{-a}^a f(x) dx = 0$, if:

[1 mark]

- (A) $f(-x) = f(x)$
- (B) $f(-x) = -f(x)$
- (C) $f(a-x) = f(x)$
- (D) $f(a-x) = -f(x)$

Correct Answer: (B) $f(-x) = -f(x)$

Solution:

Concept:

- An odd function satisfies $f(-x) = -f(x)$. Its integral over symmetric limits is zero whenever the integral exists.
- Splitting an integral at zero and substituting $x = -t$ pairs the contributions from opposite inputs.

Quick Tip to Solve: Match the symmetry centre to the limits: opposite endpoints require odd symmetry about zero.

Step 1: Pair the two halves of the interval Write

$$I = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = \int_0^a [f(-t) + f(t)] dt.$$

Here the first term follows from the substitution $x = -t$, including the reversal of its limits.

Step 2: Apply odd symmetry and eliminate alternatives If $f(-t) = -f(t)$, the integrand is zero and hence $I = 0$. Option (A) instead gives an even function, whose integral is generally twice its integral from zero to a ; the constant function 1 also disproves (C) as a guarantee. Option (D) describes antisymmetry about $a/2$, not zero: for $a = 1$, $f(x) = x - 1/2$ satisfies (D), but its integral from -1 to 1 is -1 .

Step 3: Conclusion The required condition is $f(-x) = -f(x)$. The correct option is (B).

Quick Tip to Solve: Match the symmetry centre to the limits: opposite endpoints require odd symmetry about zero.

7. $x \log x \frac{dy}{dx} + y = 2 \log x$ is an example of a:

[1 mark]

- (A) variable separable differential equation.
- (B) homogeneous differential equation.
- (C) first order linear differential equation.
- (D) differential equation whose degree is not defined.

Correct Answer: (C) first order linear differential equation.

Solution:

Concept:

- A first order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Order is the order of the highest derivative; degree is its power after expressing the equation polynomially in derivatives.

Step 1: Put the equation into linear form On an interval where $x > 0$ and $x \neq 1$, divide the given equation by $x \log x$:

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}.$$

Thus $P(x) = 1/(x \log x)$ and $Q(x) = 2/x$, both depending only on x .

Step 2: Check the other classifications Only the first derivative occurs, to the first power, so the equation has order one and degree one; the logarithm of the independent variable does not make the degree undefined. Its derivative form $y' = 2/x - y/(x \log x)$ is neither a product of a function of x with a function of y for separation nor a function solely of y/x for the standard homogeneous classification. Options (A), (B), and (D) therefore do not give its standard type.

Step 3: Conclusion This is a first order linear differential equation. The correct option is (C).

Quick Tip to Solve: Divide by the coefficient of the derivative and look for the form $y' + P(x)y = Q(x)$.

8. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, then $\vec{a}$ and $\vec{b}$ are:

[1 mark]

- (A) collinear vectors which are not parallel
- (B) parallel vectors
- (C) perpendicular vectors
- (D) unit vectors

Correct Answer: (C) perpendicular vectors

Solution:

Concept:

- Two non-zero vectors are perpendicular when their scalar product is zero.
- For components, $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$.

Step 1: Compute the scalar product Using the printed components,

$$\vec{a} \cdot \vec{b} = 2(1) + (-1)(1) + (1)(-1) = 2 - 1 - 1 = 0.$$

Both vectors are non-zero, so the zero scalar product establishes a right angle between them.

Step 2: Check the other descriptions The component ratios are 2, -1, -1, which are not equal, so the vectors are neither parallel nor collinear. Also, $|\vec{a}| = \sqrt{6}$ and $|\vec{b}| = \sqrt{3}$, so neither is a unit vector. Thus none of (A), (B), or (D) describes this pair.

Step 3: Conclusion The vectors are perpendicular. Hence, the correct option is (C).

Quick Tip to Solve: For perpendicularity, calculate the dot product before comparing component ratios.

9. If α, β and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following is not true?

[1 mark]

- (A) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
- (B) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
- (C) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
- (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Correct Answer: (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, and $n = \cos \gamma$, with $l^2 + m^2 + n^2 = 1$.
- Use $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos 2\theta = 2\cos^2 \theta - 1$.

Step 1: Verify the first three identities The direction-cosine identity gives (A). Consequently,

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 3 - 1 = 2,$$

which verifies (B). Similarly, the sum of the three double-angle cosines is $2(1) - 3 = -1$, verifying (C).

Step 2: Test the unsquared sum The identity concerns squares, not the unsquared sum. For a line

Quick Tip to Solve: Remember that direction cosines have a fixed sum of squares, not a fixed sum.

with equal positive direction cosines, $l = m = n = 1/\sqrt{3}$, their sum is $\sqrt{3}$, not 1. These are valid direction cosines since their squares sum to 1, so they provide a counterexample to (D).

Step 3: Conclusion The statement that is not true in general is (D).

Quick Tip to Solve: Remember that direction cosines have a fixed sum of squares, not a fixed sum.

10. The restrictions imposed on decision variables involved in an objective function of a linear programming problem are called: [1 mark]

- (A) feasible solutions
- (B) constraints
- (C) optimal solutions
- (D) infeasible solutions

Correct Answer: (B) constraints

Solution:

Concept:

- Decision variables describe the quantities to be chosen in a linear programming problem.
- Constraints are the linear equations or inequalities that restrict those variables, including non-negativity restrictions.

Step 1: Identify the defining term The question asks for the name of restrictions on the decision variables. Such restrictions express available resources or other conditions, for example $x + y \leq 10$ and $x, y \geq 0$. In linear programming these are called constraints.

Step 2: Distinguish the remaining terms A feasible solution is a particular choice of variables satisfying every constraint; an infeasible solution violates at least one. An optimal solution is a feasible choice giving the required maximum or minimum objective value. These are descriptions of choices, whereas the restrictions themselves are constraints.

Step 3: Conclusion The restrictions are constraints. Hence, the correct option is (B).

Quick Tip to Solve: Link restrictions to constraints; feasible and optimal describe solutions.

11. Let E and F be two events such that $P(E) = 0.1$, $P(F) = 0.3$, $P(E \cup F) = 0.4$, then $P(F | E)$ is: [1 mark]

- (A) 0.6
- (B) 0.4
- (C) 0.5
- (D) 0

Correct Answer: (D) 0

Solution:

Concept:

- The addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- For $P(E) > 0$, conditional probability is $P(F | E) = P(E \cap F) / P(E)$.

Step 1: Find the intersection probability Rearranging the addition rule and substituting the given probabilities,

$$P(E \cap F) = 0.1 + 0.3 - 0.4 = 0.$$

Thus the events have zero probability of occurring together.

Step 2: Apply conditional probability Since $P(E) = 0.1 \neq 0$, the conditional probability is defined and

$$P(F | E) = \frac{0}{0.1} = 0.$$

Options (A), (B), and (C) are positive, and each would require a positive intersection probability; they are therefore inconsistent with the given union probability.

Step 3: Conclusion $P(F | E) = 0$. Hence, the correct option is (D).

Quick Tip to Solve: When the union probability equals the sum, the intersection probability is zero.

12. If A and B are two skew symmetric matrices, then $(AB + BA)$ is: [1 mark]

- (A) a skew symmetric matrix
- (B) a symmetric matrix
- (C) a null matrix
- (D) an identity matrix

Correct Answer: (B) a symmetric matrix

Solution:

Concept:

- A skew symmetric matrix satisfies $A^T = -A$; similarly, $B^T = -B$.
- Transposition reverses product order: $(AB)^T = B^T A^T$. A matrix M is symmetric if $M^T = M$.

Quick Tip to Solve: Transpose both products and reverse their order before using skew symmetry.

Step 1: Transpose the sum Let $M = AB + BA$. Using transpose rules,

$$\begin{aligned}M^T &= B^T A^T + A^T B^T \\&= (-B)(-A) + (-A)(-B) \\&= BA + AB = AB + BA = M.\end{aligned}$$

The order reversal in each product is essential.

Step 2: Interpret the identity The equality $M^T = M$ proves symmetry. It does not imply that M is always skew symmetric, zero or the identity. For example, taking both matrices to be $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives $M = -2I$, which rules out those three universal descriptions.

Step 3: Conclusion $AB + BA$ is a symmetric matrix. Hence, the correct option is (B).

Quick Tip to Solve: Transpose both products and reverse their order before using skew symmetry.

13. If $\begin{vmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \pm 6$, then the value of k is:

[1 mark]

- (A) 2
- (B) -2
- (C) ± 2
- (D) ∓ 2

Correct Answer: (D) ∓ 2

Solution:

Concept:

- Expand a determinant along a row or column containing zeros to reduce the calculation.
- An equation with ± 6 represents both equations: determinant equal to 6 and determinant equal to -6.

Step 1: Evaluate the printed determinant Expanding along the third row, the only non-zero entry is the final 1, with positive cofactor sign. Therefore,

$$\begin{vmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ k & 0 \end{vmatrix} = 1(0 - 3k) = -3k.$$

Step 2: Solve both possible equations The printed condition gives $-3k = \pm 6$. Dividing by the negative number -3 reverses the paired signs, so $k = \mp 2$. Specifically, determinant +6 gives $k = -2$, while determinant -6 gives $k = +2$. Options (A) and (B) each give only one value. Option (C), ± 2 , describes the same unordered pair of values, but option (D), ∓ 2 , preserves their sign correspondence with the printed right-hand side. Thus (D) is the intended paired-sign answer.

Step 3: Conclusion $k = \mp 2$, with the sign opposite to the selected sign of ± 6 . Hence, the correct

Quick Tip to Solve: Expand along the bottom row and retain both signs when solving the condition.

option is (D).

Quick Tip to Solve: Expand along the bottom row and retain both signs when solving the condition.

14. The derivative of 2^x w.r.t. 3^x is:

[1 mark]

- (A) $\left(\frac{3}{2}\right)^x \frac{\log 2}{\log 3}$
(B) $\left(\frac{2}{3}\right)^x \frac{\log 3}{\log 2}$
(C) $\left(\frac{2}{3}\right)^x \frac{\log 2}{\log 3}$
(D) $\left(\frac{3}{2}\right)^x \frac{\log 3}{\log 2}$

Correct Answer: (C) $\left(\frac{2}{3}\right)^x \frac{\log 2}{\log 3}$

Solution:

Concept:

- For $a > 0$, $\frac{d}{dx} a^x = a^x \log a$.
- If $y = f(x)$ and $u = g(x)$ with $g'(x) \neq 0$, then $\frac{dy}{du} = \frac{dy/dx}{du/dx}$.

Step 1: Differentiate both functions with respect to x Put $y = 2^x$ and $u = 3^x$. Then

$$\frac{dy}{dx} = 2^x \log 2, \quad \frac{du}{dx} = 3^x \log 3.$$

The derivative of the second expression is non-zero for every real x , so the derivative ratio is valid.

Step 2: Form the derivative ratio and compare options Thus

$$\frac{dy}{du} = \frac{2^x \log 2}{3^x \log 3} = \left(\frac{2}{3}\right)^x \frac{\log 2}{\log 3}.$$

Options (A) and (D) reverse the exponential ratio. Options (B) and (D) reverse the logarithmic ratio. Only (C) has both ratios in the required order.

Step 3: Conclusion The derivative is $\left(\frac{2}{3}\right)^x \frac{\log 2}{\log 3}$. Hence, the correct option is (C).

Quick Tip to Solve: Divide the derivative of the first expression by the derivative of the second.

15. If $|\vec{a}| = 2$ and $-3 \leq k \leq 2$, then $|k\vec{a}| \in$:

[1 mark]

- (A) $[-6, 4]$
- (B) $[0, 4]$
- (C) $[4, 6]$
- (D) $[0, 6]$

Correct Answer: (D) $[0, 6]$

Solution:

Concept:

- For a real scalar k , the magnitude satisfies $|k\vec{a}| = |k| |\vec{a}|$.
- A vector magnitude is non-negative; multiplication by a negative scalar reverses direction, not the sign of its magnitude.

Step 1: Find the range of the scalar magnitude The interval $-3 \leq k \leq 2$ includes zero, so the least value of $|k|$ is zero. The greatest value is 3, attained at $k = -3$. Every value between zero and three occurs, for example by taking k in $[-3, 0]$. Thus $|k| \in [0, 3]$.

Step 2: Scale the interval and check the options Since $|\vec{a}| = 2$, we have $|k\vec{a}| = 2|k| \in [0, 6]$. Option (A) wrongly includes negative magnitudes. Option (B) misses values above four, and option (C) wrongly excludes zero and small positive magnitudes.

Step 3: Conclusion $|k\vec{a}| \in [0, 6]$. Hence, the correct option is (D).

Quick Tip to Solve: Take the absolute value of the scalar first; use zero and the endpoint with greatest absolute value.

16. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of both x -axis and z -axis, then the angle which it makes with the positive direction of y -axis is:

[1 mark]

- (A) 0
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{2}$
- (D) π

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- Direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, where the angles are measured with the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$, and each direction angle lies in $[0, \pi]$.

Step 1: Substitute the two given direction angles Here $\alpha = \gamma = \pi/4$, hence $l = n = 1/\sqrt{2}$. Therefore

$$m^2 = 1 - l^2 - n^2 = 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

Quick Tip to Solve: Square the two known direction cosines and subtract their sum from one to find the third.

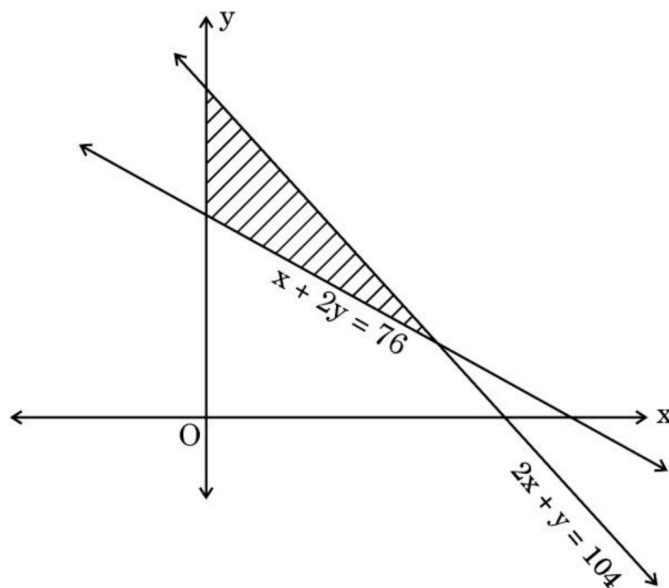
Thus $\cos \beta = m = 0$.

Step 2: Identify the permissible angle In the direction-angle interval $[0, \pi]$, cosine vanishes only at $\beta = \pi/2$. Option (A) would give $m^2 = 1$, option (B) would give $m^2 = 1/2$, and option (D) would again give $m^2 = 1$; none satisfies the sum of squares.

Step 3: Conclusion The angle with the positive y -axis is $\pi/2$. Hence, the correct option is (C).

Quick Tip to Solve: Square the two known direction cosines and subtract their sum from one to find the third.

17. Of the following, which group of constraints represents the feasible region given below? [1 mark]



(A) $x + 2y \leq 76$, $2x + y \geq 104$, $x, y \geq 0$

(B) $x + 2y \leq 76$, $2x + y \leq 104$, $x, y \geq 0$

(C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$

(D) $x + 2y \geq 76$, $2x + y \geq 104$, $x, y \geq 0$

Correct Answer: (C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$

Solution:

Concept:

- A boundary line divides the plane into two half-planes. Substitute a point in the shaded side to determine the inequality.
- The coordinate restrictions $x \geq 0$, $y \geq 0$ describe the first quadrant; solid boundaries are included.

Step 1: Test a point inside the shaded region The point $(0, 50)$ lies on the shaded vertical edge, between the intercepts $y = 38$ of $x + 2y = 76$ and $y = 104$ of $2x + y = 104$. At this point, $x + 2y = 100 > 76$, whereas $2x + y = 50 < 104$. The shaded region therefore lies above the first boundary and below the second.

Step 2: Match both inequalities and the quadrant The figure lies in the first quadrant and includes both boundary lines. Its constraints are consequently $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$. Option (A)

Quick Tip to Solve: Test an easy shaded point on the axis, such as $(0, 50)$, in each boundary equation.

reverses both inequalities, option (B) chooses the wrong side of the first line, and option (D) chooses the wrong side of the second line.

Step 3: Conclusion The feasible region is $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$. Hence, the correct option is (C).

Quick Tip to Solve: Test an easy shaded point on the axis, such as $(0, 50)$, in each boundary equation.

18. If $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then A^{-1} is:

[1 mark]

(A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(B) $30 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(C) $\frac{1}{30} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(D) $\frac{1}{30} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Solution:

Concept:

- The inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
- A diagonal matrix with nonzero diagonal entries is inverted by taking the reciprocal of each diagonal entry.

Step 1: Invert the diagonal entries The diagonal entries 2, 3, 5 are all nonzero, and $\det A = 2 \cdot 3 \cdot 5 = 30 \neq 0$. Thus

$$A^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}.$$

The off-diagonal entries remain zero.

Step 2: Verify by multiplication and reject extra factors Multiplying the proposed inverse by A gives diagonal entries $2(1/2) = 1$, $3(1/3) = 1$, $5(1/5) = 1$, so the product is I . Option (B) gives $30I$, and option (D) gives $I/30$. For option (C), the product is $\text{diag}(4, 9, 25)/30$, which is also not I .

Quick Tip to Solve: For a diagonal matrix, reciprocal entries already give the inverse; do not attach another determinant factor.

Step 3: Conclusion $A^{-1} = \text{diag}(1/2, 1/3, 1/5)$. Hence, the correct option is (A).

Quick Tip to Solve: For a diagonal matrix, reciprocal entries already give the inverse; do not attach another determinant factor.

19. Assertion (A): Every scalar matrix is a diagonal matrix.

Reason (R): In a diagonal matrix, all the diagonal elements are 0.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A diagonal matrix is square and has all off-diagonal entries zero; its diagonal entries need not be zero.
- A scalar matrix has the form cI , so all diagonal entries equal c and all off-diagonal entries are zero.

Step 1: Judge the Assertion Every scalar matrix cI has zero entries outside its main diagonal. It therefore satisfies the definition of a diagonal matrix, including when $c = 0$. The Assertion is true.

Step 2: Judge the Reason The Reason confuses diagonal and off-diagonal entries. For example, $\text{diag}(2, 3)$ is diagonal but has nonzero diagonal entries. It is the off-diagonal entries that must vanish. The Reason is false.

Step 3: Check the explanation and option codes A false Reason cannot explain the true Assertion. Options (A) and (B) incorrectly treat the Reason as true, while option (D) incorrectly treats the Assertion as false. The true-Assertion, false-Reason combination is option (C).

Step 4: Conclusion The Assertion is true, but the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Remember that diagonal matrices require zeros outside the main diagonal, not on it.

20. Assertion (A): Projection of $\vec{a}$ on $\vec{b}$ is same as projection of $\vec{b}$ on $\vec{a}$.

Reason (R): Angle between $\vec{a}$ and $\vec{b}$ is same as angle between $\vec{b}$ and $\vec{a}$ numerically.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- For nonzero vectors, the scalar projection of $\vec{a}$ on $\vec{b}$ is $\vec{a} \cdot \vec{b} / |\vec{b}| = |\vec{a}| \cos \theta$.
- The angle $\theta \in [0, \pi]$ between vectors satisfies $\cos \theta = \vec{a} \cdot \vec{b} / (|\vec{a}| |\vec{b}|)$.

Step 1: Judge the Assertion Interchanging the vectors changes the projection to $|\vec{b}| \cos \theta$, which is generally different. For example, let $\vec{a} = 2\hat{i}$ and $\vec{b} = \hat{i}$. Their angle is zero; the projection of $\vec{a}$ on $\vec{b}$ is 2, whereas the reverse projection is 1. Hence the Assertion is false.

Step 2: Judge the Reason The dot product is commutative, and the denominator $|\vec{a}| |\vec{b}|$ is unchanged on interchanging the vectors. Thus the numerical angle between the vectors is unchanged. The Reason is true.

Step 3: Check the explanation and option codes Equality of angles does not ensure equality of projections, because the projected vector magnitude also appears. Options (A), (B) and (C) all incorrectly declare the Assertion true. The false-Assertion, true-Reason combination is option (D).

Step 4: Conclusion The Assertion is false, but the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Compare $|\vec{a}| \cos \theta$ with $|\vec{b}| \cos \theta$; the same angle does not give the same magnitude.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. Evaluate: $\sec^2 \left(\tan^{-1} \frac{1}{2} \right) + \operatorname{cosec}^2 \left(\cot^{-1} \frac{1}{3} \right)$

[2 marks]

Solution:

Concept:

- If $\theta = \tan^{-1} u$, then $\tan \theta = u$ on the principal range $(-\pi/2, \pi/2)$.
- If $\phi = \cot^{-1} v$, then $\cot \phi = v$ on the principal range $(0, \pi)$.
- The identities $\sec^2 \theta = 1 + \tan^2 \theta$ and $\operatorname{cosec}^2 \phi = 1 + \cot^2 \phi$ evaluate the expression directly.

Step 1: Evaluate the secant-squared term Put $\theta = \tan^{-1}(1/2)$. By the defining property of the inverse

Quick Tip to Solve: Use $\sec^2(\tan^{-1} u) = 1 + u^2$ and $\operatorname{cosec}^2(\cot^{-1} v) = 1 + v^2$ directly.

tangent, $\tan \theta = 1/2$. Therefore

$$\sec^2\left(\tan^{-1} \frac{1}{2}\right) = 1 + \left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The inverse tangent denotes an angle, so we apply the identity to that angle, rather than treating the superscript -1 as a reciprocal.

Step 2: Evaluate the cosecant-squared term and add Put $\phi = \cot^{-1}(1/3)$. Then $\cot \phi = 1/3$, so

$$\operatorname{cosec}^2\left(\cot^{-1} \frac{1}{3}\right) = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}.$$

Adding the two values with common denominator 36 gives

$$\frac{5}{4} + \frac{10}{9} = \frac{45 + 40}{36} = \frac{85}{36}.$$

Both given inverse-function arguments are positive, so the corresponding principal angles lie in the first quadrant, where the functions used here are defined.

Step 3: Conclusion The required value is $\boxed{85/36}$.

Quick Tip to Solve: Use $\sec^2(\tan^{-1} u) = 1 + u^2$ and $\operatorname{cosec}^2(\cot^{-1} v) = 1 + v^2$ directly.

22(a). If $x = e^{x/y}$, prove that $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$.

[2 marks]

Solution:

Concept:

- Taking natural logarithms converts an exponential relation into an algebraic relation: if $x = e^u$, then $\log x = u$, with $x > 0$.
- The quotient rule gives $\frac{d}{dx}(u/v) = \frac{vu' - uv'}{v^2}$, provided $v \neq 0$.

Step 1: Express y explicitly The given relation involves x/y , so $y \neq 0$. Since an exponential is positive, $x > 0$. Taking natural logarithms on both sides gives

$$\log x = \frac{x}{y}, \quad y \log x = x.$$

Here $\log x \neq 0$: if $x = 1$, the relation would give $1 = e^{1/y}$, which cannot hold for a finite nonzero y . Hence

$$y = \frac{x}{\log x}.$$

Step 2: Differentiate by the quotient rule Differentiate the numerator and denominator with respect to x . Their derivatives are 1 and $1/x$, respectively. Thus

$$\frac{dy}{dx} = \frac{(\log x)(1) - x(1/x)}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}.$$

Quick Tip to Solve: Take logarithms first and write y as x divided by $\log x$; then apply the quotient rule directly.

The calculation is valid on each interval of the domain $x > 0, x \neq 1$.

Step 3: Conclusion Hence, $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$, as required.

Quick Tip to Solve: Take logarithms first and write y as x divided by $\log x$; then apply the quotient rule directly.

OR

22(b). Check the differentiability of $f(x) = \begin{cases} x^2 + 1, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2 \end{cases}$ **at** $x = 1$. [2 marks]

Solution:

Concept:

- A function differentiable at an interior point must first be continuous there.
- Differentiability at $x = a$ requires equality of the left and right limits of $\frac{f(x) - f(a)}{x - a}$.

Step 1: Check continuity at the joining point According to the second branch, $f(1) = 3 - 1 = 2$. The one-sided limits are

$$\lim_{x \rightarrow 1^-} f(x) = 1^2 + 1 = 2, \quad \lim_{x \rightarrow 1^+} f(x) = 3 - 1 = 2.$$

Both agree with $f(1)$, so the function is continuous at $x = 1$. Continuity allows us to proceed to the one-sided derivative test, but does not itself establish differentiability.

Step 2: Compare the one-sided derivatives Using the definition on the left,

$$f'_{-}(1) = \lim_{h \rightarrow 0^-} \frac{(1+h)^2 + 1 - 2}{h} = \lim_{h \rightarrow 0^-} (2 + h) = 2.$$

On the right, the applicable formula is $3 - x$, so

$$f'_{+}(1) = \lim_{h \rightarrow 0^+} \frac{3 - (1+h) - 2}{h} = \lim_{h \rightarrow 0^+} (-1) = -1.$$

These finite one-sided derivatives are unequal. Therefore the two-sided derivative does not exist.

Step 3: Conclusion The function is continuous but not differentiable at $x = 1$, since $f'_{-}(1) = 2 \neq -1 = f'_{+}(1)$.

Quick Tip to Solve: At a piecewise joining point, check the function value and limits before comparing the two branch slopes.

23(a). Evaluate: $\int_0^{\pi/2} \sin 2x \cos 3x \, dx$

[2 marks]

Solution:

Concept:

- The product-to-sum identity is $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$.
- An antiderivative of $\sin(kx)$ is $-\cos(kx)/k$ for a nonzero constant k .

Step 1: Convert the product to a sum Let $I = \int_0^{\pi/2} \sin 2x \cos 3x \, dx$. Applying the identity with $A = 2x$ and $B = 3x$,

$$\sin 2x \cos 3x = \frac{1}{2}[\sin 5x + \sin(-x)] = \frac{1}{2}(\sin 5x - \sin x).$$

The minus sign is essential because sine is an odd function. Therefore

$$I = \frac{1}{2} \int_0^{\pi/2} (\sin 5x - \sin x) \, dx.$$

Step 2: Integrate and substitute both limits Integrating each term gives

$$I = \frac{1}{2} \left[-\frac{\cos 5x}{5} + \cos x \right]_0^{\pi/2}.$$

At the upper limit, $\cos(5\pi/2) = \cos(\pi/2) = 0$, so the bracket is zero. At the lower limit, both cosines equal one, and the bracket is $-1/5 + 1 = 4/5$. Hence

$$I = \frac{1}{2} \left(0 - \frac{4}{5} \right) = -\frac{2}{5}.$$

Step 3: Conclusion The required integral equals $\boxed{-\frac{2}{5}}$.

Quick Tip to Solve: Use $\sin 2x \cos 3x = (\sin 5x - \sin x)/2$ and subtract the lower-limit value carefully.

OR

23(b). Given $\frac{d}{dx} F(x) = \frac{1}{\sqrt{2x - x^2}}$ **and** $F(1) = 0$, **find** $F(x)$.

[2 marks]

Solution:

Concept:

- Complete the square to use $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$, valid for $-1 < u < 1$.
- An initial value determines the constant of integration after finding an antiderivative.

Step 1: Integrate the given derivative First rewrite the expression under the radical as

$$2x - x^2 = 1 - (x - 1)^2.$$

Quick Tip to Solve: Complete the square as $2x - x^2 = 1 - (x - 1)^2$, then use the inverse-sine integral.

The given derivative is real and finite when $2x - x^2 > 0$, that is, $0 < x < 2$. Put $u = x - 1$, so $du = dx$. Integration yields

$$F(x) = \int \frac{dx}{\sqrt{1 - (x-1)^2}} = \sin^{-1}(x-1) + C.$$

Here $\sin^{-1}$ denotes the principal inverse sine function, not the reciprocal of sine.

Step 2: Use the initial value and verify Substitute $x = 1$ and $F(1) = 0$:

$$0 = \sin^{-1}(0) + C = 0 + C, \quad C = 0.$$

To check the result, the chain rule gives

$$\frac{d}{dx} \sin^{-1}(x-1) = \frac{1}{\sqrt{1 - (x-1)^2}} = \frac{1}{\sqrt{2x - x^2}}.$$

Also, its value at $x = 1$ is zero, so both conditions are satisfied.

Step 3: Conclusion $F(x) = \sin^{-1}(x-1)$ for $0 < x < 2$.

Quick Tip to Solve: Complete the square as $2x - x^2 = 1 - (x-1)^2$, then use the inverse-sine integral.

24. Find the position vector of point C which divides the line segment joining points A and B having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio $4 : 1$ externally. Further, find $|\vec{AB}| : |\vec{BC}|$. [2 marks]

Solution:

Concept:

- For position vectors $\vec{a}, \vec{b}$, external division in the ratio $AC : CB = m : n$ gives $\vec{c} = (m\vec{b} - n\vec{a}) / (m - n)$.
- The displacement vector is obtained by subtraction: $\vec{AB} = \vec{b} - \vec{a}$, and lengths are magnitudes of displacement vectors.

Step 1: Find the position vector of C Here $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$, $m = 4$ and $n = 1$. Using external division,

$$\begin{aligned} \vec{c} &= \frac{4\vec{b} - \vec{a}}{4 - 1} \\ &= \frac{4(-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{3} \\ &= \frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}. \end{aligned}$$

The denominator is a difference because the point divides externally.

Step 2: Find the required length ratio Subtracting position vectors,

$$\vec{AB} = -2\hat{i} - \hat{j} + 2\hat{k}, \quad |\vec{AB}| = \sqrt{4 + 1 + 4} = 3.$$

Quick Tip to Solve: For external division in 4:1, use $(4b - a)/3$; then observe that BC equals AB divided by 3.

Similarly,

$$\overrightarrow{BC} = \vec{c} - \vec{b} = \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{3} = \frac{1}{3}\overrightarrow{AB}.$$

Thus $|\overrightarrow{BC}| = 1$, and $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 3 : 1$. This also confirms that C lies beyond B and $AC : CB = (3 + 1) : 1 = 4 : 1$.

Step 3: Conclusion The position vector of C is $\frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}$, and $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 3 : 1$.

Quick Tip to Solve: For external division in 4:1, use $(4b - a)/3$; then observe that BC equals AB divided by 3.

25. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Prove that $|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$. State the condition under which equality holds, i.e., $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$. [2 marks]

Solution:

Concept:

- For non-zero vectors making an angle $\theta \in [0, \pi]$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.
- The sine satisfies $0 \leq \sin\theta \leq 1$, with equality to one only at $\theta = \pi/2$.

Step 1: Apply the magnitude formula Since both vectors are non-zero, their angle is defined and the product of their magnitudes is positive. Hence

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \leq |\vec{a}||\vec{b}|.$$

This follows by multiplying the bound on sine by the positive product of magnitudes.

Step 2: Check the equality condition Equality is possible precisely when $|\vec{a}||\vec{b}|\sin\theta = |\vec{a}||\vec{b}|$. Dividing by the non-zero product gives $\sin\theta = 1$, so $\theta = \pi/2$. Conversely, perpendicular vectors have sine equal to one and attain equality. Parallel vectors give zero cross product and cannot attain equality for these non-zero vectors.

Step 3: Conclusion Therefore $|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$. Equality holds if and only if $\vec{a} \perp \vec{b}$, equivalently $\vec{a} \cdot \vec{b} = 0$.

Quick Tip to Solve: Reduce a cross-product magnitude inequality to the bound on the sine of the angle between the vectors.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). If $x \cos(p + y) + \cos p \sin(p + y) = 0$, prove that $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, where p is a constant.

[3 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x , while p remains constant.
- The chain rule gives $\frac{d}{dx} \sin(p + y) = \cos(p + y)y'$ and $\frac{d}{dx} \cos(p + y) = -\sin(p + y)y'$.

Step 1: Differentiate the given relation Write $t = p + y$ for brevity, so $dt/dx = y'$. Applying the product and chain rules to $x \cos t + \cos p \sin t = 0$ gives

$$\cos t - x \sin t y' + \cos p \cos t y' = 0.$$

Collecting the terms containing the derivative,

$$(\cos p \cos t - x \sin t)y' = -\cos t.$$

The derivative of $\cos p$ is zero because p is constant.

Step 2: Use the original equation to eliminate x Multiply the differentiated equation by $\cos t$:

$$(\cos p \cos^2 t - x \sin t \cos t)y' = -\cos^2 t.$$

From the original equation, $x \cos t = -\cos p \sin t$. Substitution into the coefficient yields

$$\begin{aligned}(\cos p \cos^2 t + \cos p \sin^2 t)y' &= -\cos^2 t, \\ \cos p y' &= -\cos^2 t.\end{aligned}$$

Here we used $\sin^2 t + \cos^2 t = 1$. This derivation does not divide by either $\cos t$ or $\cos p$, so it applies wherever the implicit relation defines a differentiable function.

Step 3: Conclusion Replacing t by $p + y$, we obtain $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, as required.

Quick Tip to Solve: Multiply the differentiated equation by cosine before substitution; this avoids division by a possibly zero factor.

26(b). Find the values of a and b so that the function f defined as

$$f(x) = \begin{cases} \frac{x-2}{|x-2|} + a, & x < 2, \\ a + b, & x = 2, \\ \frac{x-2}{|x-2|} + b, & x > 2 \end{cases}$$

is a continuous function.

[3 marks]

Solution:

Concept:

- Continuity at $x = 2$ requires $\lim_{x \rightarrow 2^-} f(x) = f(2) = \lim_{x \rightarrow 2^+} f(x)$.
- For $x < 2$, $|x - 2| = -(x - 2)$; for $x > 2$, $|x - 2| = x - 2$.

Step 1: Evaluate the one-sided limits On the left of 2, $x - 2$ is negative, so

$$\frac{x-2}{|x-2|} = -1, \quad f(x) = a - 1.$$

Therefore $\lim_{x \rightarrow 2^-} f(x) = a - 1$. On the right of 2, $x - 2$ is positive, giving

$$\frac{x-2}{|x-2|} = 1, \quad f(x) = b + 1.$$

Hence $\lim_{x \rightarrow 2^+} f(x) = b + 1$. At the joining point, the middle branch gives $f(2) = a + b$; no quotient is evaluated there.

Step 2: Impose continuity and verify The left limit must equal the value at 2:

$$a - 1 = a + b \implies b = -1.$$

The right limit must also equal that value:

$$b + 1 = a + b \implies a = 1.$$

For these choices, the left branch is $-1 + 1 = 0$, the middle value is $1 - 1 = 0$, and the right branch is $1 - 1 = 0$. Thus both limits and the function value agree. The equations determine both constants uniquely: the first isolates b and the second isolates a independently. Each branch away from 2 is constant and therefore continuous; verifying the join establishes continuity on the whole real line.

Step 3: Conclusion The required values are $a = 1, b = -1$. With these values, $f(x) = 0$ for every real x .

Quick Tip to Solve: Resolve the absolute value separately on each side, then equate both limits to the explicitly defined middle value.

27(a). Find the intervals in which the function $f(x) = \frac{\log x}{x}$ is strictly increasing or strictly decreasing. [3 marks]

Solution:

Concept:

- The real logarithm requires $x > 0$, so the domain is $(0, \infty)$. Here $\log x$ denotes the natural logarithm.
- A positive derivative implies strict increase on an interval; a negative derivative implies strict decrease.

Step 1: Differentiate on the domain Using the quotient rule,

$$f'(x) = \frac{x(1/x) - \log x}{x^2} = \frac{1 - \log x}{x^2}, \quad x > 0.$$

The denominator is strictly positive throughout the domain, so the sign of the derivative depends only on $1 - \log x$. Solving $f'(x) = 0$ gives $\log x = 1$, hence the sole stationary point is $x = e$. Neither zero nor negative numbers belong to the domain and therefore cannot be included in the sign analysis.

Step 2: Determine the derivative sign on each interval For $0 < x < e$, monotonicity of the logarithm gives $\log x < 1$, and therefore $f'(x) > 0$. For $x > e$, $\log x > 1$, and therefore $f'(x) < 0$. Thus the derivative changes from positive to negative at e , consistent with a maximum there. To justify strictness, for any two points in either interval the mean value theorem expresses their change in function value as their positive separation multiplied by a derivative of the stated sign.

Step 3: Conclusion The function is strictly increasing on $(0, e)$ and strictly decreasing on (e, ∞) . The stationary endpoint e may also be included in each respective monotonic interval.

Quick Tip to Solve: Determine the logarithm domain first, then use the positive denominator to read the derivative sign directly from its numerator.

OR

27(b). Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \frac{x}{2} + \frac{2}{x}$, on the interval $[1, 2]$. [3 marks]

Solution:

Concept:

- A function continuous on a closed interval attains its absolute maximum and minimum at endpoints or interior critical points.
- The derivative sign determines monotonicity: a negative derivative throughout an interval implies that the function is strictly decreasing there.

Step 1: Differentiate the function The printed function is $f(x) = x/2 + 2/x$. It is continuous on $[1, 2]$

Quick Tip to Solve: Include both endpoints when finding absolute extrema on a closed interval, even if the derivative vanishes at an endpoint.

and differentiable on $(1, 2)$, since the denominator never vanishes there. Differentiating gives

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}.$$

The equation $f'(x) = 0$ gives $x = \pm 2$. There is no critical point in the open interval $(1, 2)$: 2 is an endpoint, while -2 is outside the specified interval.

Step 2: Check the derivative sign For $1 < x < 2$, the numerator $x^2 - 4$ is negative and the denominator $2x^2$ is positive. Consequently $f'(x) < 0$ throughout the interior. The function therefore decreases from the left endpoint to the right endpoint. The stationary value at the right endpoint must still be included in the comparison of candidates.

Step 3: Evaluate the endpoints With no interior critical point, the endpoints constitute the complete candidate list. Their values are

$$\begin{aligned} f(1) &= \frac{1}{2} + 2 = \frac{5}{2}, \\ f(2) &= \frac{2}{2} + \frac{2}{2} = 2. \end{aligned}$$

These evaluations agree with the strictly decreasing behaviour already established. Continuity on the closed interval guarantees attainment of the extrema, so comparing these two candidates gives the required absolute values.

Step 4: Conclusion The absolute maximum is $\boxed{5/2}$ at $x = 1$, and the absolute minimum is $\boxed{2}$ at $x = 2$.

Quick Tip to Solve: Include both endpoints when finding absolute extrema on a closed interval, even if the derivative vanishes at an endpoint.

28. Find: $\int \frac{x^2 + 1}{(x^2 + 2)(x^2 + 4)} dx$

[3 marks]

Solution:

Concept:

- A rational function with distinct quadratic factors can be decomposed into simpler fractions by equating coefficients of equal powers of x .
- For $a > 0$, $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}(x/a) + C$, where C is an arbitrary constant.

Step 1: Determine the partial fractions Write the integrand in the form

$$\frac{x^2 + 1}{(x^2 + 2)(x^2 + 4)} = \frac{A}{x^2 + 2} + \frac{B}{x^2 + 4}.$$

Multiplying by the denominator yields

$$x^2 + 1 = A(x^2 + 4) + B(x^2 + 2) = (A + B)x^2 + 4A + 2B.$$

Comparing the coefficient of x^2 and the constant term gives $A + B = 1$ and $4A + 2B = 1$. Subtracting

Quick Tip to Solve: Match both the quadratic coefficient and the constant term before integrating; the inverse-tangent scale factors are different here.

twice the first equation from the second gives $2A = -1$. Thus $A = -1/2$ and $B = 3/2$. This decomposition is valid for every real x , since both quadratic factors are positive.

Step 2: Integrate each standard term By linearity of integration, the required integral becomes

$$-\frac{1}{2} \int \frac{dx}{x^2 + 2} + \frac{3}{2} \int \frac{dx}{x^2 + 4}.$$

In the first term take $a = \sqrt{2}$, and in the second take $a = 2$. Applying the inverse-tangent formula gives

$$-\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{3}{4} \tan^{-1}\left(\frac{x}{2}\right) + C.$$

Only one arbitrary constant is needed after adding the two antiderivatives.

Step 3: Verify the antiderivative Differentiating the expression gives $-1/[2(x^2 + 2)] + 3/[2(x^2 + 4)]$. Combining these terms has numerator $-(x^2 + 4) + 3(x^2 + 2) = 2x^2 + 2$ over denominator $2(x^2 + 2)(x^2 + 4)$, reproducing the original integrand exactly.

Step 4: Conclusion

$$-\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{3}{4} \tan^{-1}\left(\frac{x}{2}\right) + C$$

Quick Tip to Solve: Match both the quadratic coefficient and the constant term before integrating; the inverse-tangent scale factors are different here.

29(a). Find: $\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx$

[3 marks]

Solution:

Concept:

- The identities $1 + \cos 2x = 2 \cos^2 x$ and $\sin 2x = 2 \sin x \cos x$ convert the given quotient into a combination of tangent and secant functions.
- The product rule gives $\frac{d}{dx}[e^x g(x)] = e^x [g(x) + g'(x)]$, so an integrand of this form can be integrated directly.

Step 1: Simplify the trigonometric quotient The original integrand is defined where $1 + \cos 2x \neq 0$, equivalently $\cos x \neq 0$. On any interval avoiding those zeros, the double-angle identities give

$$\begin{aligned} \frac{2 + \sin 2x}{1 + \cos 2x} &= \frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \\ &= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} \\ &= \sec^2 x + \tan x. \end{aligned}$$

Thus the integral is $\int e^x (\tan x + \sec^2 x) dx$. Keeping both terms is essential: the constant 2 in the

Quick Tip to Solve: Replace the double-angle denominator by twice cosine squared, then look for the pair $\tan x$ and its derivative secant squared.

numerator contributes $\sec^2 x$, not a constant after division.

Step 2: Recognise an exact product derivative Take $g(x) = \tan x$, for which $g'(x) = \sec^2 x$. By the product rule,

$$\frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x = e^x (\tan x + \sec^2 x).$$

This is precisely the simplified integrand, so integrating restores the product and introduces an arbitrary constant C . There is no need to integrate the two terms separately or to apply repeated integration by parts.

Step 3: Check against the printed expression Recombining the trigonometric terms gives

$$\sec^2 x + \tan x = \frac{1 + \sin x \cos x}{\cos^2 x} = \frac{2 + \sin 2x}{1 + \cos 2x}.$$

Therefore the derivative of the proposed antiderivative agrees with the original quotient, including the exponential factor. The answer holds on every interval where the printed integrand is defined; its integration constant may be chosen independently on separate such intervals.

Step 4: Conclusion

$$\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx = e^x \tan x + C$$

Quick Tip to Solve: Replace the double-angle denominator by twice cosine squared, then look for the pair $\tan x$ and its derivative secant squared.

OR

29(b). Evaluate: $\int_0^{\pi/4} \frac{1}{\sin x + \cos x} dx$

[3 marks]

Solution:

Concept:

- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ to convert the denominator into one trigonometric function.
- The standard antiderivative is $\int \sec t dt = \log |\sec t + \tan t| + C$; change the limits when substituting in a definite integral.

Step 1: Convert the integrand and change the limits Write the required integral as I . The denominator is positive throughout the interval, so there is no singularity. The angle-difference identity gives

$$I = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec(x - \pi/4) dx.$$

Put $t = x - \pi/4$. Then $dt = dx$; at $x = 0$, $t = -\pi/4$, and at $x = \pi/4$, $t = 0$. Hence

$$I = \frac{1}{\sqrt{2}} \int_{-\pi/4}^0 \sec t dt.$$

Quick Tip to Solve: Convert a sum of sine and cosine to a shifted cosine before looking for the antiderivative.

Keeping these new limits avoids mixing the original variable with the substituted variable when evaluating the antiderivative.

Step 2: Evaluate the standard integral Integrating and substituting both endpoints gives

$$\begin{aligned} I &= \frac{1}{\sqrt{2}} [\log |\sec t + \tan t|]_{-\pi/4}^0 \\ &= \frac{1}{\sqrt{2}} [\log 1 - \log(\sqrt{2} - 1)] \\ &= \frac{1}{\sqrt{2}} \log \left(\frac{1}{\sqrt{2} - 1} \right). \end{aligned}$$

We used $\sec(-\pi/4) = \sqrt{2}$ and $\tan(-\pi/4) = -1$. Rationalising the fraction yields $1/(\sqrt{2} - 1) = \sqrt{2} + 1$. Both logarithm arguments at the endpoints are positive, so the absolute-value symbols cause no sign ambiguity here.

Step 3: Conclusion The value is $\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$, with log denoting the natural logarithm.

Quick Tip to Solve: Convert a sum of sine and cosine to a shifted cosine before looking for the antiderivative.

30. Solve the following linear programming problem graphically: Maximise $z = 4x + 3y$, subject to the constraints $x + y \leq 800$, $2x + y \leq 1000$, $x \leq 400$, $x, y \geq 0$. [3 marks]

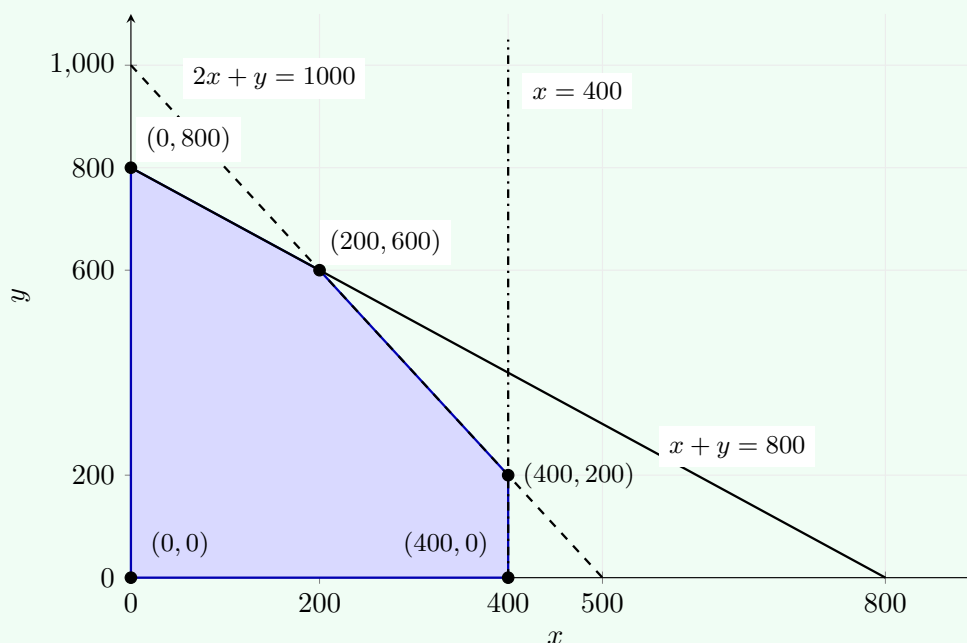
Solution:

Concept:

- Each linear inequality defines a half-plane. Their intersection in the first quadrant is the feasible region.
- A linear objective on a bounded polygon attains a maximum at a corner point. Evaluate $z = 4x + 3y$ at every feasible vertex.

Step 1: Draw the boundaries and shade the common region The line $x + y = 800$ has intercepts $(800, 0)$ and $(0, 800)$. The line $2x + y = 1000$ has intercepts $(500, 0)$ and $(0, 1000)$. Both inequalities select the side containing the origin. Draw the vertical line $x = 400$ and retain its left side, together with $x, y \geq 0$. Their common region, shown shaded, is a closed bounded polygon.

Quick Tip to Solve: Check every intersection against all inequalities before adding it to the corner-point table.



The shaded polygon is the feasible region.

Step 2: Find all feasible corner points On the axes the feasible extreme points are $(0, 0)$, $(400, 0)$ and $(0, 800)$. At $x = 400$, the boundary $2x + y = 1000$ gives $y = 200$, yielding $(400, 200)$. At the intersection of the two sloping boundaries,

$$x + y = 800, \quad 2x + y = 1000$$

subtraction gives $x = 200$, and then $y = 600$. Thus $(200, 600)$ is the remaining vertex. The intersection $(400, 400)$ of $x = 400$ with the first line is excluded because $2x + y = 1200 > 1000$.

Step 3: Compare the objective values at the vertices Substitution into the objective gives

(x, y)	$z = 4x + 3y$
$(0, 0)$	0
$(400, 0)$	1600
$(400, 200)$	2200
$(200, 600)$	2600
$(0, 800)$	2400

Since the region is bounded and nonempty, this comparison determines the global maximum. The largest listed value occurs at the intersection of the two sloping boundaries.

Step 4: Conclusion The maximum is $z = 2600$, attained at $x = 200, y = 600$.

Quick Tip to Solve: Check every intersection against all inequalities before adding it to the corner-point table.

31. The chances of P , Q and R getting selected as CEO of a company are in the ratio 4 : 1 : 2 respectively. The probabilities for the company to increase its profits from the previous year under the new CEO, P , Q or R are 0.3, 0.8 and 0.5 respectively. If the company increased the profits from the previous year, find the probability that it is due to the appointment of R as CEO. [3 marks]

Solution:

Concept:

- The law of total probability is $P(E) = \sum_i P(H_i)P(E | H_i)$, where the mutually exclusive events H_i exhaust all possibilities.
- Bayes' theorem reverses a conditional probability: $P(H_i | E) = P(H_i)P(E | H_i)/P(E)$, provided $P(E) > 0$.

Step 1: Define events and convert the selection ratio Let H_P, H_Q, H_R mean that P, Q, R , respectively, is appointed CEO, and let E mean that the company increases its profits from the previous year. Exactly one of the three is appointed, so these selection events form a partition. Their ratio 4 : 1 : 2 has total weight 7, giving

$$P(H_P) = \frac{4}{7}, \quad P(H_Q) = \frac{1}{7}, \quad P(H_R) = \frac{2}{7}.$$

The supplied success probabilities are conditional on the CEO:

$$P(E | H_P) = 0.3, \quad P(E | H_Q) = 0.8, \quad P(E | H_R) = 0.5.$$

Step 2: Calculate the total probability of increased profits Increased profits can occur under any of the three CEOs. Add the three disjoint joint probabilities:

$$\begin{aligned} P(E) &= \frac{4}{7}(0.3) + \frac{1}{7}(0.8) + \frac{2}{7}(0.5) \\ &= \frac{1.2 + 0.8 + 1}{7} = \frac{3}{7}. \end{aligned}$$

This denominator includes all possible appointments, rather than only the company succeeding under R .

Step 3: Apply Bayes' theorem for the observed profit increase The question asks for the probability of R 's appointment given that profits increased. Hence use the joint probability of R 's appointment and increased profits in the numerator:

$$P(H_R | E) = \frac{P(H_R)P(E | H_R)}{P(E)} = \frac{(2/7)(0.5)}{3/7} = \frac{1}{3}.$$

The prior probability $2/7$ alone does not answer the question, because the observation changes the relative likelihoods of the three appointments.

Step 4: Conclusion The required probability that R was appointed CEO is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: Multiply each selection probability by its success rate before normalising the required CEO's contribution.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32. A relation R on set $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ be defined as $R = \{(x, y) : x + y \text{ is an integer divisible by } 2\}$. Show that R is an equivalence relation. Also, write the equivalence class $[2]$.

[5 marks]

Solution:

Concept:

- A relation on A is an equivalence relation if it is reflexive, symmetric and transitive. Each property must hold for all the relevant elements of A .
- An integer is divisible by 2 exactly when it is even. Thus xRy means that $x + y = 2k$ for some integer k , or equivalently that x and y have the same parity.
- The equivalence class $[a]$ is the set of all elements of A related to a : $[a] = \{x \in A : xRa\}$.

Step 1: Prove reflexivity Let x be any element of A . Since every element of A is an integer, $x + x = 2x$ is an integer divisible by 2. Therefore $(x, x) \in R$ for every $x \in A$. This proves that R is reflexive, including at negative elements and at 0.

Step 2: Prove symmetry Let $x, y \in A$ and suppose that $(x, y) \in R$. By the definition of R , $x + y$ is divisible by 2. Addition is commutative, so $y + x = x + y$ is also divisible by 2. Consequently $(y, x) \in R$. Hence R is symmetric.

Step 3: Prove transitivity Let $x, y, z \in A$ with $(x, y) \in R$ and $(y, z) \in R$. There exist integers m, n such that $x + y = 2m$ and $y + z = 2n$. Adding these two equalities and subtracting $2y$ gives

$$x + z = (x + y) + (y + z) - 2y = 2(m + n - y).$$

Since m, n, y are integers, $m + n - y$ is an integer. Therefore $x + z$ is divisible by 2, which means $(x, z) \in R$. This proves transitivity; the subtraction of $2y$ is essential because the defining condition involves a sum.

Step 4: Determine the class containing 2 By definition, $[2] = \{x \in A : x + 2 \text{ is divisible by } 2\}$. Adding the even integer 2 preserves parity, so this condition holds exactly for the even elements of A . Listing only the elements actually in the given set gives $-4, -2, 0, 2, 4$. The remaining elements $-3, -1, 1, 3$ are odd and therefore cannot belong to $[2]$.

Step 5: Conclusion Since R is reflexive, symmetric and transitive, it is an equivalence relation on A . Its equivalence class containing 2 is $[2] = \{-4, -2, 0, 2, 4\}$.

Quick Tip to Solve: Recognise that an even sum means equal parity; nevertheless, explicitly verify all three relation properties before listing the even elements of the given set.

33(a). It is given that function $f(x) = x^4 - 62x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of a , hence obtain all other points where the given function $f(x)$ attains local maximum or local minimum values. [5 marks]

Solution:

Concept:

- For a differentiable function, a local extremum at an interior point $x = c$ requires $f'(c) = 0$. This necessary condition determines the unknown coefficient here.
- The second derivative test states that a stationary point is a local maximum when $f''(c) < 0$, and a local minimum when $f''(c) > 0$.
- All possible local extrema of a polynomial occur at the real roots of its first derivative, since the polynomial is differentiable for every real x .

Step 1: Use the specified local maximum to find the coefficient The given polynomial is $f(x) = x^4 - 62x^2 + ax + 9$. Differentiating with respect to x , treating a as a constant, yields

$$f'(x) = 4x^3 - 124x + a.$$

A local maximum is given at $x = 1$. Since this is an interior point of the polynomial's domain, the necessary condition is $f'(1) = 0$. Thus

$$4 - 124 + a = 0, \quad a = 120.$$

The constant term 9 does not affect the derivative or the locations of the stationary points.

Step 2: Find every stationary point for this value of a Substituting $a = 120$, the stationary-point equation becomes

$$\begin{aligned} f'(x) &= 4x^3 - 124x + 120 \\ &= 4(x^3 - 31x + 30) \\ &= 4(x - 1)(x^2 + x - 30) \\ &= 4(x - 1)(x + 6)(x - 5) = 0. \end{aligned}$$

The factor $x - 1$ follows from the known stationary point; the quadratic factor then splits using the numbers 6 and -5 , whose sum is 1 and product is -30 . Hence the complete list of stationary abscissae is $x = -6, 1, 5$. No other roots occur because this factorisation exhausts the cubic equation.

Step 3: Classify the stationary points Differentiating again gives $f''(x) = 12x^2 - 124$. At the three stationary points,

$$\begin{aligned} f''(-6) &= 12(36) - 124 = 308 > 0, \\ f''(1) &= 12 - 124 = -112 < 0, \\ f''(5) &= 12(25) - 124 = 176 > 0. \end{aligned}$$

Thus $x = -6$ and $x = 5$ give local minima, while $x = 1$ gives a local maximum. The negative second derivative at 1 also verifies that the coefficient obtained from the necessary condition really satisfies the given maximum condition.

Step 4: Obtain the coordinates of the extrema With $a = 120$, substitute into $f(x) = x^4 - 62x^2 +$

Quick Tip to Solve: After imposing $f'(1)=0$, factor the derivative using the known factor $x-1$; then use the second derivative at every root to distinguish maxima from minima.

$120x + 9$:

$$f(-6) = 1296 - 2232 - 720 + 9 = -1647,$$

$$f(1) = 1 - 62 + 120 + 9 = 68,$$

$$f(5) = 625 - 1550 + 600 + 9 = -316.$$

These values give the corresponding points on the graph, rather than only their abscissae.

Step 5: Conclusion $a = 120$. The other points of local extremum are the local minimum points $(-6, -1647)$ and $(5, -316)$. The given local maximum is $(1, 68)$, and there is no other local maximum.

Quick Tip to Solve: After imposing $f'(1)=0$, factor the derivative using the known factor $x-1$; then use the second derivative at every root to distinguish maxima from minima.

OR

33(b). The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum. [5 marks]

Solution:

Concept:

- For rectangular sides x, h , the perimeter is $2(x+h)$. Choose x as the side forming the circumference and h as the cylinder's height.
- A cylinder has circumference $2\pi r$ and volume $V = \pi r^2 h$, where r is the radius. The perimeter constraint reduces the volume to a function of one variable.
- A stationary point satisfies $V'(x) = 0$. A negative second derivative verifies a local maximum, and the sign of the first derivative determines whether it is the global maximum.

Step 1: Use the fixed perimeter Let x cm be the side wrapped around the cylinder, with the other side h cm becoming its height. Then

$$2(x+h) = 300, \quad h = 150 - x.$$

Both dimensions must be positive, so $0 < x < 150$. The circumference of the cylinder equals the wrapped side, giving $2\pi r = x$, hence $r = x/(2\pi)$. These definitions explicitly identify which side becomes the circumference and which becomes the height.

Step 2: Express and differentiate the volume Substitution into the volume formula yields

$$V(x) = \pi \left(\frac{x}{2\pi} \right)^2 (150 - x) = \frac{150x^2 - x^3}{4\pi}.$$

Therefore

$$V'(x) = \frac{300x - 3x^2}{4\pi} = \frac{3x(100 - x)}{4\pi}.$$

Setting the derivative equal to zero gives $x = 0$ or $x = 100$. The first value cannot represent a

Quick Tip to Solve: Write the circumference side first: maximize $x^2(150 - x)$, since its square controls the circular base area.

rectangular sheet of positive dimensions. Thus 100 is the only stationary value within the permissible interval.

Step 3: Check that the stationary value maximizes volume The second derivative is

$$V''(x) = \frac{300 - 6x}{4\pi}, \quad V''(100) = -\frac{75}{\pi} < 0.$$

Thus the stationary point is a local maximum. Also, $V'(x) > 0$ for $0 < x < 100$ and $V'(x) < 0$ for $100 < x < 150$; volume increases before this point and decreases afterwards. This proves that it is the absolute maximum over the possible sheets. As either dimension tends to zero, the volume tends to zero, agreeing with the derivative analysis. The other side is $h = 150 - 100 = 50$ cm. The perimeter check gives $2(100 + 50) = 300$ cm. The corresponding radius is $50/\pi$ cm, with cylinder height 50 cm.

Step 4: Conclusion The required dimensions are $100 \text{ cm} \times 50 \text{ cm}$, with the 100 cm side forming the circumference. The maximum volume is $125000/\pi \text{ cm}^3$.

Quick Tip to Solve: Write the circumference side first: maximize $x^2(150 - x)$, since its square controls the circular base area.

34. Using integration, find the area of the region enclosed between the circle $x^2 + y^2 = 16$ and the lines $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- The circle $x^2 + y^2 = 16$ has centre at the origin and radius 4. Its upper and lower boundaries are $y = \sqrt{16 - x^2}$ and $y = -\sqrt{16 - x^2}$.
- The area between two curves over $a \leq x \leq b$ is $\int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$. Here the vertical lines specify the integration limits.
- Symmetry about both coordinate axes allows the central strip's area to be calculated as four times the area in the first quadrant. The substitution $x = 4 \sin \theta$ evaluates the square-root integral.

Step 1: Identify the boundaries of the enclosed region We require the part inside the circle lying between $x = -2$ and $x = 2$. For each value of x in this interval, the vertical segment runs from the lower semicircle to the upper semicircle. Its length is

$$\sqrt{16 - x^2} - (-\sqrt{16 - x^2}) = 2\sqrt{16 - x^2}.$$

Thus the required area is

$$A = 2 \int_{-2}^2 \sqrt{16 - x^2} dx = 4 \int_0^2 \sqrt{16 - x^2} dx.$$

The integrand is even.

Step 2: Evaluate the integral by trigonometric substitution Put $x = 4 \sin \theta$, so $dx = 4 \cos \theta d\theta$. The limits $x = 0$ and $x = 2$ correspond to $\theta = 0$ and $\theta = \pi/6$. On this interval $\cos \theta$ is positive, hence

Quick Tip to Solve: Include the lower semicircle before using symmetry: the central strip has area $4 \int_0^2 \sqrt{16 - x^2} dx$.

$\sqrt{16 - x^2} = 4 \cos \theta$. Consequently,

$$\begin{aligned} A &= 64 \int_0^{\pi/6} \cos^2 \theta \, d\theta \\ &= 32 \int_0^{\pi/6} (1 + \cos 2\theta) \, d\theta \\ &= 32 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/6}. \end{aligned}$$

Here we used $\cos^2 \theta = (1 + \cos 2\theta)/2$.

Step 3: Substitute the limits and check the area Using $\sin(\pi/3) = \sqrt{3}/2$, we obtain

$$A = 32 \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) = \frac{16\pi}{3} + 8\sqrt{3}.$$

The result is positive and below the whole circle's area 16π , as expected.

Step 4: Conclusion The required area enclosed inside the circle between the two lines is $\frac{16\pi}{3} + 8\sqrt{3}$ square units.

Quick Tip to Solve: Include the lower semicircle before using symmetry: the central strip has area $4 \int_0^2 \sqrt{16 - x^2} \, dx$.

35(a). Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines. [5 marks]

Solution:

Concept:

- A line through a point $\vec{a}$ with nonzero direction vector $\vec{d}$ has vector equation $\vec{r} = \vec{a} + t\vec{d}$, where t is a real parameter.
- Perpendicular directions have zero scalar product. The cross product of two nonparallel direction vectors gives a direction perpendicular to both.

Step 1: Write the given lines in parametric form Let the common ratio on the first line be λ . Then

$$x = \lambda, \quad y = 1 + 2\lambda, \quad z = 2 + 3\lambda,$$

and its direction vector is $\vec{d}_1 = (1, 2, 3)$. On the second line the zero direction ratio means $x = 1$, rather than division by zero. With parameter μ ,

$$x = 1, \quad y = -3\mu, \quad z = 7 + 2\mu,$$

and the direction vector is $\vec{d}_2 = (0, -3, 2)$.

Quick Tip to Solve: Use the zero direction ratio to set $x=1$ immediately, then cross the two direction vectors and check both dot products.

Step 2: Determine and check the intersection Equality of x-coordinates gives $\lambda = 1$, so the first line gives $y = 3, z = 5$. On the second line, $-3\mu = 3$ gives $\mu = -1$; then $7 + 2\mu = 5$ also agrees. Thus the intersection is $P = (1, 3, 5)$. Checking all three coordinates establishes that the lines actually intersect in space.

Step 3: Find a common perpendicular direction Compute the cross product:

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & -3 & 2 \end{vmatrix} = 13\hat{i} - 2\hat{j} - 3\hat{k}.$$

This nonzero vector shows that the given directions are not parallel. Check both scalar products: $(13, -2, -3) \cdot (1, 2, 3) = 13 - 4 - 9 = 0$, and $(13, -2, -3) \cdot (0, -3, 2) = 6 - 6 = 0$. Thus this direction is perpendicular to each given line. Reversing the cross product would give the negative direction, which describes the same required line.

Step 4: Form the line through the intersection Through P , with the perpendicular direction obtained above, the vector equation is

$$\vec{r} = (\hat{i} + 3\hat{j} + 5\hat{k}) + t(13\hat{i} - 2\hat{j} - 3\hat{k}), \quad t \in \mathbb{R}.$$

At $t = 0$ it passes through the common point. Its direction has zero scalar product with each given direction, verifying both perpendicularity conditions.

Step 5: Conclusion The required line is $\boxed{\frac{x-1}{13} = \frac{y-3}{-2} = \frac{z-5}{-3}}.$

Quick Tip to Solve: Use the zero direction ratio to set $x=1$ immediately, then cross the two direction vectors and check both dot products.

OR

35(b). Two vertices of the parallelogram $ABCD$ are given as $A(-1, 2, 1)$ and $B(1, -2, 5)$. If the equation of the line passing through C and D is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between sides AB and CD . Hence, find the area of parallelogram $ABCD$. [5 marks]

Solution:

Concept:

- Opposite sides of a parallelogram are parallel. Their separation is the perpendicular distance between their supporting lines.
- For a point A and a line through P with direction vector $\vec{d}$, the perpendicular distance is $h = \frac{|\vec{AP} \times \vec{d}|}{|\vec{d}|}$. The area of a parallelogram is its base length times its perpendicular height.

Quick Tip to Solve: Read a point on CD by setting its common ratio to zero, then use the cross product for height and multiply by the length of AB .

Step 1: Find the base vector and verify parallelism The vector along side AB is

$$\overrightarrow{AB} = B - A = (1 + 1, -2 - 2, 5 - 1) = (2, -4, 4).$$

The printed equation of the line through C, D is

$$\frac{x - 4}{1} = \frac{y + 7}{-2} = \frac{z - 8}{2}.$$

Therefore a point on this line is $P = (4, -7, 8)$, and its direction is $\vec{d} = (1, -2, 2)$. Since $\overrightarrow{AB} = 2\vec{d}$, the two sides are parallel, consistent with the given parallelogram. Their distance is constant and can be obtained from any point on AB , in particular A .

Step 2: Compute the vector joining the parallel lines From $A = (-1, 2, 1)$ to $P = (4, -7, 8)$,

$$\overrightarrow{AP} = (5, -9, 7).$$

This connecting vector need not be perpendicular to the sides. Its cross product with the side direction removes the component parallel to the sides: if the perpendicular separation is h , the magnitude of the cross product is $h|\vec{d}|$. Thus it supplies the required height without assuming that AP itself is a height.

Step 3: Evaluate the cross product and distance Expanding the determinant gives

$$\begin{aligned}\overrightarrow{AP} \times \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -9 & 7 \\ 1 & -2 & 2 \end{vmatrix} \\ &= (-18 + 14)\hat{i} - (10 - 7)\hat{j} + (-10 + 9)\hat{k} \\ &= -4\hat{i} - 3\hat{j} - \hat{k}.\end{aligned}$$

Consequently,

$$|\overrightarrow{AP} \times \vec{d}| = \sqrt{16 + 9 + 1} = \sqrt{26}, \quad |\vec{d}| = \sqrt{1 + 4 + 4} = 3.$$

Dividing these magnitudes gives the perpendicular height $h = \sqrt{26}/3$. This is positive, so the two supporting lines are distinct.

Step 4: Use the side length as the base The length of the base is

$$|AB| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{36} = 6.$$

Multiplying the base by the perpendicular height gives

$$\text{Area} = 6 \left(\frac{\sqrt{26}}{3} \right) = 2\sqrt{26}.$$

Equivalently, the area is $|\overrightarrow{AB} \times \overrightarrow{AP}|$: replacing the adjacent-side vector by AP adds only a vector parallel to the base and therefore does not change the cross product magnitude.

Quick Tip to Solve: Read a point on CD by setting its common ratio to zero, then use the cross product for height and multiply by the length of AB .

Step 5: Conclusion The distance between sides AB and CD is $\frac{\sqrt{26}}{3}$ units, and the area of $ABCD$ is $2\sqrt{26}$ square units.

Quick Tip to Solve: Read a point on CD by setting its common ratio to zero, then use the cross product for height and multiply by the length of AB .

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1: Self-study helps students to build confidence in learning. It boosts the self-esteem of the learners. Recent surveys suggested that close to 50% learners were self-taught using internet resources and upskilled themselves.

A student may spend 1 hour to 6 hours in a day in upskilling self. The probability distribution of the number of hours spent by a student is given below:

$$P(X = x) = \begin{cases} kx^2, & \text{for } x = 1, 2, 3 \\ 2kx, & \text{for } x = 4, 5, 6 \\ 0, & \text{otherwise} \end{cases} \text{ where } x \text{ denotes the number of hours.} \quad [4 \text{ marks}]$$

36(i). Express the probability distribution given above in the form of a probability distribution table. [1 mark]

Solution:

Concept:

- A probability distribution table lists each possible value x of the random variable X and its probability $P(X = x)$.
- Use kx^2 for $x = 1, 2, 3$, and $2kx$ for $x = 4, 5, 6$.

Step 1: Evaluate the first branch The student can spend any of the six listed whole numbers of hours. For the first three values, substitution in the quadratic branch gives $P(X = 1) = k$, $P(X = 2) = 4k$, and $P(X = 3) = 9k$.

Step 2: Evaluate the second branch and arrange the table For the remaining values use the linear branch: $P(X = 4) = 8k$, $P(X = 5) = 10k$, and $P(X = 6) = 12k$. Thus the six values are exhaustive; all other values have probability zero. Keeping k symbolic is sufficient for the requested table.

Step 3: Conclusion

x	1	2	3	4	5	6
$P(X = x)$	k	$4k$	$9k$	$8k$	$10k$	$12k$

Quick Tip to Solve: Apply the branch condition before substituting: the formula changes at four hours.

36(ii). Find the value of k .

[1 mark]

Solution:

Concept:

- For a discrete probability distribution, the probabilities over all possible outcomes sum to one: $\sum_x P(X = x) = 1$.
- Here the probabilities are $k, 4k, 9k, 8k, 10k, 12k$, and all other probabilities vanish.

Step 1: Add the six nonzero probabilities The first branch contributes $k(1^2 + 2^2 + 3^2) = 14k$. The second branch contributes $2k(4 + 5 + 6) = 30k$. Both branches must be included because the possible numbers of hours run from one through six.

Step 2: Use normalization Consequently,

$$14k + 30k = 1 \implies 44k = 1.$$

This also makes every entry nonnegative, as required of a probability distribution. No extra probability is assigned outside these six outcomes.

Step 3: Conclusion The required constant is $k = \frac{1}{44}$.

Quick Tip to Solve: Add the probability coefficients first; their sum is the reciprocal of the normalizing constant.

36(iii)(a). Find the mean number of hours spent by the student.

[2 marks]

Solution:

Concept:

- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$; it is the probability-weighted average of its values.
- The normalization condition gives $k(1 + 4 + 9 + 8 + 10 + 12) = 1$, hence $k = 1/44$.

Step 1: Form the weighted contributions For $x = 1, 2, 3$, the terms $xP(X = x) = kx^3$ are $k, 8k, 27k$. For $x = 4, 5, 6$, the terms $xP(X = x) = 2kx^2$ are $32k, 50k, 72k$. Multiplying each probability by its own value is essential: merely adding the probabilities would give one, rather than the mean.

Step 2: Sum and substitute the normalizing constant

$$\begin{aligned} E(X) &= k(1 + 8 + 27 + 32 + 50 + 72) \\ &= 190k = \frac{190}{44} = \frac{95}{22}. \end{aligned}$$

The mean is approximately 4.318 hours. It lies between the minimum one hour and the maximum six hours, providing a useful check. A mean need not itself be an integer, even though every individual outcome here is an integer number of hours.

Quick Tip to Solve: For the mean, multiply each probability coefficient by its corresponding hour before adding.

Step 3: Conclusion The mean number of hours spent is $\boxed{\frac{95}{22} \text{ hours}}$.

Quick Tip to Solve: For the mean, multiply each probability coefficient by its corresponding hour before adding.

OR

36(iii)(b). Find $P(1 < X < 6)$.

[2 marks]

Solution:

Concept:

- For mutually exclusive values of a discrete random variable, the probability of an event is the sum of the probabilities of its included values.
- Normalization gives $44k = 1$, so $k = 1/44$. Strict inequalities exclude both endpoints.

Step 1: Identify the permitted hours The event $1 < X < 6$ contains exactly $X = 2, 3, 4, 5$. The values one and six are excluded because the inequalities are strict; there are no additional possible values between the listed whole numbers in this distribution. The relevant probabilities are $4k, 9k, 8k, 10k$, using the appropriate branch for each value.

Step 2: Add the included probabilities

$$\begin{aligned} P(1 < X < 6) &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ &= (4 + 9 + 8 + 10)k = 31k = \frac{31}{44}. \end{aligned}$$

As an independent check, the excluded endpoints have combined probability $k + 12k = 13/44$. Subtracting this from one again gives $1 - 13/44 = 31/44$, consistent with the direct sum.

Step 3: Conclusion $\boxed{P(1 < X < 6) = \frac{31}{44}}$.

Quick Tip to Solve: With strict bounds, remove the endpoint probabilities; here use one minus the probabilities at one and six.

37. Case Study - 2: A bacteria sample of certain number of bacteria is observed to grow exponentially in a given amount of time. Using exponential growth model, the rate of growth of this sample of bacteria is calculated.

The differential equation representing the growth of bacteria is given as: $\frac{dP}{dt} = kP$, where P is the population of bacteria at any time ' t '. [4 marks]

37(i). Obtain the general solution of the given differential equation and express it as an exponential function of ' t '. [2 marks]

Solution:

Concept:

- The equation $dP/dt = kP$, with constant k , is separable: for $P \neq 0$, write $dP/P = k dt$.
- The integral $\int dP/P = \ln |P|$ leads to an exponential function; a positive population uses $P > 0$.

Step 1: Separate and integrate For a nonzero population, divide the differential equation by P and integrate both sides:

$$\int \frac{dP}{P} = \int k dt \implies \ln |P| = kt + C.$$

Here C is the arbitrary constant of integration. The absolute value makes the antiderivative valid for either sign of a nonzero mathematical solution.

Step 2: Express the result exponentially and verify Exponentiation gives $|P| = e^C e^{kt}$, so an arbitrary signed constant A gives $P = Ae^{kt}$. Differentiation verifies this family: $dP/dt = kAe^{kt} = kP$. The separately excluded solution $P = 0$ is also included by taking $A = 0$. In the bacteria context $A = P(0) > 0$, and exponential growth corresponds to $k > 0$.

Step 3: Conclusion The general solution is $P(t) = Ae^{kt}$, with arbitrary constant A ; for the bacteria sample, $A = P(0)$.

Quick Tip to Solve: After integrating the logarithm, absorb its exponential into a new constant and check by differentiation.

37(ii). If population of bacteria is 1000 at $t = 0$, and 2000 at $t = 1$, find the value of k . [2 marks]

Solution:

Concept:

- The solution of $dP/dt = kP$ is $P(t) = Ae^{kt}$, where $A = P(0)$ is the initial population.
- A known later population fixes the growth constant through the ratio $P(t)/P(0) = e^{kt}$.

Step 1: Use the initial population The condition $P(0) = 1000$ gives $1000 = Ae^0 = A$. Thus the particular population model is $P(t) = 1000e^{kt}$. Substituting at time zero first removes the arbitrary constant, so the later condition can determine k directly.

Step 2: Use the population at time one At $t = 1$,

$$2000 = 1000e^k \implies e^k = 2.$$

Taking the natural logarithm on both sides gives $k = \ln 2$. This is positive, as expected for growth.

Quick Tip to Solve: Divide the later population by the initial population before taking the natural logarithm.

The time unit is the one used for the given observations; k has the corresponding inverse-time unit. Checking the model $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ gives both $P(0) = 1000$ and $P(1) = 2000$.

Step 3: Conclusion The growth constant is $k = \ln 2$ (approximately 0.6931 per unit time).

Quick Tip to Solve: Divide the later population by the initial population before taking the natural logarithm.

38. Case Study - 3: A scholarship is a sum of money provided to a student to help him or her pay for education. Some students are granted scholarships based on their academic achievements, while others are rewarded based on their financial needs.

Every year a school offers scholarships to girl children and meritorious achievers based on certain criteria. In the session 2022 - 23, the school offered monthly scholarship of ₹3,000 each to some girl students and ₹4,000 each to meritorious achievers in academics as well as sports.

In all, 50 students were given the scholarships and monthly expenditure incurred by the school on scholarships was ₹1,80,000. [4 marks]

38(i). Express the given information algebraically using matrices. [1 mark]

Solution:

Concept:

- A system of linear equations is written as $AX = B$, where A contains coefficients, X contains the unknowns and B contains the constants.
- Each row of the matrix product represents one equation; keep the variable order the same in both rows.

Step 1: Define the unknowns and form the equations Let x be the number of girl-child scholarships and y the number of meritorious-student scholarships. The two categories contain 50 recipients in total, so $x + y = 50$. Since their monthly amounts are respectively 3000 and 4000 rupees, the expenditure equation is $3000x + 4000y = 180000$.

Step 2: Arrange the equations in matrix form Using the order x, y , the coefficient rows are $(1, 1)$ and $(3000, 4000)$. Thus the required representation is

$$\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

Multiplication of the first row gives the student count, while the second row gives the monthly expenditure in rupees.

Step 3: Conclusion Hence $AX = B$, with $A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}$.

Quick Tip to Solve: Write the student-count row first, then put the scholarship amounts in the same variable order.

38(ii). Check whether the system of matrix equations so obtained is consistent or not.

[1 mark]

Solution:

Concept:

- A square coefficient matrix A is invertible when $|A| \neq 0$. Then $AX = B$ has the unique solution $X = A^{-1}B$.
- A system with a unique solution is consistent, since its equations can be satisfied simultaneously.

Step 1: Identify the coefficient matrix Let x and y denote the respective numbers of girl-child and meritorious-student scholarships. From $x + y = 50$ and $3000x + 4000y = 180000$, the coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}.$$

Step 2: Evaluate its determinant and apply the criterion The determinant is

$$|A| = (1)(4000) - (1)(3000) = 1000 \neq 0.$$

Therefore A^{-1} exists. For the given constant column $B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}$, multiplication by A^{-1} gives one definite column X , so both original equations have a common, unique solution.

Step 3: Conclusion The system is consistent and has a unique solution, because $|A| = 1000 \neq 0$.

Quick Tip to Solve: For this two-variable system, test the coefficient determinant before attempting to find the unknowns.

38(iii)(a). Find the number of scholarships of each kind given by the school, using matrices.

[2 marks]

Solution:

Concept:

- For $AX = B$ with $|A| \neq 0$, the solution is $X = A^{-1}B$.
- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

Step 1: Form the matrix equation Let x be the number of girl-child scholarships and y the number of meritorious-student scholarships. The two categories contain 50 recipients in total, so $x + y = 50$. Since their monthly amounts are respectively 3000 and 4000 rupees, the expenditure equation is $3000x + 4000y = 180000$. Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}, \quad AX = B.$$

Quick Tip to Solve: Use the same variable order throughout the inverse calculation, and check both original totals.

Step 2: Find the inverse and multiply Since $|A| = 4000 - 3000 = 1000$,

$$A^{-1} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{1000} \begin{pmatrix} 4000(50) - 180000 \\ -3000(50) + 180000 \end{pmatrix} \\ &= \begin{pmatrix} 20 \\ 30 \end{pmatrix}. \end{aligned}$$

The values also satisfy $20 + 30 = 50$ and $3000(20) + 4000(30) = 180000$, verifying both constraints.

Step 3: Conclusion The school gave 20 girl-child scholarships and 30 meritorious-student scholarships.

Quick Tip to Solve: Use the same variable order throughout the inverse calculation, and check both original totals.

OR

38(iii)(b). Had the amount of scholarship given to each girl child and meritorious student been interchanged, what would be the monthly expenditure incurred by the school? [2 marks]

Solution:

Concept:

- The scholarship counts come from $AX = B$, where the original amounts are used to form A .
- After the amounts are interchanged, expenditure is the row-column product $\begin{pmatrix} 4000 & 3000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; the recipient counts stay unchanged.

Step 1: Determine the original recipient counts Let x be the number of girl-child scholarships and y the number of meritorious-student scholarships. The two categories contain 50 recipients in total, so $x + y = 50$. Since their monthly amounts are respectively 3000 and 4000 rupees, the expenditure equation is $3000x + 4000y = 180000$. In matrix form,

$$\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

The coefficient determinant is 1000, so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 180000 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

Quick Tip to Solve: Interchange the amounts only after finding the original counts; the two groups still contain 20 and 30 recipients.

Step 2: Apply the interchanged monthly amounts Each of the 20 girl-child recipients would now receive 4000 rupees, and each of the 30 meritorious recipients would receive 3000 rupees. Thus the new monthly expenditure is

$$\begin{aligned} E &= \begin{pmatrix} 4000 & 3000 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} \\ &= 4000(20) + 3000(30) \\ &= 80000 + 90000 = 170000. \end{aligned}$$

This is 10000 rupees less than the original expenditure because the larger group now receives the smaller amount.

Step 3: Conclusion The monthly expenditure would be ₹1,70,000.

Quick Tip to Solve: Interchange the amounts only after finding the original counts; the two groups still contain 20 and 30 recipients.