

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-3-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is **not** allowed.

Section A (22 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $\begin{vmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \pm 6$, then the value of k is:

[1 mark]

- (A) 2
- (B) -2
- (C) ± 2
- (D) ∓ 2

Correct Answer: (D) ∓ 2

Solution:

Concept:

- Expand a determinant along a row with two zero entries; only its nonzero cofactor contributes.
- The paired symbols $\pm$ and $\mp$ specify opposite signs in the corresponding cases.

Step 1: Expand along the third row The third-row entry in column 3 is 1 and its cofactor sign is positive. Thus

$$\begin{vmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ k & 0 \end{vmatrix} = 1(0 - 3k) = -3k.$$

Step 2: Retain the corresponding signs From $-3k = \pm 6$, division by the negative number -3 gives $k = \mp 2$. Explicitly, determinant $+6$ gives $k = -2$, while determinant -6 gives $k = +2$. Options (A) and (B) each omit a possible value. Option (C) describes the same unordered set of values, but (D) retains the opposite sign pairing required by the displayed equation.

Step 3: Conclusion $k = \mp 2$, corresponding to the given ± 6 . Hence, the correct option is (D).

Quick Tip to Solve: Expand along the row containing two zeros, then reverse the paired signs when dividing by a negative coefficient.

2. The derivative of 5^x w.r.t. e^x is:

[1 mark]

- (A) $\left(\frac{5}{e}\right)^x \frac{1}{\log 5}$
(B) $\left(\frac{e}{5}\right)^x \frac{1}{\log 5}$
(C) $\left(\frac{5}{e}\right)^x \log 5$
(D) $\left(\frac{e}{5}\right)^x \log 5$

Correct Answer: (C) $\left(\frac{5}{e}\right)^x \log 5$

Solution:

Concept:

- For $y = y(x)$ and $u = u(x)$, $dy/du = (dy/dx)/(du/dx)$ when $du/dx \neq 0$.
- The exponential derivative is $d(a^x)/dx = a^x \log a$, where $\log$ denotes the natural logarithm.

Step 1: Differentiate both exponentials Put $y = 5^x$ and $u = e^x$. Then

$$\frac{dy}{dx} = 5^x \log 5, \quad \frac{du}{dx} = e^x.$$

Since $e^x > 0$, the required derivative is defined for every real x .

Step 2: Form the derivative quotient Therefore

$$\frac{dy}{du} = \frac{5^x \log 5}{e^x} = \left(\frac{5}{e}\right)^x \log 5.$$

Options (A) and (B) incorrectly replace the factor $\log 5$ by its reciprocal. Options (B) and (D) also invert the exponential ratio. The quotient must have 5^x in its numerator because this is the derivative of the dependent expression.

Step 3: Conclusion The required derivative is $\left(\frac{5}{e}\right)^x \log 5$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate each expression with respect to x first; divide the derivative of 5^x by that of e^x .

3. If $|\vec{a}| = 2$ and $-3 \leq k \leq 2$, then $|k\vec{a}| \in$:

[1 mark]

- (A) $[-6, 4]$
- (B) $[0, 4]$
- (C) $[4, 6]$
- (D) $[0, 6]$

Correct Answer: (D) $[0, 6]$

Solution:

Concept:

- The magnitude of a scalar multiple is $|k\vec{a}| = |k| |\vec{a}|$, so it is always nonnegative.
- For an interval containing zero, the minimum absolute value is zero and the maximum is the larger endpoint magnitude.

Step 1: Find the range of the scalar magnitude As $-3 \leq k \leq 2$, zero is allowed and $|k|$ has minimum 0. Its maximum is 3, attained at $k = -3$. Every number between 0 and 3 occurs as an absolute value, using k in $[-3, 0]$. Thus $|k| \in [0, 3]$.

Step 2: Scale by the vector magnitude Using $|\vec{a}| = 2$, we get

$$|k\vec{a}| = 2|k| \in [0, 6].$$

Option (A) wrongly treats magnitude as signed. Option (B) misses magnitudes above 4 obtained for $-3 \leq k < -2$. Option (C) excludes zero and smaller magnitudes, although $k = 0$ and values near zero are permitted.

Step 3: Conclusion $|k\vec{a}| \in [0, 6]$. Hence, the correct option is (D).

Quick Tip to Solve: Take the absolute value of the scalar before multiplying by the vector magnitude; check zero and both endpoints.

4. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of both x -axis and z -axis, then the angle which it makes with the positive direction of y -axis is:

[1 mark]

- (A) 0
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{2}$
- (D) π

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- If a line makes direction angles α, β, γ with the positive coordinate axes, its direction cosines satisfy $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Quick Tip to Solve: Square the two given direction cosines; if their sum is 1, the line is perpendicular to the remaining axis.

- Each direction angle lies in $[0, \pi]$; within this interval, $\cos \beta = 0$ gives $\beta = \pi/2$.

Step 1: Substitute the two known angles Here $\alpha = \gamma = \pi/4$, hence

$$\cos^2 \beta = 1 - \cos^2(\pi/4) - \cos^2(\pi/4) = 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

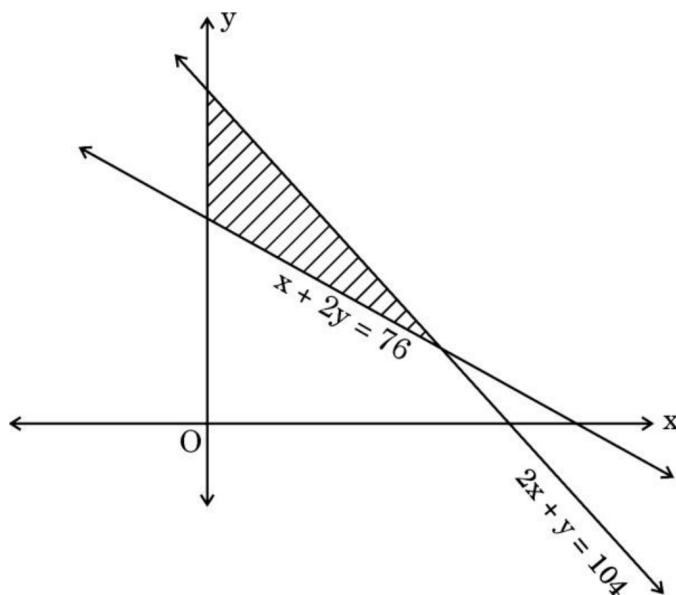
The two specified direction components already account for the entire squared unit length.

Step 2: Determine the remaining direction angle Thus $\cos \beta = 0$, so $\beta = \pi/2$. Options (A) and (D) would give a squared y-direction cosine of 1, while option (B) would give $1/2$. Each contradicts the zero value required by the direction-cosine identity.

Step 3: Conclusion The angle with the positive y-axis is $\frac{\pi}{2}$. Hence, the correct option is (C).

Quick Tip to Solve: Square the two given direction cosines; if their sum is 1, the line is perpendicular to the remaining axis.

5. Of the following, which group of constraints represents the feasible region given below? [1 mark]



- (A) $x + 2y \leq 76$, $2x + y \geq 104$, $x, y \geq 0$
 (B) $x + 2y \leq 76$, $2x + y \leq 104$, $x, y \geq 0$
 (C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$
 (D) $x + 2y \geq 76$, $2x + y \geq 104$, $x, y \geq 0$

Correct Answer: (C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$

Solution:

Concept:

- A line divides the plane into two half-planes; substitute an interior point of the shaded region to select the correct inequality.

Quick Tip to Solve: Use a point between the two y-intercepts, such as (0,50), to determine both inequality directions.

- The first-quadrant restriction is $x \geq 0, y \geq 0$; solid boundaries are included using non-strict inequalities.

Step 1: Choose a point inside the shaded region The shaded triangle lies on the positive y-axis between the intercepts 38 of $x + 2y = 76$ and 104 of $2x + y = 104$. For example, $(0, 50)$ lies on its included vertical side and between the two oblique boundaries. Substitution gives $x + 2y = 100 > 76$ and $2x + y = 50 < 104$.

Step 2: Select both half-planes The shaded region is consequently above $x + 2y = 76$ and below $2x + y = 104$, within the first quadrant. Thus the constraints are

$$x + 2y \geq 76, \quad 2x + y \leq 104, \quad x, y \geq 0.$$

Options (A) and (B) select the wrong side of the first line; option (D) selects the wrong side of the second line.

Step 3: Conclusion The constraints are $x + 2y \geq 76, 2x + y \leq 104, x, y \geq 0$. Hence, the correct option is (C).

Quick Tip to Solve: Use a point between the two y-intercepts, such as $(0, 50)$, to determine both inequality directions.

6. If $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then A^{-1} is:

[1 mark]

(A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(B) $30 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(C) $\frac{1}{30} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(D) $\frac{1}{30} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Solution:

Concept:

- The inverse A^{-1} satisfies $AA^{-1} = I$.

Quick Tip to Solve: For a diagonal matrix, take individual reciprocals and verify that each diagonal product equals 1.

- An invertible diagonal matrix is inverted by replacing each nonzero diagonal entry with its reciprocal.

Step 1: Invert each diagonal entry All three diagonal entries, 2, 3 and 5, are nonzero. Therefore

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}.$$

Multiplication by A gives diagonal entries $2(1/2) = 3(1/3) = 5(1/5) = 1$, and every off-diagonal entry remains zero.

Step 2: Check the other printed choices Option (B) multiplies this inverse by 30, producing $AA_B^{-1} = 30I$, rather than I . Option (D) divides it by 30, producing $I/30$. Option (C) gives $A/30$, whose product with A is $\text{diag}(4/30, 9/30, 25/30)$, also not the identity. Thus no determinant factor should be added to the reciprocal diagonal matrix.

Step 3: Conclusion $A^{-1} = \text{diag}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$. Hence, the correct option is (A).

Quick Tip to Solve: For a diagonal matrix, take individual reciprocals and verify that each diagonal product equals 1.

7. For any square matrix A , $(A - A')$ is always:

[1 mark]

- (A) an identity matrix
- (B) a null matrix
- (C) a skew symmetric matrix
- (D) a symmetric matrix

Correct Answer: (C) a skew symmetric matrix

Solution:

Concept:

- A matrix B is skew symmetric if $B' = -B$, where the prime denotes transpose.
- Transposition obeys $(A - B)' = A' - B'$ and $(A')' = A$.

Step 1: Transpose the given difference Let $B = A - A'$. Applying the transpose rules gives

$$B' = (A - A')' = A' - A = -(A - A') = -B.$$

This is exactly the defining condition for a skew symmetric matrix.

Step 2: Distinguish the other matrix types The diagonal entries of $A - A'$ are zero, but its off-diagonal entries need not be zero, so it is not always a null matrix. It is not an identity matrix. It is symmetric only in the special case $A = A'$, when the difference vanishes; symmetry is not guaranteed for an arbitrary square matrix.

Step 3: Conclusion $A - A'$ is always a skew symmetric matrix. Hence, the correct option is (C).

Quick Tip to Solve: Transpose the difference and compare it directly with its negative; this proves skew symmetry for any square matrix.

8. A function $f : R \rightarrow A$ defined as $f(x) = x^2 + 1$ is onto, if A is:

[1 mark]

- (A) $(-\infty, \infty)$
- (B) $(1, \infty)$
- (C) $[1, \infty)$
- (D) $[-1, \infty)$

Correct Answer: (C) $[1, \infty)$

Solution:

Concept:

- A function is onto when its range equals its codomain: each element of the codomain has a preimage.
- For real x , $x^2 \geq 0$, and equality is attained at $x = 0$.

Step 1: Determine the range Since $x^2 \geq 0$, $f(x) = x^2 + 1 \geq 1$. The endpoint is included because $f(0) = 1$. Conversely, for each $y \geq 1$, the real input $x = \sqrt{y-1}$ gives $f(x) = y$. Thus every value in $[1, \infty)$ is attained.

Step 2: Compare the codomains Options (A) and (D) include values below 1, which have no preimages. Option (B) excludes the value 1 even though the stated domain includes 0, so it cannot be the codomain of this function as defined. Option (C) equals the range exactly.

Step 3: Conclusion $A = [1, \infty)$. Hence, the correct option is (C).

Quick Tip to Solve: Find the minimum first; include its endpoint when a real input actually attains it.

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix such that $\text{adj } A = A$. Then, $(a + b + c + d)$ is equal to: [1 mark]

- (A) $2a$
- (B) $2b$
- (C) $2c$
- (D) 0

Correct Answer: (A) $2a$

Solution:

Concept:

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its adjugate is $\text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- Equal matrices have equal entries in corresponding positions.

Step 1: Equate corresponding entries The condition $\text{adj } A = A$ gives $d = a$, $-b = b$ and $-c = c$. Therefore $b = c = 0$, while $d = a$. The condition imposes no requirement that $a = 0$.

Step 2: Evaluate the requested sum Substitution gives

$$a + b + c + d = a + 0 + 0 + a = 2a.$$

Quick Tip to Solve: For a 2 by 2 adjugate, swap the diagonal entries and negate the off-diagonal entries.

Options (B) and (C) both reduce to 0 because $b = c = 0$, and option (D) is also 0. These do not hold for arbitrary a ; for example, $A = I$ satisfies the condition but has entry sum 2.

Step 3: Conclusion $a + b + c + d = 2a$. Hence, the correct option is (A).

Quick Tip to Solve: For a 2 by 2 adjugate, swap the diagonal entries and negate the off-diagonal entries.

10. A function $f(x) = |1 - x + |x||$ is:

[1 mark]

- (A) discontinuous at $x = 1$ only
- (B) discontinuous at $x = 0$ only
- (C) discontinuous at $x = 0, 1$
- (D) continuous everywhere

Correct Answer: (D) continuous everywhere

Solution:

Concept:

- The absolute-value function is continuous on the real line; sums and compositions of continuous functions remain continuous.
- Use $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$.

Step 1: Remove both absolute values For $x \geq 0$, the inner expression is $1 - x + x = 1$. For $x < 0$, it is $1 - 2x > 0$. Hence

$$f(x) = \begin{cases} 1 - 2x, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Each branch is continuous throughout its interval.

Step 2: Check the junction and options At $x = 0$, both one-sided limits are 1, equal to $f(0)$. Thus there is no discontinuity at 0. At $x = 1$, the function is constant in a neighbourhood and is continuous. Options (A), (B) and (C) therefore incorrectly assert discontinuities; a corner at 0 affects differentiability, not continuity.

Step 3: Conclusion The function is continuous everywhere. Hence, the correct option is (D).

Quick Tip to Solve: Check continuity by joining branch values; a change of slope alone is not a discontinuity.

11. The point of inflexion of a function $f(x)$ is the point where:

[1 mark]

- (A) $f'(x) = 0$ and $f'(x)$ changes its sign from positive to negative from left to right of that point.
- (B) $f'(x) = 0$ and $f'(x)$ changes its sign from negative to positive from left to right of that point.
- (C) $f'(x) = 0$ and $f'(x)$ does not change its sign from left to right of that point.
- (D) $f'(x) \neq 0$.

Correct Answer: No uniquely correct printed option: an inflexion is identified by a change of concavity.

Solution:

Concept:

- A point of inflexion is a point where the graph changes concavity.
- When the second derivative exists on both sides, a change in its sign verifies this change of concavity; a stationary point is not required.

Step 1: Test options A and B A positive-to-negative change in f' with $f' = 0$ is the first derivative test for a local maximum, as for $f(x) = -x^2$ at 0. A negative-to-positive change is the test for a local minimum, as for $f(x) = x^2$ at 0. Neither condition describes an inflexion.

Step 2: Test options C and D Option (C) can occur at a stationary inflexion: $f(x) = x^3$ has $f'(0) = 0$, with $f'(x) = 3x^2$ positive on both sides. But it is not a general criterion: $f(x) = x + x^3$ has an inflexion at 0 while $f'(0) = 1$. Conversely, a nonzero first derivative does not imply inflexion; $f(x) = x^2$ has $f'(1) = 2$ but no change of concavity there. Thus (D) also fails as a criterion.

Step 3: Conclusion No printed option gives a valid general criterion. An inflexion requires a change of concavity; the question has no uniquely defensible option.

Quick Tip to Solve: Distinguish the sign change of the first derivative from a change in concavity.

12. If $g(x)$ is a continuous function satisfying $g(-x) = -g(x)$, then $\int_0^{2a} g(x) dx$ is equal to:

[1 mark]

- (A) 0
- (B) $2 \int_0^a g(x) dx$
- (C) $\int_{-a}^a g(x) dx$
- (D) $-\int_{-2a}^0 g(x) dx$

Correct Answer: (D) $-\int_{-2a}^0 g(x) dx$

Solution:

Concept:

- The identity $g(-x) = -g(x)$ says that g is odd.
- For an odd continuous function, integrals over reflected intervals have opposite signs.

Step 1: Reflect the negative interval Put $x = -t$ in the integral occurring in option (D). Then

Quick Tip to Solve: For an odd integrand, reflect both limits through zero and reverse the sign of the integral.

$dx = -dt$, and its limits become $2a$ and 0 . Therefore

$$\int_{-2a}^0 g(x) dx = \int_0^{2a} g(-t) dt = - \int_0^{2a} g(t) dt.$$

Negating this expression gives exactly the requested integral.

Step 2: Reject the other proposed identities Oddness makes $\int_{-a}^a g(x) dx = 0$, but the interval from 0 to $2a$ is not symmetric about 0 . For example, take $g(x) = x$ and $a = 1$. The requested integral is 2 , whereas (A) and (C) are 0 and (B) is $2 \int_0^1 x dx = 1$. Thus only (D) is an identity under the given condition.

Step 3: Conclusion $\int_0^{2a} g(x) dx = - \int_{-2a}^0 g(x) dx$. Hence, the correct option is (D).

Quick Tip to Solve: For an odd integrand, reflect both limits through zero and reverse the sign of the integral.

13. $x \log x \frac{dy}{dx} + y = 2 \log x$ is an example of a:

[1 mark]

- (A) variable separable differential equation.
- (B) homogeneous differential equation.
- (C) first order linear differential equation.
- (D) differential equation whose degree is not defined.

Correct Answer: (C) first order linear differential equation.

Solution:

Concept:

- A first order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Its degree is the power of the highest derivative when the equation is polynomial in derivatives.

Step 1: Write the standard linear form On any interval with $x > 0$ and $x \neq 1$, divide by $x \log x$ to obtain

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}.$$

Thus $P(x) = 1/(x \log x)$ and $Q(x) = 2/x$ depend only on the independent variable. Both y and its first derivative occur to the first power.

Step 2: Compare the remaining classifications The equation is not of the separable form $dy/dx = F(x)G(y)$, since the expression $(2 \log x - y)/(x \log x)$ does not factor in that way. It is not homogeneous in the usual $F(y/x)$ sense either. The original equation is polynomial in dy/dx , so its degree is defined and equals 1; the logarithm of x does not invalidate the degree.

Step 3: Conclusion It is a first order linear differential equation. Hence, the correct option is (C).

Quick Tip to Solve: Arrange the equation as derivative plus a function of x times y before classifying it.

14. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, then $\vec{a}$ and $\vec{b}$ are:

[1 mark]

- (A) collinear vectors which are not parallel
- (B) parallel vectors
- (C) perpendicular vectors
- (D) unit vectors

Correct Answer: (C) perpendicular vectors

Solution:

Concept:

- For nonzero vectors, $\vec{a} \cdot \vec{b} = 0$ is the test for perpendicularity.
- Parallel vectors must be scalar multiples; a unit vector has magnitude 1.

Step 1: Calculate the scalar product Using the components of the printed vectors,

$$\vec{a} \cdot \vec{b} = 2(1) + (-1)(1) + (1)(-1) = 2 - 1 - 1 = 0.$$

Both vectors are nonzero. Their angle θ therefore satisfies $|\vec{a}||\vec{b}|\cos\theta = 0$, giving $\theta = 90^\circ$.

Step 2: Check the other options The component ratios are $2/1 = 2$, $(-1)/1 = -1$ and $1/(-1) = -1$, so the vectors are not scalar multiples and (B) fails. Collinear nonzero vectors have parallel directions, so (A) cannot describe these vectors. Also, $|\vec{a}| = \sqrt{6}$ and $|\vec{b}| = \sqrt{3}$, neither equal to 1; hence (D) fails.

Step 3: Conclusion The vectors are perpendicular. Hence, the correct option is (C).

Quick Tip to Solve: Use the dot product for perpendicularity; component ratios instead test parallelism.

15. If α, β and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following is not true?

[1 mark]

- (A) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
- (B) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
- (C) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
- (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Correct Answer: (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, with $l^2 + m^2 + n^2 = 1$.
- Use $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos 2\theta = 2\cos^2 \theta - 1$.

Step 1: Test the first three identities Option (A) is the standard direction-cosine identity. Therefore

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 3 - 1 = 2,$$

Quick Tip to Solve: Convert sums of sine squares and double angles into the sum of cosine squares; test an unsquared sum with equal direction cosines.

so (B) is true. Also

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2(1) - 3 = -1,$$

so (C) is true.

Step 2: Test the unsquared sum The identity for squares does not impose the same value on the sum itself. For a line equally inclined to the positive axes, $l = m = n = 1/\sqrt{3}$, which satisfies the squared identity, but $l + m + n = \sqrt{3} \neq 1$. Thus (D) fails in general.

Step 3: Conclusion The statement that is not true in general is $\cos \alpha + \cos \beta + \cos \gamma = 1$. Hence, the correct option is (D).

Quick Tip to Solve: Convert sums of sine squares and double angles into the sum of cosine squares; test an unsquared sum with equal direction cosines.

16. The restrictions imposed on decision variables involved in an objective function of a linear programming problem are called: [1 mark]

- (A) feasible solutions
- (B) constraints
- (C) optimal solutions
- (D) infeasible solutions

Correct Answer: (B) constraints

Solution:

Concept:

- Decision variables represent quantities chosen in a linear programming problem.
- Constraints are the linear equations or inequalities restricting those variables; feasible solutions satisfy every constraint.

Step 1: Identify the role of restrictions An objective function, such as $Z = ax + by$, describes the quantity to maximize or minimize. Conditions such as $x + y \leq 10$ or $x \geq 0$ restrict the allowed values of the decision variables. Such restrictions are called constraints.

Step 2: Distinguish restrictions from solutions A feasible solution is a choice of variables satisfying all the restrictions, so (A) names a solution rather than a restriction. An optimal solution is a feasible choice giving the best objective value, ruling out (C). An infeasible solution violates at least one restriction, ruling out (D). Option (B) is the term for the restrictions themselves.

Step 3: Conclusion The restrictions are called constraints. Hence, the correct option is (B).

Quick Tip to Solve: Ask whether the term names a rule or a choice of variable values: rules are constraints, while choices are solutions.

17. Let E and F be two events such that $P(E) = 0.1$, $P(F) = 0.3$, $P(E \cup F) = 0.4$, then $P(F | E)$ is: [1 mark]

- (A) 0.6
- (B) 0.4
- (C) 0.5
- (D) 0

Correct Answer: (D) 0

Solution:

Concept:

- Addition rule: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- For $P(E) > 0$, conditional probability is $P(F | E) = P(F \cap E) / P(E)$.

Step 1: Find the intersection probability Using the given union probability,

$$P(E \cap F) = 0.1 + 0.3 - 0.4 = 0.$$

Thus the events have no common outcomes of positive probability; their union probability equals the sum of their individual probabilities.

Step 2: Use the correct conditional denominator The condition is that E occurs, so divide by $P(E) = 0.1$, which is nonzero:

$$P(F | E) = \frac{P(E \cap F)}{P(E)} = \frac{0}{0.1} = 0.$$

The positive values in (A), (B) and (C) would require a positive intersection probability and therefore contradict the supplied union probability.

Step 3: Conclusion $P(F | E) = 0$. Hence, the correct option is (D).

Quick Tip to Solve: Before dividing for a conditional probability, recover the intersection from the union; here the two individual probabilities already sum to the union.

18. If $A = [a_{ij}]$ is an identity matrix, then which of the following is true?

[1 mark]

(A) $a_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$

(B) $a_{ij} = 1, \forall i, j$

(C) $a_{ij} = 0, \forall i, j$

(D) $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Correct Answer: (D) $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Solution:

Concept:

- An identity matrix is a square matrix whose main diagonal entries are 1 and all other entries are 0.
- For an entry a_{ij} , the condition $i = j$ identifies the main diagonal; $i \neq j$ identifies off-diagonal entries.

Step 1: Translate the definition into entry conditions On the main diagonal, the row and column indices agree, so $a_{ii} = 1$. When the indices differ, the entry is off the main diagonal and equals zero. Thus

$$a_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

This is precisely the rule printed in (D).

Step 2: Check the other entry rules Option (A) reverses the diagonal and off-diagonal values. Option (B) makes every entry one, which introduces nonzero off-diagonal entries in orders greater than one. Option (C) makes all diagonal entries zero, giving a zero matrix rather than an identity. None states the defining identity-matrix rule.

Step 3: Conclusion The identity-matrix entry rule is the one in option (D). Hence, the correct option is (D).

Quick Tip to Solve: For an identity matrix, equal indices mean one; different indices mean zero. Check these two cases separately.

19. Assertion (A): Projection of $\vec{a}$ on $\vec{b}$ is same as projection of $\vec{b}$ on $\vec{a}$.

Reason (R): Angle between $\vec{a}$ and $\vec{b}$ is same as angle between $\vec{b}$ and $\vec{a}$ numerically.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- The scalar projection of nonzero $\vec{a}$ on $\vec{b}$ is $\vec{a} \cdot \vec{b} / |\vec{b}| = |\vec{a}| \cos \theta$.
- The reverse projection is $|\vec{b}| \cos \theta$; the angle θ between the vectors is symmetric.

Step 1: Judge Assertion (A) The projections depend on different magnitudes. For example, take $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{i}$. Their angle is zero, so the projection of $\vec{a}$ on $\vec{b}$ is 2, whereas the reverse projection is 3. Therefore the claimed equality does not hold generally and Assertion (A) is false.

Step 2: Judge Reason (R) For nonzero vectors,

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}.$$

Swapping the vectors leaves both numerator and denominator unchanged. Their angle in $[0, \pi]$ is consequently the same numerically. Reason (R) is true.

Step 3: Check the explanatory relation Equality of angles supplies the same cosine factor but does not supply equal vector magnitudes. It therefore cannot establish equality of the two projections. Options (A) and (B) require a true assertion, while (C) also declares the reason false; all three conflict with the checks above.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: When reversing a scalar projection, the denominator changes to the magnitude of the new target vector; equal angles alone do not imply equal projections.

20. Assertion (A): Every scalar matrix is a diagonal matrix.

Reason (R): In a diagonal matrix, all the diagonal elements are 0.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A scalar matrix has the form kI_n : every off-diagonal entry is zero and each diagonal entry is the same scalar k .
- A diagonal matrix requires zero off-diagonal entries; its diagonal entries may be any numbers.

Step 1: Judge Assertion (A) Every scalar matrix has all entries away from the main diagonal equal to zero. This satisfies the defining condition for a diagonal matrix, regardless of the common diagonal value k . Therefore every scalar matrix is a diagonal matrix and Assertion (A) is true.

Step 2: Judge Reason (R) A diagonal matrix does not require its diagonal entries to vanish. For example,

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

is diagonal even though both diagonal entries are nonzero. The zeros must occur off the diagonal. Thus Reason (R) confuses diagonal and off-diagonal entries and is false.

Step 3: Check the explanatory relation A false reason cannot explain the true assertion. Options (A) and (B) both call the reason true, so they fail. Option (D) calls the assertion false, so it also fails. The actual explanation is that scalar matrices satisfy the zero off-diagonal condition required of diagonal matrices.

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: In definitions of diagonal and scalar matrices, check off-diagonal zeros first; a scalar matrix additionally has equal diagonal entries.

21(a). Evaluate: $\int_0^{\pi/2} \sin 2x \cos 3x \, dx$

[2 marks]

Solution:

Concept:

- The product-to-sum identity is $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.
- For nonzero constant k , $\int \sin(kx) \, dx = -\cos(kx)/k + C$; use the stated bounds after integrating.

Quick Tip to Solve: Use product-to-sum before integration and retain the minus from $\sin(2x-3x)$; both upper-limit cosine terms are zero here.

Step 1: Convert the product into a sum Let $I = \int_0^{\pi/2} \sin 2x \cos 3x \, dx$. With $A = 2x$ and $B = 3x$,

$$\sin 2x \cos 3x = \frac{1}{2} \{ \sin 5x + \sin(-x) \} = \frac{1}{2} (\sin 5x - \sin x).$$

The negative sign is essential because $\sin(-x) = -\sin x$. Hence

$$I = \frac{1}{2} \int_0^{\pi/2} (\sin 5x - \sin x) \, dx.$$

Step 2: Integrate and substitute both limits An antiderivative of the integrand inside the last integral is $-\cos 5x/5 + \cos x$. Therefore

$$\begin{aligned} I &= \frac{1}{2} \left[-\frac{\cos 5x}{5} + \cos x \right]_0^{\pi/2} \\ &= \frac{1}{2} \left[\left(-\frac{\cos(5\pi/2)}{5} + \cos(\pi/2) \right) - \left(-\frac{\cos 0}{5} + \cos 0 \right) \right] \\ &= \frac{1}{2} \left[0 - \left(-\frac{1}{5} + 1 \right) \right] = -\frac{2}{5}. \end{aligned}$$

Both cosine terms vanish at the upper limit, while the lower-limit bracket equals $4/5$.

Step 3: Conclusion

$$\boxed{\int_0^{\pi/2} \sin 2x \cos 3x \, dx = -\frac{2}{5}}.$$

Quick Tip to Solve: Use product-to-sum before integration and retain the minus from $\sin(2x-3x)$; both upper-limit cosine terms are zero here.

OR

21(b). Given $\frac{d}{dx}F(x) = \frac{1}{\sqrt{2x-x^2}}$ **and** $F(1) = 0$, **find** $F(x)$.

[2 marks]

Solution:

Concept:

- Complete the square: $2x - x^2 = 1 - (x - 1)^2$. The real derivative exists for $0 < x < 2$.
- The standard antiderivative is $\int du/\sqrt{1-u^2} = \sin^{-1} u + C$, where C is an arbitrary constant.

Step 1: Rewrite and integrate the derivative The quadratic in the denominator is positive between its roots 0 and 2. On this interval put $u = x - 1$, so that $du = dx$. Then

$$F(x) = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C.$$

The shift has derivative 1, so no extra numerical factor is required.

Quick Tip to Solve: Complete the square before choosing an inverse-trigonometric antiderivative; then use $F(1)$ to fix the constant.

Step 2: Use the given initial value and check At $x = 1$, the inverse-sine argument is zero. Thus $0 = F(1) = \sin^{-1} 0 + C = C$. Differentiating the resulting expression gives

$$\frac{d}{dx} \sin^{-1}(x-1) = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}},$$

which agrees with the prescribed derivative on its real domain.

Step 3: Conclusion $F(x) = \sin^{-1}(x-1), \quad 0 < x < 2$.

Quick Tip to Solve: Complete the square before choosing an inverse-trigonometric antiderivative; then use $F(1)$ to fix the constant.

Section B (8 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

22. Find the position vector of point C which divides the line segment joining points A and B having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio $4 : 1$ externally. Further, find $|\vec{AB}| : |\vec{BC}|$. [2 marks]

Solution:

Concept:

- For external division in the ratio $AC : CB = m : n$, the position vector is $\vec{c} = (m\vec{b} - n\vec{a})/(m - n)$.
- The displacement vector is obtained by subtracting the initial position vector from the final one; its modulus gives distance.

Step 1: Apply the external section formula Here $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$, $m = 4$ and $n = 1$. Therefore

$$\begin{aligned} \vec{c} &= \frac{4\vec{b} - \vec{a}}{3} \\ &= \frac{4(-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{3} \\ &= \frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}. \end{aligned}$$

The subtraction in the formula reflects external, rather than internal, division.

Step 2: Find the two displacement lengths Subtracting position vectors gives

$$\vec{AB} = \vec{b} - \vec{a} = -2\hat{i} - \hat{j} + 2\hat{k}, \quad |\vec{AB}| = \sqrt{4 + 1 + 4} = 3.$$

Also $\vec{BC} = \vec{c} - \vec{b} = (-2\hat{i} - \hat{j} + 2\hat{k})/3$, so $|\vec{BC}| = 1$. In particular, C lies beyond B , and $AC = AB + BC = 4$, consistent with the stated external ratio.

Step 3: Conclusion $\vec{c} = \frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}$, and $|\vec{AB}| : |\vec{BC}| = 3 : 1$.

Quick Tip to Solve: For external division 4:1, use a denominator of 4-1; the segment AB is the difference AC-CB.

23. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a} + \vec{b} - \vec{c} = \vec{0}$, find the angle between vectors $\vec{a}$ and $\vec{c}$. [2 marks]

Solution:

Concept:

- For vectors $\vec{u}, \vec{v}$, $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}$.
- The dot product satisfies $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$, where $0 \leq \theta \leq \pi$ is their angle.

Step 1: Express the third unit vector as a difference The given relation is $\vec{a} + \vec{b} = \vec{c}$. To obtain the requested angle directly, rearrange this as $\vec{b} = \vec{c} - \vec{a}$. Taking the squared modulus on both sides and using all three unit lengths gives

$$1 = |\vec{c} - \vec{a}|^2 = 1 + 1 - 2\vec{a} \cdot \vec{c}.$$

Hence $2\vec{a} \cdot \vec{c} = 1$, or $\vec{a} \cdot \vec{c} = 1/2$.

Step 2: Convert the dot product into an angle Let θ be the angle between $\vec{a}$ and $\vec{c}$. Since both vectors have modulus 1,

$$\cos \theta = \frac{\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{c}|} = \frac{1}{2}.$$

The angle between two nonzero vectors lies in $[0, \pi]$, where this cosine gives $\theta = \pi/3$. The angle $2\pi/3$ instead belongs to $\vec{a}$ and $\vec{b}$, so it is not the angle requested here.

Step 3: Conclusion The required angle is $\boxed{\pi/3 = 60^\circ}$.

Quick Tip to Solve: Isolate b as c-a before squaring; this produces the dot product of the two vectors whose angle is asked.

24. Find the value of $\left[\sin^2 \left\{ \cos^{-1} \left(\frac{3}{5} \right) \right\} + \tan^2 \left\{ \sec^{-1}(3) \right\} \right]$. [2 marks]

Solution:

Concept:

- If $\alpha = \cos^{-1}(3/5)$, then $\cos \alpha = 3/5$ and $\sin^2 \alpha = 1 - \cos^2 \alpha$.
- The identity $\tan^2 \beta = \sec^2 \beta - 1$ applies when $\sec \beta$ is defined; for $\beta = \sec^{-1} 3$, $\sec \beta = 3$.

Step 1: Evaluate the sine-squared term Write $\alpha = \cos^{-1}(3/5)$. Its principal value is in $[0, \pi]$, and its cosine is exactly $3/5$. The Pythagorean identity therefore gives

$$\sin^2 \left\{ \cos^{-1} \left(\frac{3}{5} \right) \right\} = 1 - \left(\frac{3}{5} \right)^2 = \frac{16}{25}.$$

There is no need to evaluate the inverse cosine numerically because only the square of its sine is required.

Step 2: Evaluate the tangent-squared term and add Put $\beta = \sec^{-1} 3$. Since 3 lies in the real domain

Quick Tip to Solve: Use complementary square identities directly; inverse-trigonometric angles need not be calculated.

of inverse secant, the defining relation gives $\sec \beta = 3$. Hence

$$\tan^2\{\sec^{-1} 3\} = \sec^2 \beta - 1 = 9 - 1 = 8.$$

Combining the two exact values with denominator 25 yields $16/25 + 8 = (16 + 200)/25 = 216/25$. Squaring the secant is essential; the identity does not give $3 - 1$.

Step 3: Conclusion The value of the given expression is $\boxed{216/25}$.

Quick Tip to Solve: Use complementary square identities directly; inverse-trigonometric angles need not be calculated.

25(a). If $x = e^{x/y}$, prove that $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$.

[2 marks]

Solution:

Concept:

- Taking natural logarithms of $x = e^{x/y}$ gives $\log x = x/y$ for real admissible values, with $x > 0$.
- The quotient rule is $(u/v)' = (uv' - uv)/v^2$; also $d(\log x)/dx = 1/x$.

Step 1: Express y as a function of x The exponential relation implies $x > 0$, and $y \neq 0$ is needed for x/y . Taking the natural logarithm yields

$$\log x = \frac{x}{y} \implies y \log x = x.$$

The value $x = 1$ is inadmissible: it would require $0 = 1/y$, impossible for a finite nonzero y . Therefore $\log x \neq 0$, and we may write $y = x/\log x$.

Step 2: Differentiate using the quotient rule On either interval $(0, 1)$ or $(1, \infty)$, both numerator and denominator are differentiable and the denominator is nonzero. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\log x)(1) - x(1/x)}{(\log x)^2} \\ &= \frac{\log x - 1}{(\log x)^2}. \end{aligned}$$

The factor x cancels against $1/x$ in the second term of the numerator. This calculation follows the original exponential equation exactly, without treating y as a constant.

Step 3: Conclusion Hence $\boxed{\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}}$, for $x > 0, x \neq 1$.

Quick Tip to Solve: Take natural logarithms first and isolate $y = x/\log x$; the quotient rule then gives the required form immediately.

25(b). Check the differentiability of $f(x) = \begin{cases} x^2 + 1, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2 \end{cases}$ at $x = 1$.

[2 marks]

Solution:**Concept:**

- Continuity at an interior point requires the left limit, right limit and function value to agree.
- Differentiability at $x = a$ requires equal finite one-sided limits of $\frac{f(x) - f(a)}{x - a}$. Continuity alone is insufficient.

Step 1: Check continuity at the joining point The printed second branch includes $x = 1$, so $f(1) = 3 - 1 = 2$. From the first branch, $\lim_{x \rightarrow 1^-} f(x) = 1^2 + 1 = 2$; from the second branch, $\lim_{x \rightarrow 1^+} f(x) = 3 - 1 = 2$. Thus both limits equal the function value, and the function is continuous at the joining point.

Step 2: Compute the one-sided derivatives For $x < 1$, the difference quotient uses $f(x) = x^2 + 1$:

$$f'_-(1) = \lim_{x \rightarrow 1^-} \frac{x^2 + 1 - 2}{x - 1} = \lim_{x \rightarrow 1^-} (x + 1) = 2.$$

For $x > 1$, use the other branch:

$$f'_+(1) = \lim_{x \rightarrow 1^+} \frac{3 - x - 2}{x - 1} = \lim_{x \rightarrow 1^+} (-1) = -1.$$

The two finite slopes disagree, so a unique derivative cannot exist at this point.

Step 3: Conclusion Although f is continuous at $x = 1$, it is not differentiable there because $f'_-(1) = 2 \neq -1 = f'_+(1)$.

Quick Tip to Solve: At a piecewise joining point, check continuity first and then compare both one-sided derivatives.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Find: $\int \frac{x^{-1}}{(\log x)^2 - 5 \log x + 4} dx$.

[3 marks]

Solution:**Concept:**

- Use $t = \log x$, so $dt = dx/x$; here $\log$ denotes the natural logarithm and $x > 0$.
- A rational function with distinct linear factors in its denominator can be integrated by partial fractions.
- The standard formula is $\int dt/(t - a) = \log |t - a| + C$, on any interval avoiding $t = a$.

Quick Tip to Solve: Whenever the numerator is dx/x and the denominator is a polynomial in $\log x$, substitute $t = \log x$ first.

Step 1: Substitute for the logarithm and factor The numerator printed in the question is x^{-1} . Therefore the entire differential factor is dx/x , which exactly equals dt when $t = \log x$. The integral becomes

$$I = \int \frac{dt}{t^2 - 5t + 4} = \int \frac{dt}{(t-1)(t-4)}.$$

The two numbers have sum 5 and product 4, establishing this factorisation. The original integrand is defined for $x > 0$, except at $x = e$ and $x = e^4$.

Step 2: Resolve into simple fractions and integrate Write

$$\frac{1}{(t-1)(t-4)} = \frac{A}{t-1} + \frac{B}{t-4}.$$

Multiplication by the denominator gives $1 = A(t-4) + B(t-1)$. At $t = 1$, $A = -1/3$; at $t = 4$, $B = 1/3$. Hence

$$I = \frac{1}{3} \int \left(\frac{1}{t-4} - \frac{1}{t-1} \right) dt = \frac{1}{3} (\log |t-4| - \log |t-1|) + C.$$

The absolute values retain the correct real antiderivative on each interval of definition.

Step 3: Restore the variable and check Substitute $t = \log x$. Differentiating the resulting expression gives

$$\frac{1}{3x} \left(\frac{1}{\log x - 4} - \frac{1}{\log x - 1} \right) = \frac{1}{x((\log x)^2 - 5 \log x + 4)},$$

which is precisely the printed integrand.

Step 4: Conclusion

$$I = \frac{1}{3} \log \left| \frac{\log x - 4}{\log x - 1} \right| + C$$

This holds on any interval with $x > 0$ excluding e and e^4 .

Quick Tip to Solve: Whenever the numerator is dx/x and the denominator is a polynomial in $\log x$, substitute $t = \log x$ first.

27(a). Find: $\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx$

[3 marks]

Solution:

Concept:

- Use $\sin 2x = 2 \sin x \cos x$ and $1 + \cos 2x = 2 \cos^2 x$.
- The product rule gives $\frac{d}{dx}[e^x g(x)] = e^x [g(x) + g'(x)]$.
- For $g(x) = \tan x$, $g'(x) = \sec^2 x$; the expressions require $\cos x \neq 0$.

Step 1: Simplify the trigonometric fraction Apply the double-angle identities to both numerator and

Quick Tip to Solve: Simplify double angles first, then look for the exponential factor multiplied by a function plus its derivative.

denominator:

$$\frac{2 + \sin 2x}{1 + \cos 2x} = \frac{2 + 2 \sin x \cos x}{2 \cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} = \sec^2 x + \tan x.$$

Splitting the fraction is valid wherever the given denominator is nonzero, equivalently wherever $\cos x \neq 0$. Thus the integral is $I = \int e^x (\sec^2 x + \tan x) dx$.

Step 2: Recognise a product derivative The two trigonometric terms are a function and its derivative: $\tan x$ differentiates to $\sec^2 x$. By the product rule,

$$\frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x = e^x (\tan x + \sec^2 x).$$

This matches the whole transformed integrand, including the multiplying exponential. Therefore it can be integrated directly as the derivative of this product.

Step 3: Verify against the original form Combine the terms in the derivative over $\cos^2 x$:

$$e^x (\tan x + \sec^2 x) = e^x \frac{\sin x \cos x + 1}{\cos^2 x} = e^x \frac{2 + \sin 2x}{1 + \cos 2x}.$$

This reverses the simplification exactly, verifying the antiderivative on each interval where the original function is defined.

Step 4: Conclusion

$$\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx = e^x \tan x + C$$

Quick Tip to Solve: Simplify double angles first, then look for the exponential factor multiplied by a function plus its derivative.

OR

27(b). Evaluate: $\int_0^{\pi/4} \frac{1}{\sin x + \cos x} dx$

[3 marks]

Solution:

Concept:

- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$.
- The antiderivative is $\int \sec u du = \log |\sec u + \tan u| + C$, where $\cos u \neq 0$.
- After a substitution in a definite integral, convert both limits before evaluating.

Step 1: Transform the denominator and limits Let $u = x - \pi/4$, so $du = dx$. At $x = 0$, $u = -\pi/4$; at $x = \pi/4$, $u = 0$. The addition identity gives

$$\sin x + \cos x = \sqrt{2} \cos u, \quad I = \frac{1}{\sqrt{2}} \int_{-\pi/4}^0 \sec u du.$$

Quick Tip to Solve: Rewrite $\sin x + \cos x$ as a shifted cosine and convert both limits to avoid endpoint sign errors.

On this closed interval $\cos u > 0$, so the denominator never vanishes and the integral is finite.

Step 2: Evaluate using the secant antiderivative For completeness, the logarithmic formula follows from

$$\frac{d}{du} \log |\sec u + \tan u| = \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u} = \sec u.$$

Hence

$$I = \frac{1}{\sqrt{2}} [\log |\sec u + \tan u|]_{-\pi/4}^0.$$

At the upper limit the argument is $1 + 0 = 1$. At the lower limit it is $\sqrt{2} - 1$, which is positive. Thus

$$I = \frac{1}{\sqrt{2}} (0 - \log(\sqrt{2} - 1)).$$

Step 3: Simplify the endpoint expression Rationalising shows $1/(\sqrt{2} - 1) = \sqrt{2} + 1$, since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$. Consequently $-\log(\sqrt{2} - 1) = \log(\sqrt{2} + 1)$. The final value is positive, consistent with the positive original integrand throughout $[0, \pi/4]$. This also checks the order of the substituted limits.

Step 4: Conclusion

$$\int_0^{\pi/4} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

Quick Tip to Solve: Rewrite $\sin x + \cos x$ as a shifted cosine and convert both limits to avoid endpoint sign errors.

28. Solve the following linear programming problem graphically: Minimize $z = 600x + 400y$, subject to the constraints $x + y \geq 8$, $x + 2y \leq 16$, $4x + y \leq 29$, $x, y \geq 0$. [3 marks]

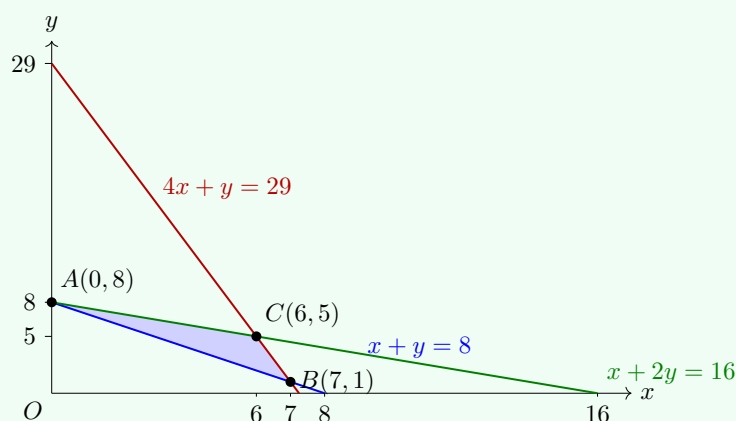
Solution:

Concept:

- The feasible region is the common region satisfying all the inequalities, including the first-quadrant restrictions $x, y \geq 0$.
- For a linear objective $z = 600x + 400y$, an optimum on a bounded feasible polygon occurs at a corner point. Evaluate the objective at every vertex.

Step 1: Draw the boundary lines and select the common region The boundary $x + y = 8$ passes through $(8, 0)$ and $(0, 8)$; select the side away from the origin. The line $x + 2y = 16$ passes through $(16, 0)$ and $(0, 8)$; select its origin side. The line $4x + y = 29$ passes through $(29/4, 0)$ and $(0, 29)$; select its origin side. Restrict all these half-planes to $x, y \geq 0$. Their common region is the shaded triangle ABC in the graph.

Quick Tip to Solve: For this graph, the two lines with y-intercept 8 meet on the y-axis; check that point before evaluating the other two intersections.



The shaded triangle ABC is the feasible region.

Step 2: Find all vertices of the feasible triangle The first two lines intersect at $A = (0, 8)$. Solving $x + y = 8$ and $4x + y = 29$, subtraction gives $3x = 21$, so $B = (7, 1)$. For the remaining pair, use $y = 8 - x/2$ in $4x + y = 29$: $7x/2 = 21$, hence $C = (6, 5)$. Each vertex satisfies the third constraint as well. The nonnegative axes contribute no further vertex: on $y = 0$, the conditions $x \geq 8$ and $x \leq 29/4$ conflict.

Step 3: Evaluate the objective at each corner Evaluate the objective at each vertex:

$$z(A) = 600(0) + 400(8) = 3200,$$

$$z(B) = 600(7) + 400(1) = 4600,$$

$$z(C) = 600(6) + 400(5) = 5600.$$

Since the feasible triangle is bounded, its least corner value is the global minimum.

Step 4: Conclusion The minimum value is $z_{\min} = 3200$, attained at $x = 0, y = 8$.

Quick Tip to Solve: For this graph, the two lines with y-intercept 8 meet on the y-axis; check that point before evaluating the other two intersections.

29. The chances of P , Q and R getting selected as CEO of a company are in the ratio $4 : 1 : 2$ respectively. The probabilities for the company to increase its profits from the previous year under the new CEO, P , Q or R are 0.3, 0.8 and 0.5 respectively. If the company increased the profits from the previous year, find the probability that it is due to the appointment of R as CEO. [3 marks]

Solution:

Concept:

- Bayes' theorem gives $P(R | E) = P(R)P(E | R)/P(E)$, where E is the event that the company increases its profits.
- The CEO selection events P, Q, R are mutually exclusive and exhaustive. Thus total probability gives $P(E) = \sum_{S=P,Q,R} P(S)P(E | S)$.

Step 1: Convert the selection ratio into prior probabilities The total number of ratio parts is $4 + 1 + 2 =$

Quick Tip to Solve: Multiply the ratio weights 4, 1, 2 by 0.3, 0.8, 0.5; the R weight is 1 and the total is 3, giving $1/3$ directly.

7. Hence the selection probabilities are

$$P(P) = \frac{4}{7}, \quad P(Q) = \frac{1}{7}, \quad P(R) = \frac{2}{7}.$$

These are prior probabilities, before learning whether the profit increased. The given conditional probabilities are $P(E | P) = 3/10$, $P(E | Q) = 4/5$, and $P(E | R) = 1/2$.

Step 2: Find the probability of increased profits The increased-profit event may occur under any one of the three CEOs. Apply the total probability theorem:

$$\begin{aligned} P(E) &= \frac{4}{7} \cdot \frac{3}{10} + \frac{1}{7} \cdot \frac{4}{5} + \frac{2}{7} \cdot \frac{1}{2} \\ &= \frac{6}{35} + \frac{4}{35} + \frac{5}{35} = \frac{15}{35} = \frac{3}{7}. \end{aligned}$$

The joint probability that R is appointed and the profits increase is $P(R \cap E) = (2/7)(1/2) = 1/7$.

Step 3: Condition on the observed profit increase We require the probability of the cause R , given the observed event E . Therefore Bayes' theorem gives

$$P(R | E) = \frac{P(R \cap E)}{P(E)} = \frac{1/7}{3/7} = \frac{1}{3}.$$

The denominator includes profit increases under all three possible appointments, rather than just under R .

Step 4: Conclusion The required probability that the increase in profits is due to R being appointed is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: Multiply the ratio weights 4, 1, 2 by 0.3, 0.8, 0.5; the R weight is 1 and the total is 3, giving 1/3 directly.

30(a). If $x \cos(p + y) + \cos p \sin(p + y) = 0$, prove that $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, where p is a constant.

[3 marks]

Solution:

Concept:

- When $y = y(x)$ and p is constant, the chain rule gives $\frac{d}{dx} \sin(p + y) = \cos(p + y)y'$ and $\frac{d}{dx} \cos(p + y) = -\sin(p + y)y'$.
- Use the original implicit equation to eliminate x , and then apply $\sin^2 \theta + \cos^2 \theta = 1$.

Step 1: Differentiate the given relation Put $\theta = p + y$, so $d\theta/dx = dy/dx$ because p is constant. Differentiating $x \cos \theta + \cos p \sin \theta = 0$ by the product and chain rules yields

$$\cos \theta - x \sin \theta \frac{dy}{dx} + \cos p \cos \theta \frac{dy}{dx} = 0.$$

Quick Tip to Solve: Multiply the differentiated equation by $\cos(p + y)$, then replace $x \cos(p + y)$ using the original relation; this avoids unsafe division.

Thus

$$(\cos p \cos \theta - x \sin \theta) \frac{dy}{dx} = -\cos \theta.$$

Step 2: Eliminate x without dividing by a trigonometric factor Multiply the differentiated equation by $\cos \theta$:

$$(\cos p \cos^2 \theta - x \sin \theta \cos \theta) \frac{dy}{dx} = -\cos^2 \theta.$$

From the given equation, $x \cos \theta = -\cos p \sin \theta$. Substitute this product into the bracket, obtaining

$$\cos p (\cos^2 \theta + \sin^2 \theta) \frac{dy}{dx} = -\cos^2 \theta.$$

This substitution avoids dividing by $\cos \theta$, so no exceptional case is lost when that factor vanishes.

Step 3: Use the identity and restore the angle Since $\cos^2 \theta + \sin^2 \theta = 1$, the last equation reduces to $\cos p \, dy/dx = -\cos^2 \theta$. Restoring $\theta = p + y$ gives precisely the required relation at every point on a differentiable branch of the given implicit curve.

Step 4: Conclusion Hence $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, as required.

Quick Tip to Solve: Multiply the differentiated equation by $\cos(p+y)$, then replace $x \cos(p+y)$ using the original relation; this avoids unsafe division.

OR

30(b). Find the value of a and b so that the function f defined as

$$f(x) = \begin{cases} \frac{x-2}{|x-2|} + a, & x < 2, \\ a + b, & x = 2, \\ \frac{x-2}{|x-2|} + b, & x > 2 \end{cases}$$

is a continuous function.

[3 marks]

Solution:

Concept:

- Continuity at a point c requires $\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$. The two limits must agree with the assigned function value.
- For $x < 2$, $|x - 2| = 2 - x$; for $x > 2$, $|x - 2| = x - 2$. Thus the absolute-value ratio must be simplified separately on the two intervals.

Step 1: Simplify the two branches For $x < 2$, the numerator is negative, so

$$\frac{x-2}{|x-2|} = \frac{x-2}{2-x} = -1.$$

Quick Tip to Solve: Simplify the absolute-value ratio on each side, then compare each limit with the assigned value at the joining point.

Therefore the left branch is the constant $a - 1$. For $x > 2$, numerator and denominator both equal $x - 2$, giving the constant right branch $b + 1$. These constant functions are continuous on their respective open intervals, so only the joining point $x = 2$ needs checking.

Step 2: Impose continuity at the joining point The one-sided limits and prescribed value are

$$\lim_{x \rightarrow 2^-} f(x) = a - 1, \quad f(2) = a + b, \quad \lim_{x \rightarrow 2^+} f(x) = b + 1.$$

Equating the left limit to the value gives $a - 1 = a + b$, hence $b = -1$. Equating the right limit to the value gives $b + 1 = a + b$, hence $a = 1$. Both conditions are necessary: merely equating the left and right limits would leave the assigned value unchecked.

Step 3: Verify the resulting function Substituting these constants into the three branches gives $a - 1 = 0$, $a + b = 0$ and $b + 1 = 0$. Hence the function equals zero for every real x , including the joining point. A constant function is continuous everywhere, so the values obtained satisfy the full requirement, rather than only equality of the one-sided limits.

Step 4: Conclusion The required values are $a = 1, b = -1$.

Quick Tip to Solve: Simplify the absolute-value ratio on each side, then compare each limit with the assigned value at the joining point.

31(a). Find the intervals in which the function $f(x) = \frac{\log x}{x}$ is strictly increasing or strictly decreasing. [3 marks]

Solution:

Concept:

- The real logarithm is defined for $x > 0$; here $\log x$ denotes the natural logarithm, with derivative $1/x$.
- If f is differentiable and $f'(x) > 0$ throughout an interval, it is strictly increasing there; if $f'(x) < 0$, it is strictly decreasing.

Step 1: Differentiate on the domain The domain of $f(x) = \log x/x$ is $(0, \infty)$. Applying the quotient rule with numerator $\log x$ and denominator x , we obtain

$$f'(x) = \frac{x(1/x) - (\log x)(1)}{x^2} = \frac{1 - \log x}{x^2}.$$

The denominator is strictly positive throughout the domain, so only the numerator determines the sign of the derivative.

Step 2: Find the critical point and derivative signs The derivative vanishes when $1 - \log x = 0$, or $\log x = 1$, giving $x = e$. Since the logarithm is strictly increasing, $0 < x < e$ implies $\log x < 1$ and thus $f'(x) > 0$. Likewise, $x > e$ implies $\log x > 1$ and $f'(x) < 0$. There are no other zeros or undefined derivative values within the domain, so these two intervals exhaust the sign analysis.

Step 3: Translate the signs into strict monotonicity By the derivative test, any two domain points

Quick Tip to Solve: For a quotient with a positive squared denominator, make a sign chart only for the derivative numerator.

$u < v < e$ satisfy $f(u) < f(v)$, and any two points $e < u < v$ satisfy $f(u) > f(v)$. This follows directly from the mean value theorem applied between the points. The stationary point e separates the intervals and is a maximum; its derivative being zero does not reverse the strict inequalities on either side. The usual open intervals of monotonicity are stated below; including the endpoint e also preserves strict monotonicity on each side.

Step 4: Conclusion The function is strictly increasing on $(0, e)$ and strictly decreasing on (e, ∞) .

Quick Tip to Solve: For a quotient with a positive squared denominator, make a sign chart only for the derivative numerator.

OR

31(b). Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \frac{x}{2} + \frac{2}{x}$, on the interval $[1, 2]$. [3 marks]

Solution:

Concept:

- A continuous function on a closed interval attains an absolute maximum and minimum. Compare its values at both endpoints and at all stationary points in the interval.
- For $f(x) = x/2 + 2/x$, differentiation gives $f'(x) = 1/2 - 2/x^2$, valid when $x \neq 0$.

Step 1: Check continuity and find stationary candidates The interval $[1, 2]$ does not contain zero, so both terms of f are continuous throughout it and the absolute-extrema test applies. Differentiating,

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}.$$

The equation $f'(x) = 0$ gives $x^2 = 4$, hence $x = \pm 2$. Only $x = 2$ belongs to the given interval, and it is already an endpoint. There is no stationary point in $(1, 2)$.

Step 2: Compare the endpoint values Evaluate the function at every candidate in the interval:

$$f(1) = \frac{1}{2} + 2 = \frac{5}{2}, \quad f(2) = \frac{2}{2} + \frac{2}{2} = 2.$$

The value at $x = 1$ exceeds that at $x = 2$. Since all possible endpoint and stationary candidates have been considered, this comparison identifies the absolute extrema. The other algebraic root, -2 , must be excluded because it lies outside the interval.

Step 3: Confirm the behavior between endpoints For $1 < x < 2$, $x^2 - 4 < 0$ while $2x^2 > 0$, so $f'(x) < 0$. The function therefore strictly decreases across the interior of the interval. Its greatest value is attained at the left endpoint, and its least value at the right endpoint. In particular, the zero derivative at $x = 2$ is consistent with the endpoint minimum and does not require a point to the right within the specified interval.

Step 4: Conclusion The absolute maximum is $5/2$ at $x = 1$, and the absolute minimum is 2 at $x = 2$.

Quick Tip to Solve: Always retain both endpoints in a closed-interval extrema problem, even when a stationary point coincides with an endpoint.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). It is given that function $f(x) = x^4 - 62x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of 'a', hence obtain all other points where the given function $f(x)$ attains local maximum or local minimum values. [5 marks]

Solution:

Concept:

- At an interior local extremum of a differentiable function, its first derivative vanishes. Thus the given local maximum at $x = 1$ implies $f'(1) = 0$.
- At a stationary point $x = c$, $f''(c) > 0$ ensures a strict local minimum, while $f''(c) < 0$ ensures a strict local maximum. A zero first derivative alone does not classify an extremum.

Step 1: Determine the constant Differentiate the given polynomial, treating a as a constant:

$$f'(x) = 4x^3 - 124x + a.$$

At the given local maximum, $f'(1) = 0$. Hence

$$4 - 124 + a = 0, \quad a = 120.$$

Also, $f''(x) = 12x^2 - 124$, so $f''(1) = -112 < 0$. Thus the value of a obtained from the necessary condition does satisfy the specified local maximum condition, rather than merely giving a stationary point.

Step 2: Obtain all stationary points With $a = 120$, factor the derivative using the known root $x = 1$:

$$\begin{aligned} f'(x) &= 4(x^3 - 31x + 30) \\ &= 4(x - 1)(x^2 + x - 30) \\ &= 4(x - 1)(x + 6)(x - 5). \end{aligned}$$

Therefore the only stationary points are $x = -6, 1, 5$. Every local extremum must be in this list, since the polynomial is differentiable everywhere.

Step 3: Classify the extrema and calculate their ordinates Apply the second derivative test at the other two stationary points:

$$f''(-6) = 12(36) - 124 = 308 > 0, \quad f''(5) = 12(25) - 124 = 176 > 0.$$

Thus both are strict local minima. Using $f(x) = x^4 - 62x^2 + 120x + 9$, their values are

$$\begin{aligned} f(-6) &= 1296 - 2232 + 720 + 9 = -1647, \\ f(5) &= 625 - 1550 + 600 + 9 = -316. \end{aligned}$$

The given maximum has value $f(1) = 1 - 62 + 120 + 9 = 68$. As the second derivative is nonzero at every stationary point, no further test is needed. In particular, neither of the remaining points is a

Quick Tip to Solve: Determine the parameter using the given stationary point first, then use its known linear factor to factor the cubic derivative efficiently.

local maximum.

Step 4: Conclusion $a = 120$. The other extrema are local minima at $(-6, -1647)$ and $(5, -316)$. The given point $(1, 68)$ is the only local maximum; there is no other local maximum.

Quick Tip to Solve: Determine the parameter using the given stationary point first, then use its known linear factor to factor the cubic derivative efficiently.

OR

32(b). The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum. [5 marks]

Solution:

Concept:

- The perimeter of a rectangle with sides x and h is $2(x + h)$. Rolling the sheet makes one side the circumference $2\pi r$ and the other side the cylinder's height h .
- A cylinder has volume $V = \pi r^2 h$, where r is its radius. Express the volume in one variable using the perimeter constraint and maximise it on the interval where both side lengths are positive.

Step 1: Express the cylinder dimensions through one side Let x cm be the side that becomes the cylinder's circumference, and let h cm be the other side, which becomes its height. The specified perimeter gives

$$2(x + h) = 300, \quad h = 150 - x.$$

Since the rectangle has positive side lengths, $0 < x < 150$. The rolling condition gives

$$2\pi r = x, \quad r = \frac{x}{2\pi}.$$

Step 2: Form the volume and locate its stationary point Substituting both expressions into the cylinder volume formula gives

$$V(x) = \pi \left(\frac{x}{2\pi} \right)^2 (150 - x) = \frac{150x^2 - x^3}{4\pi}.$$

Differentiating with respect to x ,

$$V'(x) = \frac{300x - 3x^2}{4\pi} = \frac{3x(100 - x)}{4\pi}.$$

The stationary equation gives $x = 0$ or $x = 100$. The value $x = 0$ cannot represent a rectangular sheet of positive dimensions, so the only admissible stationary point is $x = 100$ cm.

Step 3: Verify that this is the maximum and recover the other side The second derivative is

$$V''(x) = \frac{300 - 6x}{4\pi}, \quad V''(100) = -\frac{75}{\pi} < 0.$$

Hence the stationary point is a strict local maximum. To verify that it is the largest volume throughout

Quick Tip to Solve: Name the circumference side before differentiating: its square appears in volume, so it should be twice the height at the maximum under this perimeter constraint.

the feasible interval, note that $V'(x) > 0$ for $0 < x < 100$ and $V'(x) < 0$ for $100 < x < 150$. Thus volume increases up to this point and decreases afterwards. It also tends to zero at either endpoint, where one dimension degenerates. The remaining side is $h = 150 - 100 = 50$ cm. The resulting radius is $50/\pi$ cm and the maximum volume is $125000/\pi \text{ cm}^3$. The dimensions also check the original constraint: $2(100 + 50) = 300$ cm.

Step 4: Conclusion The required rectangular sheet measures $100 \text{ cm} \times 50 \text{ cm}$, with the 100 cm side forming the circumference and the 50 cm side forming the height.

Quick Tip to Solve: Name the circumference side before differentiating: its square appears in volume, so it should be twice the height at the maximum under this perimeter constraint.

33. Find the area of the region bounded by the lines $x - 2y = 4$, $x = -1$, $x = 6$ and x -axis, using integration. [5 marks]

Solution:

Concept:

- The area between a curve and the x -axis is the integral of its vertical distance from that axis: $A = \int |y| dx$. The given vertical lines determine the limits of integration.
- When a curve crosses the axis, split the interval there. The strip height is $-y$ below the axis and y above the axis, so each portion contributes positive area.

Step 1: Express the line in terms of x From $x - 2y = 4$, the ordinate on the line is

$$y = \frac{x - 4}{2}.$$

The region lies between this line and $y = 0$, with vertical boundaries $x = -1$ and $x = 6$. A vertical strip of width dx therefore has height $|(x - 4)/2|$.

Step 2: Find the crossing and form the area integrals At the point where the line meets the x -axis, $y = 0$; hence $x = 4$. Since $-1 < 4 < 6$, we split the region at this value. For $-1 \leq x \leq 4$, the line is below the axis and the strip height is $(4 - x)/2$. For $4 \leq x \leq 6$, the line is above the axis and the height is $(x - 4)/2$. Thus,

$$A = \int_{-1}^4 \frac{4 - x}{2} dx + \int_4^6 \frac{x - 4}{2} dx.$$

Integrating the signed ordinate over the whole interval without this split would subtract the left-hand area rather than add it.

Step 3: Evaluate the area below the axis The first portion has area

$$\begin{aligned} A_1 &= \left[2x - \frac{x^2}{4} \right]_{-1}^4 \\ &= (8 - 4) - \left(-2 - \frac{1}{4} \right) \\ &= 4 + \frac{9}{4} = \frac{25}{4}. \end{aligned}$$

Quick Tip to Solve: Locate the x -intercept first and split at $x = 4$; use positive strip heights on both sides before adding their integrals.

Step 4: Evaluate the area above the axis and add The second portion has area

$$\begin{aligned} A_2 &= \left[\frac{x^2}{4} - 2x \right]_4^6 \\ &= (9 - 12) - (4 - 8) = 1. \end{aligned}$$

Hence the sum of the two non-overlapping portions is $A_1 + A_2 = 25/4 + 1 = 29/4$.

Step 5: Conclusion The required area is $\boxed{\frac{29}{4}}$ square units.

Quick Tip to Solve: Locate the x-intercept first and split at $x = 4$; use positive strip heights on both sides before adding their integrals.

34(a). Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines. [5 marks]

Solution:

Concept:

- A line through $\mathbf{a}$ with direction vector $\mathbf{b}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. At an intersection, the coordinates from the two parametric representations must agree.
- A direction vector perpendicular to two non-parallel vectors can be obtained by their cross product. Perpendicularity can be checked using zero dot products.

Step 1: Write both lines in parametric form Let the common ratio of the first line be λ . Then its coordinates and direction vector are

$$(x, y, z) = (\lambda, 1 + 2\lambda, 2 + 3\lambda), \quad \mathbf{b}_1 = (1, 2, 3).$$

Let the parameter on the second line be μ . Its direction ratios are $(0, -3, 2)$, so its parametric equations are

$$(x, y, z) = (1, -3\mu, 7 + 2\mu), \quad \mathbf{b}_2 = (0, -3, 2).$$

The zero direction ratio means that x stays fixed at 1 on this line; the printed symmetric notation is interpreted through these parametric equations.

Step 2: Determine and verify the point of intersection Equating the x -coordinates gives $\lambda = 1$. Equating the y -coordinates now gives $1 + 2 = -3\mu$, so $\mu = -1$. The corresponding z -coordinates are $2 + 3(1) = 5$ and $7 + 2(-1) = 5$, which agree. Thus the two given lines meet at $P = (1, 3, 5)$.

Quick Tip to Solve: A zero direction ratio fixes that coordinate: use $x = 1$ to find the intersection quickly, then cross the two direction vectors.

Step 3: Find a direction perpendicular to both lines The cross product of the two direction vectors is

$$\begin{aligned}\mathbf{b}_1 \times \mathbf{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & -3 & 2 \end{vmatrix} \\ &= \hat{i}(4 + 9) - \hat{j}(2 - 0) + \hat{k}(-3 - 0) \\ &= 13\hat{i} - 2\hat{j} - 3\hat{k}.\end{aligned}$$

The required direction ratios are 13, -2, -3.

Step 4: Form the line and check perpendicularity Using the point $P = (1, 3, 5)$, the line has vector equation

$$\mathbf{r} = (\hat{i} + 3\hat{j} + 5\hat{k}) + t(13\hat{i} - 2\hat{j} - 3\hat{k}), \quad t \in \mathbb{R}.$$

Its direction satisfies

$$\begin{aligned}(13, -2, -3) \cdot (1, 2, 3) &= 13 - 4 - 9 = 0, \\ (13, -2, -3) \cdot (0, -3, 2) &= 0 + 6 - 6 = 0.\end{aligned}$$

Therefore it is perpendicular to both given lines. At $t = 0$, it passes through their intersection.

Step 5: Conclusion The required equation is $\boxed{\frac{x-1}{13} = \frac{y-3}{-2} = \frac{z-5}{-3}}.$

Quick Tip to Solve: A zero direction ratio fixes that coordinate: use $x = 1$ to find the intersection quickly, then cross the two direction vectors.

OR

34(b). Two vertices of the parallelogram $ABCD$ are given as $A(-1, 2, 1)$ and $B(1, -2, 5)$. If the equation of the line passing through C and D is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between sides AB and CD . Hence, find the area of parallelogram $ABCD$. [5 marks]

Solution:

Concept:

- Opposite sides of a parallelogram are parallel. The vector $\overrightarrow{AB}$ therefore gives a direction parallel to both AB and CD .
- For a point P on one of two parallel lines with common direction vector $\mathbf{d}$, their perpendicular distance is $h = \frac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$, where A lies on the other line.
- The area of a parallelogram is base times perpendicular height: $\mathcal{A} = |AB|h$. Equivalently, it is the magnitude of the cross product of a base vector and a vector joining the two parallel sides.

Step 1: Identify the parallel directions and a point on the opposite side The given vertices yield

$$\overrightarrow{AB} = (1 + 1, -2 - 2, 5 - 1) = (2, -4, 4) = 2(1, -2, 2).$$

Quick Tip to Solve: Choose the zero-parameter point on the given line; divide the cross-product magnitude by the direction-vector length for the height.

The direction vector of the printed line through C and D is $\mathbf{d} = (1, -2, 2)$. Thus the line AB is parallel to CD , as required. Setting the common parameter of the given line equal to zero gives a convenient point $P = (4, -7, 8)$ on CD . It need not be one of the vertices: any point on this line can be used to calculate its perpendicular separation from AB . Hence

$$\overrightarrow{AP} = (4 + 1, -7 - 2, 8 - 1) = (5, -9, 7).$$

Step 2: Calculate the perpendicular distance between the sides The component of $\overrightarrow{AP}$ perpendicular to the common line direction is the height between the parallel sides. The cross product eliminates its parallel component:

$$\overrightarrow{AP} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -9 & 7 \\ 1 & -2 & 2 \end{vmatrix} = -4\mathbf{i} - 3\mathbf{j} - \mathbf{k}.$$

Its magnitude is $\sqrt{16 + 9 + 1} = \sqrt{26}$, while $|\mathbf{d}| = \sqrt{1 + 4 + 4} = 3$. Therefore the perpendicular distance is

$$h = \frac{\sqrt{26}}{3} \text{ units.}$$

Using the whole direction vector $\overrightarrow{AB} = 2\mathbf{d}$ would multiply both the numerator and denominator by two and give the same height.

Step 3: Use the base and the perpendicular height to find the area The length of the side joining the given vertices is

$$|AB| = \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{36} = 6.$$

Thus the base-height formula gives

$$\mathcal{A} = 6 \times \frac{\sqrt{26}}{3} = 2\sqrt{26} \text{ square units.}$$

This also agrees with $|\overrightarrow{AB} \times \overrightarrow{AP}| = 2\sqrt{26}$, providing a direct check of the computed area.

Step 4: Conclusion The distance between sides AB and CD is $\boxed{\frac{\sqrt{26}}{3} \text{ units}}$, and the area of parallelogram $ABCD$ is $\boxed{2\sqrt{26} \text{ square units}}$.

Quick Tip to Solve: Choose the zero-parameter point on the given line; divide the cross-product magnitude by the direction-vector length for the height.

35. A relation R on set $A = \{x : -10 \leq x \leq 10, x \in \mathbb{Z}\}$ is defined as $R = \{(x, y) : (x - y) \text{ is divisible by } 5\}$. Show that R is an equivalence relation. Also, write the equivalence class $[5]$. [5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive. Each property must hold for all relevant elements of the given set.
- An integer n is divisible by 5 precisely when $n = 5k$ for some integer k . Zero and negative multiples also satisfy this definition.
- The equivalence class of $a \in A$ is $[a] = \{x \in A : (a, x) \in R\}$; it contains exactly those elements of the underlying set related to a .

Step 1: Establish reflexivity on the given set Here $A = \{-10, -9, \dots, 9, 10\}$ and $(x, y) \in R$ means that $x - y$ is divisible by 5. Let x be any element of A . Then

$$x - x = 0 = 5 \times 0.$$

Since 0 is an integer, 5 divides this difference. Thus $(x, x) \in R$ for every $x \in A$, which establishes that R is reflexive. In particular, there is no exception at zero or at either endpoint of the set.

Step 2: Establish symmetry Let $x, y \in A$ and suppose $(x, y) \in R$. By the defining condition, there is an integer k such that $x - y = 5k$. Reversing the order of the difference gives

$$y - x = -(x - y) = -5k = 5(-k).$$

As $-k$ is also an integer, $y - x$ is divisible by 5. Therefore $(y, x) \in R$. This proves the required implication for every related pair, so R is symmetric.

Step 3: Establish transitivity Let $x, y, z \in A$ and suppose both $(x, y) \in R$ and $(y, z) \in R$. Then, for some integers m, n ,

$$x - y = 5m, \quad y - z = 5n.$$

Adding these equations cancels the middle element:

$$x - z = (x - y) + (y - z) = 5m + 5n = 5(m + n).$$

The sum $m + n$ is an integer, so 5 divides $x - z$. Hence $(x, z) \in R$, proving transitivity. The three established properties together show that R is an equivalence relation on A .

Step 4: Determine the equivalence class of five By definition,

$$[5] = \{x \in A : 5 - x \text{ is divisible by } 5\}.$$

The difference $5 - x$ is a multiple of 5 exactly when x is a multiple of 5. Thus $x = 5r$ for an integer r . Applying the bounds of the underlying set gives

$$-10 \leq 5r \leq 10 \implies -2 \leq r \leq 2.$$

Quick Tip to Solve: After proving the three properties using integer multiples, list only the multiples of five within the bounds of the original set.

The possible integer values are therefore $r = -2, -1, 0, 1, 2$, giving $x = -10, -5, 0, 5, 10$. Every one of these elements lies in A and satisfies the relation with 5; other multiples of 5 lie outside the stated set and must be excluded.

Step 5: Conclusion The relation R is reflexive, symmetric and transitive, hence an equivalence relation. Its equivalence class at 5 is $[5] = \{-10, -5, 0, 5, 10\}$.

Quick Tip to Solve: After proving the three properties using integer multiples, list only the multiples of five within the bounds of the original set.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1: A bacteria sample of certain number of bacteria is observed to grow exponentially in a given amount of time. Using exponential growth model, the rate of growth of this sample of bacteria is calculated.

The differential equation representing the growth of bacteria is given as: $\frac{dP}{dt} = kP$, where P is the population of bacteria at any time ' t '. [4 marks]

36(i). Obtain the general solution of the given differential equation and express it as an exponential function of ' t '. [2 marks]

Solution:

Concept:

- Separation of variables puts the population variable P on one side and time t on the other.
- For a positive population, $\int dP/P = \ln P$; the constant k is the proportional growth rate.

Step 1: Separate the variables The population is positive in the bacteria model. Dividing $dP/dt = kP$ by P and multiplying by dt gives

$$\frac{dP}{P} = k dt.$$

Thus the equation is separable, with the growth parameter k constant in time.

Step 2: Integrate and convert to exponential form Integrating both sides gives

$$\int \frac{dP}{P} = \int k dt \implies \ln P = kt + C_1.$$

Exponentiating, $P = e^{kt+C_1} = e^{C_1}e^{kt}$. Set $C = e^{C_1}$, the arbitrary positive initial population. More generally, the mathematical family allows any real C , including the equilibrium solution $P = 0$. Differentiating Ce^{kt} gives $kCe^{kt} = kP$, verifying the equation.

Step 3: Conclusion The general solution is $P(t) = Ce^{kt}$. For the population model, $C = P(0) > 0$.

Quick Tip to Solve: After exponentiating the integration constant, rename it C ; it equals the initial population.

36(ii). If population of bacteria is 1000 at $t = 0$, and 2000 at $t = 1$, find the value of k .

[2 marks]

Solution:

Concept:

- The exponential growth model is $P(t) = Ce^{kt}$, where $C = P(0)$.
- Taking natural logarithms of a growth ratio isolates the constant k .

Step 1: Use the initial population Integrating $dP/dt = kP$ by separating variables gives $\ln P = kt + C_1$, and hence $P(t) = Ce^{kt}$. At $t = 0$, the exponential factor is $e^0 = 1$, so

$$1000 = Ce^0 = C.$$

The population function therefore has the form $P(t) = 1000e^{kt}$.

Step 2: Use the population after one time unit At $t = 1$, substitute the given population 2000:

$$2000 = 1000e^k \implies e^k = 2.$$

Taking natural logarithms yields $k = \ln 2$. As a check, this gives $P(t) = 1000e^{t \ln 2} = 1000 2^t$, which has population 1000 initially and 2000 at $t = 1$. The positive value agrees with the observed growth.

Step 3: Conclusion $k = \ln 2$, approximately 0.6931 per unit time.

Quick Tip to Solve: Divide the later population by the initial population before taking the natural logarithm.

37. Case Study - 2: A scholarship is a sum of money provided to a student to help him or her pay for education. Some students are granted scholarships based on their academic achievements, while others are rewarded based on their financial needs.

Every year a school offers scholarships to girl children and meritorious achievers based on certain criteria. In the session 2022 - 23, the school offered monthly scholarship of ₹3,000 each to some girl students and ₹4,000 each to meritorious achievers in academics as well as sports.

In all, 50 students were given the scholarships and monthly expenditure incurred by the school on scholarships was ₹1,80,000.

[4 marks]

37(i). Express the given information algebraically using matrices.

[1 mark]

Solution:

Concept:

- A matrix equation $AX = B$ records the coefficients, unknowns and constants of simultaneous linear equations.
- Let x count girl-child scholarships and y count meritorious scholarships.

Step 1: Form the two equations The total number of recipients is 50, so $x + y = 50$. The monthly payments are 3000 rupees and 4000 rupees respectively; thus $3000x + 4000y = 180000$. Keep the unknowns in the same order in both equations.

Step 2: Arrange coefficients as a matrix Place the coefficients of x, y in consecutive columns and the

Quick Tip to Solve: Use one row for the student count and one for the monthly cost, keeping the column order fixed.

two totals in a column vector:

$$\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

Multiplying each row by the unknown column reproduces the corresponding equation.

Step 3: Conclusion The required representation is $AX = B$, with $A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}$.

Quick Tip to Solve: Use one row for the student count and one for the monthly cost, keeping the column order fixed.

37(ii). Check whether the system of matrix equations so obtained is consistent or not.

[1 mark]

Solution:

Concept:

- A square coefficient matrix with nonzero determinant is invertible.
- If A is invertible, $AX = B$ has the unique solution $X = A^{-1}B$, so it is consistent.

Step 1: Compute the determinant The coefficient matrix of the equations $x + y = 50$ and $3000x + 4000y = 180000$ is $A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}$. Its determinant is

$$|A| = 1(4000) - 1(3000) = 1000.$$

It is therefore nonzero.

Step 2: Apply the consistency criterion Since $|A| \neq 0$, the inverse A^{-1} exists. Multiplying $AX = B$ by this inverse determines exactly one vector $X = A^{-1}B$; hence the two equations can be satisfied simultaneously. There is no contradiction between the number of recipients and the stated total expenditure.

Step 3: Conclusion The system is consistent and has a unique solution.

Quick Tip to Solve: For this square system, a nonzero coefficient determinant immediately establishes consistency.

37(iii)(a). Find the number of scholarships of each kind given by the school, using matrices.

[2 marks]

Solution:

Concept:

- Write the scholarship equations as $AX = B$, with x girl-child and y meritorious scholarships.

Quick Tip to Solve: Check both the recipient total and the weighted payment total after multiplying by the inverse.

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when the determinant is nonzero.

Step 1: Form the matrix equation and inverse The count and expenditure equations are $x + y = 50$ and $3000x + 4000y = 180000$. Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}, \quad B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

Since $|A| = 4000 - 3000 = 1000$,

$$A^{-1} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix}.$$

Step 2: Multiply by the inverse and check Using $X = A^{-1}B$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1000} \begin{pmatrix} 4000(50) - 180000 \\ -3000(50) + 180000 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

These are nonnegative whole numbers. They satisfy $20 + 30 = 50$ and $3000(20) + 4000(30) = 60000 + 120000 = 180000$, as required.

Step 3: Conclusion The school gave 20 girl-child scholarships and 30 meritorious scholarships.

Quick Tip to Solve: Check both the recipient total and the weighted payment total after multiplying by the inverse.

OR

37(iii)(b). Had the amount of scholarship given to each girl child and meritorious student been interchanged, what would be the monthly expenditure incurred by the school? [2 marks]

Solution:

Concept:

- Let x and y be the counts of girl-child and meritorious scholarships respectively.
- After the amounts are interchanged, the counts stay fixed and expenditure is $E = 4000x + 3000y$.

Step 1: Determine the original recipient counts The original conditions give

$$\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

The coefficient determinant is 1000. Multiplication by its inverse gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 180000 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

Quick Tip to Solve: Interchange the payment rates only; retain the original numbers of recipients in each category.

Step 2: Calculate the expenditure with interchanged amounts Each of the 20 girl students now receives 4000 rupees and each of the 30 meritorious students receives 3000 rupees. Hence

$$E = \begin{pmatrix} 4000 & 3000 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = 80000 + 90000 = 170000.$$

There are more meritorious recipients, so interchanging the rates reduces the total by $1000(30 - 20) = 10000$ rupees, confirming the calculation.

Step 3: Conclusion The monthly expenditure would be ₹1,70,000.

Quick Tip to Solve: Interchange the payment rates only; retain the original numbers of recipients in each category.

38. Case Study - 3: Self-study helps students to build confidence in learning. It boosts the self-esteem of the learners. Recent surveys suggested that close to 50% learners were self-taught using internet resources and upskilled themselves.

A student may spend 1 hour to 6 hours in a day in upskilling self. The probability distribution of the number of hours spent by a student is given below:

$$P(X = x) = \begin{cases} kx^2, & \text{for } x = 1, 2, 3 \\ 2kx, & \text{for } x = 4, 5, 6 \\ 0, & \text{otherwise} \end{cases} \text{ where } x \text{ denotes the number of hours.} \quad [4 \text{ marks}]$$

38(i). Express the probability distribution given above in the form of a probability distribution table. [1 mark]

Solution:

Concept:

- A probability distribution table lists every possible value of the random variable X and its corresponding probability.
- Use $P(X = x) = kx^2$ for $x = 1, 2, 3$, and $P(X = x) = 2kx$ for $x = 4, 5, 6$.

Step 1: Evaluate the first branch For the first three possible durations, substitution in kx^2 gives $P(X = 1) = k$, $P(X = 2) = 4k$, and $P(X = 3) = 9k$.

Step 2: Evaluate the second branch For the remaining durations, substitution in $2kx$ gives $P(X = 4) = 8k$, $P(X = 5) = 10k$, and $P(X = 6) = 12k$. These six values exhaust the stated possible durations; the probability is zero at all other values.

Step 3: Conclusion The required probability distribution table is

x	1	2	3	4	5	6
$P(X = x)$	k	$4k$	$9k$	$8k$	$10k$	$12k$

Quick Tip to Solve: Check the branch boundary carefully: use the square only through 3, and use twice the value from 4 onwards.

38(ii). Find the value of k .

[1 mark]

Solution:

Concept:

- The probabilities of all possible values of a discrete random variable sum to 1.
- Here the probabilities are $k, 4k, 9k, 8k, 10k, 12k$, corresponding respectively to $X = 1, 2, 3, 4, 5, 6$.

Step 1: Apply normalization Since these six outcomes exhaust the possible values of X ,

$$k + 4k + 9k + 8k + 10k + 12k = 1.$$

The first three terms come from kx^2 , and the last three from $2kx$.

Step 2: Solve for the constant Adding the coefficients gives $1 + 4 + 9 + 8 + 10 + 12 = 44$, so $44k = 1$. Thus $k = 1/44$. This is positive and every individual probability lies between zero and one, as required for a probability distribution.

Step 3: Conclusion $k = \frac{1}{44}$.

Quick Tip to Solve: Normalize the complete distribution before computing any mean or event probability.

38(iii)(a). Find the mean number of hours spent by the student.

[2 marks]

Solution:

Concept:

- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$; each possible duration is weighted by its probability.
- Normalization gives $k(1 + 4 + 9 + 8 + 10 + 12) = 1$, hence $k = 1/44$.

Step 1: Form the probability-weighted sum The six possible durations are 1, 2, 3, 4, 5, 6 hours. Their probabilities are respectively $k, 4k, 9k, 8k, 10k, 12k$. Therefore

$$\begin{aligned} E(X) &= 1(k) + 2(4k) + 3(9k) \\ &\quad + 4(8k) + 5(10k) + 6(12k). \end{aligned}$$

It is essential to multiply each probability by its corresponding value of X ; simply averaging the six durations would ignore their unequal probabilities.

Step 2: Evaluate the mean The contributions have coefficients 1, 8, 27, 32, 50, 72. Their sum is 190, giving

$$E(X) = 190k = \frac{190}{44} = \frac{95}{22}.$$

The value is approximately 4.318, which lies within the range from 1 to 6 hours and is closer to the larger durations, whose probabilities are greater.

Step 3: Conclusion The mean number of hours spent is $\frac{95}{22}$ hours (approximately 4.32 hours).

Quick Tip to Solve: For a mean, multiply the probability-table row by the duration row before adding; retain the exact fraction.

38(iii)(b). Find $P(1 < X < 6)$.

[2 marks]

Solution:

Concept:

- For a discrete random variable, an event probability is the sum of the probabilities of the possible values satisfying the event.
- The strict inequalities $1 < X < 6$ include $X = 2, 3, 4, 5$, while excluding both endpoint values 1 and 6.

Step 1: Identify and add the required probabilities The duration takes only the integer values 1, 2, 3, 4, 5, 6. Thus the event in the question is the disjoint union of $X = 2$, $X = 3$, $X = 4$, and $X = 5$. Using the appropriate branch of the distribution for each value,

$$\begin{aligned} P(1 < X < 6) &= 4k + 9k + 8k + 10k \\ &= 31k. \end{aligned}$$

The value 3 uses kx^2 , whereas 4 uses $2kx$.

Step 2: Normalize and evaluate The sum of all six probabilities is $44k = 1$, so $k = 1/44$. Consequently the event has probability $31/44$. As a check, its complement consists of the two excluded endpoints, with total probability $k + 12k = 13/44$; subtracting this from one also gives $31/44$.

Step 3: Conclusion $P(1 < X < 6) = \frac{31}{44}$.

Quick Tip to Solve: For strict inequalities on integer-valued X , list the included integers first to avoid including the endpoints.