

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-3-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is **not** allowed.

Section A (26 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The value of $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ is:

[1 mark]

- (A) 0
(B) 2
(C) 7
(D) -2

Correct Answer: (A) 0

Solution:

Concept:

- A determinant is zero if two columns are proportional: if $C_1 = kC_2$, its columns are linearly dependent.
- For a 3×3 determinant, adding a multiple of one column to another does not change its value.

Step 1: Compare the first two columns The first column is $(8, 12, 16)^T$ and the second is $(2, 3, 4)^T$. Each entry in the first is four times the corresponding entry in the second. Thus $C_1 = 4C_2$, independently of the entries in the third column.

Step 2: Use the determinant property and check the choices Apply $C_1 \rightarrow C_1 - 4C_2$. The first column becomes $(0, 0, 0)^T$, so expansion along it gives zero. The nonzero proposed values 2, 7 and -2 cannot occur when two columns are proportional.

Step 3: Conclusion The determinant equals 0. Hence, the correct option is (A).

Quick Tip to Solve: Before expanding a determinant, compare columns; here the ratio 4 is constant in all three rows.

2. If $y = \sin^{-1} x$, then $\frac{d^2y}{dx^2}$ is:

[1 mark]

- (A) $\sec y$
(B) $\sec y \tan y$
(C) $\sec^2 y \tan y$
(D) $\tan^2 y \sec y$

Correct Answer: (C) $\sec^2 y \tan y$

Solution:

Concept:

- For $y = \sin^{-1} x$, the principal range is $[-\pi/2, \pi/2]$; differentiation applies for $-1 < x < 1$.
- The chain rule gives $\frac{d}{dx}(\sec y) = \sec y \tan y \frac{dy}{dx}$.

Quick Tip to Solve: When differentiating an expression in y with respect to x, multiply by dy/dx ; here this adds a second secant factor.

Step 1: Find the first derivative Write $x = \sin y$. Differentiating with respect to x gives $1 = \cos y \, dy/dx$. On the interior of the principal range, $\cos y > 0$, so $dy/dx = \sec y$.

Step 2: Differentiate again and compare options Using the chain rule,

$$\frac{d^2y}{dx^2} = \sec y \tan y \frac{dy}{dx} = \sec^2 y \tan y.$$

Option (A) is only the first derivative; (B) omits the factor dy/dx , and (D) incorrectly squares $\tan y$ instead of $\sec y$.

Step 3: Conclusion $\frac{d^2y}{dx^2} = \sec^2 y \tan y$. Hence, the correct option is (C).

Quick Tip to Solve: When differentiating an expression in y with respect to x , multiply by dy/dx ; here this adds a second secant factor.

3. If $|\vec{a}| = 2$ and $-3 \leq k \leq 2$, then $|k\vec{a}| \in$:

[1 mark]

- (A) $[-6, 4]$
- (B) $[0, 4]$
- (C) $[4, 6]$
- (D) $[0, 6]$

Correct Answer: (D) $[0, 6]$

Solution:

Concept:

- For a real scalar k , the magnitude of a scaled vector is $|k\vec{a}| = |k| |\vec{a}|$.
- A magnitude is nonnegative; a negative scalar reverses direction but does not create a negative length.

Step 1: Find the possible scalar magnitudes Since $-3 \leq k \leq 2$, the interval contains zero. Hence the least value of $|k|$ is zero, attained at $k = 0$, and its greatest value is 3, attained at $k = -3$. All values between 0 and 3 are attained.

Step 2: Scale and check the intervals As $|\vec{a}| = 2$, we have $|k\vec{a}| = 2|k|$, whose complete range is $[0, 6]$. Option (A) incorrectly permits negative magnitudes. Option (B) misses the value 6 at $k = -3$; option (C) excludes zero and other small magnitudes.

Step 3: Conclusion $|k\vec{a}| \in [0, 6]$. Hence, the correct option is (D).

Quick Tip to Solve: Take the absolute value of the scalar before multiplying by the vector magnitude; check both endpoints and zero.

4. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of both x -axis and z -axis, then the angle which it makes with the positive direction of y -axis is: [1 mark]

- (A) 0
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{2}$
- (D) π

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- If a line makes angles α, β, γ with the positive coordinate axes, its direction cosines are $\cos \alpha, \cos \beta, \cos \gamma$.
- Direction cosines satisfy $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, with each direction angle in $[0, \pi]$.

Step 1: Substitute the two known angles Here $\alpha = \gamma = \pi/4$. Therefore

$$\cos^2 \beta = 1 - \cos^2(\pi/4) - \cos^2(\pi/4) = 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

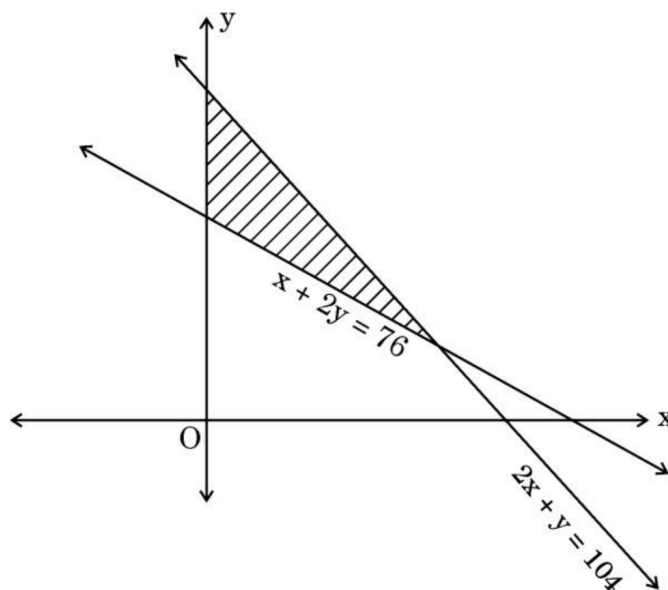
The given two direction cosines already exhaust the sum of squares.

Step 2: Determine the angle and test the choices Thus $\cos \beta = 0$. In the direction-angle range $[0, \pi]$, this gives $\beta = \pi/2$. Angles 0 and π have squared cosine 1, while $\pi/4$ has squared cosine $1/2$; each would make the total exceed 1.

Step 3: Conclusion The angle with the positive y -axis is $\pi/2$. Hence, the correct option is (C).

Quick Tip to Solve: Two direction angles of $\pi/4$ contribute one-half each, leaving zero for the third squared direction cosine.

5. Of the following, which group of constraints represents the feasible region given below? [1 mark]



(A) $x + 2y \leq 76$, $2x + y \geq 104$, $x, y \geq 0$

(B) $x + 2y \leq 76$, $2x + y \leq 104$, $x, y \geq 0$

(C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$

(D) $x + 2y \geq 76$, $2x + y \geq 104$, $x, y \geq 0$

Correct Answer: (C) $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$

Solution:

Concept:

- A linear inequality selects one side of its boundary line; a point in the shaded region determines the correct sign.
- The first-quadrant restrictions are $x \geq 0$ and $y \geq 0$; a solid boundary is included using $\leq$ or $\geq$.

Step 1: Test a point inside the shaded region The shading lies between the two lines beside the positive y -axis. Take $(0, 50)$, between the intercepts 38 and 104. At this point, $x + 2y = 100 > 76$, whereas $2x + y = 50 < 104$. Thus the shaded sides require $x + 2y \geq 76$ and $2x + y \leq 104$.

Step 2: Include the quadrant restrictions and compare The entire shaded triangle is in the first quadrant, so $x, y \geq 0$. Option (A) reverses both required signs, (B) chooses the wrong side of $x + 2y = 76$, and (D) chooses the wrong side of $2x + y = 104$. Only (C) selects their shaded intersection.

Step 3: Conclusion The constraints are $x + 2y \geq 76$, $2x + y \leq 104$, $x, y \geq 0$. Hence, the correct option is (C).

Quick Tip to Solve: Test $(0, 50)$, which lies between the two y -intercepts; its substitutions immediately determine both inequality signs.

6. If $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then A^{-1} is:

[1 mark]

(A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(B) $30 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

(C) $\frac{1}{30} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(D) $\frac{1}{30} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$

Solution:

Concept:

- An inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.
- For a diagonal matrix with nonzero diagonal entries, the inverse has their reciprocals on the diagonal.

Step 1: Invert the nonzero diagonal entries The diagonal entries of A are 2, 3 and 5. Their product is 30, so A is invertible. Set

$$B = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}.$$

Multiplication of diagonal matrices multiplies corresponding diagonal entries.

Step 2: Verify the identity and eliminate the other choices Thus $AB = BA = \text{diag}(1, 1, 1) = I$. Printed option (B), $30B$, instead gives $A(30B) = 30I$. Option (D), $B/30$, gives $I/30$. Option (C), $A/30$, gives $A^2/30 = \text{diag}(4, 9, 25)/30$, which is also not I .

Step 3: Conclusion $A^{-1} = \text{diag}(1/2, 1/3, 1/5)$. Hence, the correct option is (A).

Quick Tip to Solve: For a diagonal matrix, reciprocate each nonzero diagonal entry and multiply once to verify the identity.

7. If $A = [a_{ij}]$ is an identity matrix, then which of the following is true?

[1 mark]

(A) $a_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}$

(B) $a_{ij} = 1, \forall i, j$

(C) $a_{ij} = 0, \forall i, j$

(D) $a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

Correct Answer: (D) $a_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

Solution:

Concept:

- An identity matrix has 1 in each principal diagonal position and 0 in every other position.
- The row and column indices agree on the principal diagonal: $i = j$; off the diagonal, $i \neq j$.

Step 1: Translate the definition into indexed entries In $A = [a_{ij}]$, the index i identifies the row and j the column. An identity matrix leaves vectors unchanged on multiplication. Its diagonal entries must therefore be 1, while off-diagonal entries must be 0 so that different components are not mixed.

Step 2: Compare all the proposed rules Option (D) states $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ when $i = j$, exactly the identity rule. Option (A) interchanges these values. Option (B) sets off-diagonal entries to 1 as well, and option (C) sets even diagonal entries to 0; neither is the general identity-matrix rule.

Step 3: Conclusion $a_{ij} = 1$ for $i = j$, and $a_{ij} = 0$ for $i \neq j$. Hence, the correct option is (D).

Quick Tip to Solve: Read $i=j$ as a diagonal position; the identity has ones there and zeros at every position with unequal indices.

8. Let Z denote the set of integers, then function $f : Z \rightarrow Z$ defined as $f(x) = x^3 - 1$ is:

[1 mark]

(A) both one-one and onto

(B) one-one but not onto

(C) onto but not one-one

(D) neither one-one nor onto

Correct Answer: (B) one-one but not onto

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ implies $a = b$ for every pair of inputs in its domain.
- A function $f : Z \rightarrow Z$ is onto only if every integer in the codomain has an integer preimage.

Step 1: Check whether equal outputs force equal inputs For integers a, b , suppose $f(a) = f(b)$. Then $a^3 - 1 = b^3 - 1$, so $a^3 = b^3$. The real cube function is strictly increasing, and hence $a = b$. The given

Quick Tip to Solve: To disprove onto Z , choose an integer output whose value plus one is not a perfect cube; output 1 works.

function is therefore one-one.

Step 2: Check the integer codomain and compare the options To obtain the integer output 1, we would need $x^3 - 1 = 1$, or $x^3 = 2$. No integer satisfies this: $1^3 = 1 < 2 < 8 = 2^3$. Thus 1 has no integer preimage and the function is not onto. Options (A) and (C) falsely claim it is onto; (D) falsely denies injectivity.

Step 3: Conclusion The function is one-one but not onto. Hence, the correct option is (B).

Quick Tip to Solve: To disprove onto $\mathbb{Z}$, choose an integer output whose value plus one is not a perfect cube; output 1 works.

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix such that $\text{adj } A = A$. Then, $(a + b + c + d)$ is equal to: [1 mark]

- (A) $2a$
- (B) $2b$
- (C) $2c$
- (D) 0

Correct Answer: (A) $2a$

Solution:

Concept:

- The adjugate of a 2×2 matrix is obtained by interchanging its diagonal entries and changing the signs of its off-diagonal entries.
- Two matrices are equal when their corresponding entries are equal.

Step 1: Write the adjugate For the given matrix,

$$\text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Equating this to $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives $d = a$, $-b = b$ and $-c = c$. Thus $b = c = 0$; the diagonal entry a is unrestricted.

Step 2: Evaluate and compare the options Substitution gives

$$a + b + c + d = a + 0 + 0 + a = 2a.$$

Options (B) and (C) are both zero because the off-diagonal entries vanish. Option (D) is also zero, but the sum need not vanish: for example, $A = I$ satisfies the condition and has sum 2. Hence those options are not the general expression required.

Step 3: Conclusion The required sum is $\boxed{2a}$. Hence, the correct option is (A).

Quick Tip to Solve: For a 2×2 adjugate equation, compare the off-diagonal entries first: each must vanish.

10. A function $f(x) = |1 - x + |x||$ is:

[1 mark]

- (A) discontinuous at $x = 1$ only
- (B) discontinuous at $x = 0$ only
- (C) discontinuous at $x = 0, 1$
- (D) continuous everywhere

Correct Answer: (D) continuous everywhere

Solution:

Concept:

- The function $|x|$ is continuous for every real x . Sums and compositions of continuous functions are continuous.
- To remove an absolute value, determine the sign of its argument separately on each relevant interval.

Step 1: Obtain a piecewise expression If $x \geq 0$, then $|x| = x$, so $1 - x + |x| = 1$. If $x < 0$, then $|x| = -x$, so $1 - x + |x| = 1 - 2x > 0$. The outer absolute value therefore changes neither branch:

$$f(x) = \begin{cases} 1 - 2x, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Step 2: Check the possible joining point Both branches are continuous on their own intervals. At $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = 1 = \lim_{x \rightarrow 0^+} f(x) = f(0).$$

Thus the join is continuous. The point $x = 1$ lies in the constant branch, so it is also continuous. Options (A), (B) and (C) all assert a discontinuity which does not occur.

Step 3: Conclusion The function is continuous everywhere on $\mathbb{R}$. Hence, the correct option is (D).

Quick Tip to Solve: Resolve the inner absolute value first; check continuity where its sign changes, at $x = 0$.

11. The rate of change of surface area of a sphere with respect to its radius r , when $r = 4$ cm, is:

[1 mark]

- (A) $64\pi \text{ cm}^2/\text{cm}$
- (B) $48\pi \text{ cm}^2/\text{cm}$
- (C) $32\pi \text{ cm}^2/\text{cm}$
- (D) $16\pi \text{ cm}^2/\text{cm}$

Correct Answer: (C) $32\pi \text{ cm}^2/\text{cm}$

Solution:

Concept:

- The surface area of a sphere of radius r is $S = 4\pi r^2$.
- The rate of change of surface area with respect to radius is dS/dr , with units of area divided by

Quick Tip to Solve: Differentiate $4\pi r^2$ before substituting the radius; substituting first gives area, not rate.

length.

Step 1: Differentiate with respect to radius Treat S as a function of r and apply the power rule:

$$\frac{dS}{dr} = 4\pi \frac{d}{dr}(r^2) = 8\pi r.$$

This is a derivative with respect to radius; no time derivative or value of dr/dt is needed.

Step 2: Substitute the given radius At $r = 4$ cm,

$$\left. \frac{dS}{dr} \right|_{r=4} = 8\pi(4) = 32\pi \text{ cm}^2/\text{cm}.$$

The value 64π in option (A) is the surface area itself at this radius, not its rate of change. Options (B) and (D), 48π and 16π , do not equal the evaluated derivative.

Step 3: Conclusion The rate is $32\pi \text{ cm}^2/\text{cm}$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate $4\pi r^2$ before substituting the radius; substituting first gives area, not rate.

12. $\int_{-a}^a f(x) dx = 0$, if:

[1 mark]

(A) $f(-x) = f(x)$

(B) $f(-x) = -f(x)$

(C) $f(a-x) = f(x)$

(D) $f(a-x) = -f(x)$

Correct Answer: (B) $f(-x) = -f(x)$

Solution:

Concept:

- An odd function satisfies $f(-x) = -f(x)$ wherever both sides are defined.
- For an integrable function on $[-a, a]$, split the integral at zero and substitute $x = -t$ in its negative half.

Step 1: Combine the two halves For $a > 0$, substitution in the negative half gives

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= \int_0^a (f(-t) + f(t)) dt. \end{aligned}$$

If $f(-t) = -f(t)$, the integrand is identically zero.

Step 2: Test the other conditions Condition (A) states evenness; it gives twice the integral over $[0, a]$, which need not vanish. The constant function $f(x) = 1$ satisfies both (A) and (C), but its integral is $2a$. Condition (D) describes antisymmetry about $a/2$, rather than zero. For example, $f(x) = x - a/2$ satisfies (D), while its integral on $[-a, a]$ is $-a^2$, not zero.

Quick Tip to Solve: For symmetric limits $[-a, a]$, pair $f(x)$ with $f(-x)$; oddness makes each pair cancel.

Step 3: Conclusion The required condition is $f(-x) = -f(x)$. Hence, the correct option is (B).

Quick Tip to Solve: For symmetric limits $[-a, a]$, pair $f(x)$ with $f(-x)$; oddness makes each pair cancel.

13. $x \log x \frac{dy}{dx} + y = 2 \log x$ is an example of a:

[1 mark]

- (A) variable separable differential equation.
- (B) homogeneous differential equation.
- (C) first order linear differential equation.
- (D) differential equation whose degree is not defined.

Correct Answer: (C) first order linear differential equation.

Solution:

Concept:

- A first order linear differential equation has the form $dy/dx + P(x)y = Q(x)$, where P and Q depend only on x .
- The order is the order of the highest derivative; the degree is its power when the equation is polynomial in derivatives.

Step 1: Put the equation into standard form On an interval with $x > 0$ and $x \neq 1$, divide by $x \log x$:

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}.$$

Thus $P(x) = 1/(x \log x)$ and $Q(x) = 2/x$. The unknown function and its first derivative occur only to the first power, with coefficients depending on x .

Step 2: Check the competing classifications Written as $dy/dx = 2/x - y/(x \log x)$, the right side is not the usual separable product of a function of x and a function of y , nor a function solely of y/x for a homogeneous equation. Its derivative occurs polynomially to power one, so its degree is defined and equals 1; option (D) is false. The standard linear form directly identifies option (C).

Step 3: Conclusion It is a first order linear differential equation. Hence, the correct option is (C).

Quick Tip to Solve: Divide by the coefficient of dy/dx , then identify $P(x)$ and $Q(x)$ in the linear form.

14. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, then $\vec{a}$ and $\vec{b}$ are:

[1 mark]

- (A) collinear vectors which are not parallel
- (B) parallel vectors
- (C) perpendicular vectors
- (D) unit vectors

Correct Answer: (C) perpendicular vectors

Solution:

Concept:

- For nonzero vectors $\vec{a}$ and $\vec{b}$, perpendicularity is equivalent to $\vec{a} \cdot \vec{b} = 0$.
- In an orthonormal Cartesian basis, the dot product is the sum of products of corresponding components.

Step 1: Calculate the dot product The component triples are $(2, -1, 1)$ and $(1, 1, -1)$. Hence

$$\vec{a} \cdot \vec{b} = 2(1) + (-1)(1) + (1)(-1) = 2 - 1 - 1 = 0.$$

Both vectors are nonzero, so the zero dot product determines their angle as 90° .

Step 2: Check the remaining options The component ratios for proportionality would be $2/1 = 2$, $(-1)/1 = -1$ and $1/(-1) = -1$; these are unequal. The vectors are therefore neither parallel nor collinear, excluding (A) and (B). Their magnitudes are $\sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ and $\sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$, not 1, so (D) is false.

Step 3: Conclusion The vectors are perpendicular. Hence, the correct option is (C).

Quick Tip to Solve: For a perpendicularity option, compute the dot product first; cancellation immediately settles it.

15. If α, β and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following is not true?

[1 mark]

- (A) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
- (B) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
- (C) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
- (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Correct Answer: (D) $\cos \alpha + \cos \beta + \cos \gamma = 1$

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$ and $n = \cos \gamma$, with $l^2 + m^2 + n^2 = 1$.
- Use $\sin^2 \theta = 1 - \cos^2 \theta$ and $\cos 2\theta = 2\cos^2 \theta - 1$ to obtain related identities.

Step 1: Verify the three squared identities Option (A) is exactly the direction-cosine identity. It

Quick Tip to Solve: Retain the squares in the direction-cosine identity; the unsquared sum is not fixed for all lines.

implies

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 3 - 1 = 2,$$

verifying (B). It also gives

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2(1) - 3 = -1,$$

verifying (C).

Step 2: Test the unsquared sum The identity concerns squares, so it imposes no fixed value of $l + m + n$. For a line equally inclined to the three positive axes, take $l = m = n = 1/\sqrt{3}$. These satisfy $l^2 + m^2 + n^2 = 1$, but

$$l + m + n = \frac{3}{\sqrt{3}} = \sqrt{3} \neq 1.$$

Thus (D) is not a generally true identity, although it can hold for special lines such as the positive x -axis.

Step 3: Conclusion The statement that is not true in general is $\cos \alpha + \cos \beta + \cos \gamma = 1$. Hence, the correct option is (D).

Quick Tip to Solve: Retain the squares in the direction-cosine identity; the unsquared sum is not fixed for all lines.

16. The restrictions imposed on decision variables involved in an objective function of a linear programming problem are called: [1 mark]

- (A) feasible solutions
- (B) constraints
- (C) optimal solutions
- (D) infeasible solutions

Correct Answer: (B) constraints

Solution:

Concept:

- Decision variables are the unknown quantities whose values determine the objective function in a linear programming problem.
- Constraints are the linear equalities or inequalities restricting the permissible values of the decision variables.

Step 1: Identify the term from its definition A restriction such as $2x + y \leq 10$ limits which choices of x and y are allowed. Likewise, non-negativity restrictions such as $x \geq 0$ and $y \geq 0$ constrain the variables. The restrictions themselves are called constraints, irrespective of whether the objective is maximised or minimised.

Step 2: Distinguish restrictions from solutions A feasible solution is a particular assignment of values satisfying every constraint. An infeasible solution violates at least one constraint. An optimal solution is a feasible assignment giving the best value of the objective function. Thus options (A), (C) and

Quick Tip to Solve: Keep the restriction separate from a point satisfying it: an inequality is a constraint, a point is a solution.

(D) describe assignments of values, whereas the question asks for the restrictions imposed on those values.

Step 3: Conclusion The restrictions are called constraints. Hence, the correct option is (B).

Quick Tip to Solve: Keep the restriction separate from a point satisfying it: an inequality is a constraint, a point is a solution.

17. Let E and F be two events such that $P(E) = 0.1$, $P(F) = 0.3$, $P(E \cup F) = 0.4$, then $P(F | E)$ is: [1 mark]

- (A) 0.6
- (B) 0.4
- (C) 0.5
- (D) 0

Correct Answer: (D) 0

Solution:

Concept:

- For two events, the addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- Conditional probability is $P(F | E) = \frac{P(F \cap E)}{P(E)}$, provided $P(E) > 0$.

Step 1: Find the intersection probability Substitute the given probabilities in the addition rule:

$$P(E \cap F) = 0.1 + 0.3 - 0.4 = 0.$$

Thus, there is no positive probability of both events occurring together.

Step 2: Apply the conditional probability formula Since $P(E) = 0.1 > 0$, the conditional probability is defined. Hence

$$P(F | E) = \frac{0}{0.1} = 0.$$

The denominator is the probability of the conditioning event E , not that of F . The positive values 0.6, 0.4 and 0.5 in options (A), (B) and (C) cannot equal a quotient with zero numerator.

Step 3: Conclusion $P(F | E) = 0$. Hence, the correct option is (D).

Quick Tip to Solve: First subtract the union probability from the sum: when the difference is zero, either defined conditional probability is zero.

18. If A and B are two skew symmetric matrices, then $(AB + BA)$ is:

[1 mark]

- (A) a skew symmetric matrix
- (B) a symmetric matrix
- (C) a null matrix
- (D) an identity matrix

Correct Answer: (B) a symmetric matrix

Solution:

Concept:

- A skew symmetric matrix satisfies $A^T = -A$; similarly, $B^T = -B$.
- Transposition reverses a product: $(AB)^T = B^T A^T$. A matrix M is symmetric when $M^T = M$.

Step 1: Transpose the given sum Let $M = AB + BA$. Using the transpose rules,

$$\begin{aligned} M^T &= (AB)^T + (BA)^T \\ &= B^T A^T + A^T B^T \\ &= (-B)(-A) + (-A)(-B) \\ &= BA + AB = AB + BA = M. \end{aligned}$$

The order of each product reverses, but the sum contains both orders.

Step 2: Test the other classifications The equality $M^T = M$ proves symmetry. It does not require M to be zero or the identity. For example, take

$$A = B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $AB + BA = 2A^2 = -2I$, which is neither null nor identity nor skew symmetric. Thus (A), (C) and (D) are not valid general conclusions.

Step 3: Conclusion $AB + BA$ is a **symmetric matrix**. Hence, the correct option is (B).

Quick Tip to Solve: When transposing this sum, reverse each product and replace each skew symmetric transpose by its negative; the two negatives cancel.

19. Assertion (A): For any non-zero unit vector $\vec{a}$, $\vec{a} \cdot (-\vec{a}) = (-\vec{a}) \cdot \vec{a} = -1$.

Reason (R): Angle between $\vec{a}$ and $(-\vec{a})$ is $\frac{\pi}{2}$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- For any vector $\vec{a}$, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$. A unit vector has magnitude 1.
- The negative vector $-\vec{a}$ has the same magnitude as $\vec{a}$, but points in the opposite direction.

Step 1: Check the Assertion By linearity of the scalar product,

$$\vec{a} \cdot (-\vec{a}) = -\vec{a} \cdot \vec{a} = -|\vec{a}|^2 = -1.$$

The scalar product is commutative, so $(-\vec{a}) \cdot \vec{a} = -1$ also. Therefore the Assertion is true.

Step 2: Check the Reason A non-zero vector and its negative are oppositely directed; their angle is π , not $\pi/2$. Equivalently,

$$\cos \theta = \frac{-1}{1 \cdot 1} = -1 \Rightarrow \theta = \pi.$$

Hence the Reason is false.

Step 3: Decide whether the Reason explains the Assertion A false statement cannot explain the true Assertion. An angle $\pi/2$ would give a zero scalar product, contradicting the value -1 . Options (A) and (B) wrongly declare the Reason true, while (D) wrongly declares the Assertion false.

Step 4: Conclusion The Assertion is true, but the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Remember that negating a non-zero vector reverses its direction, giving angle π and a negative squared magnitude as the dot product.

20. Assertion (A): Every scalar matrix is a diagonal matrix.

Reason (R): In a diagonal matrix, all the diagonal elements are 0.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A diagonal matrix is a square matrix whose off-diagonal entries are zero; its diagonal entries can be arbitrary.
- A scalar matrix has the form kI , where I is the identity matrix and all diagonal entries are the same scalar k .

Step 1: Check the Assertion In a scalar matrix kI , every off-diagonal entry is zero. It therefore meets the definition of a diagonal matrix. The additional condition of equal diagonal entries makes it scalar. Hence every scalar matrix is diagonal and the Assertion is true.

Step 2: Check the Reason It is the off-diagonal entries of a diagonal matrix that must be zero, not the diagonal entries. For instance,

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

is diagonal, although neither of its diagonal entries is zero. Thus the Reason is false.

Step 3: Assess the explanatory relation The Reason misstates the definition and cannot explain the true Assertion. A scalar matrix is diagonal because its off-diagonal entries vanish. Options (A) and (B) incorrectly regard the Reason as true; option (D) incorrectly rejects the Assertion.

Step 4: Conclusion The Assertion is true, but the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Check which entries must vanish: diagonal matrices require zeros away from the main diagonal; scalar matrices additionally require equal diagonal entries.

21. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors. If θ is the angle between $\vec{a}$ and $(2\vec{a}+3\vec{b}+6\vec{c})$, find the value of $\cos \theta$.

[2 marks]

Solution:

Concept:

- For non-zero vectors $\vec{u}$ and $\vec{v}$, $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$.
- Mutually perpendicular unit vectors satisfy $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$ and $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} = 0$.

Quick Tip to Solve: For an orthonormal triple, divide the coefficient along the first vector by the square root of the sum of all three squared coefficients.

Step 1: Find the scalar product Write $\vec{v} = 2\vec{a} + 3\vec{b} + 6\vec{c}$. Then

$$\begin{aligned}\vec{a} \cdot \vec{v} &= 2(\vec{a} \cdot \vec{a}) + 3(\vec{a} \cdot \vec{b}) + 6(\vec{a} \cdot \vec{c}) \\ &= 2(1) + 3(0) + 6(0) = 2.\end{aligned}$$

Only the component parallel to $\vec{a}$ contributes to this product.

Step 2: Calculate the magnitude of the resultant vector Expanding $\vec{v} \cdot \vec{v}$, all mixed products vanish because the three vectors are mutually perpendicular. Thus

$$|\vec{v}|^2 = 4|\vec{a}|^2 + 9|\vec{b}|^2 + 36|\vec{c}|^2 = 4 + 9 + 36 = 49.$$

Therefore $|\vec{v}| = 7$; also $|\vec{a}| = 1$. The coefficients must be combined by their squares, not by ordinary addition.

Step 3: Use the angle formula Substitute the scalar product and the magnitudes:

$$\cos \theta = \frac{2}{1 \cdot 7}.$$

The positive scalar product shows that the required angle is acute, consistent with the positive value obtained.

Step 4: Conclusion $\cos \theta = \frac{2}{7}$.

Quick Tip to Solve: For an orthonormal triple, divide the coefficient along the first vector by the square root of the sum of all three squared coefficients.

22. Evaluate: $\cot^2 \{ \operatorname{cosec}^{-1} 3 \} + \sin^2 \left\{ \cos^{-1} \left(\frac{1}{3} \right) \right\}$

[2 marks]

Solution:

Concept:

- If $\alpha = \operatorname{cosec}^{-1} 3$, then $\operatorname{cosec} \alpha = 3$. Use $\operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha$.
- If $\beta = \cos^{-1}(1/3)$, then $\cos \beta = 1/3$. The identity $\sin^2 \beta + \cos^2 \beta = 1$ gives the other squared ratio.

Step 1: Evaluate the first squared ratio Put $\alpha = \operatorname{cosec}^{-1} 3$. Since 3 lies in the domain of inverse cosecant, this is defined and $\operatorname{cosec} \alpha = 3$. Therefore

$$\cot^2 \alpha = \operatorname{cosec}^2 \alpha - 1 = 3^2 - 1 = 8.$$

There is no need to find the angle itself.

Step 2: Evaluate the second squared ratio Put $\beta = \cos^{-1}(1/3)$, with principal value $\beta \in [0, \pi]$. Then $\cos \beta = 1/3$, so

$$\sin^2 \beta = 1 - \cos^2 \beta = 1 - \left(\frac{1}{3} \right)^2 = \frac{8}{9}.$$

Quick Tip to Solve: Use cosecant squared minus one for the first term and one minus cosine squared for the second; neither inverse angle needs explicit evaluation.

The argument must be squared before subtraction; replacing the squared cosine by $1/3$ would give an incorrect value.

Step 3: Add the two results The requested expression is the sum of the squared ratios already found. Using the common denominator 9,

$$8 + \frac{8}{9} = \frac{72 + 8}{9}.$$

Both terms are non-negative, as expected for squares of real trigonometric ratios.

Step 4: Conclusion $\cot^2\{\operatorname{cosec}^{-1} 3\} + \sin^2\{\cos^{-1}(1/3)\} = \frac{80}{9}.$

Quick Tip to Solve: Use cosecant squared minus one for the first term and one minus cosine squared for the second; neither inverse angle needs explicit evaluation.

23(a). If $x = e^{x/y}$, prove that $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}.$

[2 marks]

Solution:

Concept:

- Natural logarithms undo the exponential: if $x = e^u$, then $\log x = u$, with $x > 0$.
- The quotient rule gives $\frac{d}{dx}(u/v) = \frac{vu' - uv'}{v^2}$, where the denominator is nonzero, and $\frac{d}{dx} \log x = 1/x$.

Step 1: Express y as a function of x The given relation $x = e^{x/y}$ requires $y \neq 0$ and implies $x > 0$. Taking the natural logarithm on both sides gives

$$\log x = \frac{x}{y}, \quad y \log x = x.$$

The value $x = 1$ is impossible in the original relation: it would require $1/y = 0$ for finite y . Hence $\log x \neq 0$, and we may write

$$y = \frac{x}{\log x}.$$

Step 2: Differentiate using the quotient rule Differentiate the expression for y , taking $u = x$, $v = \log x$, so that $u' = 1$ and $v' = 1/x$. Then

$$\frac{dy}{dx} = \frac{(\log x)(1) - x(1/x)}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}.$$

The calculation is valid throughout each interval $(0, 1)$ and $(1, \infty)$, where the original relation defines a finite value of y .

Step 3: Conclusion Thus $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$, as required.

Quick Tip to Solve: Take logarithms first and isolate $y = x/\log x$; this turns the implicit exponential relation into one quotient-rule calculation.

23(b). Check the differentiability of $f(x) = \begin{cases} x^2 + 1, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2 \end{cases}$ at $x = 1$.

[2 marks]

Solution:**Concept:**

- A function is differentiable at an interior point only if its left-hand and right-hand derivatives exist and are equal.
- The one-sided derivatives at a are the limits of $\frac{f(a+h) - f(a)}{h}$ as $h \rightarrow 0^-$ and $h \rightarrow 0^+$, respectively. Continuity alone does not imply differentiability.

Step 1: Check the value and continuity at the joining point The second branch contains $x = 1$, so $f(1) = 3 - 1 = 2$. From the first branch,

$$\lim_{x \rightarrow 1^-} f(x) = 1^2 + 1 = 2, \quad \lim_{x \rightarrow 1^+} f(x) = 3 - 1 = 2.$$

Both limits equal $f(1)$, so the function is continuous at 1. We must still compare its slopes on the two sides.

Step 2: Calculate both one-sided derivatives For negative h sufficiently close to zero, $1 + h < 1$, and the quadratic branch applies:

$$f'_-(1) = \lim_{h \rightarrow 0^-} \frac{(1+h)^2 + 1 - 2}{h} = \lim_{h \rightarrow 0^-} (2+h) = 2.$$

For positive h sufficiently close to zero, the linear branch applies:

$$f'_+(1) = \lim_{h \rightarrow 0^+} \frac{3 - (1+h) - 2}{h} = \lim_{h \rightarrow 0^+} (-1) = -1.$$

Each one-sided derivative exists, but their values differ, so the two-sided derivative has no common limit.

Step 3: Conclusion Since $f'_-(1) = 2 \neq -1 = f'_+(1)$, f is not differentiable at $x = 1$, although it is continuous there.

Quick Tip to Solve: At a piecewise junction, first use the branch containing the point to find $f(1)$, then compare the derivative limits from both sides.

Section B (4 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

24(a). Evaluate: $\int_0^{\pi/2} \sin 2x \cos 3x \, dx$

[2 marks]

Solution:

Concept:

- The product-to-sum identity is $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$; also $\sin(-t) = -\sin t$.
- For a nonzero constant k , $\int \sin(kx) \, dx = -\cos(kx)/k$. A definite integral is the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Replace the product by a sum Let I denote the given integral. Apply the identity with $A = 2x$ and $B = 3x$. Because $A - B = -x$, its second term is negative:

$$\sin 2x \cos 3x = \frac{1}{2}[\sin 5x + \sin(-x)] = \frac{1}{2}(\sin 5x - \sin x).$$

Thus $I = \frac{1}{2} \int_0^{\pi/2} (\sin 5x - \sin x) \, dx$.

Step 2: Integrate and substitute the limits Integrating the two sine terms separately yields

$$I = \frac{1}{2} \left[-\frac{\cos 5x}{5} + \cos x \right]_0^{\pi/2}.$$

At the upper limit, both $\cos(5\pi/2)$ and $\cos(\pi/2)$ equal zero. At the lower limit, both cosines equal one. Therefore

$$I = \frac{1}{2} \left[0 - \left(-\frac{1}{5} + 1 \right) \right] = -\frac{1}{2} \cdot \frac{4}{5} = -\frac{2}{5}.$$

The negative result is allowed: this integral measures signed area, and the integrand need not be positive on the interval.

Step 3: Conclusion Hence $\boxed{\int_0^{\pi/2} \sin 2x \cos 3x \, dx = -\frac{2}{5}}.$

Quick Tip to Solve: Keep the sign in $\sin(2x-3x)=\sin(-x)=-\sin x$; both cosine terms vanish at the upper limit.

OR

24(b). Given $\frac{d}{dx} F(x) = \frac{1}{\sqrt{2x-x^2}}$ **and** $F(1) = 0$, **find** $F(x)$.

[2 marks]

Solution:

Concept:

- Completing the square gives $2x - x^2 = 1 - (x-1)^2$, which matches a standard inverse-trigonometric integral.

Quick Tip to Solve: Complete the square $2x - x^2 = 1 - (x-1)^2$ before integrating; the condition at $x=1$ then makes the constant zero.

- The formula $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$ holds for $-1 < u < 1$. An initial value determines the constant of integration C .

Step 1: Integrate the given derivative Since $F'(x) = 1/\sqrt{2x-x^2}$, the real derivative is defined on $0 < x < 2$, where the radicand is positive. Rewrite it by completing the square:

$$F(x) = \int \frac{dx}{\sqrt{1-(x-1)^2}}.$$

Put $u = x - 1$, so that $du = dx$. The standard integral then gives

$$F(x) = \sin^{-1}(x-1) + C.$$

Here $\sin^{-1}$ denotes the principal inverse sine, with values in $[-\pi/2, \pi/2]$, rather than the reciprocal of sine.

Step 2: Use the prescribed value at $x=1$ Substitute $x = 1$ into the integrated expression and use $F(1) = 0$:

$$0 = \sin^{-1}(0) + C = 0 + C, \quad C = 0.$$

To verify the result, differentiate it on $(0, 2)$:

$$\frac{d}{dx} \sin^{-1}(x-1) = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}.$$

It therefore satisfies both the given differential relation and the initial condition.

Step 3: Conclusion The required function is $F(x) = \sin^{-1}(x-1)$ for $0 < x < 2$.

Quick Tip to Solve: Complete the square $2x - x^2 = 1 - (x-1)^2$ before integrating; the condition at $x=1$ then makes the constant zero.

25. Find the position vector of point C which divides the line segment joining points A and B having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio $4 : 1$ externally. Further, find $|\vec{AB}| : |\vec{BC}|$. [2 marks]

Solution:

Concept:

- If C divides AB externally in the ratio $m : n$, its position vector is $\vec{c} = (m\vec{b} - n\vec{a})/(m - n)$, where $\vec{a}$ and $\vec{b}$ locate A and B .
- A displacement vector is the destination position vector minus the starting position vector. Its length is $\sqrt{v_x^2 + v_y^2 + v_z^2}$; scalar multiples have lengths in the ratio of the absolute scalars.

Step 1: Apply the external section formula Here $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$, and $AC : CB = 4 : 1$.

Quick Tip to Solve: For external division $4:1$, C lies beyond B and $AB=AC-BC$, so $AB:BC=(4-1):1$ without a second distance calculation.

Since the division is external, use subtraction in both numerator and denominator:

$$\begin{aligned}\vec{c} &= \frac{4\vec{b} - \vec{a}}{4 - 1} \\ &= \frac{4(-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{3} \\ &= \frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}.\end{aligned}$$

This point lies beyond B , because $\vec{c} = \vec{b} + \frac{1}{3}(\vec{b} - \vec{a})$.

Step 2: Compare the lengths of AB and BC Subtract the position vectors to obtain

$$\vec{AB} = \vec{b} - \vec{a} = -2\hat{i} - \hat{j} + 2\hat{k}, \quad |\vec{AB}| = \sqrt{4 + 1 + 4} = 3.$$

Likewise,

$$\vec{BC} = \vec{c} - \vec{b} = \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{3} = \frac{1}{3}\vec{AB}, \quad |\vec{BC}| = 1.$$

Thus the lengths can be compared directly. As a check, $AC = AB + BC = 4$ for the collinear order A, B, C , giving the stated external ratio $AC : CB = 4 : 1$.

Step 3: Conclusion The position vector is $\vec{OC} = \frac{-5\hat{i} + 2\hat{j} + 5\hat{k}}{3}$, and $|\vec{AB}| : |\vec{BC}| = 3 : 1$.

Quick Tip to Solve: For external division 4:1, C lies beyond B and $AB = AC - BC$, so $AB : BC = (4 - 1) : 1$ without a second distance calculation.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Solve the following linear programming problem graphically: Maximize $z = x + y$ subject to constraints $2x + 5y \leq 100$, $8x + 5y \leq 200$, $x \geq 0$, $y \geq 0$. [3 marks]

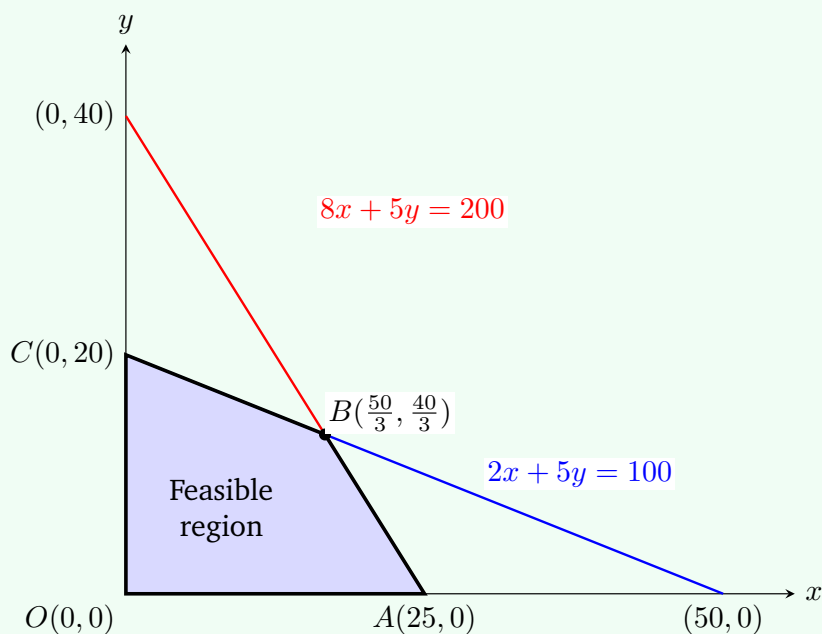
Solution:

Concept:

- A linear inequality determines a half-plane. The conditions $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant.
- For a bounded feasible polygon, a linear objective $z = x + y$ attains its maximum at a corner point. Here x, y are the decision variables and z is the quantity to maximize.

Step 1: Plot the boundary lines and shade the feasible region Replace the inequalities by equations. The line $2x + 5y = 100$ passes through $(50, 0)$ and $(0, 20)$; the line $8x + 5y = 200$ passes through $(25, 0)$ and $(0, 40)$. Since $(0, 0)$ satisfies both inequalities, choose the side containing the origin for each line. Their common portion in the first quadrant is the shaded polygon $OABC$ shown below. Every point in this polygon satisfies all four constraints, including its boundary.

Quick Tip to Solve: Subtract the two boundary equations first: their identical y -coefficients immediately give the intersection's x -coordinate.



The shaded polygon OABC is the feasible region.

Step 2: Find the intersection and evaluate the objective At the intersection B , subtracting the two boundary equations gives

$$\begin{aligned}(8x + 5y) - (2x + 5y) &= 200 - 100, \\ 6x &= 100, \quad x = \frac{50}{3}, \\ 5y &= 100 - \frac{100}{3} = \frac{200}{3}, \quad y = \frac{40}{3}.\end{aligned}$$

Thus the vertices are $O = (0, 0)$, $A = (25, 0)$, $B = (50/3, 40/3)$, and $C = (0, 20)$. Evaluate $z = x + y$ at each:

Corner point	$z = x + y$
$(0, 0)$	0
$(25, 0)$	25
$(50/3, 40/3)$	30
$(0, 20)$	20

The feasible region is nonempty and bounded, so comparison of these corner values establishes the optimum.

Step 3: Conclusion The maximum value is $z_{\max} = 30$, attained at $(x, y) = (50/3, 40/3)$.

Quick Tip to Solve: Subtract the two boundary equations first: their identical y -coefficients immediately give the intersection's x -coordinate.

27. The chances of P , Q and R getting selected as CEO of a company are in the ratio 4 : 1 : 2 respectively. The probabilities for the company to increase its profits from the previous year under the new CEO, P , Q or R are 0.3, 0.8 and 0.5 respectively. If the company increased the profits from the previous year, find the probability that it is due to the appointment of R as CEO. [3 marks]

Solution:

Concept:

- Let E_P, E_Q, E_R be the events that P, Q, R respectively is appointed CEO. They are mutually exclusive and exhaustive, so their probabilities sum to one.
- If H denotes increased profits, the total probability theorem gives $P(H) = \sum_{i=P,Q,R} P(E_i)P(H | E_i)$.
- Bayes' theorem reverses the conditioning: $P(E_R | H) = P(E_R)P(H | E_R)/P(H)$. The observed event H changes the relative likelihood of each appointment.

Step 1: Convert the appointment ratio into probabilities The total of the ratio terms is $4 + 1 + 2 = 7$. Hence

$$P(E_P) = \frac{4}{7}, \quad P(E_Q) = \frac{1}{7}, \quad P(E_R) = \frac{2}{7}.$$

The given profit probabilities are conditional on the appointed CEO:

$$P(H | E_P) = \frac{3}{10}, \quad P(H | E_Q) = \frac{4}{5}, \quad P(H | E_R) = \frac{1}{2}.$$

In particular, the probability that R is appointed and profits increase is $P(E_R \cap H) = (2/7)(1/2) = 1/7$.

Step 2: Calculate the probability of increased profits Each possible appointment contributes to the observed increase. By the total probability theorem,

$$\begin{aligned} P(H) &= \frac{4}{7} \cdot \frac{3}{10} + \frac{1}{7} \cdot \frac{4}{5} + \frac{2}{7} \cdot \frac{1}{2} \\ &= \frac{6}{35} + \frac{4}{35} + \frac{5}{35} = \frac{3}{7}. \end{aligned}$$

These terms cover all possibilities because exactly one of the three people becomes CEO.

Step 3: Apply Bayes' theorem for R's appointment The question conditions on the fact that profits increased, so use the joint probability for R and the increase as numerator, and the probability of the increase as denominator:

$$P(E_R | H) = \frac{P(E_R \cap H)}{P(H)} = \frac{1/7}{3/7} = \frac{1}{3}.$$

Step 4: Conclusion Given that the company's profits increased, the probability that R was appointed CEO is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: Multiply each appointment ratio term by its profit probability, then divide R 's weighted term by the sum of all three weighted terms.

28(a). If $x \cos(p + y) + \cos p \sin(p + y) = 0$, prove that $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, where p is a constant.

[3 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x , while p is constant. The chain rule gives $\frac{d}{dx} \sin(p + y) = \cos(p + y) \frac{dy}{dx}$.
- The product rule is $\frac{d}{dx}(uv) = u'v + uv'$. The identity $\sin^2 \theta + \cos^2 \theta = 1$ simplifies the resulting derivative.

Step 1: Differentiate the given relation implicitly Put $\theta = p + y$, so $d\theta/dx = dy/dx$, since p is constant. The given relation is

$$x \cos \theta + \cos p \sin \theta = 0.$$

Applying the product and chain rules to both terms gives

$$\cos \theta - x \sin \theta \frac{dy}{dx} + \cos p \cos \theta \frac{dy}{dx} = 0.$$

Here the first term comes from differentiating x , while the other terms include the derivative of y through the trigonometric arguments.

Step 2: Use the original equation to eliminate x Multiply the differentiated equation by $\cos \theta$ and collect the derivative terms:

$$\cos^2 \theta + (-x \sin \theta \cos \theta + \cos p \cos^2 \theta) \frac{dy}{dx} = 0.$$

From the original relation, $x \cos \theta = -\cos p \sin \theta$. Therefore

$$\begin{aligned} -x \sin \theta \cos \theta + \cos p \cos^2 \theta &= \cos p \sin^2 \theta + \cos p \cos^2 \theta \\ &= \cos p. \end{aligned}$$

Substitution yields $\cos^2 \theta + \cos p \frac{dy}{dx} = 0$. This manipulation does not divide by either cosine, so it also avoids introducing unnecessary nonzero restrictions.

Step 3: Conclusion Replacing θ by $p + y$, we obtain $\cos p \frac{dy}{dx} = -\cos^2(p + y)$, as required.

Quick Tip to Solve: Multiply the differentiated equation by $\cos(p+y)$, then use the original equation; this avoids division and exposes $\sin$ squared plus $\cos$ squared.

28(b). Find the value of a and b so that function f defined as:

$$f(x) = \begin{cases} \frac{x-2}{|x-2|} + a, & \text{if } x < 2, \\ a + b, & \text{if } x = 2, \\ \frac{x-2}{|x-2|} + b, & \text{if } x > 2 \end{cases}$$

is a continuous function.

[3 marks]

Solution:

Concept:

- Continuity at $x = c$ requires $\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$; the two one-sided limits must equal the assigned value.
- The absolute value satisfies $|x - 2| = 2 - x$ for $x < 2$ and $|x - 2| = x - 2$ for $x > 2$, so the quotient has opposite signs on the two sides.

Step 1: Simplify the branches on either side of two For $x < 2$, the quantity $x - 2$ is negative. Therefore,

$$\frac{x-2}{|x-2|} = \frac{x-2}{2-x} = -1, \quad f(x) = a - 1.$$

For $x > 2$, the quantity $x - 2$ is positive, giving

$$\frac{x-2}{|x-2|} = 1, \quad f(x) = b + 1.$$

Each of these constant branches is continuous throughout its own interval. Thus only the joining point $x = 2$ needs to be checked; substituting directly in the quotient there would be invalid.

Step 2: Match the one-sided limits with the assigned value The simplified branches give

$$\lim_{x \rightarrow 2^-} f(x) = a - 1, \quad \lim_{x \rightarrow 2^+} f(x) = b + 1.$$

The middle branch specifies $f(2) = a + b$. Continuity consequently requires both equations

$$a - 1 = a + b, \quad b + 1 = a + b.$$

Cancelling a in the first equation gives $b = -1$; cancelling b in the second gives $a = 1$.

Step 3: Verify continuity everywhere With these constants, the left branch has value $a - 1 = 0$, the right branch has value $b + 1 = 0$, and the assigned middle value is $a + b = 0$. Thus the resulting function equals zero for every real x , confirming that no discontinuity remains at the joining point or elsewhere.

Step 4: Conclusion The required constants are $a = 1, b = -1$. For these values, $f(x) = 0$ on all of $\mathbb{R}$, so f is continuous.

Quick Tip to Solve: Replace $(x - 2)/|x - 2|$ by -1 on the left and $+1$ on the right, then equate each limit separately to $a + b$.

29(a). Find the intervals in which the function $f(x) = \frac{\log x}{x}$ is strictly increasing or strictly decreasing.

[3 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where $f'(x) > 0$ and strictly decreasing where $f'(x) < 0$.
- For the natural logarithm, $\frac{d}{dx} \log x = \frac{1}{x}$, with domain $x > 0$. The quotient rule is $(u/v)' = (vu' - uv')/v^2$.

Step 1: State the domain and differentiate The logarithm is defined for positive x , and the denominator must be nonzero. Hence the domain is $(0, \infty)$. Apply the quotient rule with $u = \log x$ and $v = x$:

$$f'(x) = \frac{x(1/x) - (\log x)(1)}{x^2} = \frac{1 - \log x}{x^2}.$$

Since $x^2 > 0$ everywhere in the domain, the sign of the derivative depends only on the numerator $1 - \log x$.

Step 2: Locate the point where the derivative changes sign Solving $f'(x) = 0$ gives

$$1 - \log x = 0 \implies \log x = 1 \implies x = e.$$

There are no other stationary points in the domain. The natural logarithm is strictly increasing, so $\log x < 1$ when $0 < x < e$, while $\log x > 1$ when $x > e$.

Step 3: Apply the derivative sign test On $(0, e)$, the numerator is positive, giving $f'(x) > 0$; consequently the function is strictly increasing there. On (e, ∞) , the numerator is negative, giving $f'(x) < 0$; consequently it is strictly decreasing there. The derivative vanishes only at e , where the change from positive to negative also confirms the peak value $f(e) = 1/e$. The interval $x \leq 0$ is excluded because the original logarithm is undefined there.

Step 4: Conclusion f is strictly increasing on $(0, e)$ and strictly decreasing on (e, ∞) .

Quick Tip to Solve: For $\log x/x$, ignore the positive denominator x^2 when checking the derivative sign; solve $\log x = 1$ first.

OR

29(b). Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \frac{x}{2} + \frac{2}{x}$, on the interval $[1, 2]$.

[3 marks]

Solution:

Concept:

- A continuous function on a closed bounded interval attains its absolute maximum and minimum. Compare its values at the endpoints and at every stationary point inside the interval.

Quick Tip to Solve: On a closed interval, include both endpoints even if the derivative vanishes at one of them; here the derivative is negative everywhere inside.

- For $f(x) = x/2 + 2/x$, the derivative is $f'(x) = 1/2 - 2/x^2$. A negative derivative throughout the interior establishes strict decrease.

Step 1: Verify continuity and find stationary points The function is continuous on $[1, 2]$, since its only undefined point is $x = 0$, outside the given interval. Differentiating gives

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}.$$

Thus the possible stationary points satisfy

$$x^2 - 4 = 0 \implies x = 2 \text{ or } x = -2.$$

The point -2 lies outside the interval; 2 is already its right endpoint. There is therefore no stationary point strictly between the endpoints.

Step 2: Check the direction of variation For $1 < x < 2$, we have $x^2 < 4$, so the numerator $x^2 - 4$ is negative, whereas $2x^2$ is positive. Hence $f'(x) < 0$ throughout $(1, 2)$. The function decreases as x moves from 1 to 2 . The zero derivative at the endpoint 2 does not change this monotonicity on the interior.

Step 3: Compare the endpoint values Evaluate the function at both candidates:

$$\begin{aligned} f(1) &= \frac{1}{2} + 2 = \frac{5}{2}, \\ f(2) &= \frac{2}{2} + \frac{2}{2} = 2. \end{aligned}$$

Because the function decreases across the entire interval, every value between the endpoints lies between 2 and $5/2$. This verifies that the endpoint comparison gives the absolute extrema, rather than merely local extrema.

Step 4: Conclusion The absolute maximum value is $\boxed{5/2}$, attained at $\boxed{x = 1}$. The absolute minimum value is $\boxed{2}$, attained at $\boxed{x = 2}$.

Quick Tip to Solve: On a closed interval, include both endpoints even if the derivative vanishes at one of them; here the derivative is negative everywhere inside.

30. Find: $\int \frac{\sqrt{x}}{(x+1)(x-1)} dx$

[3 marks]

Solution:

Concept:

- Use $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t dt$, to convert a rational expression involving $\sqrt{x}$ into a rational function of t .
- Standard antiderivatives are $\int dt/(1+t^2) = \tan^{-1} t + C$ and $\int dt/(t-a) = \log|t-a| + C$, where C is an arbitrary constant.

Quick Tip to Solve: After putting $t = \sqrt{x}$, use $2t^2 = (t^2 + 1) + (t^2 - 1)$ to split the integrand immediately into two standard forms.

Step 1: Remove the square root by substitution Let I denote the required integral. Put $t = \sqrt{x}$; then $x = t^2$ and $dx = 2t dt$. Thus

$$I = \int \frac{t(2t)}{(t^2 + 1)(t^2 - 1)} dt = \int \frac{2t^2}{(t^2 + 1)(t^2 - 1)} dt.$$

For real values of the original integrand, $x \geq 0$ and $x \neq 1$. The substitution gives $t \geq 0$, with $t \neq 1$; calculations for the antiderivative are made on the open intervals avoiding $x = 1$.

Step 2: Decompose the rational expression The numerator can be written as $2t^2 = (t^2 + 1) + (t^2 - 1)$. Therefore,

$$\frac{2t^2}{(t^2 + 1)(t^2 - 1)} = \frac{1}{t^2 - 1} + \frac{1}{t^2 + 1}.$$

Factor $t^2 - 1 = (t - 1)(t + 1)$ and split the first term:

$$\frac{1}{t^2 - 1} = \frac{1}{2} \left(\frac{1}{t - 1} - \frac{1}{t + 1} \right).$$

Bringing the two fractions back over a common denominator verifies the minus sign and the factor $1/2$.

Step 3: Integrate the standard terms and substitute back Integrating the decomposed expression gives

$$\begin{aligned} I &= \frac{1}{2} \log |t - 1| - \frac{1}{2} \log |t + 1| + \tan^{-1} t + C \\ &= \frac{1}{2} \log \left| \frac{t - 1}{t + 1} \right| + \tan^{-1} t + C. \end{aligned}$$

Now replace t by $\sqrt{x}$. The absolute value in the logarithm permits the expression on both sides of the excluded point $x = 1$. The integration constant may be chosen separately on the disconnected intervals of the domain.

Step 4: Conclusion

$$I = \frac{1}{2} \log \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + \tan^{-1} \sqrt{x} + C.$$

Quick Tip to Solve: After putting $t = \sqrt{x}$, use $2t^2 = (t^2 + 1) + (t^2 - 1)$ to split the integrand immediately into two standard forms.

31(a). Find: $\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx$

[3 marks]

Solution:

Concept:

- Double-angle identities give $1 + \cos 2x = 2 \cos^2 x$ and $\sin 2x = 2 \sin x \cos x$, so the trigonometric fraction can be expressed using $\tan x$ and $\sec^2 x$.

Quick Tip to Solve: Simplify the double-angle fraction first; then pair $\tan x$ with its derivative $\sec^2 x$ to recognise $e^x f(x)$.

- The product rule implies $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Thus, an integrand of this form is integrated by recognising a product derivative.

Step 1: Simplify the trigonometric factor Let the required indefinite integral be I . Using the double-angle identities, wherever the printed denominator is nonzero,

$$\begin{aligned}\frac{2 + \sin 2x}{1 + \cos 2x} &= \frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \\ &= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} \\ &= \sec^2 x + \tan x.\end{aligned}$$

Consequently, $I = \int e^x(\sec^2 x + \tan x) dx$. The two terms are a function and its derivative: $\frac{d}{dx}(\tan x) = \sec^2 x$.

Step 2: Recognise and verify the product derivative Apply the product rule directly:

$$\frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x = e^x(\tan x + \sec^2 x).$$

This is exactly the simplified integrand, so its antiderivative follows without integration by parts. The identity is valid on each interval where $\cos x \neq 0$, which is also the domain of the original integrand. The arbitrary constant accounts for all antiderivatives on such an interval.

Step 3: Conclusion Therefore, $\int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx = e^x \tan x + C$.

Quick Tip to Solve: Simplify the double-angle fraction first; then pair $\tan x$ with its derivative $\sec^2 x$ to recognise $e^x f(x)$.

OR

31(b). Evaluate: $\int_0^{\pi/4} \frac{1}{\sin x + \cos x} dx$

[3 marks]

Solution:

Concept:

- The identity $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ converts the denominator to a single trigonometric function.
- For a definite integral, a substitution changes both the integrand and the limits. The standard antiderivative is $\int \sec t dt = \ln |\sec t + \tan t| + C$.

Step 1: Substitute a shifted angle and change the limits Write the required integral as I . On the interval $[0, \pi/4]$, the denominator $\sin x + \cos x$ is positive, so the integral is proper. Since

$$\sin x + \cos x = \sqrt{2} \cos(x - \pi/4),$$

Quick Tip to Solve: Combine $\sin x + \cos x$ into one shifted cosine and change both limits before using the secant integral.

set $t = x - \pi/4$. Then $dt = dx$. The lower limit $x = 0$ becomes $t = -\pi/4$, and the upper limit $x = \pi/4$ becomes $t = 0$. Hence

$$I = \frac{1}{\sqrt{2}} \int_{-\pi/4}^0 \sec t \, dt.$$

Step 2: Apply the antiderivative and simplify the logarithm Using the standard integral of the secant,

$$\begin{aligned} I &= \frac{1}{\sqrt{2}} [\ln |\sec t + \tan t|]_{-\pi/4}^0 \\ &= \frac{1}{\sqrt{2}} [\ln 1 - \ln(\sqrt{2} - 1)] \\ &= \frac{1}{\sqrt{2}} \ln \left(\frac{1}{\sqrt{2} - 1} \right). \end{aligned}$$

Here $\sec(-\pi/4) = \sqrt{2}$ and $\tan(-\pi/4) = -1$. Since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, the reciprocal inside the logarithm equals $\sqrt{2} + 1$. This gives a positive value, consistent with integrating a positive function over the stated interval.

Step 3: Conclusion The required value is $\frac{\ln(\sqrt{2} + 1)}{\sqrt{2}}$.

Quick Tip to Solve: Combine $\sin x + \cos x$ into one shifted cosine and change both limits before using the secant integral.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines. [5 marks]

Solution:

Concept:

- A line through (x_0, y_0, z_0) with direction vector (a, b, c) has coordinates $(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$. A zero direction component fixes its coordinate.
- The cross product of two nonparallel direction vectors is perpendicular to both. Perpendicularity is checked by a zero scalar product.

Step 1: Parametrise the lines and find their intersection Taking independent parameters λ and μ , the given lines are

$$L_1 : (x, y, z) = (\lambda, 1 + 2\lambda, 2 + 3\lambda),$$

$$L_2 : (x, y, z) = (1, -3\mu, 7 + 2\mu).$$

In L_2 , the zero direction ratio means $x = 1$, not division by zero. Equality of the x -coordinates at the intersection requires $\lambda = 1$. Hence L_1 gives $y = 3$ and $z = 5$. From L_2 , $-3\mu = 3$ gives $\mu = -1$, and $7 + 2\mu = 5$ confirms the same z -coordinate. Thus all three coordinates agree and the intersection is

Quick Tip to Solve: Use the zero direction ratio to fix x and find the intersection, then cross the two direction vectors and check both dot products.

$$P = (1, 3, 5).$$

Step 2: Obtain the perpendicular direction The direction vectors of the given lines are $\vec{u} = (1, 2, 3)$ and $\vec{v} = (0, -3, 2)$. Their cross product is

$$\begin{aligned}\vec{n} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & -3 & 2 \end{vmatrix} \\ &= [2(2) - 3(-3)]\hat{i} - [1(2) - 3(0)]\hat{j} \\ &\quad + [1(-3) - 2(0)]\hat{k} \\ &= 13\hat{i} - 2\hat{j} - 3\hat{k}.\end{aligned}$$

This is a nonzero direction. In particular,

$$\vec{n} \cdot \vec{u} = 13 - 4 - 9 = 0, \quad \vec{n} \cdot \vec{v} = 0 + 6 - 6 = 0.$$

Both scalar products vanish, establishing that this direction is perpendicular to each given line.

Step 3: Write the line through the intersection Using point P and direction ratios $(13, -2, -3)$, the required line has parametric equations

$$x = 1 + 13t, \quad y = 3 - 2t, \quad z = 5 - 3t.$$

Setting $t = 0$ gives the intersection, and eliminating t gives the symmetric form. Because the line passes through the common point, the perpendicular directions establish perpendicular intersecting lines, as required.

Step 4: Conclusion The required equation is $\frac{x-1}{13} = \frac{y-3}{-2} = \frac{z-5}{-3}$.

Quick Tip to Solve: Use the zero direction ratio to fix x and find the intersection, then cross the two direction vectors and check both dot products.

OR

32(b). Two vertices of the parallelogram $ABCD$ are given as $A(-1, 2, 1)$ and $B(1, -2, 5)$. If the equation of the line passing through C and D is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between sides AB and CD . Hence, find the area of parallelogram $ABCD$. [5 marks]

Solution:

Concept:

- Parallel lines have proportional direction vectors. The denominators in a symmetric line equation give its direction vector.
- For two parallel lines, the perpendicular distance is $h = \frac{|\vec{AP} \times \mathbf{d}|}{|\mathbf{d}|}$, where A lies on the first line, P lies on the second and $\mathbf{d}$ is their common direction vector.

Quick Tip to Solve: Take the zero-parameter point on the second line and divide the cross-product magnitude by the length of its direction vector; then multiply by the actual base length.

- The area of a parallelogram is $\text{Area} = |AB|h$, where h is its perpendicular height.

Step 1: Determine the common direction and base length Using the given vertices,

$$\overrightarrow{AB} = (1+1, -2-2, 5-1) = (2, -4, 4) = 2(1, -2, 2).$$

The direction of CD is $\mathbf{d} = (1, -2, 2)$, proportional to $\overrightarrow{AB}$; hence the sides are parallel. Their base length is $|AB| = \sqrt{4+16+16} = 6$, and $|\mathbf{d}| = \sqrt{1+4+4} = 3$.

Step 2: Choose a point on the opposite line Set the common parameter in the equation of CD equal to zero. This gives $P = (4, -7, 8)$, a point on that line. Therefore $\overrightarrow{AP} = (4+1, -7-2, 8-1) = (5, -9, 7)$.

Step 3: Calculate the perpendicular separation Expanding the cross product carefully gives

$$\begin{aligned}\overrightarrow{AP} \times \mathbf{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -9 & 7 \\ 1 & -2 & 2 \end{vmatrix} \\ &= -4\hat{i} - 3\hat{j} - \hat{k}.\end{aligned}$$

Its magnitude is $\sqrt{16+9+1} = \sqrt{26}$. Hence the perpendicular separation of the parallel sides is

$$h = \frac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|} = \frac{\sqrt{26}}{3}.$$

Step 4: Use base times height for the area With AB as base, the height is the calculated perpendicular distance. Hence $\text{Area}(ABCD) = 6 \left(\frac{\sqrt{26}}{3} \right) = 2\sqrt{26}$ square units.

Step 5: Conclusion The distance is $\boxed{\frac{\sqrt{26}}{3}}$ units; the area is $\boxed{2\sqrt{26}}$ square units.

Quick Tip to Solve: Take the zero-parameter point on the second line and divide the cross-product magnitude by the length of its direction vector; then multiply by the actual base length.

33. Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{a\}$. Find the value of 'a' such that the function $f : A \rightarrow B$ defined by $f(x) = \frac{x-2}{x-3}$ is onto. Also, check whether the given function is one-one or not. [5 marks]

Solution:

Concept:

- A function $f : A \rightarrow B$ is onto when each $y \in B$ equals $f(x)$ for some $x \in A$. Thus the range must equal the stated codomain.
- A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1, x_2 in its domain.
- For every $x \in A$, $x - 3 \neq 0$, so cross-multiplication is valid.

Quick Tip to Solve: Rewrite the fraction as $1 + 1/(x-3)$ to identify the missing range value, then solve $y = f(x)$ to verify every other value has a valid preimage.

Step 1: Determine the excluded output value For $x \in A = \mathbb{R} \setminus \{3\}$,

$$f(x) = \frac{x-2}{x-3} = 1 + \frac{1}{x-3}.$$

Since $1/(x-3) \neq 0$, $f(x) \neq 1$. Thus the range excludes 1.

Step 2: Show that every other real value is attained Let $y \neq 1$ be any real number and solve $y = f(x)$:

$$\begin{aligned}y(x-3) &= x-2, \\(y-1)x &= 3y-2, \\x &= \frac{3y-2}{y-1}.\end{aligned}$$

Since $y \neq 1$, this is defined. Moreover,

$$x-3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0,$$

so this preimage belongs to A . Substitution gives $f(x) = y$. Therefore the range is exactly $\mathbb{R} \setminus \{1\}$.

Step 3: Match the range to the codomain The proposed codomain is $B = \mathbb{R} \setminus \{a\}$. For f to be onto, this set must equal the range $\mathbb{R} \setminus \{1\}$, so $a = 1$. If $a \neq 1$, then $1 \in B$ has no preimage, so the function cannot be onto.

Step 4: Prove the function is one-one Take $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Both denominators are nonzero, so cross-multiplication gives

$$\begin{aligned}\frac{x_1-2}{x_1-3} &= \frac{x_2-2}{x_2-3}, \\(x_1-2)(x_2-3) &= (x_2-2)(x_1-3), \\x_1x_2 - 3x_1 - 2x_2 + 6 &= x_1x_2 - 3x_2 - 2x_1 + 6, \\-x_1 + x_2 &= 0.\end{aligned}$$

Therefore $x_1 = x_2$, proving injectivity throughout the stated domain.

Step 5: Conclusion The required value is $a = 1$. With $B = \mathbb{R} \setminus \{1\}$, the function is onto and is also one-one; hence it is bijective.

Quick Tip to Solve: Rewrite the fraction as $1 + 1/(x-3)$ to identify the missing range value, then solve $y = f(x)$ to verify every other value has a valid preimage.

34(a). It is given that function $f(x) = x^4 - 62x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of 'a', hence obtain all other points where the given function $f(x)$ attains local maximum or local minimum values. [5 marks]

Solution:

Concept:

- If a differentiable function has a local extremum at an interior point $x = c$, the necessary condition is $f'(c) = 0$. This identifies stationary points, which must then be classified.
- The second derivative test gives a local maximum when $f''(c) < 0$ and a local minimum when $f''(c) > 0$, provided $f'(c) = 0$.
- A point on the graph has coordinates $(c, f(c))$. The abscissa c locates the extremum; its ordinate is obtained by substitution into the original function.

Step 1: Use the given local maximum to determine the constant The polynomial $f(x) = x^4 - 62x^2 + ax + 9$ is differentiable for every real x . Since it has a local maximum at the interior point $x = 1$, its derivative must vanish there. Differentiating gives

$$f'(x) = 4x^3 - 124x + a.$$

Consequently,

$$f'(1) = 4 - 124 + a = 0 \implies a = 120.$$

This uses the necessary stationary-point condition; the nature of the point will be verified next.

Step 2: Find every stationary point of the resulting function Substitute the determined constant into the derivative and factor:

$$\begin{aligned} f'(x) &= 4x^3 - 124x + 120 \\ &= 4(x^3 - 31x + 30) \\ &= 4(x - 1)(x^2 + x - 30) \\ &= 4(x - 1)(x + 6)(x - 5). \end{aligned}$$

Thus the stationary abscissae are $x = -6, 1, 5$. The factorisation lists all real zeros of the derivative, so no other stationary points can occur.

Step 3: Classify all stationary points The second derivative is $f''(x) = 12x^2 - 124$. Its values are

$$f''(-6) = 308 > 0, \quad f''(1) = -112 < 0, \quad f''(5) = 176 > 0.$$

Therefore $x = -6$ and $x = 5$ are points of local minimum, while $x = 1$ is a point of local maximum, consistent with the given condition. A polynomial is differentiable everywhere on its domain $\mathbb{R}$; every local extremum must be stationary. Hence this classification is exhaustive.

Step 4: Calculate the corresponding values on the graph With $a = 120$, use $f(x) = x^4 - 62x^2 + 120x + 9$:

$$\begin{aligned} f(-6) &= 1296 - 2232 + 720 + 9 = -1647, \\ f(1) &= 1 - 62 + 120 + 9 = 68, \\ f(5) &= 625 - 1550 + 600 + 9 = -316. \end{aligned}$$

Quick Tip to Solve: Use the known stationary root $x = 1$ to factor the cubic derivative, then classify each root with f'' instead of solving the cubic afresh.

These ordinates distinguish the actual graph points from the abscissae used in the derivative tests.

Step 5: Conclusion $a = 120$. The other extrema are local minima at $x = -6$ and $x = 5$, namely $(-6, -1647)$ and $(5, -316)$. There is no other local maximum; the given one is $(1, 68)$.

Quick Tip to Solve: Use the known stationary root $x = 1$ to factor the cubic derivative, then classify each root with f'' instead of solving the cubic afresh.

OR

34(b). The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum. [5 marks]

Solution:

Concept:

- For a rectangle with side lengths x and y , its perimeter is $2(x + y)$. The given perimeter therefore imposes a relation between the two sides.
- When a rectangular sheet forms the curved surface of a cylinder, one side is the circumference $2\pi r$ and the other is the height h . The cylinder volume is $V = \pi r^2 h$.
- Expressing volume in one variable allows its maximum to be found from $V' = 0$, with the derivative sign or second derivative confirming its nature over the physically allowed domain.

Step 1: Express the cylinder dimensions in terms of one side Let the side that becomes the circumference be x cm, and let the other side, which becomes the cylinder height, be y cm. From the given perimeter,

$$2(x + y) = 300 \implies y = 150 - x.$$

Both side lengths must be positive, so $0 < x < 150$. Since $2\pi r = x$, the cylinder has

$$r = \frac{x}{2\pi}, \quad h = 150 - x.$$

Choosing the circumference explicitly avoids interchanging it with the height while applying the volume formula.

Step 2: Form and differentiate the volume function Substituting the radius and height gives the volume, in cubic centimetres, as

$$V(x) = \pi \left(\frac{x}{2\pi} \right)^2 (150 - x) = \frac{150x^2 - x^3}{4\pi}.$$

Hence

$$V'(x) = \frac{300x - 3x^2}{4\pi} = \frac{3x(100 - x)}{4\pi}.$$

Solving $V'(x) = 0$ gives $x = 0$ or $x = 100$. The first value is outside the admissible domain and would produce a degenerate cylinder. Thus $x = 100$ is the only admissible stationary point.

Quick Tip to Solve: Name the circumference side first: for a fixed side sum S , maximise $x^2(S - x)$, which gives circumference $2S/3$ and height $S/3$.

Step 3: Verify that the stationary point gives the maximum volume The second derivative is

$$V''(x) = \frac{300 - 6x}{4\pi}, \quad V''(100) = \frac{-300}{4\pi} < 0.$$

Therefore $x = 100$ gives a local maximum. Moreover, throughout $0 < x < 100$, both factors x and $100 - x$ are positive, so $V'(x) > 0$. Throughout $100 < x < 150$, $V'(x) < 0$. The volume increases up to 100 and decreases thereafter, proving that this is the maximum over all admissible dimensions. The limiting volumes at $x = 0$ and $x = 150$ are zero, as expected for degenerate cylinders.

Step 4: Recover the sheet dimensions and check the perimeter At the maximum, the other side is $y = 150 - 100 = 50$ cm. The perimeter check is

$$2(100 + 50) = 300 \text{ cm.}$$

The corresponding cylinder has $r = 50/\pi$ cm and $h = 50$ cm, so its maximum volume is

$$V_{\max} = \pi \left(\frac{50}{\pi} \right)^2 (50) = \frac{125000}{\pi} \text{ cm}^3.$$

Thus the longer side must form the circumference and the shorter side the height to realise this maximum.

Step 5: Conclusion The rectangular sheet should have dimensions $100 \text{ cm} \times 50 \text{ cm}$, with the 100 cm side forming the circumference of the cylinder and the 50 cm side its height.

Quick Tip to Solve: Name the circumference side first: for a fixed side sum S , maximise $x^2(S - x)$, which gives circumference $2S/3$ and height $S/3$.

35. Using integration, find the area of the region enclosed between the curve $y = \sqrt{4 - x^2}$ and the lines $x = -1$, $x = 1$ and the x -axis. [5 marks]

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$, the x -axis and vertical lines $x = a$, $x = b$ is $A = \int_a^b f(x) dx$, measured in square units.
- The curve $y = \sqrt{4 - x^2}$ is the upper semicircle of $x^2 + y^2 = 4$. Its ordinate is non-negative on the required interval.
- For an even function f , symmetry gives $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$. The substitution $x = 2 \sin \theta$ removes the square root.

Step 1: Identify the required strip and integral The vertical boundaries are $x = -1$ and $x = 1$. The upper boundary is $y = \sqrt{4 - x^2}$ and the lower boundary is $y = 0$. A vertical strip of width dx therefore has area $\sqrt{4 - x^2} dx$. The integrand is even, so the two halves about the y -axis have equal areas:

$$A = \int_{-1}^1 \sqrt{4 - x^2} dx = 2 \int_0^1 \sqrt{4 - x^2} dx.$$

Quick Tip to Solve: Exploit symmetry first: integrate from 0 to 1, double it, and use $x = 2 \sin \theta$ with upper angle $\pi/6$.

The limits select the central portion of the semicircle.

Step 2: Substitute to evaluate the definite integral Put $x = 2 \sin \theta$, choosing $0 \leq \theta \leq \pi/6$. Then $dx = 2 \cos \theta d\theta$ and, because $\cos \theta \geq 0$ on this interval,

$$\sqrt{4 - x^2} = \sqrt{4(1 - \sin^2 \theta)} = 2 \cos \theta.$$

When $x = 0$, $\theta = 0$; when $x = 1$, $\sin \theta = 1/2$, so $\theta = \pi/6$. Transform both bounds along with the integrand:

$$\begin{aligned} A &= 2 \int_0^{\pi/6} (2 \cos \theta)(2 \cos \theta) d\theta \\ &= 8 \int_0^{\pi/6} \cos^2 \theta d\theta \\ &= 4 \int_0^{\pi/6} (1 + \cos 2\theta) d\theta \\ &= [4\theta + 2 \sin 2\theta]_0^{\pi/6}. \end{aligned}$$

Step 3: Apply the limits and check the scale Using $\sin(\pi/3) = \sqrt{3}/2$, we obtain

$$A = 4 \left(\frac{\pi}{6} \right) + 2 \left(\frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} + \sqrt{3}.$$

The lower-limit terms vanish. The positive answer represents an area above the x -axis. As a check, the height is between $\sqrt{3}$ and 2 over width 2, and the computed area lies between $2\sqrt{3}$ and 4.

Step 4: Conclusion The enclosed area is $\boxed{\frac{2\pi}{3} + \sqrt{3} \text{ square units}}.$

Quick Tip to Solve: Exploit symmetry first: integrate from 0 to 1, double it, and use $x = 2 \sin \theta$ with upper angle $\pi/6$.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1: A scholarship is a sum of money provided to a student to help him or her pay for education. Some students are granted scholarships based on their academic achievements, while others are rewarded based on their financial needs.

Every year a school offers scholarships to girl children and meritorious achievers based on certain criteria. In the session 2022 - 23, the school offered monthly scholarship of ₹3,000 each to some girl students and ₹4,000 each to meritorious achievers in academics as well as sports.

In all, 50 students were given the scholarships and monthly expenditure incurred by the school on scholarships was ₹1,80,000. [4 marks]

36(i). Express the given information algebraically using matrices. [1 mark]

Solution:

Concept:

- A linear system can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of totals.
- The order of unknowns in X must agree with the order of coefficients in every row of A .

Step 1: Define the two unknowns Let x be the number of girl students receiving the monthly scholarship of ₹3,000, and y be the number of meritorious achievers receiving ₹4,000. The given total number and total monthly expenditure give $x + y = 50$ and $3000x + 4000y = 180000$, respectively.

Step 2: Arrange the equations as a matrix product Taking the columns in the order x, y , the number equation supplies the row $(1, 1)$, and the expenditure equation supplies $(3000, 4000)$. Thus

$$A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

Multiplying A by X recovers both original equations in the same order.

Step 3: Conclusion The required matrix equation is $\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$

Quick Tip to Solve: Use one row for the student count and one for expenditure; keep the girl-scholarship column first in both rows.

36(ii). Check whether the system of matrix equations so obtained is consistent or not. [1 mark]

Solution:

Concept:

- For a square coefficient matrix A , a non-zero determinant implies that A^{-1} exists and $AX = B$ has a unique solution.

Quick Tip to Solve: For these two equations, check $4000 - 3000$; its non-zero value immediately establishes consistency and uniqueness.

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $|A| = ad - bc$. A unique solution makes the system consistent.

Step 1: Compute the determinant of the coefficient matrix The equations from the information are $x + y = 50$ and $3000x + 4000y = 180000$. Hence

$$A = \begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix}, \quad |A| = (1)(4000) - (1)(3000) = 1000.$$

The determinant is non-zero; the expenditure equation is therefore independent of the count equation.

Step 2: Apply the consistency criterion Since $|A| \neq 0$, the coefficient matrix is invertible. Multiplication of $AX = B$ by A^{-1} yields $X = A^{-1}B$, which exists for the given totals and is unique. Thus the equations can be satisfied simultaneously; there is no conflict between the student count and the expenditure data.

Step 3: Conclusion The system is consistent and has a unique solution because $|A| = 1000 \neq 0$.

Quick Tip to Solve: For these two equations, check $4000 - 3000$; its non-zero value immediately establishes consistency and uniqueness.

36(iii)(a). Find the number of scholarships of each kind given by the school, using matrices. [2 marks]

Solution:

Concept:

- If $AX = B$ and $|A| \neq 0$, solve using $X = A^{-1}B$. Here x and y count the two scholarship categories.
- The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

Step 1: Form the matrix system and find its inverse Let x count the girl students receiving ₹3,000 each, and y count the meritorious achievers receiving ₹4,000 each. Then

$$\begin{pmatrix} 1 & 1 \\ 3000 & 4000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 180000 \end{pmatrix}.$$

The determinant is $4000 - 3000 = 1000 \neq 0$, so

$$A^{-1} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix}.$$

Quick Tip to Solve: Keep x tied to the 3000-rupee category, then verify the inverse-matrix answer against both the count and expenditure totals.

Step 2: Multiply the inverse by the totals Using the matrix method,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 180000 \end{pmatrix} \\ = \frac{1}{1000} \begin{pmatrix} 200000 - 180000 \\ -150000 + 180000 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

Both counts are non-negative integers. They total $20 + 30 = 50$, and their monthly cost is $3000(20) + 4000(30) = 180000$, agreeing with both given conditions.

Step 3: Conclusion The school gave 20 girl-student scholarships and 30 meritorious-achiever scholarships.

Quick Tip to Solve: Keep x tied to the 3000-rupee category, then verify the inverse-matrix answer against both the count and expenditure totals.

OR

36(iii)(b). Had the amount of scholarship given to each girl child and meritorious student been interchanged, what would be the monthly expenditure incurred by the school? [2 marks]

Solution:

Concept:

- The original scholarship counts follow from $AX = B$ using $X = A^{-1}B$, where x counts girl-student scholarships and y counts merit scholarships.
- When only the scholarship amounts are interchanged, the category counts remain the same. The new expenditure is the row-column product $\begin{pmatrix} 4000 & 3000 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Step 1: Determine the original numbers of scholarships The original information gives $x + y = 50$ and $3000x + 4000y = 180000$. The coefficient determinant is 1000, and therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1000} \begin{pmatrix} 4000 & -1 \\ -3000 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 180000 \end{pmatrix} \\ = \frac{1}{1000} \begin{pmatrix} 20000 \\ 30000 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}.$$

Thus the girl-student group has 20 scholarship recipients and the meritorious group has 30.

Step 2: Apply the interchanged amounts After interchange each of the 20 girl students receives ₹4,000, and each of the 30 meritorious achievers receives ₹3,000. Hence the new monthly expenditure

Quick Tip to Solve: Interchange the two amounts while keeping the counts 20 and 30 fixed; swapping the counts as well would undo the change.

E is

$$\begin{aligned} E &= \begin{pmatrix} 4000 & 3000 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} \\ &= 4000(20) + 3000(30) \\ &= 80000 + 90000 = 170000. \end{aligned}$$

As a check, the smaller group gains ₹1,000 per student while the larger group loses the same amount per student. The net change is $1000(20 - 30) = -10000$, consistent with the original expenditure decreasing by ₹10,000.

Step 3: Conclusion The monthly expenditure after interchange would be ₹1,70,000.

Quick Tip to Solve: Interchange the two amounts while keeping the counts 20 and 30 fixed; swapping the counts as well would undo the change.

37. Case Study - 2: Self-study helps students to build confidence in learning. It boosts the self-esteem of the learners. Recent surveys suggested that close to 50% learners were self-taught using internet resources and upskilled themselves.

A student may spend 1 hour to 6 hours in a day in upskilling self. The probability distribution of the number of hours spent by a student is given below:

$$P(X = x) = \begin{cases} kx^2, & \text{for } x = 1, 2, 3 \\ 2kx, & \text{for } x = 4, 5, 6 \\ 0, & \text{otherwise} \end{cases} \text{ where } x \text{ denotes the number of hours.} \quad [4 \text{ marks}]$$

37(i). Express the probability distribution given above in the form of a probability distribution table. [1 mark]

Solution:

Concept:

- A discrete probability distribution lists each possible value of X and its probability $P(X = x)$. Here X is the number of hours spent studying.
- Use kx^2 for $x = 1, 2, 3$ and $2kx$ for $x = 4, 5, 6$; the coefficient changes after three hours.

Step 1: Evaluate the first branch For the first three possible values, substitute separately into kx^2 :

$$P(X = 1) = k, \quad P(X = 2) = 4k, \quad P(X = 3) = 9k.$$

These are probabilities, so the row must contain the resulting expressions, rather than just the formula kx^2 .

Step 2: Evaluate the second branch and tabulate For the remaining values, $P(X = 4) = 8k$, $P(X = 5) = 10k$ and $P(X = 6) = 12k$. Therefore the required table is

x	1	2	3	4	5	6
$P(X = x)$	k	$4k$	$9k$	$8k$	$10k$	$12k$

All other values have probability zero.

Quick Tip to Solve: At $x=4$ switch to $2kx$; continuing kx squared would give the wrong fourth probability.

Step 3: Conclusion The probability row corresponding to $x = 1, 2, 3, 4, 5, 6$ is $k, 4k, 9k, 8k, 10k, 12k$, respectively.

Quick Tip to Solve: At $x=4$ switch to $2kx$; continuing kx squared would give the wrong fourth probability.

37(ii). Find the value of k .

[1 mark]

Solution:

Concept:

- The sum of probabilities of all possible values of a discrete random variable is one: $\sum_x P(X = x) = 1$.
- Here the six probabilities are $k, 4k, 9k, 8k, 10k, 12k$, obtained from the appropriate branch of the given formula.

Step 1: Apply the total probability condition The outcomes $X = 1, 2, 3, 4, 5, 6$ are mutually exclusive and exhaust all possible study durations. Thus

$$k + 4k + 9k + 8k + 10k + 12k = 1.$$

The first three coefficients arise from squares, while the last three arise from twice the number of hours.

Step 2: Solve and check admissibility Adding the coefficients gives $(1 + 4 + 9) + (8 + 10 + 12) = 14 + 30 = 44$. Hence

$$44k = 1 \implies k = \frac{1}{44}.$$

This positive value makes every probability nonnegative, and their sum is exactly one.

Step 3: Conclusion The required normalising constant is $k = \frac{1}{44}$.

Quick Tip to Solve: Add the six coefficients first: their total is 44, so the normalising constant is its reciprocal.

37(iii)(a). Find the mean number of hours spent by the student.

[2 marks]

Solution:

Concept:

- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$; each outcome is weighted by its probability.
- Normalisation gives $44k = 1$, hence $k = 1/44$. The probabilities are $k, 4k, 9k, 8k, 10k, 12k$ for $x = 1, 2, 3, 4, 5, 6$.

Step 1: Obtain the probability weights The two branches give $P(X = x) = kx^2$ for the first three

Quick Tip to Solve: For the mean, multiply each probability coefficient by its own x before adding; normalise the result by 44.

values and $P(X = x) = 2kx$ for the last three. Their total is

$$k(1 + 4 + 9 + 8 + 10 + 12) = 44k = 1,$$

so each coefficient must be divided by 44. This ensures that the weighted sum describes the average study duration.

Step 2: Compute the expectation Multiply every number of hours by its associated probability and add:

$$\begin{aligned} E(X) &= 1(k) + 2(4k) + 3(9k) + 4(8k) + 5(10k) + 6(12k) \\ &= k(1 + 8 + 27 + 32 + 50 + 72) \\ &= 190k = \frac{190}{44} = \frac{95}{22}. \end{aligned}$$

The sum is weighted by the probabilities; simply averaging the six possible values would incorrectly assume equal likelihoods. The result lies between the smallest and largest possible durations, as expected.

Step 3: Conclusion The mean number of hours is $\boxed{\frac{95}{22} \text{ hours}}$, approximately 4.32 hours.

Quick Tip to Solve: For the mean, multiply each probability coefficient by its own x before adding; normalise the result by 44.

OR

37(iii)(b). Find $P(1 < X < 6)$.

[2 marks]

Solution:

Concept:

- For a discrete random variable, an event probability is the sum of probabilities of the values satisfying the event.
- The inequalities $1 < X < 6$ are strict, so only $X = 2, 3, 4, 5$ are included. Normalisation gives $k = 1/44$.

Step 1: Identify the admissible values and constant The integer-valued study duration can take values 1, 2, 3, 4, 5, 6. The strict inequalities exclude both endpoints, leaving 2, 3, 4, 5. From the total probability condition,

$$k + 4k + 9k + 8k + 10k + 12k = 1, \quad 44k = 1,$$

giving $k = 1/44$. Each of the four selected probabilities must use the branch applicable to that value.

Step 2: Add the required probabilities and verify The events for the four distinct values are mutually

Quick Tip to Solve: For strict inequalities exclude both endpoints; here the complement contains only $X=1$ and $X=6$.

exclusive. Thus

$$\begin{aligned}P(1 < X < 6) &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\&= 4k + 9k + 8k + 10k = 31k = \frac{31}{44}.\end{aligned}$$

As a check, the excluded endpoint probabilities total $k + 12k = 13/44$; subtracting this from one also gives $31/44$.

Step 3: Conclusion The required probability is $P(1 < X < 6) = \frac{31}{44}$.

Quick Tip to Solve: For strict inequalities exclude both endpoints; here the complement contains only $X=1$ and $X=6$.

38. Case Study - 3: A bacteria sample of certain number of bacteria is observed to grow exponentially in a given amount of time. Using exponential growth model, the rate of growth of this sample of bacteria is calculated.

The differential equation representing the growth of bacteria is given as: $\frac{dP}{dt} = kP$, where P is the population of bacteria at any time t . [4 marks]

38(i). Obtain the general solution of the given differential equation and express it as an exponential function of ' t '. [2 marks]

Solution:

Concept:

- A separable differential equation allows terms involving the dependent variable and independent variable to be placed on opposite sides.
- For positive population P , $\int dP/P = \ln P$. Here k is constant and t is time.

Step 1: Separate the variables and integrate The given equation is $dP/dt = kP$. Since a bacterial population is positive, divide by P and multiply by dt :

$$\frac{dP}{P} = k dt.$$

Integrating both sides gives

$$\int \frac{dP}{P} = \int k dt \implies \ln P = kt + C,$$

where C is the constant of integration. For the unrestricted mathematical equation one may write $\ln |P|$ on a nonzero solution interval.

Step 2: Express the solution exponentially and check Exponentiating the population equation yields

$$P = e^{kt+C} = e^C e^{kt} = A e^{kt},$$

where A is a constant and $A = P(0) > 0$ for this population model. Differentiating gives $dP/dt =$

Quick Tip to Solve: After integration absorb the exponential of the integration constant into one multiplier A ; it is the initial population.

$kAe^{kt} = kP$, verifying the result. As a mathematical solution family, allowing arbitrary real A , including $A = 0$, also includes the identically zero solution that was excluded by division.

Step 3: Conclusion The general solution in exponential form is $P(t) = Ae^{kt}$, with A an arbitrary constant (positive for the bacterial population).

Quick Tip to Solve: After integration absorb the exponential of the integration constant into one multiplier A ; it is the initial population.

38(ii). If population of bacteria is 1000 at $t = 0$, and 2000 at $t = 1$, find the value of k .

[2 marks]

Solution:

Concept:

- The solution of $dP/dt = kP$ is $P(t) = Ae^{kt}$, where $A = P(0)$ and k is the constant growth rate.
- To determine constants, substitute the initial and later populations; use the natural logarithm because the exponential has base e .

Step 1: Use the initial population At $t = 0$, the exponential equals one. Substituting the first observation into the general solution gives

$$1000 = Ae^{k \cdot 0} = A.$$

Hence the population model becomes

$$P(t) = 1000e^{kt}.$$

The initial observation fixes the multiplier independently of the growth constant; the later observation is needed to determine k .

Step 2: Use the population at one time unit At $t = 1$, the second observation gives

$$2000 = 1000e^k \implies e^k = 2.$$

Taking natural logarithms on both sides,

$$\ln(e^k) = \ln 2 \implies k = \ln 2.$$

This is positive, consistent with a growing sample. Substitution gives $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$, which reproduces both $P(0) = 1000$ and $P(1) = 2000$.

Step 3: Conclusion The growth constant is $k = \ln 2$, approximately 0.6931 per unit of time.

Quick Tip to Solve: Divide the later population by the initial population before taking the natural logarithm: here the ratio is 2.