

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-4-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question Paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question Paper is divided into **five** Sections – Section **A, B, C, D** and **E**.
- (iii) In **Section A** – Questions no. **1** to **18** are Multiple Choice Questions (MCQs) and Questions no. **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B** – Questions no. **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C** – Questions no. **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D** – Questions no. **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In **Section E** – Questions no. **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section **B**, **3** questions in Section **C**, **3** questions in Section **D** and **2** questions in Section **E**.
- (ix) Use of calculators is not allowed.

Section A (20 marks)

This section consists of 20 multiple choice questions of 1 mark each. $20 \times 1 = 20$

1. If $\begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a scalar matrix, then the value of $a + 2b + 3c + 4d$ is: [1 mark]

- (A) 0
- (B) 5
- (C) 10
- (D) 25

Correct Answer: (D) 25

Solution:

Concept:

- A scalar matrix is a diagonal matrix whose diagonal entries are all equal: $A = \lambda I$, where I is the identity matrix.
- Every off-diagonal entry of a scalar matrix is zero; corresponding entries of equal matrices are equal.

Step 1: Compare the diagonal and off-diagonal entries The third diagonal entry is 5, so the common diagonal value is $\lambda = 5$. Consequently, $a = d = 5$. The entries b and c lie off the main diagonal, giving $b = c = 0$.

Step 2: Substitute in the required expression Use all four entries with their stated coefficients:

$$a + 2b + 3c + 4d = 5 + 2(0) + 3(0) + 4(5) = 5 + 20.$$

Option (A) would discard the nonzero diagonal entries. Option (B) counts only one diagonal contribution, and option (C) adds the two diagonal entries without the coefficient of d .

Step 3: Conclusion The required value is 25. Hence, the correct option is (D).

Quick Tip to Solve: Read the fixed diagonal entry first: it sets both a and d , while b and c immediately vanish.

2. Given that $A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$, matrix A is:

[1 mark]

(A) $7 \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

(B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

(C) $\frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

(D) $\frac{1}{49} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Solution:

Concept:

- For an invertible matrix $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, its inverse is $M^{-1} = \frac{1}{ps - qr} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$.
- If $A^{-1} = M/7$, then $A = 7M^{-1}$, since an inverse must give the identity matrix on multiplication.

Step 1: Invert the matrix inside the scalar factor Put $M = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$. Its determinant is

$$|M| = 2(2) - 1(-3) = 7.$$

Thus $M^{-1} = \frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$.

Step 2: Account for the factor and check the choices Inverting the outside factor gives

$$A = 7M^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

Multiplying the displayed matrix by the given A^{-1} gives I . Options (A), (C) and (D) have extra factors 7, $1/7$ and $1/49$, respectively, so their products with A^{-1} would be these multiples of I , not I .

Step 3: Conclusion $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Keep the outer $1/7$ separate: its reciprocal cancels the determinant factor $1/7$ when you invert.

3. If $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, then the value of $I - A + A^2 - A^3 + \dots$ is:

[1 mark]

(A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

Solution:

Concept:

- Matrix powers are formed using row-by-column multiplication, not by squaring entries separately.
- If $A^2 = O$, where O is the zero matrix, then $A^n = O$ for every integer $n \geq 2$.

Step 1: Compute the square of the matrix Multiply the given matrix by itself:

$$A^2 = \begin{pmatrix} 4 - 4 & 2 - 2 \\ -8 + 8 & -4 + 4 \end{pmatrix} = O.$$

Therefore all powers from the second onward vanish. Every partial sum containing $I - A$ has the same value, so no convergence formula is needed.

Step 2: Subtract the matrix from the identity The expression reduces to

$$I - A = \begin{pmatrix} 1 - 2 & 0 - 1 \\ 0 - (-4) & 1 - (-2) \end{pmatrix}.$$

This has first row $(-1, -1)$ and second row $(4, 3)$. Option (B) disagrees with these entries; (C) and (D) also fail because the nonzero off-diagonal entries of $-A$ remain.

Step 3: Conclusion The value is $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$. Hence, the correct option is (A).

Quick Tip to Solve: Before using any series identity, compute A^2 ; here its zero value removes every higher power at once.

4. If $A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 2 & 3 \\ 5 & 1 & -1 \end{pmatrix}$, then the value of $|A(\text{adj } A)|$ is:

[1 mark]

- (A) $100 I$
- (B) $10 I$
- (C) 10
- (D) 1000

Correct Answer: (D) 1000

Solution:

Concept:

- For a square matrix A , $A \text{ adj } A = |A|I$, where $\text{adj } A$ is its adjugate and I is the identity matrix of the same order.
- The determinant of λI_n is λ^n . Thus the determinant of a scalar matrix depends on its order.

Step 1: Expand the determinant along the first row The two zeros make expansion particularly short:

$$|A| = (-2) \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = (-2)(-2 - 3) = 10.$$

Therefore $A \text{ adj } A = 10I_3$.

Step 2: Take the determinant of the product matrix The vertical bars in the question ask for a determinant, so evaluate

$$|A \text{ adj } A| = |10I_3| = 10 \cdot 10 \cdot 10.$$

Option (B) gives the product matrix itself, while (A) is also a matrix rather than the required scalar. Option (C) omits the third power required for a matrix of order three.

Step 3: Conclusion $|A \text{ adj } A| = 1000$. Hence, the correct option is (D).

Quick Tip to Solve: Distinguish $A \text{ adj } A$ from its determinant: for order three, cube $|A|$ after finding it.

5. Given that $\begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -2 & 0 \end{pmatrix} = 0$, the value of x is:

[1 mark]

- (A) -4
- (B) -2
- (C) 2
- (D) 4

Correct Answer: (C) 2

Solution:

Concept:

- In matrix multiplication, each entry is the sum of products of the entries of one row with the corresponding entries of one column.

Quick Tip to Solve: The zero second column gives no restriction; solve only $4 - 2x = 0$ from the first column.

- A row matrix is the zero matrix only when each entry is zero; here the symbol 0 denotes the zero row matrix.

Step 1: Multiply the row by each column The first column gives $1(4) + x(-2) = 4 - 2x$. The second column gives $1(0) + x(0) = 0$. Thus

$$\begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 - 2x & 0 \end{pmatrix}.$$

Step 2: Equate the nontrivial entry to zero The second entry is already zero for every x , so only the first entry imposes a condition:

$$4 - 2x = 0 \implies 2x = 4.$$

For options (A), (B) and (D), the first entry becomes 12, 8 and -4 , respectively. None makes the product zero.

Step 3: Conclusion $x = 2$. Hence, the correct option is (C).

Quick Tip to Solve: The zero second column gives no restriction; solve only $4 - 2x = 0$ from the first column.

6. Derivative of e^{2x} with respect to e^x , is:

[1 mark]

- (A) e^x
- (B) $2e^x$
- (C) $2e^{2x}$
- (D) $2e^{3x}$

Correct Answer: (B) $2e^x$

Solution:

Concept:

- If y and u are differentiable functions of x , then $\frac{dy}{du} = \frac{dy/dx}{du/dx}$, provided $du/dx \neq 0$.
- The chain rule gives $\frac{d}{dx}e^{ax} = ae^{ax}$, where a is constant.

Step 1: Differentiate both functions with respect to x Set $y = e^{2x}$ and $u = e^x$. Their derivatives are

$$\frac{dy}{dx} = 2e^{2x}, \quad \frac{du}{dx} = e^x.$$

Since $e^x > 0$ for all real x , dividing by du/dx is valid everywhere.

Step 2: Divide the derivatives The requested derivative is

$$\frac{dy}{du} = \frac{2e^{2x}}{e^x} = 2e^{2x-x}.$$

Option (A) misses the factor 2, (C) is the derivative with respect to x , and (D) multiplies the derivatives instead of dividing them.

Quick Tip to Solve: Set $u = e^x$; then $e^{2x} = u^2$, so differentiating with respect to u immediately gives $2u$.

Step 3: Conclusion The derivative of e^{2x} with respect to e^x is $2e^x$. Hence, the correct option is (B).

Quick Tip to Solve: Set $u = e^x$; then $e^{2x} = u^2$, so differentiating with respect to u immediately gives $2u$.

7. For what value of k , the function given below is continuous at $x = 0$?

$$f(x) = \begin{cases} \frac{\sqrt{4+x}-2}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

[1 mark]

(A) 0

(B) $\frac{1}{4}$

(C) 1

(D) 4

Correct Answer: (B) $\frac{1}{4}$

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0)$; here the assigned value is $f(0) = k$.
- Rationalising a difference of square roots uses $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$.

Step 1: Rationalise the expression for nonzero x Direct substitution gives the indeterminate form $0/0$. For $x \neq 0$ near zero, multiply numerator and denominator by the conjugate:

$$\frac{\sqrt{4+x}-2}{x} = \frac{(4+x)-4}{x(\sqrt{4+x}+2)} = \frac{1}{\sqrt{4+x}+2}.$$

Step 2: Evaluate the limit and impose continuity The new denominator is continuous and nonzero near zero, so the limits from both sides agree:

$$\lim_{x \rightarrow 0} f(x) = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

Hence k must equal this limit. Option (A) incorrectly treats the original zero numerator alone; (C) and (D) do not equal the computed limit, with (D) giving its reciprocal.

Step 3: Conclusion Continuity holds for $k = \frac{1}{4}$. Hence, the correct option is (B).

Quick Tip to Solve: Rationalise before substituting $x = 0$; cancel x only in the nonzero branch, then match the limit to k .

8. The value of $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ is:

[1 mark]

- (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{2}$
- (D) $\frac{\pi}{18}$

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- The standard antiderivative is $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C$ for $-a < x < a$, where $a > 0$.
- When the denominator vanishes at an endpoint, evaluate the integral by the one-sided limit of integrals with an interior endpoint.

Step 1: Apply the standard integral formula Here $9 - x^2 = 3^2 - x^2$, so $a = 3$. For $0 < b < 3$,

$$\int_0^b \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\left(\frac{x}{3}\right) \right]_0^b = \sin^{-1}\left(\frac{b}{3}\right).$$

The lower endpoint contributes $\sin^{-1}(0) = 0$.

Step 2: Take the upper endpoint limit As $b \rightarrow 3^-$, the argument $b/3 \rightarrow 1^-$. The inverse sine is continuous at 1, giving

$$\lim_{b \rightarrow 3^-} \sin^{-1}(b/3) = \sin^{-1}(1) = \frac{\pi}{2}.$$

This finite limit also confirms convergence. Options (A), (B) and (D) are smaller angles whose sines are not 1, so none matches the printed upper endpoint.

Step 3: Conclusion The integral equals $\frac{\pi}{2}$. Hence, the correct option is (C).

Quick Tip to Solve: Divide the upper limit by $a = 3$ before taking inverse sine: here $3/3 = 1$, so the angle is $\pi/2$.

9. The general solution of the differential equation $x dy + y dx = 0$ is:

[1 mark]

- (A) $xy = c$
- (B) $x + y = c$
- (C) $x^2 + y^2 = c^2$
- (D) $\log y = \log x + c$

Correct Answer: (A) $xy = c$

Solution:

Concept:

- The product rule gives $d(xy) = x dy + y dx$. Recognising an exact differential can avoid separating variables.
- If $dF = 0$, then $F = c$, where c is an arbitrary constant.

Step 1: Recognise the product derivative The left-hand side of the given equation is exactly the differential of the product xy . Thus the equation becomes

$$d(xy) = 0.$$

Integrating both sides gives $xy = c$, with one arbitrary constant as expected for a first-order equation.

Step 2: Check the printed alternatives Differentiating option (A) returns $x dy + y dx = 0$. Option (B) instead gives $dx + dy = 0$, while option (C) gives $x dx + y dy = 0$. Option (D), $\log y = \log x + c$, gives $x dy - y dx = 0$; the sign is wrong. Only (A) reproduces the given equation.

Step 3: Conclusion The general solution is $xy = c$. Hence, the correct option is (A).

Quick Tip to Solve: For $x dy + y dx$, spot $d(xy)$ immediately; a minus sign would instead suggest a quotient.

10. The integrating factor of the differential equation $(x + 2y^2) \frac{dy}{dx} = y$ ($y > 0$) is:

[1 mark]

- (A) $\frac{1}{x}$
- (B) x
- (C) y
- (D) $\frac{1}{y}$

Correct Answer: (D) $\frac{1}{y}$

Solution:

Concept:

- A linear equation in x as a function of y has the form $\frac{dx}{dy} + P(y)x = Q(y)$.
- Its integrating factor is $e^{\int P(y) dy}$. The independent variable in this formula must be y .

Step 1: Write the equation in linear form Invert the given derivative to treat x as the dependent

Quick Tip to Solve: When the equation is linear in x , first write dx/dy ; integrate its coefficient with respect to y .

variable. Since $y > 0$, divide by y to obtain

$$\frac{dx}{dy} = \frac{x + 2y^2}{y}, \quad \frac{dx}{dy} - \frac{x}{y} = 2y.$$

Hence $P(y) = -1/y$.

Step 2: Compute and verify the factor Using $y > 0$, the integrating factor is

$$e^{\int -1/y dy} = e^{-\log y} = \frac{1}{y}.$$

Indeed, multiplication gives $\frac{1}{y} \frac{dx}{dy} - \frac{x}{y^2} = 2$, whose left side is $\frac{d}{dy}(x/y)$. The factor y would correspond to the opposite sign of P ; x and $1/x$ use the wrong independent variable for this linear form.

Step 3: Conclusion The integrating factor is $\boxed{1/y}$. Hence, the correct option is (D).

Quick Tip to Solve: When the equation is linear in x , first write dx/dy ; integrate its coefficient with respect to y .

11. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$, then the angle between $2\vec{a}$ and $-\vec{b}$ is:

[1 mark]

- (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{3}$
- (C) $\frac{5\pi}{6}$
- (D) $\frac{11\pi}{6}$

Correct Answer: (C) $\frac{5\pi}{6}$

Solution:

Concept:

- For nonzero vectors $\vec{u}, \vec{v}$, the angle $\theta \in [0, \pi]$ satisfies $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$.
- Positive scaling preserves direction; multiplying one vector by a negative number reverses its direction.

Step 1: Evaluate the dot product and magnitudes Choose $\vec{u} = 2\vec{a}$ and $\vec{v} = -\vec{b}$. Then

$$\vec{u} \cdot \vec{v} = -2\sqrt{3}, \quad |\vec{u}| = 2, \quad |\vec{v}| = 2.$$

Consequently, $\cos \theta = (-2\sqrt{3})/(2 \cdot 2) = -\sqrt{3}/2$.

Step 2: Choose the allowed angle The unique angle in $[0, \pi]$ with this cosine is $5\pi/6$. The angles $\pi/6$ and $\pi/3$ are acute and have positive cosine, so options (A) and (B) fail. Option (D), $11\pi/6$, is outside the range of the angle between two vectors and also has positive cosine.

Quick Tip to Solve: Find the acute angle between $\vec{a}$ and $\vec{b}$, then subtract it from π when just one vector is reversed.

Step 3: Conclusion The required angle is $\boxed{5\pi/6}$. Hence, the correct option is (C).

Quick Tip to Solve: Find the acute angle between $\vec{a}$ and $\vec{b}$, then subtract it from π when just one vector is reversed.

12. The vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = -3\hat{i} + 4\hat{j} + 4\hat{k}$ represent the sides of: [1 mark]

- (A) an equilateral triangle
- (B) an obtuse-angled triangle
- (C) an isosceles triangle
- (D) a right-angled triangle

Correct Answer: (D) a right-angled triangle

Solution:

Concept:

- Three directed vectors forming successive sides of a triangle have sum zero.
- A triangle is right-angled when the square of its longest side equals the sum of the squares of the other two sides.

Step 1: Verify closure and find the side lengths The component sums are

$$\vec{a} + \vec{b} + \vec{c} = (2 + 1 - 3)\hat{i} + (-1 - 3 + 4)\hat{j} + (1 - 5 + 4)\hat{k} = \vec{0}.$$

Their squared magnitudes are $|\vec{a}|^2 = 4 + 1 + 1 = 6$, $|\vec{b}|^2 = 1 + 9 + 25 = 35$ and $|\vec{c}|^2 = 9 + 16 + 16 = 41$.

Step 2: Apply the converse of Pythagoras The longest side has squared length 41, and $6 + 35 = 41$. Thus the side lengths satisfy Pythagoras exactly. All three lengths are distinct, excluding equilateral and isosceles triangles. An obtuse triangle would require the largest squared length to exceed the sum of the other two, whereas equality holds here.

Step 3: Conclusion The vectors represent the sides of a right-angled triangle. Hence, the correct option is (D).

Quick Tip to Solve: Compare squared magnitudes directly: $6 + 35 = 41$ establishes a right angle without computing any square roots.

13. Let $\vec{a}$ be any vector such that $|\vec{a}| = a$. The value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$ is:

[1 mark]

- (A) a^2
- (B) $2a^2$
- (C) $3a^2$
- (D) 0

Correct Answer: (B) $2a^2$

Solution:

Concept:

- Write $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$; then $a_x^2 + a_y^2 + a_z^2 = |\vec{a}|^2 = a^2$.
- The identity $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$ applies to each unit basis vector.

Step 1: Evaluate the three squared cross products Since the basis vectors are unit vectors,

$$|\vec{a} \times \hat{i}|^2 = a^2 - a_x^2,$$

$$|\vec{a} \times \hat{j}|^2 = a^2 - a_y^2,$$

$$|\vec{a} \times \hat{k}|^2 = a^2 - a_z^2.$$

Each term removes the component parallel to the corresponding axis.

Step 2: Add and compare the options Adding these expressions gives

$$3a^2 - (a_x^2 + a_y^2 + a_z^2) = 3a^2 - a^2 = 2a^2.$$

Thus each squared component contributes twice overall. The expressions a^2 , $3a^2$ and 0 are not the general identity; the zero vector may give coincident numerical values, but the formula valid for every vector is $2a^2$.

Step 3: Conclusion The required sum is $2a^2$. Hence, the correct option is (B).

Quick Tip to Solve: In the sum of cross products with all three coordinate axes, every squared component appears exactly twice.

14. The vector equation of a line passing through the point $(1, -1, 0)$ and parallel to Y -axis is:

[1 mark]

- (A) $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} - \hat{j})$
- (B) $\vec{r} = \hat{i} - \hat{j} + \lambda\hat{j}$
- (C) $\vec{r} = \hat{i} - \hat{j} + \lambda\hat{k}$
- (D) $\vec{r} = \lambda\hat{j}$

Correct Answer: (B) $\vec{r} = \hat{i} - \hat{j} + \lambda\hat{j}$

Solution:

Concept:

- A line through position vector $\vec{a}$ and parallel to a nonzero direction vector $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, $\lambda \in \mathbb{R}$.

Quick Tip to Solve: Check both the fixed point and the direction: being parallel to the Y -axis alone does not identify the required line.

- The vector $\hat{j}$ points along the Y -axis, and the point $(1, -1, 0)$ has position vector $\hat{i} - \hat{j}$.

Step 1: Use the given point and direction Substitute the position and direction vectors into the line formula:

$$\vec{r} = \hat{i} - \hat{j} + \lambda \hat{j}.$$

In coordinates this is $x = 1, y = -1 + \lambda, z = 0$. Only the y -coordinate changes, as required for a line parallel to the Y -axis.

Step 2: Check the other proposed lines Option (A) has direction $\hat{i} - \hat{j}$, so its x -coordinate also changes. Option (C) has direction $\hat{k}$ and is parallel to the Z -axis. Option (D) is the Y -axis itself, where $x = 0$, so it misses $(1, -1, 0)$. Option (B) satisfies both requirements.

Step 3: Conclusion The line is $\boxed{\vec{r} = \hat{i} - \hat{j} + \lambda \hat{j}}, \lambda \in \mathbb{R}$. Hence, the correct option is (B).

Quick Tip to Solve: Check both the fixed point and the direction: being parallel to the Y -axis alone does not identify the required line.

15. The lines $\frac{1-x}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{2x-3}{2p} = \frac{y}{-1} = \frac{z-4}{7}$ are perpendicular to each other for p equal to: [1 mark]

- (A) $-\frac{1}{2}$
 (B) $\frac{1}{2}$
 (C) 2
 (D) 3

Correct Answer: (C) 2

Solution:

Concept:

- Direction ratios must be read after each numerator is reduced to $x - x_0, y - y_0$ or $z - z_0$.
- Perpendicular directions have zero dot product: $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

Step 1: Extract the correct direction ratios For the first line, $(1-x)/2 = (x-1)/(-2)$, so its direction ratios are $(-2, 3, 1)$. For the second,

$$(2x-3)/(2p) = (x-3/2)/p,$$

so its direction ratios are $(p, -1, 7)$. The coefficient 2 in the numerator must be cancelled before reading the first direction ratio.

Step 2: Set the dot product to zero The perpendicularity condition is

$$(-2)p + 3(-1) + 1(7) = 0, \quad -2p + 4 = 0.$$

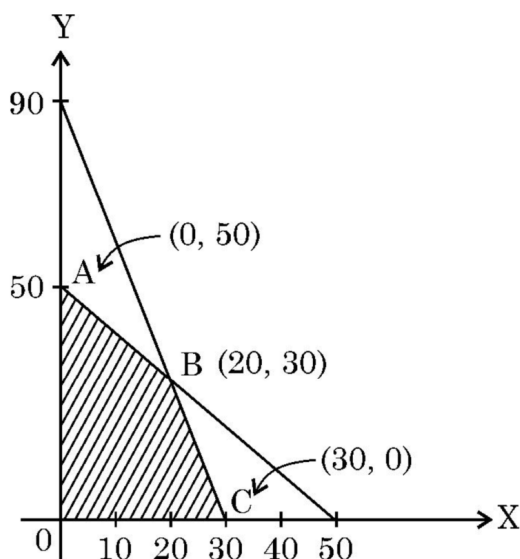
It gives $p = 2$. As a check on every option, the dot product $4 - 2p$ is 5, 3, 0 and -2 at $p = -1/2, 1/2, 2, 3$, respectively. Only the third value makes the directions perpendicular.

Quick Tip to Solve: Normalise the x -numerators first: $1-x$ changes the sign, while $2x-3$ halves its denominator.

Step 3: Conclusion The required value is $p = 2$. Hence, the correct option is (C).

Quick Tip to Solve: Normalise the x -numerators first: $1 - x$ changes the sign, while $2x - 3$ halves its denominator.

16. The maximum value of $Z = 4x + y$ for a L.P.P. whose feasible region is given below is: [1 mark]



- (A) 50
- (B) 110
- (C) 120
- (D) 170

Correct Answer: (C) 120

Solution:

Concept:

- A linear objective function on a bounded polygonal feasible region attains its maximum at a corner point.
- For $Z = 4x + y$, substitute each feasible vertex from the supplied graph, including the origin.

Step 1: Identify the vertices of the shaded region The graph shows the feasible polygon with vertices $O(0, 0)$, $A(0, 50)$, $B(20, 30)$ and $C(30, 0)$. The intercepts $(0, 90)$ and $(50, 0)$ belong to extensions of boundary lines; they lie outside the shaded feasible region and must not be tested as feasible corners.

Step 2: Evaluate the objective at every vertex The objective values are

$$Z(O) = 4(0) + 0 = 0,$$

$$Z(A) = 4(0) + 50 = 50,$$

$$Z(B) = 4(20) + 30 = 110,$$

$$Z(C) = 4(30) + 0 = 120.$$

Quick Tip to Solve: Test the corners of the shaded polygon only; line intercepts outside the shading are not feasible candidates.

Comparing all four values establishes the largest feasible value. Thus 50 and 110 are attained but are not maxima, while 170 exceeds every feasible corner value and is impossible.

Step 3: Conclusion The maximum is $Z_{\max} = 120$, attained at $(30, 0)$. Hence, the correct option is (C).

Quick Tip to Solve: Test the corners of the shaded polygon only; line intercepts outside the shading are not feasible candidates.

17. The probability distribution of a random variable X is:

X	0	1	2	3	4
$P(X)$	0.1	k	$2k$	k	0.1

where k is some unknown constant.

The probability that the random variable X takes the value 2 is:

[1 mark]

- (A) $\frac{1}{5}$
 (B) $\frac{2}{5}$
 (C) $\frac{4}{5}$
 (D) 1

Correct Answer: (B) $\frac{2}{5}$

Solution:

Concept:

- For a discrete random variable, the probabilities of all possible values sum to 1: $\sum_x P(X = x) = 1$.
- The entry in the probability row under a particular value of X gives the probability of that value.

Step 1: Use the total probability The five values listed exhaust the distribution. Hence

$$0.1 + k + 2k + k + 0.1 = 1.$$

Combining the constant terms and the multiples of k gives $0.2 + 4k = 1$, so $4k = 0.8$ and $k = 0.2 = \frac{1}{5}$.

Step 2: Read the entry for the required value The probability corresponding to $X = 2$ is $2k$, not k . Therefore

$$P(X = 2) = 2 \left(\frac{1}{5} \right) = \frac{2}{5}.$$

Option (A) gives only k ; option (C) is the combined probability of $X = 1, 2, 3$; option (D) is the total probability of the entire distribution.

Step 3: Conclusion $P(X = 2) = \frac{2}{5}$. Hence, the correct option is (B).

Quick Tip to Solve: First normalise the entire row; then use the entry $2k$ directly below $X = 2$.

18. The function $f(x) = kx - \sin x$ is strictly increasing for:

[1 mark]

- (A) $k > 1$
- (B) $k < 1$
- (C) $k > -1$
- (D) $k < -1$

Correct Answer: (A) $k > 1$

Solution:

Concept:

- A sufficient condition for a differentiable function to be strictly increasing throughout an interval is $f'(x) > 0$ there.
- For real x , $-1 \leq \cos x \leq 1$; the largest possible value of $\cos x$ is 1.

Step 1: Differentiate and find a uniform lower bound For $f(x) = kx - \sin x$, with k constant,

$$f'(x) = k - \cos x \geq k - 1.$$

If $k > 1$, this lower bound is positive, so the derivative is positive for every real x .

Step 2: Check the other printed ranges Options (B) and (C) both permit $k = 0$. Then $f'(0) = -1$, so neither range guarantees an increasing function. For option (D), $k < -1$ gives $f'(x) \leq k + 1 < 0$, so the function decreases. Although the boundary value $k = 1$ also gives a strictly increasing function, it is not a separate printed option: its derivative $1 - \cos x$ is nonnegative and is positive except at isolated points. Among the stated ranges, only (A) guarantees strict increase throughout.

Step 3: Conclusion The printed sufficient range is $k > 1$. Hence, the correct option is (A).

Quick Tip to Solve: Compare k with the maximum of $\cos x$, which is 1, to make $k - \cos x$ positive everywhere.

19. Assertion (A): The relation $R = \{(x, y) : (x + y) \text{ is a prime number and } x, y \in \mathbb{N}\}$ is not a reflexive relation.

Reason (R): The number ' $2n$ ' is composite for all natural numbers n .

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A relation on $\mathbb{N}$ is reflexive only if (n, n) belongs to it for every $n \in \mathbb{N}$. One missing diagonal pair disproves reflexivity.

Quick Tip to Solve: Test $n = 1$ against the universal claim in the reason, and $n = 2$ against reflexivity of the relation.

- A prime number has exactly two positive divisors. In particular, 2 is prime, while every even integer greater than 2 is composite.

Step 1: Judge the assertion A diagonal pair (n, n) is in R precisely when $n + n = 2n$ is prime. Taking $n = 2$ gives $2 + 2 = 4$, which is composite; thus $(2, 2) \notin R$. Since not every diagonal pair belongs to R , the relation is not reflexive. Assertion (A) is true.

Step 2: Judge the reason The reason claims that $2n$ is composite for every natural number. But $n = 1$ gives $2n = 2$, a prime number. This single counterexample disproves its universal claim. Reason (R) is false.

Step 3: Check the explanatory link and codes The assertion needs only one natural number whose diagonal pair is absent; it does not require every diagonal pair to be absent. The false reason therefore cannot explain the true assertion. Options (A) and (B) require a true reason, while (D) requires a false assertion; all three are excluded.

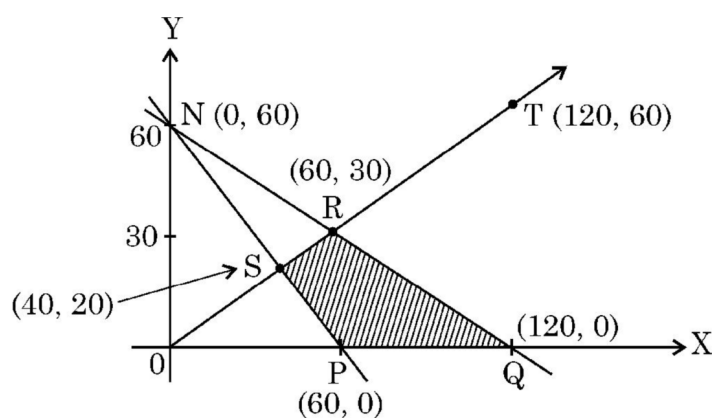
Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Test $n = 1$ against the universal claim in the reason, and $n = 2$ against reflexivity of the relation.

20. Assertion (A): The corner points of the bounded feasible region of a L.P.P. are shown below. The maximum value of $Z = x + 2y$ occurs at infinite points.

Reason (R): The optimal solution of a LPP having bounded feasible region must occur at corner points.

[1 mark]



- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
 (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
 (C) Assertion (A) is true, but Reason (R) is false.
 (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- For a linear objective on a nonempty bounded feasible polygon, an optimum is attained at a corner point.
- If adjacent corner points have the same optimal objective value, linearity makes every point of their connecting feasible edge optimal too.

Step 1: Judge the assertion The shaded polygon has vertices $S(40, 20)$, $R(60, 30)$, $Q(120, 0)$ and $P(60, 0)$. For $Z = x + 2y$, their values are respectively 80, 120, 120, 60. Thus the largest corner value is 120, attained at adjacent vertices R and Q . Every point on their feasible edge satisfies $x + 2y = 120$. Hence there are infinitely many maximisers, and the assertion is true.

Step 2: Judge the reason The reason states the corner-point principle: an optimum on a bounded feasible polygon is attained at a corner. This is true.

Step 3: Check the explanatory link The corner-point principle alone allows a unique best corner. Infinitely many optimisers here follow from the extra fact that adjacent corners R and Q tie for the maximum. Thus the true reason does not explain the true assertion. Thus (A), (C) and (D) are excluded.

Step 4: Conclusion The maximum is 120 along RQ . Both statements are true, but the reason does not explain the assertion. Hence, the correct option is (B).

Quick Tip to Solve: When two adjacent vertices tie for the largest Z , check their whole joining edge for infinitely many optimal points.

Section B (10 marks)

In this section there are 5 very short answer type questions of 2 marks each.

21(a). Express $\tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the simplest form.

[2 marks]

Solution:

Concept:

- The half-angle identity $\frac{1 + \sin x}{\cos x} = \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$ applies when the denominators are nonzero.
- The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$. Thus $\tan^{-1}(\tan \theta) = \theta$ only when θ lies in this interval.

Step 1: Rewrite the argument as a tangent On the given interval $-\pi/2 < x < \pi/2$, we have $\cos x > 0$ and $1 - \sin x > 0$. Rationalising gives

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x(1 + \sin x)}{1 - \sin^2 x} = \frac{1 + \sin x}{\cos x}.$$

Put $u = x/2$. Using $1 + \sin x = (\cos u + \sin u)^2$ and $\cos x = (\cos u - \sin u)(\cos u + \sin u)$, this ratio is

$$\frac{\cos u + \sin u}{\cos u - \sin u} = \tan \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

Quick Tip to Solve: Rationalise with $1 + \sin x$, use the half-angle identity, and verify the resulting angle lies in the principal range.

The cancelled factor is nonzero on the stated interval.

Step 2: Check the principal branch before simplifying Divide the given interval by 2 and add $\pi/4$:

$$-\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{4} \implies 0 < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}.$$

This lies entirely within the principal range of inverse tangent. Therefore no addition or subtraction of π is needed when applying $\tan^{-1}$ to the tangent expression.

Step 3: Conclusion

$$\boxed{\tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right) = \frac{\pi}{4} + \frac{x}{2}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Quick Tip to Solve: Rationalise with $1 + \sin x$, use the half-angle identity, and verify the resulting angle lies in the principal range.

OR

21(b). Find the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [2 marks]

Solution:

Concept:

- The principal ranges are $\tan^{-1} t \in (-\pi/2, \pi/2)$, $\cos^{-1} t \in [0, \pi]$ and $\sin^{-1} t \in [-\pi/2, \pi/2]$.
- Use standard trigonometric values together with each inverse function's principal range to select its unique angle.

Step 1: Evaluate the inverse tangent and inverse cosine Since $\tan(\pi/4) = 1$ and $\pi/4$ belongs to the principal range of inverse tangent,

$$\tan^{-1}(1) = \frac{\pi}{4}.$$

A cosine value of $-1/2$ selects the second-quadrant angle in $[0, \pi]$, so

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

The angle $4\pi/3$ also has cosine $-1/2$, but it is outside this principal range.

Step 2: Evaluate the inverse sine and add The sine value $-1/\sqrt{2}$ corresponds to the negative angle $-\pi/4$ in the principal range $[-\pi/2, \pi/2]$. Hence

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

Substituting all three principal values into the given sum gives

$$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{4}.$$

Quick Tip to Solve: Use a second-quadrant angle for inverse cosine of $-1/2$ and a negative angle for inverse sine of $-1/\sqrt{2}$.

The inverse tangent and inverse sine terms cancel. Each inverse function is evaluated on its own branch before the values are added; there is no further inverse function outside the sum.

Step 3: Conclusion The required value is $\boxed{\frac{2\pi}{3}}$.

Quick Tip to Solve: Use a second-quadrant angle for inverse cosine of $-1/2$ and a negative angle for inverse sine of $-1/\sqrt{2}$.

22(a). If $y = \cos^3(\sec^2 2t)$, find $\frac{dy}{dt}$.

[2 marks]

Solution:

Concept:

- The chain rule differentiates a composite function one layer at a time: if $y = F(u)$ and $u = G(t)$, then $\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dt}$.
- The required elementary derivatives are $\frac{d}{du}(\cos u) = -\sin u$ and $\frac{d}{dv}(\sec v) = \sec v \tan v$; powers are differentiated by the power rule.

Step 1: Differentiate the outer power and cosine Put $u = \sec^2(2t)$. Then $y = (\cos u)^3$, so the power rule followed by the chain rule gives

$$\frac{dy}{dt} = -3 \cos^2 u \sin u \frac{du}{dt}.$$

The negative sign comes from differentiating the cosine, while the exponent three contributes the factor 3.

Step 2: Differentiate the inner squared secant Here $u = (\sec(2t))^2$. Differentiate both the square and the angle $2t$:

$$\begin{aligned} \frac{du}{dt} &= 2 \sec(2t) [\sec(2t) \tan(2t)] \cdot 2 \\ &= 4 \sec^2(2t) \tan(2t). \end{aligned}$$

Substitute this expression in the outer derivative and replace u by $\sec^2(2t)$. The derivative is defined wherever the original secant is defined, namely $\cos(2t) \neq 0$.

Step 3: Conclusion

$$\boxed{\frac{dy}{dt} = -12 \cos^2(\sec^2(2t)) \sin(\sec^2(2t)) \sec^2(2t) \tan(2t).}$$

Quick Tip to Solve: Work from the outer cube inward; keep the factor 2 from the angle $2t$ to obtain the coefficient -12 .

22(b). If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

[2 marks]

Solution:

Concept:

- Taking natural logarithms converts a variable exponent into a product: $\log(x^y) = y \log x$, for $x > 0$.
- In implicit differentiation, y is a function of x . The product rule gives $\frac{d}{dx}(y \log x) = \frac{dy}{dx} \log x + \frac{y}{x}$.

Step 1: Take logarithms and obtain a relation for y From $x^y = e^{x-y}$, taking natural logarithms gives

$$y \log x = x - y, \quad y(1 + \log x) = x.$$

Therefore $y = \frac{x}{1 + \log x}$. The denominator cannot vanish: if $x = e^{-1}$, the preceding relation would say $0 = x$, which is impossible. Thus we work with $x > 0$, $x \neq e^{-1}$.

Step 2: Differentiate implicitly and eliminate y Differentiating $y \log x = x - y$ with respect to x ,

$$\begin{aligned} \frac{dy}{dx} \log x + \frac{y}{x} &= 1 - \frac{dy}{dx}, \\ (1 + \log x) \frac{dy}{dx} &= 1 - \frac{y}{x} \\ &= 1 - \frac{1}{1 + \log x} \\ &= \frac{\log x}{1 + \log x}. \end{aligned}$$

Dividing by the nonzero factor $1 + \log x$ establishes the required expression.

Step 3: Conclusion Hence, for the stated logarithmic domain,

$$\boxed{\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}}.$$

Quick Tip to Solve: After taking logarithms, use $y(1 + \log x) = x$ to replace y/x ; this removes y from the derivative immediately.

23. Find the interval in which the function $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing.

[2 marks]

Solution:

Concept:

- The derivative determines monotonicity: $f'(x) < 0$ on an interval implies that f is strictly decreasing there.
- A zero derivative at an isolated point does not necessarily interrupt strict decrease. Check whether the sign of f' actually changes on crossing that point.

Quick Tip to Solve: Factor $f'(x) = 4x^2(x - 3)$; the squared factor cannot change sign, so do not split the interval of decrease at $x = 0$.

Step 1: Differentiate and factor the derivative For $f(x) = x^4 - 4x^3 + 10$,

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3).$$

The stationary points are $x = 0$ and $x = 3$. For every $x \neq 0$, the factor $4x^2$ is positive, so the sign of $f'(x)$ is the sign of $x - 3$.

Step 2: Examine the signs across the stationary points When $x < 0$ or $0 < x < 3$, we have $x - 3 < 0$, and hence $f'(x) < 0$. When $x > 3$, the derivative is positive. At $x = 0$, the derivative vanishes but remains negative on both sides; thus the function continues decreasing through zero. In particular, if $a < 0 < b < 3$, the mean value theorem on $[a, 0]$ and $[0, b]$ gives $f(a) > f(0) > f(b)$. Therefore the two negative-derivative intervals join into one interval of strict decrease.

Step 3: Conclusion The function is strictly decreasing on $(-\infty, 3)$. The derivative is negative on $(-\infty, 0) \cup (0, 3)$; its isolated zero at 0 does not break the decrease. The endpoint 3 may also be included, giving $(-\infty, 3]$.

Quick Tip to Solve: Factor $f'(x) = 4x^2(x - 3)$; the squared factor cannot change sign, so do not split the interval of decrease at $x = 0$.

24. The volume of a cube is increasing at the rate of $6 \text{ cm}^3/\text{s}$. How fast is the surface area of cube increasing, when the length of an edge is 8 cm ? [2 marks]

Solution:

Concept:

- For a cube of edge length a , the volume is $V = a^3$ and the total surface area is $S = 6a^2$.
- Related rates are found by differentiating with respect to time t . Since a changes with time, the chain rule introduces $\frac{da}{dt}$ in both derivatives.

Step 1: Calculate how fast the edge length is increasing Differentiate the volume relation before substituting the instantaneous edge length:

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}.$$

At the instant $a = 8 \text{ cm}$, the given volume rate is $6 \text{ cm}^3/\text{s}$. Therefore,

$$6 = 3(8)^2 \frac{da}{dt}, \quad \frac{da}{dt} = \frac{6}{192} = \frac{1}{32} \text{ cm/s}.$$

Step 2: Relate the surface-area rate to the edge rate The cube has six square faces, so differentiating $S = 6a^2$ gives

$$\frac{dS}{dt} = 12a \frac{da}{dt}.$$

Substitution of the same instantaneous values yields the product $12 \times 8 \times \frac{1}{32}$. Its units are $\text{cm} \times \text{cm/s} = \text{cm}^2/\text{s}$, as required for a rate of change of area. Both the edge length and its rate are positive, so the

Quick Tip to Solve: Eliminate the edge rate directly: $\frac{dS}{dt} = \frac{4}{a} \frac{dV}{dt}$, then substitute $a = 8$ and $dV/dt = 6$.

area is increasing.

Step 3: Conclusion The surface area is increasing at

$$\frac{dS}{dt} = 3 \text{ cm}^2/\text{s}.$$

Quick Tip to Solve: Eliminate the edge rate directly: $\frac{dS}{dt} = \frac{4}{a} \frac{dV}{dt}$, then substitute $a = 8$ and $dV/dt = 6$.

25. Find: $\int \frac{1}{x(x^2 - 1)} dx$.

[2 marks]

Solution:

Concept:

- Split a rational integrand into simpler fractions whose numerators match derivatives of the denominators.
- The logarithmic rule is $\int \frac{g'(x)}{g(x)} dx = \log |g(x)| + C$, on any interval where $g(x) \neq 0$.

Step 1: Decompose the integrand Since $1 = x^2 - (x^2 - 1)$, for $x \neq 0, \pm 1$ we can write

$$\frac{1}{x(x^2 - 1)} = \frac{x^2 - (x^2 - 1)}{x(x^2 - 1)} = \frac{x}{x^2 - 1} - \frac{1}{x}.$$

Combining the fractions again produces numerator $x^2 - x^2 + 1 = 1$, verifying the decomposition and its minus sign.

Step 2: Integrate the two logarithmic forms In the first term take $u = x^2 - 1$, so $du = 2x dx$. Thus

$$\begin{aligned} \int \frac{1}{x(x^2 - 1)} dx &= \frac{1}{2} \int \frac{2x}{x^2 - 1} dx - \int \frac{1}{x} dx \\ &= \frac{1}{2} \log |x^2 - 1| - \log |x| + C. \end{aligned}$$

Absolute values allow these real logarithmic expressions wherever their arguments are nonzero. As a check, differentiating the two logarithms gives $\frac{x}{x^2 - 1} - \frac{1}{x}$, exactly the original integrand.

Step 3: Conclusion

$$\int \frac{1}{x(x^2 - 1)} dx = \frac{1}{2} \log |x^2 - 1| - \log |x| + C.$$

This holds on each interval excluding $-1, 0, 1$; C is an arbitrary constant on that interval.

Quick Tip to Solve: Use $1 = x^2 - (x^2 - 1)$ to split the fraction, then supply the factor $1/2$ required by the derivative of $x^2 - 1$.

Section C (18 marks)

In this section there are 6 short answer type questions of 3 marks each.

26. Given that $y = (\sin x)^x \cdot x^{\sin x} + a^x$, find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- Logarithmic differentiation simplifies a product whose bases and exponents both depend on x : differentiate the logarithm first, then multiply by the original function.
- The product rule is $(uv)' = u'v + uv'$. For a positive constant a , $\frac{d}{dx}(a^x) = a^x \ln a$.

Step 1: Separate the variable-power product Put $u = (\sin x)^x x^{\sin x}$, so that $y = u + a^x$. On intervals where $x > 0$ and $\sin x > 0$, both logarithms used below are defined. Taking natural logarithms changes the product into a sum and brings the variable exponents down:

$$\ln u = x \ln(\sin x) + \sin x \ln x.$$

We differentiate u this way because treating either exponent as constant would omit terms.

Step 2: Differentiate both logarithmic terms Apply the product rule separately to the two terms on the right. The chain rule gives $(\ln(\sin x))' = \cos x / \sin x = \cot x$. Hence

$$\frac{u'}{u} = \ln(\sin x) + x \cot x + \cos x \ln x + \frac{\sin x}{x}.$$

In particular, the derivative of $\sin x \ln x$ contains both $\cos x \ln x$ and $\sin x/x$. Multiply through by u , then use $y' = u' + a^x \ln a$.

Step 3: Conclusion Therefore, for the real-valued expression on the above domain and constant $a > 0$,

$$\begin{aligned} \frac{dy}{dx} &= (\sin x)^x x^{\sin x} \left[\ln(\sin x) + x \cot x \right. \\ &\quad \left. + \cos x \ln x + \frac{\sin x}{x} \right] + a^x \ln a. \end{aligned}$$

Quick Tip to Solve: Take the logarithm of the product term only; the added term a^x must be differentiated separately.

27(a). Evaluate: $\int_0^{\pi/4} \frac{x dx}{1 + \cos 2x + \sin 2x}$

[3 marks]

Solution:

Concept:

- Reflection of limits gives $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. Adding reflected integrals can remove a linear factor in the numerator.

Quick Tip to Solve: First reflect x to $\pi/4 - x$: the sine and cosine swap, so adding the two integrals removes the numerator x .

- Use $1 + \cos 2x = 2 \cos^2 x$, $\sin 2x = 2 \sin x \cos x$, and $d(\tan x) = \sec^2 x \, dx$.

Step 1: Reflect the integral about the midpoint Let the required integral be I and put $a = \pi/4$. Under the reflection $x \mapsto \pi/4 - x$, the cosine and sine terms interchange:

$$\cos(\pi/2 - 2x) = \sin 2x, \quad \sin(\pi/2 - 2x) = \cos 2x.$$

The denominator therefore remains the same, and

$$I = \int_0^{\pi/4} \frac{\pi/4 - x}{1 + \cos 2x + \sin 2x} \, dx.$$

Add this expression to the original integral. The terms containing x in the numerators cancel, giving

$$2I = \frac{\pi}{4} J, \quad J = \int_0^{\pi/4} \frac{dx}{1 + \cos 2x + \sin 2x}.$$

Step 2: Evaluate the remaining integral Factor the denominator as $2 \cos x (\cos x + \sin x)$. Dividing numerator and denominator by $\cos^2 x$, which is nonzero throughout the interval, gives

$$J = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 x}{1 + \tan x} \, dx.$$

Set $t = \tan x$. The limits become 0 and 1, so

$$J = \frac{1}{2} \int_0^1 \frac{dt}{1+t} = \frac{1}{2} [\ln(1+t)]_0^1 = \frac{1}{2} \ln 2.$$

The denominator stays positive on the integration interval, so all the substitutions and endpoint evaluations are valid.

Step 3: Conclusion Since $I = \frac{\pi}{8} J$, the required value is

$$\boxed{\int_0^{\pi/4} \frac{x \, dx}{1 + \cos 2x + \sin 2x} = \frac{\pi \ln 2}{16}}.$$

Quick Tip to Solve: First reflect x to $\pi/4 - x$: the sine and cosine swap, so adding the two integrals removes the numerator x .

27(b). Find: $\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right] dx$

[3 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Thus an integrand of this form has antiderivative $e^x f(x) + C$.
- To differentiate a power of $1+x^2$, combine the power rule with the chain-rule factor $2x$. Here $1+x^2 > 0$ for every real x .

Step 1: Recognise a function and its derivative Choose the simpler term inside the brackets as

$$f(x) = \frac{x}{\sqrt{1+x^2}} = x(1+x^2)^{-1/2}.$$

Differentiate using the product rule:

$$\begin{aligned} f'(x) &= (1+x^2)^{-1/2} + x \left[-\frac{1}{2}(1+x^2)^{-3/2}(2x) \right] \\ &= \frac{1+x^2-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}. \end{aligned}$$

This is exactly the other term in the printed integrand. The cancellation in the numerator is why this choice of f is useful.

Step 2: Apply the exponential product identity The given integral can now be written as

$$\int e^x [f'(x) + f(x)] dx.$$

The product rule identifies its integrand as the derivative of $e^x f(x)$. Indeed, differentiating that product produces $e^x f(x)$ from differentiating the exponential and $e^x f'(x)$ from differentiating f . Both terms have the positive signs required in the question, so no integration by parts or additional substitution is necessary.

Step 3: Conclusion Therefore,

$$\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right] dx = \frac{xe^x}{\sqrt{1+x^2}} + C,$$

where C is an arbitrary real constant. The formula holds for all real x .

Quick Tip to Solve: Differentiate $x/\sqrt{1+x^2}$ first; its derivative is the other bracketed term, revealing the $e^x(f + f')$ pattern.

28. Find: $\int \frac{3x+5}{\sqrt{x^2+2x+4}} dx$

[3 marks]

Solution:

Concept:

- When a linear numerator occurs over the square root of a quadratic, express it as a multiple of the derivative of that quadratic plus a constant.
- Complete the square and use $\int \frac{du}{\sqrt{u^2+a^2}} = \ln|u + \sqrt{u^2+a^2}| + C$, where $a > 0$.

Step 1: Split the numerator using the derivative Write $Q = x^2 + 2x + 4$, whose derivative is $Q' = 2x + 2$. Express the numerator as

$$3x + 5 = \frac{3}{2}(2x + 2) + 2.$$

Thus, denoting the required integral by I ,

$$I = \frac{3}{2} \int \frac{2x+2}{\sqrt{x^2+2x+4}} dx + 2 \int \frac{dx}{\sqrt{x^2+2x+4}}.$$

The first integral is a direct substitution: putting $t = Q$ gives $\int t^{-1/2} dt = 2\sqrt{t}$. Including its coefficient, this contribution is $3\sqrt{x^2+2x+4}$.

Step 2: Complete the square for the second integral Since $x^2 + 2x + 4 = (x+1)^2 + 3$, put $u = x+1$, so $du = dx$. The remaining integral has the standard form with $a = \sqrt{3}$:

$$\int \frac{dx}{\sqrt{x^2+2x+4}} = \ln|x+1 + \sqrt{x^2+2x+4}| + C.$$

Multiply this expression by the remaining coefficient 2. As a check, the derivatives of the two contributions are respectively $3(x+1)/\sqrt{Q}$ and $2/\sqrt{Q}$; their sum has numerator $3x+5$, exactly as required.

Step 3: Conclusion Combining both contributions into one antiderivative,

$$I = 3\sqrt{x^2+2x+4} + 2 \ln|x+1 + \sqrt{x^2+2x+4}| + C.$$

Here C is arbitrary. The quadratic is $(x+1)^2 + 3 > 0$, and the logarithm's argument is positive for all real x , so this result is valid on the entire real line.

Quick Tip to Solve: Use $3x+5 = 3(x+1) + 2$: the first term matches the derivative of the square root, and the constant term gives the logarithm.

29(a). Find the particular solution of the differential equation $\frac{dy}{dx} = y \cot 2x$, given that $y\left(\frac{\pi}{4}\right) = 2$. [3 marks]

Solution:

Concept:

- A separable differential equation can be integrated after collecting all factors involving y with dy , and all factors involving x with dx .
- The chain rule gives $\int \cot 2x \, dx = \frac{1}{2} \ln |\sin 2x| + C$. The factor $\frac{1}{2}$ accounts for the derivative of $2x$.

Step 1: Separate and integrate The prescribed value $y(\pi/4) = 2$ is nonzero, so we separate variables on the solution interval containing the initial point:

$$\frac{dy}{y} = \cot 2x \, dx.$$

Integrating both sides yields

$$\ln |y| = \frac{1}{2} \ln |\sin 2x| + C.$$

The interval containing $\pi/4$ on which the coefficient $\cot 2x$ is continuous is $0 < x < \pi/2$. Here $\sin 2x > 0$, so exponentiating gives $y = K\sqrt{\sin 2x}$, where K is a constant.

Step 2: Apply the initial condition Substitute $x = \pi/4$ and $y = 2$ into the integrated relation:

$$2 = K\sqrt{\sin(\pi/2)} = K.$$

Thus the initial condition selects the positive branch. Squaring the relation without checking this sign would also allow a negative branch, which cannot satisfy the prescribed value.

Step 3: Verify the differential equation Differentiating the selected expression gives

$$\frac{d}{dx} \left(2\sqrt{\sin 2x} \right) = \frac{2 \cos 2x}{\sqrt{\sin 2x}} = 2\sqrt{\sin 2x} \cot 2x.$$

This is exactly $y \cot 2x$, and substitution of $x = \pi/4$ gives the required value 2.

Step 4: Conclusion The particular solution is $y = 2\sqrt{\sin 2x}$, on the interval $0 < x < \pi/2$ containing the initial point.

Quick Tip to Solve: Retain the factor $\frac{1}{2}$ when integrating $\cot 2x$, and use $y(\pi/4) = 2$ to select the positive square root.

29(b). Find the particular solution of the differential equation $(xe^{y/x} + y) dx = x dy$, given that $y = 1$ when $x = 1$. [3 marks]

Solution:

Concept:

- An equation of the form $dy/dx = F(y/x)$ is homogeneous. Put $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$.
- After separation, integrate using $\int e^{-v} dv = -e^{-v}$ and $\int dx/x = \ln|x|$. The initial condition determines the arbitrary constant.

Step 1: Use the homogeneous substitution Divide the printed equation by $x dx$, with $x \neq 0$:

$$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}.$$

The right side depends only on y/x . Set $y = vx$, substitute its derivative, and cancel v :

$$v + x \frac{dv}{dx} = e^v + v \implies e^{-v} dv = \frac{dx}{x}.$$

Step 2: Integrate and determine the constant Integration gives $-e^{-v} = \ln|x| + C$, or equivalently

$$e^{-y/x} + \ln|x| = C_1.$$

At $x = 1$, $y = 1$, so

$$C_1 = e^{-1} + \ln 1 = \frac{1}{e}.$$

Since the initial point has positive x , the solution interval lies in $x > 0$. Solving the relation for y uses the positive quantity $e^{-1} - \ln x$, so the corresponding interval is $0 < x < e^{1/e}$.

Step 3: Check the integrated relation Differentiate $e^{-y/x} + \ln x = e^{-1}$ implicitly:

$$-e^{-y/x} \frac{xy' - y}{x^2} + \frac{1}{x} = 0.$$

Hence $xy' - y = xe^{y/x}$, which rearranges to the given differential equation. At $x = 1$, the relation gives $e^{-y} = e^{-1}$, and therefore $y = 1$, confirming the initial condition.

Step 4: Conclusion The particular solution is $\boxed{e^{-y/x} + \ln x = \frac{1}{e}}$. Equivalently, $y = -x \ln(e^{-1} - \ln x)$, for $0 < x < e^{1/e}$.

Quick Tip to Solve: With $y = vx$, the two v terms cancel immediately. Keep the minus sign in $\int e^{-v} dv$ when determining the constant.

30. Solve the following linear programming problem graphically:

Maximise $Z = 2x + 3y$

subject to the constraints:

$$x + y \leq 6$$

$$x \geq 2$$

$$y \leq 3$$

$$x, y \geq 0$$

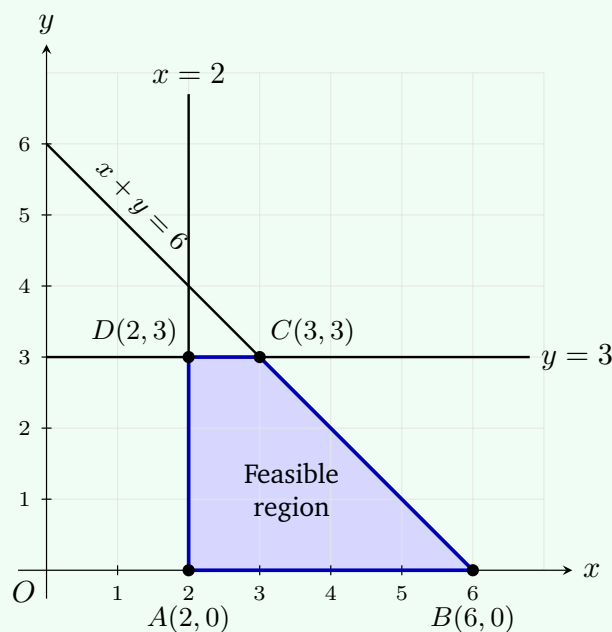
[3 marks]

Solution:

Concept:

- The feasible region is the common part of all half-planes specified by the constraints.
- A linear objective function on a nonempty bounded polygon attains its maximum at a corner point; evaluate $Z = 2x + 3y$ at every vertex.

Step 1: Draw the boundary lines and shade their intersection Draw $x + y = 6$ through $(0, 6)$ and $(6, 0)$, selecting the side containing the origin. Draw $x = 2$ and select its right side; draw $y = 3$ and select the side below it. Restrict to $y \geq 0$; $x \geq 0$ already follows from $x \geq 2$. Their common region is the shaded quadrilateral $ABCD$. Solid boundaries indicate that equality is included.



The shaded quadrilateral is the feasible region; its vertices are labelled with coordinates.

Step 2: Find the corner points On $y = 0$, the boundary intersections are $A = (2, 0)$ and $B = (6, 0)$. On $y = 3$, solving $x + y = 6$ gives $C = (3, 3)$, while $x = 2$ gives $D = (2, 3)$. All four satisfy every constraint. The intersection $(2, 4)$ is excluded because it violates $y \leq 3$.

Step 3: Compare the objective values The feasible polygon is bounded, so the corner-point method

Quick Tip to Solve: Include $x = 2$ and $y = 3$ when forming the feasible polygon; the intercept $(0, 6)$ only helps draw the line and is not feasible.

applies directly:

Corner point	$Z = 2x + 3y$
$A(2, 0)$	4
$B(6, 0)$	12
$C(3, 3)$	15
$D(2, 3)$	13

The largest value is attained at C . Neither adjacent vertex has this value, so the optimum is unique.

Step 4: Conclusion The maximum value is $Z_{\max} = 15$, attained at $x = 3, y = 3$.

Quick Tip to Solve: Include $x = 2$ and $y = 3$ when forming the feasible polygon; the intercept $(0, 6)$ only helps draw the line and is not feasible.

31(a). A card from a well shuffled deck of 52 playing cards is lost. From the remaining cards of the pack, a card is drawn at random and is found to be a King. Find the probability of the lost card being a King. [3 marks]

Solution:

Concept:

- Bayes' theorem updates the probability of an event after related information is observed. If L means that the lost card is a King and K means that the drawn card is a King, we need $P(L | K)$.
- The alternatives L and L^c are mutually exclusive and exhaustive. Hence $P(K) = P(L)P(K | L) + P(L^c)P(K | L^c)$.

Step 1: Define the events and their prior probabilities There are four Kings and 48 other cards in a deck of 52. Since the deck is well shuffled, each card is equally likely to be lost. Therefore,

$$P(L) = \frac{4}{52}, \quad P(L^c) = \frac{48}{52}.$$

These probabilities describe the lost card before using the information about the card subsequently drawn.

Step 2: Find the conditional probabilities of drawing a King After one card is lost, exactly 51 cards remain. If the lost card was a King, three Kings remain; if it was not a King, all four Kings remain. Thus,

$$P(K | L) = \frac{3}{51}, \quad P(K | L^c) = \frac{4}{51}.$$

In particular, the observation of a King is slightly less likely when another King has already been lost.

Step 3: Apply Bayes' theorem The probability of both losing a King and then drawing a King must be

Quick Tip to Solve: Once the observed King is excluded, the lost card is equally likely to be any of the other 51 cards, of which three are Kings: $3/51 = 1/17$.

divided by the total probability of drawing a King:

$$\begin{aligned}P(L | K) &= \frac{P(L)P(K | L)}{P(L)P(K | L) + P(L^c)P(K | L^c)} \\&= \frac{\frac{4}{52} \cdot \frac{3}{51}}{\frac{4}{52} \cdot \frac{3}{51} + \frac{48}{52} \cdot \frac{4}{51}} \\&= \frac{12}{12 + 192} = \frac{12}{204} = \frac{1}{17}.\end{aligned}$$

Step 4: Conclusion Given that the card drawn from the remaining pack is a King, the probability that the lost card is also a King is $\frac{1}{17}$.

Quick Tip to Solve: Once the observed King is excluded, the lost card is equally likely to be any of the other 51 cards, of which three are Kings: $3/51 = 1/17$.

OR

31(b). A biased die is twice as likely to show an even number as an odd number. If such a die is thrown twice, find the probability distribution of the number of sixes. Also, find the mean of the distribution. [3 marks]

Solution:

Concept:

- For the biased die, assign probability k to each odd face and $2k$ to each even face; the probabilities of all six outcomes add to one.
- For two independent throws, the number X of sixes has probabilities $P(X = r) = \binom{2}{r} p^r (1 - p)^{2-r}$, where $p = P(6)$ and $r = 0, 1, 2$.
- The mean is $E(X) = \sum_r r P(X = r)$, or np for a binomial distribution with n throws.

Step 1: Find the probability of a six on one throw The odd outcomes are 1, 3, 5, each with probability k , and the even outcomes are 2, 4, 6, each with probability $2k$. Normalising these six probabilities gives

$$3k + 3(2k) = 1 \implies k = \frac{1}{9}.$$

Since six is even,

$$p = P(6) = \frac{2}{9}, \quad 1 - p = \frac{7}{9}.$$

Step 2: Calculate the probabilities of zero, one and two sixes The possible numbers of sixes are 0, 1, 2. Multiply probabilities for independent throws. Exactly one six can occur in either order, giving a factor

Quick Tip to Solve: Give each odd face weight 1 and each even face weight 2. Divide by total weight 9, then use $(q^2, 2pq, p^2)$ for zero, one and two sixes.

of two:

$$P(X = 0) = \left(\frac{7}{9}\right)^2 = \frac{49}{81},$$

$$P(X = 1) = 2 \left(\frac{2}{9}\right) \left(\frac{7}{9}\right) = \frac{28}{81},$$

$$P(X = 2) = \left(\frac{2}{9}\right)^2 = \frac{4}{81}.$$

Their sum is $(49 + 28 + 4)/81 = 1$, as required.

Step 3: Compute the mean of the distribution Weight each possible value by its probability:

$$\begin{aligned} E(X) &= 0 \cdot \frac{49}{81} + 1 \cdot \frac{28}{81} + 2 \cdot \frac{4}{81} \\ &= \frac{36}{81} = \frac{4}{9}. \end{aligned}$$

The check $np = 2(2/9) = 4/9$ agrees.

Step 4: Conclusion The probability distribution and mean are

x	0	1	2
$P(X = x)$	$\frac{49}{81}$	$\frac{28}{81}$	$\frac{4}{81}$

$E(X) = \frac{4}{9}.$

Quick Tip to Solve: Give each odd face weight 1 and each even face weight 2. Divide by total weight 9, then use $(q^2, 2pq, p^2)$ for zero, one and two sixes.

Section D (20 marks)

In the section there are 4 long answer type questions of 5 marks each.

32(a). Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, X -axis and the ordinates $x = -2$ and $x = 2$, using integration. [5 marks]

Solution:

Concept:

- Write $|x| = -x$ for $x < 0$, and $|x| = x$ for $x \geq 0$, before sketching $y = x|x|$.
- Geometric area is $\int_a^b |f(x)| dx$, where $y = f(x)$ is the curve. Split at a sign change and use $\int x^2 dx = x^3/3 + C$.

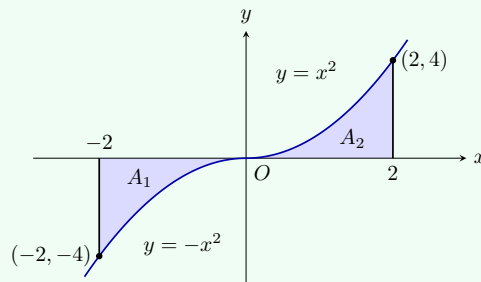
Step 1: Sketch the two branches The definition of absolute value gives

$$y = x|x| = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

The left branch is below the X -axis and the right branch is above it. Both meet at $O(0,0)$ with a

Quick Tip to Solve: For this origin-symmetric graph, double $\int_0^2 x^2 dx$; do not let the negative branch cancel the positive area.

horizontal tangent. Plot $(-2, -4)$, $(-1, -1)$, $(1, 1)$ and $(2, 4)$ to obtain the sketch. Since $f(-x) = -f(x)$, the graph has symmetry about the origin. The shaded regions are bounded by the curve, the axis and the ordinates $x = -2$ and $x = 2$.



Graph of $y = x|x|$; the two shaded regions give the required area.

Step 2: Calculate each positive area On $[-2, 0]$, subtract the lower boundary $y = -x^2$ from the upper boundary $y = 0$. On $[0, 2]$, subtract $y = 0$ from $y = x^2$. Thus,

$$A_1 = \int_{-2}^0 \{0 - (-x^2)\} dx = \left[\frac{x^3}{3} \right]_{-2}^0 = 0 - \left(-\frac{8}{3} \right) = \frac{8}{3},$$

$$A_2 = \int_0^2 (x^2 - 0) dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}.$$

The curve meets the axis only at the origin, so no further splitting of either interval is needed. Both quantities are positive geometric areas. Their equality also follows from origin symmetry. Directly integrating $x|x|$ over $[-2, 2]$ would give zero, because it subtracts the lower region instead of adding its area.

Step 3: Conclusion The graph is sketched above. The required area is

$$A = A_1 + A_2 = \frac{8}{3} + \frac{8}{3} = \frac{16}{3} \text{ square units.}$$

Quick Tip to Solve: For this origin-symmetric graph, double $\int_0^2 x^2 dx$; do not let the negative branch cancel the positive area.

OR

32(b). Using integration, find the area bounded by the ellipse $9x^2 + 25y^2 = 225$, the lines $x = -2$, $x = 2$, and the X -axis. [5 marks]

Solution:

Concept:

- If a curve lies above the X -axis, its area between $x = a$ and $x = b$ is $\int_a^b y dx$. The upper half of an ellipse gives the nonnegative ordinate used in this formula.

Quick Tip to Solve: First take the positive ordinate $y = (3/5)\sqrt{25 - x^2}$, then double the integral from 0 to 2; do not integrate signed ordinates from both halves of the ellipse.

- For an even function, $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$. Here the ellipse and the two bounding ordinates are symmetric about the Y -axis.

Step 1: Express the upper boundary as a function of x Divide the ellipse equation by 225:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

For the region above the X -axis, take

$$y = 3\sqrt{1 - \frac{x^2}{25}} = \frac{3}{5}\sqrt{25 - x^2}.$$

The ordinate is positive for $-2 \leq x \leq 2$. The corresponding bounded region below the X -axis has the same area by reflection.

Step 2: Set up the area integral using symmetry Since $\sqrt{25 - x^2}$ is even, the required area is

$$A = \frac{3}{5} \int_{-2}^2 \sqrt{25 - x^2} dx = \frac{6}{5} \int_0^2 \sqrt{25 - x^2} dx.$$

The factor 2 accounts for the left and right portions of this region.

Step 3: Evaluate the integral by trigonometric substitution Put $x = 5 \sin \theta$, so $dx = 5 \cos \theta d\theta$. The limits become 0 and $\alpha = \sin^{-1}(2/5)$, where $0 < \alpha < \pi/2$. Thus the square root is $5 \cos \theta$, and

$$\begin{aligned} A &= 30 \int_0^\alpha \cos^2 \theta d\theta \\ &= 15 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^\alpha. \end{aligned}$$

Since $\sin \alpha = 2/5$, we have $\cos \alpha = \sqrt{21}/5$ and $\sin 2\alpha = 4\sqrt{21}/25$. The identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ gives the antiderivative in the second line. The lower endpoint contributes zero. Substitute these values in the expression above, taking α in radians.

Step 4: Conclusion The required area is

$$15 \sin^{-1} \left(\frac{2}{5} \right) + \frac{6\sqrt{21}}{5} \text{ square units.}$$

Quick Tip to Solve: First take the positive ordinate $y = (3/5)\sqrt{25 - x^2}$, then double the integral from 0 to 2; do not integrate signed ordinates from both halves of the ellipse.

33(a). Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \rightarrow B$, defined by $f(x) = \frac{x-3}{x-5}$. Show that f is one-one and onto. [5 marks]

Solution:

Concept:

- A function $f : A \rightarrow B$ is one-one (injective) if $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies $x_1 = x_2$.
- It is onto (surjective) if every $y \in B$ has at least one preimage $x \in A$ with $f(x) = y$. Solve for this preimage and check that it belongs to the stated domain.

Step 1: Check that the stated codomain is valid For $x \in A$, the denominator is nonzero and $f(x)$ is real. Also,

$$f(x) = \frac{x-3}{x-5} = 1 + \frac{2}{x-5}.$$

Since $2/(x-5) \neq 0$, we have $f(x) \neq 1$; hence $f(x) \in B$.

Step 2: Prove that equal images have equal preimages Let $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Since both denominators are nonzero,

$$\begin{aligned}\frac{x_1-3}{x_1-5} &= \frac{x_2-3}{x_2-5}, \\ (x_1-3)(x_2-5) &= (x_2-3)(x_1-5), \\ x_1x_2 - 5x_1 - 3x_2 + 15 &= x_1x_2 - 5x_2 - 3x_1 + 15, \\ -2x_1 + 2x_2 &= 0.\end{aligned}$$

This forces $x_1 = x_2$, establishing injectivity on the entire domain.

Step 3: Construct a preimage for an arbitrary codomain element Choose any $y \in B$; thus $y \neq 1$. Solving $y = (x-3)/(x-5)$ gives

$$yx - 5y = x - 3, \quad x(y-1) = 5y-3, \quad x = \frac{5y-3}{y-1}.$$

This is real because $y-1 \neq 0$. Moreover,

$$x-5 = \frac{5y-3-5(y-1)}{y-1} = \frac{2}{y-1} \neq 0,$$

so the constructed x is in A , not merely in $\mathbb{R}$. Finally,

$$x-3 = \frac{2y}{y-1}, \quad f(x) = \frac{2y/(y-1)}{2/(y-1)} = y.$$

Thus every $y \in B$ has a preimage in A , establishing surjectivity. The exclusion of 1 from the codomain is essential: if it were included, it would have no preimage. Also, merely solving for x would not suffice without checking that this value avoids the excluded domain element.

Step 4: Conclusion Hence $f : A \rightarrow B$, defined by $f(x) = (x-3)/(x-5)$, is both one-one and onto, and therefore is a bijection.

Quick Tip to Solve: Rewrite $f(x) = 1 + 2/(x-5)$ to spot the excluded output immediately. When proving onto, always check $x-5 = 2/(y-1) \neq 0$ for your constructed preimage.

33(b). Check whether the relation S in the set of real numbers $\mathbb{R}$ defined by $S = \{(a, b) : \text{where } a - b + \sqrt{2} \text{ is an irrational number}\}$ is reflexive, symmetric or transitive. [5 marks]

Solution:

Concept:

- A relation S on $\mathbb{R}$ is reflexive if $(a, a) \in S$ for every real number a .
- It is symmetric if $(a, b) \in S$ always implies $(b, a) \in S$. It is transitive if $(a, b) \in S$ and $(b, c) \in S$ always imply $(a, c) \in S$.
- To disprove symmetry or transitivity, one counterexample is sufficient. A nonzero rational multiple of $\sqrt{2}$ is irrational, whereas 0 is rational.

Step 1: Test reflexivity for an arbitrary real number The membership condition is that $a - b + \sqrt{2}$ must be irrational. For any $a \in \mathbb{R}$, substitute $b = a$:

$$a - a + \sqrt{2} = \sqrt{2}.$$

Since $\sqrt{2}$ is irrational, $(a, a) \in S$. This calculation holds for every real a , so it verifies the universal requirement for reflexivity.

Step 2: Test symmetry using a reversed pair Choose $a = \sqrt{2}$ and $b = 0$. For the ordered pair $(\sqrt{2}, 0)$,

$$a - b + \sqrt{2} = \sqrt{2} - 0 + \sqrt{2} = 2\sqrt{2},$$

which is irrational. Thus $(\sqrt{2}, 0) \in S$. But for the reversed pair,

$$b - a + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0,$$

which is rational. Therefore $(0, \sqrt{2}) \notin S$. The required implication fails, so symmetry does not hold. The added $\sqrt{2}$ is unchanged when the pair is reversed; one must not simply negate the whole original expression.

Step 3: Test transitivity with two successive related pairs Take $a = 0$, $b = \frac{\sqrt{2}}{2}$ and $c = \sqrt{2}$. Then

$$\begin{aligned} a - b + \sqrt{2} &= 0 - \frac{\sqrt{2}}{2} + \sqrt{2} = \frac{\sqrt{2}}{2}, \\ b - c + \sqrt{2} &= \frac{\sqrt{2}}{2} - \sqrt{2} + \sqrt{2} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Both values are irrational. Hence $(a, b) \in S$ and $(b, c) \in S$, satisfying both premises of transitivity. However,

$$a - c + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0$$

is rational, so $(a, c) \notin S$. This triple disproves transitivity. In particular, two irrational membership expressions cannot be combined by assuming their sum, or their sum minus $\sqrt{2}$, remains irrational.

Step 4: Conclusion The relation S is reflexive, but neither symmetric nor transitive.

Quick Tip to Solve: For symmetry, use the pair $(\sqrt{2}, 0)$; for transitivity, use $0, \sqrt{2}/2, \sqrt{2}$, so the last membership expression cancels to zero.

34. If $A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$, find A^{-1} and hence solve the following system of equations:

$$2x + y - 3z = 13$$

$$3x + 2y + z = 4$$

$$x + 2y - z = 8$$

[5 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant left after deleting row i and column j .
- Write a linear system as $AX = B$, with X the column of unknowns. Multiplication on the left by A^{-1} gives $X = A^{-1}B$.

Step 1: Calculate the determinant of the coefficient matrix Expanding the given matrix along its first row,

$$\begin{aligned} |A| &= 2 \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 2(-2 - 2) - (-3 - 1) - 3(6 - 2) \\ &= -8 + 4 - 12 = -16. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the system has a unique solution.

Step 2: Find all cofactors and transpose their matrix Applying the alternating cofactor signs gives

$$\begin{aligned} C_{11} &= 2(-1) - 1(2) = -4, & C_{12} &= -[3(-1) - 1(1)] = 4, \\ C_{13} &= 3(2) - 2(1) = 4, & C_{21} &= -[1(-1) - (-3)(2)] = -5, \\ C_{22} &= 2(-1) - (-3)(1) = 1, & C_{23} &= -[2(2) - 1(1)] = -3, \\ C_{31} &= 1(1) - (-3)(2) = 7, & C_{32} &= -[2(1) - (-3)(3)] = -11, \\ C_{33} &= 2(2) - 1(3) = 1. \end{aligned}$$

Therefore the cofactor matrix and its transpose are

$$C = \begin{pmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{pmatrix}, \quad \text{adj } A = C^T = \begin{pmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{pmatrix}.$$

Step 3: Form the inverse and apply it to the system Dividing the adjugate by the determinant,

$$A^{-1} = \frac{1}{-16} \begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & 11 \\ -4 & 3 & -1 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by -16 ; then check the signs of z by substituting into all three original equations.

The coefficient order in each equation is x, y, z , so the given system is

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \\ 8 \end{pmatrix}.$$

Using the inverse found above,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{16} \begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & 11 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 13 \\ 4 \\ 8 \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 52 + 20 - 56 \\ -52 - 4 + 88 \\ -52 + 12 - 8 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 16 \\ 32 \\ -48 \end{pmatrix}. \end{aligned}$$

Step 4: Verify the values in all three equations The entries above give $x = 1, y = 2, z = -3$. Substitution checks every original equation:

$$\begin{aligned} 2(1) + 2 - 3(-3) &= 2 + 2 + 9 = 13, \\ 3(1) + 2(2) + (-3) &= 3 + 4 - 3 = 4, \\ 1 + 2(2) - (-3) &= 1 + 4 + 3 = 8. \end{aligned}$$

Thus all three right-hand sides are reproduced with the same ordered triple.

Step 5: Conclusion

$$A^{-1} = \frac{1}{16} \begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & 11 \\ -4 & 3 & -1 \end{pmatrix}, \quad x = 1, y = 2, z = -3.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by -16 ; then check the signs of z by substituting into all three original equations.

35(a). Find the distance between the line $\frac{x}{2} = \frac{2y-6}{4} = \frac{1-z}{-1}$ and another line parallel to it passing through the point $(4, 0, -5)$. [5 marks]

Solution:

Concept:

- A line in vector form is $\vec{r} = \vec{a} + t\vec{d}$, where $\vec{a}$ is the position vector of a point on the line and $\vec{d} \neq \vec{0}$ is its direction vector.
- The distance between parallel lines equals the perpendicular distance from any point on one line

Quick Tip to Solve: First rewrite the ratios as $x/2 = (y-3)/2 = (z-1)/1$; using $(2, 4, -1)$ directly as direction ratios gives the wrong distance.

to the other. If A and B are points on the respective lines, then $D = \frac{|\vec{AB} \times \vec{d}|}{|\vec{d}|}$.

- The cross-product formula removes the component of $\vec{AB}$ parallel to the lines, leaving the length of its perpendicular component.

Step 1: Convert the given line to its standard form Put the three printed ratios equal to t . Then

$$x = 2t, \quad 2y - 6 = 4t, \quad 1 - z = -t,$$

so $y = 3 + 2t$ and $z = 1 + t$. Thus the line is

$$\vec{r} = (0\hat{i} + 3\hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} + \hat{k}).$$

It passes through $A = (0, 3, 1)$ and has direction vector $\vec{d} = 2\hat{i} + 2\hat{j} + \hat{k}$. In particular, the last direction component is positive because both signs in $(1 - z)/(-1)$ reverse.

Step 2: Represent the parallel line and joining vector The second line passes through $B = (4, 0, -5)$ and has the same direction vector. Its vector equation is

$$\vec{r} = (4\hat{i} - 5\hat{k}) + s(2\hat{i} + 2\hat{j} + \hat{k}).$$

Therefore

$$\vec{AB} = 4\hat{i} - 3\hat{j} - 6\hat{k}, \quad |\vec{d}| = \sqrt{2^2 + 2^2 + 1^2} = 3.$$

Since translating a point along a parallel line does not change its perpendicular separation from the other line, we may use this particular point B .

Step 3: Evaluate the perpendicular component Compute the required cross product:

$$\begin{aligned} \vec{AB} \times \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & -6 \\ 2 & 2 & 1 \end{vmatrix} \\ &= 9\hat{i} - 16\hat{j} + 14\hat{k}. \end{aligned}$$

Hence its magnitude is $\sqrt{9^2 + (-16)^2 + 14^2} = \sqrt{533}$, to be divided by $|\vec{d}| = 3$.

Step 4: Check using the projection identity Independently, $|\vec{AB}|^2 = 16 + 9 + 36 = 61$ and $\vec{AB} \cdot \vec{d} = 8 - 6 - 6 = -4$. Therefore the squared length perpendicular to the direction is

$$|\vec{AB}|^2 - \frac{(\vec{AB} \cdot \vec{d})^2}{|\vec{d}|^2} = 61 - \frac{16}{9} = \frac{533}{9}.$$

This agrees with the cross-product calculation and is positive, so the parallel lines are distinct.

Step 5: Conclusion The distance between the two parallel lines is $\boxed{\frac{\sqrt{533}}{3} \text{ units}}.$

Quick Tip to Solve: First rewrite the ratios as $x/2 = (y-3)/2 = (z-1)/1$; using $(2, 4, -1)$ directly as direction ratios gives the wrong distance.

35(b). If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-7}$ are perpendicular to each other, find the value of k and hence write the vector equation of a line perpendicular to these two lines and passing through the point $(3, -4, 7)$. [5 marks]

Solution:

Concept:

- For a line $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the vector $a\hat{i} + b\hat{j} + c\hat{k}$ gives its direction.
- Two nonzero direction vectors are perpendicular when their scalar product is zero: $\vec{d}_1 \cdot \vec{d}_2 = 0$.
- The vector product $\vec{d}_1 \times \vec{d}_2$ is perpendicular to both vectors. A line through a point with position vector $\vec{p}$, in nonzero direction $\vec{n}$, has equation $\vec{r} = \vec{p} + \lambda\vec{n}$, $\lambda \in \mathbb{R}$.

Step 1: Identify the two direction vectors From the denominators of the given symmetric equations, take

$$\vec{d}_1 = -3\hat{i} + 2k\hat{j} + 2\hat{k}, \quad \vec{d}_2 = 3k\hat{i} + \hat{j} - 7\hat{k}.$$

The coordinates in the numerators specify points on the lines, while the denominators alone determine the angle between their directions. Therefore the stated perpendicularity condition applies directly to these two vectors.

Step 2: Apply the perpendicularity condition to determine the parameter Their scalar product must vanish:

$$\begin{aligned} (-3)(3k) + (2k)(1) + (2)(-7) &= 0, \\ -9k + 2k - 14 &= 0, \\ -7k - 14 &= 0, \\ k &= -2. \end{aligned}$$

Substituting this value gives the nonzero vectors

$$\vec{d}_1 = -3\hat{i} - 4\hat{j} + 2\hat{k}, \quad \vec{d}_2 = -6\hat{i} + \hat{j} - 7\hat{k}.$$

Their dot product is $18 - 4 - 14 = 0$, confirming the required right angle.

Step 3: Find a direction perpendicular to both given directions The required direction can be obtained from their vector product:

$$\begin{aligned} \vec{n} &= \vec{d}_1 \times \vec{d}_2 \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -4 & 2 \\ -6 & 1 & -7 \end{vmatrix} \\ &= (28 - 2)\hat{i} - (21 + 12)\hat{j} + (-3 - 24)\hat{k} \\ &= 26\hat{i} - 33\hat{j} - 27\hat{k}. \end{aligned}$$

This is nonzero. Either this vector or any nonzero scalar multiple of it describes the same line direction, so reversing all three signs also gives an equivalent answer.

Quick Tip to Solve: Determine k with the dot product first, substitute it in both direction vectors, and only then take their cross product for the new line.

Step 4: Use the given point and verify the direction The prescribed point $P = (3, -4, 7)$ has position vector $\vec{p} = 3\hat{i} - 4\hat{j} + 7\hat{k}$. Insert $\vec{p}$ and the computed $\vec{n}$ in $\vec{r} = \vec{p} + \lambda\vec{n}$. The parameter value $\lambda = 0$ gives exactly P . Also,

$$\vec{n} \cdot \vec{d}_1 = -78 + 132 - 54 = 0,$$

$$\vec{n} \cdot \vec{d}_2 = -156 - 33 + 189 = 0.$$

Thus the constructed line has a direction at right angles to each of the two given line directions, as required.

Step 5: Conclusion $k = -2$, and the required vector equation is

$$\vec{r} = (3\hat{i} - 4\hat{j} + 7\hat{k}) + \lambda(26\hat{i} - 33\hat{j} - 27\hat{k}), \quad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: Determine k with the dot product first, substitute it in both direction vectors, and only then take their cross product for the new line.

Section E (12 marks)

In this section, there are 3 case study based questions of 4 marks each.

36. Case Study - 1: A store has been selling calculators at ₹350 each. A market survey indicates that a reduction in price (p) of calculator increases the number of units (x) sold. The relation between the price and quantity sold is given by the demand function $p = 450 - \frac{1}{2}x$. [4 marks]

36(i). Determine the number of units (x) that should be sold to maximise the revenue $R(x) = xp(x)$. Also, verify the result. [2 marks]

Solution:

Concept:

- Revenue is the product of the number of calculators sold and the selling price per calculator: $R(x) = xp(x)$.
- An interior maximum occurs at a stationary point $R'(x) = 0$. The second derivative test confirms a maximum when $R''(x) < 0$.

Step 1: Form the revenue function The demand function gives $p(x) = 450 - \frac{x}{2}$. Therefore, the revenue as a function of the number sold is

$$R(x) = x \left(450 - \frac{x}{2} \right) = 450x - \frac{x^2}{2}.$$

Non-negative quantity and price give the meaningful domain $0 \leq x \leq 900$.

Step 2: Locate and verify the maximum Differentiating with respect to x ,

$$R'(x) = 450 - x.$$

Quick Tip to Solve: For this linear demand, write $R = 450x - x^2/2$ first; differentiating the price alone cannot locate maximum revenue.

The stationary-point equation is $450 - x = 0$, giving $x = 450$, which lies in the domain. Also,

$$R''(x) = -1 < 0.$$

Thus the stationary point is a maximum. Indeed, $R'(x) > 0$ before 450 and $R'(x) < 0$ after 450, so revenue increases and then decreases throughout the feasible domain.

Step 3: Conclusion The store should sell 450 calculators to maximise its revenue. The result is verified by $R''(450) = -1 < 0$.

Quick Tip to Solve: For this linear demand, write $R = 450x - x^2/2$ first; differentiating the price alone cannot locate maximum revenue.

36(ii). What rebate in price of calculator should the store give to maximise the revenue? [2 marks]

Solution:

Concept:

- The revenue function is $R(x) = xp(x)$, where x is the number sold and $p(x)$ is the price per calculator.
- The rebate per calculator is its original selling price minus the price that maximises revenue.

Step 1: Find the revenue-maximising quantity Using the given demand relation,

$$R(x) = x \left(450 - \frac{x}{2} \right) = 450x - \frac{x^2}{2}.$$

Hence $R'(x) = 450 - x = 0$ gives $x = 450$. Since $R''(x) = -1 < 0$, this quantity produces the maximum revenue. It is feasible because both the number sold and its corresponding price are positive.

Step 2: Calculate the required reduction in price At this quantity, the selling price per calculator must be

$$p(450) = 450 - \frac{450}{2} = 225.$$

The store originally charges ₹ 350 per calculator. Subtract the new price from that actual original price:

$$\text{Rebate} = 350 - 225 = 125.$$

The number 450 in the demand function is the price intercept, not the original selling price from which the rebate is measured.

Step 3: Conclusion The store should give a rebate of ₹ 125 per calculator, reducing the selling price to ₹ 225.

Quick Tip to Solve: After finding the optimum price of 225, subtract it from the given original price 350, not from the demand intercept 450.

37. Case Study - 2: An instructor at the astronomical centre shows three among the brightest stars in a particular constellation. Assume that the telescope is located at $O(0, 0, 0)$ and the three stars have their locations at the points D , A and V having position vectors $2\hat{i} + 3\hat{j} + 4\hat{k}$, $7\hat{i} + 5\hat{j} + 8\hat{k}$ and $-3\hat{i} + 7\hat{j} + 11\hat{k}$ respectively. [4 marks]

37(i). How far is the star V from star A ? [1 mark]

Solution:

Concept:

- The displacement from A to V is $\overrightarrow{AV} = \overrightarrow{OV} - \overrightarrow{OA}$.
- The distance is the magnitude of the displacement: $|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$.

Step 1: Find the displacement between the stars Using $A = (7, 5, 8)$ and $V = (-3, 7, 11)$, subtract corresponding coordinates:

$$\overrightarrow{AV} = (-3 - 7)\hat{i} + (7 - 5)\hat{j} + (11 - 8)\hat{k} = -10\hat{i} + 2\hat{j} + 3\hat{k}.$$

Step 2: Take the magnitude The separation is a scalar and is therefore non-negative. Square all three displacement components and add:

$$|\overrightarrow{AV}| = \sqrt{(-10)^2 + 2^2 + 3^2} = \sqrt{113}.$$

Reversing the displacement would change its signs but would leave this distance unchanged.

Step 3: Conclusion The star V is $\boxed{\sqrt{113} \text{ units}}$ from star A .

Quick Tip to Solve: Subtract the coordinates of A from those of V before squaring; distances from the telescope do not give the separation.

37(ii). Find a unit vector in the direction of $\overrightarrow{DA}$. [1 mark]

Solution:

Concept:

- The vector from D to A is $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD}$.
- A unit vector in the direction of a non-zero vector $\vec{u}$ is $\vec{u}/|\vec{u}|$; divide by the positive magnitude to retain its direction.

Step 1: Find the directed vector Subtract the coordinates of the initial point $D = (2, 3, 4)$ from those of the final point $A = (7, 5, 8)$:

$$\overrightarrow{DA} = (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} = 5\hat{i} + 2\hat{j} + 4\hat{k}.$$

Step 2: Determine its normalising factor The magnitude of this vector is

$$|\overrightarrow{DA}| = \sqrt{5^2 + 2^2 + 4^2} = \sqrt{45} = 3\sqrt{5}.$$

Dividing each component by this positive number preserves the direction from D to A and gives

Quick Tip to Solve: For $\overrightarrow{DA}$, take final minus initial, $A - D$; using $D - A$ gives the opposite direction.

magnitude $\sqrt{45}/\sqrt{45} = 1$.

Step 3: Conclusion A unit vector in the direction of $\overrightarrow{DA}$ is $\boxed{\frac{5\hat{i} + 2\hat{j} + 4\hat{k}}{3\sqrt{5}}}$.

Quick Tip to Solve: For $\overrightarrow{DA}$, take final minus initial, $A - D$; using $D - A$ gives the opposite direction.

37(iii)(a). Find the measure of $\angle VDA$.

[2 marks]

Solution:

Concept:

- The angle $\angle VDA$ has vertex D , so it is the angle between $\overrightarrow{DV}$ and $\overrightarrow{DA}$.
- For non-zero vectors $\vec{u}, \vec{v}$, $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$, where $0 \leq \theta \leq \pi$.

Step 1: Form both vectors from the vertex With $D = (2, 3, 4)$, $V = (-3, 7, 11)$ and $A = (7, 5, 8)$, the required vectors are

$$\begin{aligned}\overrightarrow{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \overrightarrow{DA} &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Both point away from D , as required for the interior angle at that vertex.

Step 2: Apply the dot-product formula Their dot product and magnitudes are

$$\begin{aligned}\overrightarrow{DV} \cdot \overrightarrow{DA} &= (-5)(5) + (4)(2) + (7)(4) = 11, \\ |\overrightarrow{DV}| &= \sqrt{25 + 16 + 49} = \sqrt{90}, \\ |\overrightarrow{DA}| &= \sqrt{25 + 4 + 16} = \sqrt{45}.\end{aligned}$$

Hence

$$\cos \angle VDA = \frac{11}{\sqrt{90}\sqrt{45}} = \frac{11}{45\sqrt{2}}.$$

The dot product is positive, so the angle is acute, which also checks the sign of the cosine.

Step 3: Conclusion $\angle VDA = \cos^{-1}\left(\frac{11}{45\sqrt{2}}\right)$.

Quick Tip to Solve: The middle letter identifies the vertex: subtract D from both V and A before taking the dot product.

37(iii)(b). What is the projection of vector $\overrightarrow{DV}$ on vector $\overrightarrow{DA}$?

[2 marks]

Solution:

Concept:

- The scalar projection of $\vec{u}$ on a non-zero vector $\vec{v}$ is $\vec{u} \cdot \hat{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$.
- The denominator is the magnitude of the vector onto which the projection is taken; here that vector is $\overrightarrow{DA}$.

Step 1: Find the relevant displacement vectors Using the given position vectors, subtract the coordinates of D from those of each endpoint:

$$\begin{aligned}\overrightarrow{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \overrightarrow{DA} &= (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} \\ &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Step 2: Resolve the first vector along the second Compute the dot product and the magnitude of the direction vector:

$$\begin{aligned}\overrightarrow{DV} \cdot \overrightarrow{DA} &= (-5)(5) + (4)(2) + (7)(4) = 11, \\ |\overrightarrow{DA}| &= \sqrt{25 + 4 + 16} = 3\sqrt{5}.\end{aligned}$$

The required projection is therefore

$$\frac{\overrightarrow{DV} \cdot \overrightarrow{DA}}{|\overrightarrow{DA}|} = \frac{11}{3\sqrt{5}}.$$

The positive sign means that the component is along $\overrightarrow{DA}$, rather than opposite to it.

Step 3: Conclusion The projection of $\overrightarrow{DV}$ on $\overrightarrow{DA}$ is $\boxed{\frac{11}{3\sqrt{5}} \text{ units}}.$

Quick Tip to Solve: For projection of DV on DA , divide their dot product by $|DA|$ only; dividing by both magnitudes gives a cosine.

38. Case Study - 3: Rohit, Jaspreet and Alia appeared for an interview for three vacancies in the same post. The probability of Rohit's selection is $\frac{1}{5}$, Jaspreet's selection is $\frac{1}{3}$ and Alia's selection is $\frac{1}{4}$. The event of selection is independent of each other.

[4 marks]

38(i). What is the probability that at least one of them is selected?

[1 mark]

Solution:

Concept:

- For an event E , its complement has probability $P(\overline{E}) = 1 - P(E)$.

Quick Tip to Solve: For "at least one", multiply the rejection probabilities and subtract from 1; adding individual selection probabilities would double-count overlaps.

- Independent selection events have independent complements. Thus the probability that nobody is selected is the product of the three rejection probabilities.

Step 1: Find the probability of no selection Let R, J, A denote the selection of Rohit, Jaspreet and Alia, respectively. Their rejection probabilities are $4/5, 2/3, 3/4$. Independence gives

$$P(\overline{R} \cap \overline{J} \cap \overline{A}) = \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{2}{5}.$$

Step 2: Use the complementary event At least one selection means one, two or three selections; the only excluded outcome is no selection. Hence

$$P(R \cup J \cup A) = 1 - P(\overline{R} \cap \overline{J} \cap \overline{A}) = 1 - \frac{2}{5}.$$

Step 3: Conclusion The probability that at least one of them is selected is $\boxed{\frac{3}{5}}$.

Quick Tip to Solve: For “at least one”, multiply the rejection probabilities and subtract from 1; adding individual selection probabilities would double-count overlaps.

38(ii). Find $P(G | \overline{H})$ where G is the event of Jaspreet’s selection and $\overline{H}$ denotes the event that Rohit is not selected. [1 mark]

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, provided $P(F) > 0$.
- Independence of two events also gives independence of either event and the complement of the other. Knowing Rohit’s rejection therefore does not affect Jaspreet’s selection probability.

Step 1: Identify the conditioning event Here G denotes Jaspreet’s selection and $\overline{H}$ denotes Rohit’s rejection. Thus

$$P(G) = \frac{1}{3}, \quad P(\overline{H}) = 1 - \frac{1}{5} = \frac{4}{5}.$$

Step 2: Apply independence in the conditional formula Since the selection events are independent, G and $\overline{H}$ are independent. Their joint probability is $(1/3)(4/5) = 4/15$, so

$$P(G | \overline{H}) = \frac{P(G \cap \overline{H})}{P(\overline{H})} = \frac{4/15}{4/5}.$$

Step 3: Conclusion The required conditional probability is $\boxed{P(G | \overline{H}) = \frac{1}{3}}$.

Quick Tip to Solve: When selections are independent, conditioning on another person’s selection or rejection leaves the original selection probability unchanged.

38(iii)(a). Find the probability that exactly one of them is selected.

[2 marks]

Solution:

Concept:

- For independent events, multiply probabilities to find the probability of a specified combination of selections and rejections.
- Exactly one selection is the union of three mutually exclusive cases: only Rohit, only Jaspreet, or only Alia. Add their probabilities because these cases cannot occur together.

Step 1: List the selection and rejection probabilities Let R, J, A denote the selection of Rohit, Jaspreet and Alia. The given probabilities and their complements are

$$\begin{aligned}P(R) &= \frac{1}{5}, & P(\bar{R}) &= \frac{4}{5}, \\P(J) &= \frac{1}{3}, & P(\bar{J}) &= \frac{2}{3}, \\P(A) &= \frac{1}{4}, & P(\bar{A}) &= \frac{3}{4}.\end{aligned}$$

Each case of exactly one selection must include two rejection factors.

Step 2: Calculate the three disjoint cases By independence,

$$\begin{aligned}P(R \cap \bar{J} \cap \bar{A}) &= \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{10}, \\P(\bar{R} \cap J \cap \bar{A}) &= \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{5}, \\P(\bar{R} \cap \bar{J} \cap A) &= \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{15}.\end{aligned}$$

These three cases exhaust exactly one selection, so their probabilities are added without an overlap correction.

Step 3: Conclusion The probability of exactly one selection is

$$\frac{1}{10} + \frac{1}{5} + \frac{2}{15} = \frac{3 + 6 + 4}{30} = \boxed{\frac{13}{30}}.$$

Quick Tip to Solve: Write one selection factor and two rejection factors in each product; the three selection probabilities differ, so a single binomial formula does not apply.

38(iii)(b). Find the probability that exactly two of them are selected.

[2 marks]

Solution:

Concept:

- Independence allows multiplication of selection and rejection probabilities for a specified outcome involving the three candidates.
- Exactly two selections gives three mutually exclusive cases, according to which one person is rejected. Their probabilities are added because the cases are disjoint.

Step 1: Identify the three possible pairs Write R, J, A for the events of Rohit's, Jaspreet's and Alia's selection. Their selection probabilities are $1/5, 1/3, 1/4$, and the rejection probabilities are $4/5, 2/3, 3/4$, respectively. Exactly two selections means R, J selected with A rejected; R, A selected with J rejected; or J, A selected with R rejected. The rejection condition excludes the outcome in which all three are selected.

Step 2: Multiply within each case Using independence for each complete outcome,

$$\begin{aligned} P(R \cap J \cap \bar{A}) &= \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{20}, \\ P(R \cap \bar{J} \cap A) &= \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{30}, \\ P(\bar{R} \cap J \cap A) &= \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{15}. \end{aligned}$$

Every outcome with exactly two selected candidates belongs to precisely one of these cases, so no subtraction for overlap is needed.

Step 3: Conclusion The probability of exactly two selections is

$$\frac{1}{20} + \frac{1}{30} + \frac{1}{15} = \frac{3+2+4}{60} = \frac{3}{20}.$$

Quick Tip to Solve: For exactly two of three independent selections, each product must contain two selection probabilities and one rejection probability; cycle through the rejected person.