

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-4-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question Paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question Paper is divided into **five** Sections – Section **A, B, C, D** and **E**.
- (iii) In **Section A** – Questions no. **1 to 18** are Multiple Choice Questions (MCQs) and Questions no. **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B** – Questions no. **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C** – Questions no. **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D** – Questions no. **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In **Section E** – Questions no. **36 to 38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section **B**, **3** questions in Section **C**, **3** questions in Section **D** and **2** questions in Section **E**.
- (ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section consists of 20 multiple choice questions of 1 mark each. $20 \times 1 = 20$

1. The lines $\frac{1-x}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{2x-3}{2p} = \frac{y}{-1} = \frac{z-4}{7}$ are perpendicular to each other for p equal to: [1 mark]

- (A) $-\frac{1}{2}$
- (B) $\frac{1}{2}$
- (C) 2
- (D) 3

Correct Answer: (C) 2

Solution:

Concept:

- A direction vector is obtained by expressing each coordinate in terms of the common parameter.
- Perpendicular lines have direction vectors with zero scalar product.

Step 1: Read the directions Put the first common ratio equal to t . Then $x = 1 - 2t$, $y = 1 + 3t$, $z = t$, so a direction vector is $(-2, 3, 1)$. For the second line, put the ratio equal to s : $x = \frac{3}{2} + ps$, $y = -s$, $z = 4 + 7s$. Its direction vector is $(p, -1, 7)$; the coefficient of x must be accounted for.

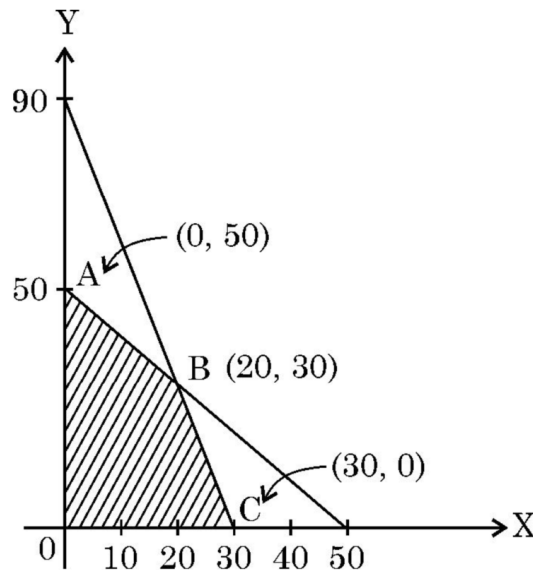
Step 2: Apply perpendicularity The scalar product is $-2p + 3(-1) + 1(7) = -2p + 4$. Thus $-2p + 4 = 0$. The options $-\frac{1}{2}, \frac{1}{2}, 2, 3$ give scalar products 5, 3, 0, -2, respectively; only the third makes the directions perpendicular.

Step 3: Conclusion $p = 2$. Hence, the correct option is (C).

Quick Tip to Solve: First isolate x in a symmetric ratio containing $2x$; its direction coefficient is p , not $2p$.

2. The maximum value of $Z = 4x + y$ for a L.P.P. whose feasible region is given below is:

[1 mark]



- (A) 50
- (B) 110
- (C) 120
- (D) 170

Correct Answer: (C) 120

Solution:

Concept:

- A linear objective on a bounded polygonal feasible region attains its maximum at a vertex.
- Evaluate $Z = 4x + y$ only at feasible corner points.

Step 1: Identify feasible vertices The shaded region in the printed graph has corner points $O(0,0)$, $A(0,50)$, $B(20,30)$ and $C(30,0)$. The intercept $(0,90)$ belongs to a constraint line, but lies outside the shaded feasible region.

Step 2: Compare objective values At these vertices, the objective values are

$$\begin{aligned} Z(O) &= 0, & Z(A) &= 50, \\ Z(B) &= 4(20) + 30 = 110, & Z(C) &= 4(30) + 0 = 120. \end{aligned}$$

The option 50 is the value at A and 110 is the value at B; neither is largest. The value 170 is not obtained as the maximum on this feasible polygon.

Step 3: Conclusion The maximum value is 120, attained at $(30,0)$. Hence, the correct option is (C).

Quick Tip to Solve: Use the shaded polygon vertices, not every intercept of the boundary lines.

3. The probability distribution of a random variable X is:

X	0	1	2	3	4
$P(X)$	0.1	k	$2k$	k	0.1

where k is some unknown constant.

The probability that the random variable X takes the value 2 is:

[1 mark]

- (A) $\frac{1}{5}$
(B) $\frac{2}{5}$
(C) $\frac{4}{5}$
(D) 1

Correct Answer: (B) $\frac{2}{5}$

Solution:

Concept:

- All probabilities in a discrete distribution sum to 1.
- The probability of a value is the entry directly beneath that value.

Step 1: Determine the constant The five mutually exclusive values exhaust the distribution, so

$$0.1 + k + 2k + k + 0.1 = 1.$$

Combining terms gives $4k = 0.8$, hence $k = 0.2 = \frac{1}{5}$. All listed probabilities are now nonnegative and their sum is one.

Step 2: Select the required entry For $X = 2$, the table gives $P(X = 2) = 2k = 0.4$. The option $\frac{1}{5}$ is k itself and belongs to $X=1$ or $X=3$; $\frac{4}{5}$ is the total of the three middle entries, while 1 is the probability of the entire sample space.

Step 3: Conclusion $P(X = 2) = \frac{2}{5}$. Hence, the correct option is (B).

Quick Tip to Solve: Normalize the entire row first, then use $2k$ rather than k for the column $X=2$.

4. If $A = [a_{ij}] = \begin{pmatrix} 2 & -1 & 5 \\ 1 & 3 & 2 \\ 5 & 0 & 4 \end{pmatrix}$ and C_{ij} is the cofactor of element a_{ij} , then the value of $a_{21} \cdot C_{11} + a_{22} \cdot C_{12} + a_{23} \cdot C_{13}$ is: [1 mark]

- (A) -57
(B) 0
(C) 9
(D) 57

Correct Answer: (B) 0

Solution:

Concept:

- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the corresponding minor.
- The scalar product of one row with the cofactors of a different row is zero.

Step 1: Calculate first-row cofactors Using the given matrix,

$$C_{11} = 3(4) - 2(0) = 12, \quad C_{12} = -(1(4) - 2(5)) = 6, \quad C_{13} = 1(0) - 3(5) = -15.$$

The minus sign for the middle cofactor is essential.

Step 2: Use the second-row entries The required expression is $1(12) + 3(6) + 2(-15) = 12 + 18 - 30$. Equivalently it is the determinant formed by replacing the first row by the second row, and therefore has two equal rows. The values -57, 9 and 57 do not equal this cross-row cofactor sum; in particular -57 is the determinant of the original matrix.

Step 3: Conclusion The required value is 0. Hence, the correct option is (B).

Quick Tip to Solve: Match row indices: row 2 dotted with row 1 cofactors is zero without expanding the determinant.

5. If $A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$ and $A^2 - kA - 5I = O$, then the value of k is: [1 mark]

- (A) 3
(B) 5
(C) 7
(D) 9

Correct Answer: (B) 5

Solution:

Concept:

- Matrix equality requires equality of corresponding entries.
- For a square matrix, multiply rows by columns to compute A^2 ; I is the identity matrix.

Quick Tip to Solve: Use an off-diagonal entry because the identity matrix contributes zero there.

Step 1: Compute the square With the printed matrix $A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$, direct multiplication gives

$$A^2 = \begin{pmatrix} 1+9 & 3+12 \\ 3+12 & 9+16 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ 15 & 25 \end{pmatrix}.$$

Step 2: Compare entries and verify The upper-right entry of $A^2 - kA - 5I = O$ gives $15 - 3k = 0$. Substituting $k = 5$, the diagonal entries are $10 - 5 - 5 = 0$ and $25 - 20 - 5 = 0$, and both off-diagonal entries vanish. The options 3, 7 and 9 leave nonzero off-diagonal entries 6, -6 and -12, respectively.

Step 3: Conclusion $k = 5$. Hence, the correct option is (B).

Quick Tip to Solve: Use an off-diagonal entry because the identity matrix contributes zero there.

6. If $e^{x^2y} = c$, then $\frac{dy}{dx}$ is:

[1 mark]

- (A) $\frac{xe^{x^2y}}{2y}$
- (B) $-\frac{2y}{x}$
- (C) $\frac{2y}{x}$
- (D) $\frac{x}{2y}$

Correct Answer: (B) $-\frac{2y}{x}$

Solution:

Concept:

- Implicit differentiation differentiates both sides with respect to x , treating y as a function of x .
- For e^u , the chain rule gives derivative $e^u u'$.

Step 1: Differentiate the equation The printed equation is $e^{x^2y} = c$, with c constant. Using the chain rule followed by the product rule,

$$e^{x^2y} \left(2xy + x^2 \frac{dy}{dx} \right) = 0.$$

The exponential is strictly positive, so its factor may be cancelled.

Step 2: Isolate the derivative On a differentiable branch with $x \neq 0$, $2xy + x^2 y' = 0$ gives $y' = -2y/x$. Option C has the opposite sign, option D reverses the ratio and changes the factor, and option A retains an exponential that should cancel. At $x=0$ the displayed derivative formula is undefined, so it describes the ordinary nonzero- x branches.

Step 3: Conclusion $\frac{dy}{dx} = -\frac{2y}{x}$, for $x \neq 0$. Hence, the correct option is (B).

Quick Tip to Solve: Cancel the nonzero exponential before isolating y prime; differentiate x squared times y as a product.

7. The value of constant c that makes the function f defined by $f(x) = \begin{cases} x^2 - c^2, & x < 4 \\ cx + 20, & x \geq 4 \end{cases}$ continuous for all real numbers is: [1 mark]

- (A) -2
 (B) -1
 (C) 0
 (D) 2

Correct Answer: (A) -2

Solution:

Concept:

- Each polynomial branch is continuous on its own interval.
- At the joining point, continuity requires the left limit, right limit and function value to agree.

Step 1: Match the branches at four For $x < 4$, the left limit is $16 - c^2$. For $x \geq 4$, the function value and right limit are both $4c + 20$. Thus continuity at the sole joining point requires $16 - c^2 = 4c + 20$.

Step 2: Solve and test the options Rearranging gives $c^2 + 4c + 4 = 0$, or $(c + 2)^2 = 0$. For $c = -2$, both branches meet at the value 12. The differences $16 - c^2 - (4c + 20)$ for $c = -1, 0$ and 2 are $-1, -4$ and -16 , respectively, so those choices fail continuity. Since the remaining points lie within polynomial branches, this condition ensures continuity everywhere.

Step 3: Conclusion $c = -2$. Hence, the correct option is (A).

Quick Tip to Solve: Check only the joining point $x=4$ and equate the two polynomial values there.

8. The value of $\int_{-1}^1 |x| dx$ is: [1 mark]

- (A) -2
 (B) -1
 (C) 1
 (D) 2

Correct Answer: (C) 1

Solution:

Concept:

- The absolute value is $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$.
- An even integrand satisfies $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

Step 1: Split where the sign changes The integrand is nonnegative on the entire interval. Using its piecewise definition,

$$\int_{-1}^1 |x| dx = \int_{-1}^0 (-x) dx + \int_0^1 x dx.$$

Treating the integrand as x throughout would incorrectly allow cancellation.

Quick Tip to Solve: Exploit evenness: twice the integral of x from zero to one gives the answer immediately.

Step 2: Evaluate both contributions The first integral is $[-x^2/2]_{-1}^0 = 1/2$, and the second is $[x^2/2]_0^1 = 1/2$. Their sum is therefore positive. The negative options -2 and -1 cannot be areas under a nonnegative function; the value 2 would double the actual two triangular areas once again.

Step 3: Conclusion $\int_{-1}^1 |x| dx = 1$. Hence, the correct option is (C).

Quick Tip to Solve: Exploit evenness: twice the integral of x from zero to one gives the answer immediately.

9. The number of arbitrary constants in the particular solution of the differential equation $\log \left(\frac{dy}{dx} \right) = 3x + 4y$; $y(0) = 0$ is/are: [1 mark]

- (A) 2
- (B) 1
- (C) 0
- (D) 3

Correct Answer: (C) 0

Solution:

Concept:

- A general solution of a first-order differential equation contains one arbitrary constant.
- A particular solution is obtained after the given initial condition fixes the arbitrary constant.

Step 1: Identify the solution requested The equation contains only the first derivative, so its general solution contains one arbitrary constant. However, the question specifically asks about the particular solution satisfying $y(0) = 0$, rather than the unrestricted general solution.

Step 2: Apply the initial condition Writing $dy/dx = e^{3x+4y}$ gives $e^{-4y} dy = e^{3x} dx$. Integration yields $-e^{-4y}/4 = e^{3x}/3 + C$. At $x = 0, y = 0$, this gives $C = -7/12$. Thus the constant is fixed, and is no longer arbitrary. Option (B) counts the general solution; (A) and (D) incorrectly introduce additional constants.

Step 3: Conclusion The particular solution contains 0 arbitrary constants. Hence, the correct option is (C).

Quick Tip to Solve: Distinguish a general solution from a particular solution: apply the initial condition before counting arbitrary constants.

10. If $\begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a scalar matrix, then the value of $a + 2b + 3c + 4d$ is:

[1 mark]

- (A) 0
- (B) 5
- (C) 10
- (D) 25

Correct Answer: (D) 25

Solution:

Concept:

- A scalar matrix has the form kI : every diagonal entry equals k , and every off-diagonal entry is zero.
- Compare corresponding entries of the given matrix with $5I_3$.

Step 1: Determine the entries The third diagonal entry is 5, so the scalar multiplier is 5. Hence $a = d = 5$. The off-diagonal entries b and c must both vanish. Merely requiring a diagonal matrix would not establish equality of its diagonal entries.

Step 2: Evaluate the expression Substitution gives

$$a + 2b + 3c + 4d = 5 + 2(0) + 3(0) + 4(5) = 25.$$

The values 0, 5, 10 in (A), (B), and (C) fail to include the complete weighted contribution $4d$.

Step 3: Conclusion The required value is 25. Hence, the correct option is (D).

Quick Tip to Solve: Use the known diagonal entry 5 to fix all diagonal entries, then set the off-diagonal entries to zero.

11. If $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, then the value of $I - A + A^2 - A^3 + \dots$ is:

[1 mark]

(A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

Solution:

Concept:

- If $A^2 = O$, then $A^n = O$ for every integer $n \geq 2$.
- The identity matrix I has diagonal entries 1 and off-diagonal entries 0.

Step 1: Calculate the square Multiply rows by columns:

$$A^2 = \begin{pmatrix} 4-4 & 2-2 \\ -8+8 & -4+4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus every higher power also vanishes. Although the expression has infinitely many terms, only its first two terms can contribute.

Step 2: Simplify the expression Therefore

$$I - A + A^2 - A^3 + \dots = I - A = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}.$$

Option (B) is $I + A$, which has the wrong sign. Option (C) discards the first two terms, while (D) discards the nonzero term $-A$.

Step 3: Conclusion The value is $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$. Hence, the correct option is (A).

Quick Tip to Solve: Calculate the square first; a zero square makes every subsequent power vanish immediately.

12. Given that $A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$, matrix A is:

[1 mark]

- (A) $7 \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(C) $\frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(D) $\frac{1}{49} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Solution:

Concept:

- For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ when the determinant is nonzero.
- For a nonzero scalar k , $(kM)^{-1} = k^{-1}M^{-1}$.

Step 1: Invert the inner matrix Put $M = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$. Its determinant is $2(2) - 1(-3) = 7$, so

$$M^{-1} = \frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

Notice that the off-diagonal signs change when forming the adjugate.

Step 2: Account for the scalar factor Since $A^{-1} = M/7$, inversion gives $A = 7M^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$.

Multiplying this matrix by the supplied A^{-1} gives I . The extra factors 7, $1/7$, $1/49$ in (A), (C), and (D) would instead give scalar multiples of I .

Step 3: Conclusion $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Invert the outside scalar as well as the matrix; the factor 7 cancels the determinant factor here.

13. The integrating factor of the differential equation $(x + 2y^2)\frac{dy}{dx} = y$ ($y > 0$) is:

[1 mark]

- (A) $\frac{1}{x}$
- (B) x
- (C) y
- (D) $\frac{1}{y}$

Correct Answer: (D) $\frac{1}{y}$

Solution:

Concept:

- A linear equation $dx/dy + P(y)x = Q(y)$ has integrating factor $e^{\int P(y) dy}$.
- Treat x as the dependent variable when the differential equation is linear in x .

Step 1: Rewrite in linear form The equation $(x + 2y^2)dy/dx = y$ becomes

$$\frac{dx}{dy} = \frac{x + 2y^2}{y}, \quad \frac{dx}{dy} - \frac{x}{y} = 2y.$$

Thus it is linear in x as a function of y , and its coefficient is $P(y) = -1/y$.

Step 2: Find the integrating factor Because $y > 0$,

$$\text{I.F.} = e^{\int (-1/y) dy} = e^{-\log y} = \frac{1}{y}.$$

Option (C) would result from reversing the sign of P . Options (A) and (B) use x , whereas the independent variable in the linear form is y .

Step 3: Conclusion The integrating factor is $1/y$. Hence, the correct option is (D).

Quick Tip to Solve: When the equation is not linear in y , invert the derivative and check whether it is linear in x .

14. A vector perpendicular to the line $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$ is:

[1 mark]

- (A) $5\hat{i} + \hat{j} + 6\hat{k}$
- (B) $\hat{i} + 3\hat{j} + 5\hat{k}$
- (C) $2\hat{i} - 2\hat{j}$
- (D) $9\hat{i} - 3\hat{j}$

Correct Answer: (B) $\hat{i} + 3\hat{j} + 5\hat{k}$

Solution:

Concept:

- In $\vec{r} = \vec{r}_0 + \lambda\vec{d}$, the vector $\vec{d}$ gives the direction of the line.
- A nonzero vector $\vec{v}$ is perpendicular to this direction exactly when $\vec{v} \cdot \vec{d} = 0$.

Quick Tip to Solve: Extract the coefficient of the line parameter and test dot products; ignore the position vector.

Step 1: Identify the direction The direction vector is $\vec{d} = 3\hat{i} - \hat{j}$, with components $(3, -1, 0)$. The vector $\hat{i} + \hat{j} - \hat{k}$ locates a point on the line and does not determine its direction. For $\vec{v} = p\hat{i} + q\hat{j} + r\hat{k}$, the perpendicularity condition is $3p - q = 0$.

Step 2: Test all the options The dot products for (A), (B), (C), and (D) are respectively

$$3(5) - 1 = 14, \quad 3(1) - 3 = 0, \quad 3(2) - (-2) = 8, \quad 3(9) - (-3) = 30.$$

Only (B) has zero dot product. Its $\hat{k}$ component makes no contribution because the line direction has zero $\hat{k}$ component.

Step 3: Conclusion A perpendicular vector is $\hat{i} + 3\hat{j} + 5\hat{k}$. Hence, the correct option is (B).

Quick Tip to Solve: Extract the coefficient of the line parameter and test dot products; ignore the position vector.

15. The vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = -3\hat{i} + 4\hat{j} + 4\hat{k}$ represent the sides of: [1 mark]

- (A) an equilateral triangle
- (B) an obtuse-angled triangle
- (C) an isosceles triangle
- (D) a right-angled triangle

Correct Answer: (D) a right-angled triangle

Solution:

Concept:

- Three directed side vectors close a triangle when their sum is zero.
- A triangle is right-angled when the sum of the squares of its two shorter sides equals the square of its longest side.

Step 1: Check closure and lengths The sum is $\vec{a} + \vec{b} + \vec{c} = (2 + 1 - 3, -1 - 3 + 4, 1 - 5 + 4) = \vec{0}$. The squared side lengths are

$$|\vec{a}|^2 = 4 + 1 + 1 = 6, \quad |\vec{b}|^2 = 1 + 9 + 25 = 35, \quad |\vec{c}|^2 = 9 + 16 + 16 = 41.$$

They are all different, excluding equilateral and isosceles triangles.

Step 2: Classify the angle The longest side has squared length 41, and $6 + 35 = 41$. Thus Pythagoras gives a right angle, excluding the obtuse alternative. Directly, $\vec{a} \cdot \vec{b} = 2 + 3 - 5 = 0$, which also confirms perpendicular side directions.

Step 3: Conclusion The vectors represent the sides of a right-angled triangle. Hence, the correct option is (D).

Quick Tip to Solve: Compare squared side lengths directly; avoid square roots when applying Pythagoras to vectors.

16. Let $\vec{a}$ be any vector such that $|\vec{a}| = a$. The value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$ is:

[1 mark]

- (A) a^2
- (B) $2a^2$
- (C) $3a^2$
- (D) 0

Correct Answer: (B) $2a^2$

Solution:

Concept:

- For $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, its squared magnitude is $a^2 = x^2 + y^2 + z^2$.
- The magnitude of a cross product with a coordinate unit vector depends on the two components perpendicular to that axis.

Step 1: Find the three squared magnitudes Writing $\vec{a} = (x, y, z)$ gives $\vec{a} \times \hat{i} = (0, z, -y)$, $\vec{a} \times \hat{j} = (-z, 0, x)$, and $\vec{a} \times \hat{k} = (y, -x, 0)$. Their squared magnitudes are therefore $y^2 + z^2$, $z^2 + x^2$, and $x^2 + y^2$, respectively.

Step 2: Add the expressions Each component square occurs twice, so the sum is

$$2x^2 + 2y^2 + 2z^2 = 2a^2.$$

Option (A) counts each component once; (C) wrongly assumes every cross product has magnitude a . Option (D) is not true for a general nonzero vector, even though all expressions vanish when $a = 0$.

Step 3: Conclusion The required value is $2a^2$. Hence, the correct option is (B).

Quick Tip to Solve: Expand into Cartesian components and count each squared component: it appears in two cross products.

17. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$, then the angle between $2\vec{a}$ and $-\vec{b}$ is:

[1 mark]

- (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{3}$
- (C) $\frac{5\pi}{6}$
- (D) $\frac{11\pi}{6}$

Correct Answer: (C) $\frac{5\pi}{6}$

Solution:

Concept:

- For nonzero vectors, $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$, where $0 \leq \theta \leq \pi$.
- A positive scalar preserves direction; a negative scalar reverses it.

Quick Tip to Solve: Reversing one vector replaces an angle θ by $\pi - \theta$; positive scaling does not change it.

Step 1: Apply the dot product formula For $\vec{u} = 2\vec{a}$ and $\vec{v} = -\vec{b}$, the dot product is $-2\sqrt{3}$, while the magnitudes are 2 and 2. Hence

$$\cos \theta = \frac{-2\sqrt{3}}{2 \times 2} = -\frac{\sqrt{3}}{2}.$$

Step 2: Check the printed choices In the permitted angle range, this cosine corresponds to $5\pi/6$. Choices (A) and (B) have positive cosine, whereas (D) lies outside $[0, \pi]$ and also has positive cosine. Thus only (C) meets both conditions.

Step 3: Conclusion The angle is $5\pi/6$. Hence, the correct option is (C).

Quick Tip to Solve: Reversing one vector replaces an angle θ by $\pi - \theta$; positive scaling does not change it.

18. The function $f(x) = kx - \sin x$ is strictly increasing for:

[1 mark]

- (A) $k > 1$
- (B) $k < 1$
- (C) $k > -1$
- (D) $k < -1$

Correct Answer: (A) $k > 1$

Solution:

Concept:

- A sufficient condition for strict increase on the real line is $f'(x) > 0$ at every point.
- The bound $-1 \leq \cos x \leq 1$ controls the smallest possible derivative.

Step 1: Differentiate and use the bound

$$f'(x) = k - \cos x \geq k - 1.$$

If $k > 1$, this lower bound is positive, so the function is strictly increasing for every real x .

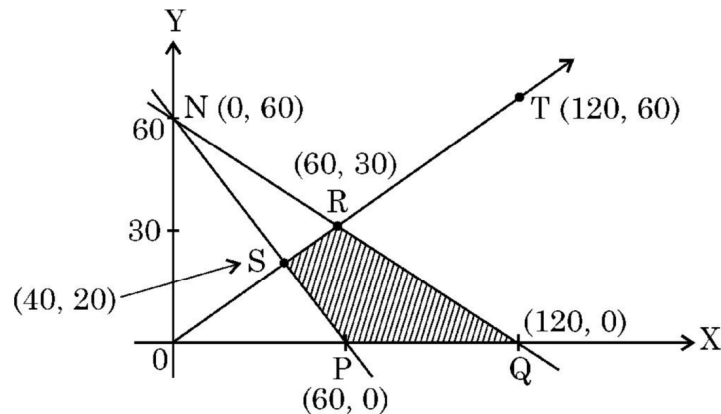
Step 2: Check the other conditions Each of (B), (C) and (D) allows a value of $k < 1$, for which $f'(0) = k - 1 < 0$, so it cannot ensure increase on the whole real line. For example, $k = 0$ satisfies (B) and (C), while $k = -2$ satisfies (D). At the boundary $k = 1$, the function is also strictly increasing because its derivative is nonnegative with only isolated zeros; that boundary is absent from the offered conditions.

Step 3: Conclusion Among the printed conditions, $k > 1$ guarantees strict increase. Hence, the correct option is (A).

Quick Tip to Solve: For a derivative involving $k - \cos x$, use the largest cosine value to find its smallest value.

19. Assertion (A): The corner points of the bounded feasible region of a L.P.P. are shown below. The maximum value of $Z = x + 2y$ occurs at infinite points.

Reason (R): The optimal solution of a LPP having bounded feasible region must occur at corner points. [1 mark]



- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
 (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
 (C) Assertion (A) is true, but Reason (R) is false.
 (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- For a linear objective on a bounded polygonal feasible region, an optimum is attained at a vertex.
- If adjacent vertices share the optimal value, every point of their connecting edge shares it.

Step 1: Judge the assertion The vertices shown are $S(40, 20)$, $R(60, 30)$, $P(60, 0)$, $Q(120, 0)$. Their objective values are

$$Z(S) = 80, \quad Z(R) = 120, \quad Z(P) = 60, \quad Z(Q) = 120.$$

Thus the maximum is 120. Every point on the feasible edge RQ has $x + 2y = 120$, giving infinitely many optimal points. Assertion (A) is true.

Step 2: Judge the reason The corner-point theorem guarantees that a bounded feasible region has an optimal corner point when optimizing a linear objective. This is the standard meaning of the stated reason, so Reason (R) is true.

Step 3: Decide whether the reason explains the assertion The existence of an optimal corner point alone does not imply infinitely many optimizers. Here infinitude follows from two adjacent optimal vertices and linearity along their connecting edge, information the reason does not state.

Step 4: Conclusion Both statements are true, but the reason does not explain the assertion. Hence, the correct option is (B).

Quick Tip to Solve: Equal optimal values at adjacent vertices mean their entire connecting feasible edge is optimal.

20. Assertion (A): The relation $R = \{(x, y) : (x + y) \text{ is a prime number and } x, y \in \mathbb{N}\}$ is not a reflexive relation.

Reason (R): The number ' $2n$ ' is composite for all natural numbers n .

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- Reflexivity on $\mathbb{N}$ requires $(n, n) \in R$ for every natural number n .
- A prime number has exactly two positive divisors; 2 is prime.

Step 1: Judge the assertion For a diagonal pair (n, n) , the required sum is $2n$. Taking $n = 2$ gives 4, which is not prime, so $(2, 2) \notin R$. A single missing diagonal pair disproves reflexivity. Therefore Assertion (A) is true.

Step 2: Judge the reason The universal statement fails at $n = 1$: $2n = 2$, which is prime rather than composite. Although $2n$ is composite for every integer $n > 1$, it is not composite for all natural numbers. Reason (R) is false.

Step 3: Select the assertion-reason code Because the reason is false, it cannot explain the true assertion. Options (A) and (B) incorrectly declare the reason true, while option (D) incorrectly declares the assertion false.

Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: To disprove reflexivity find one missing diagonal pair; to test a universal claim check $n = 1$ first.

Section B (10 marks)

In this section there are 5 very short answer type questions of 2 marks each.

21. The volume of a cube is increasing at the rate of $6 \text{ cm}^3/\text{s}$. How fast is the surface area of cube increasing, when the length of an edge is 8 cm ?

[2 marks]

Solution:

Concept:

- For a cube of edge a , volume $V = a^3$ and total surface area $S = 6a^2$.
- Related rates follow by differentiating with respect to time t , since the edge length changes with time.

Step 1: Find the rate of change of the edge The given volume rate is $dV/dt = 6 \text{ cm}^3/\text{s}$. Differentiating

Quick Tip to Solve: Eliminate the edge rate directly: for a cube, $dS/dt = (4/a)(dV/dt)$.

the volume relation gives

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}.$$

At $a = 8$ cm,

$$6 = 3(8)^2 \frac{da}{dt} \implies \frac{da}{dt} = \frac{1}{32} \text{ cm/s}.$$

The positive sign agrees with an expanding cube.

Step 2: Differentiate the total surface area All six faces contribute to $S = 6a^2$. Thus

$$\frac{dS}{dt} = 12a \frac{da}{dt} = 12(8) \left(\frac{1}{32} \right) = 3 \text{ cm}^2/\text{s}.$$

Equivalently, dividing the two rate equations gives $dS/dt = (4/a)(dV/dt)$, which yields the same value and confirms the substitution.

Step 3: Conclusion The surface area is increasing at $\boxed{3 \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Eliminate the edge rate directly: for a cube, $dS/dt = (4/a)(dV/dt)$.

22(a). Express $\tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the simplest form.

[2 marks]

Solution:

Concept:

- The half-angle identity gives $\tan(\pi/4 + x/2) = (1 + \sin x)/\cos x$.
- The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$; an inverse cancellation requires the angle to lie there.

Step 1: Rewrite the fraction On the given interval, $\cos x > 0$ and $1 - \sin x > 0$, so all divisions below are valid. Rationalizing the denominator gives

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x(1 + \sin x)}{1 - \sin^2 x} = \frac{1 + \sin x}{\cos x} = \tan \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

The last equality is the usual tangent half-angle identity.

Step 2: Check the principal branch before cancellation The stipulated interval implies

$$-\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{4} \implies 0 < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}.$$

Therefore this angle belongs to the principal range of inverse tangent. Applying $\tan^{-1}$ to its tangent returns the angle itself, with no addition or subtraction of π .

Step 3: Conclusion The simplest form is $\boxed{\frac{\pi}{4} + \frac{x}{2}}$, for $-\pi/2 < x < \pi/2$.

Quick Tip to Solve: After converting a ratio into a tangent, check the resulting angle against the principal inverse-tangent range.

22(b). Find the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

[2 marks]

Solution:**Concept:**

- The principal ranges of $\tan^{-1} u$, $\cos^{-1} u$ and $\sin^{-1} u$ are respectively $(-\pi/2, \pi/2)$, $[0, \pi]$ and $[-\pi/2, \pi/2]$.
- Choose each inverse-function value in its own principal range before adding the angles.

Step 1: Evaluate the positive argument Since $\tan(\pi/4) = 1$ and $\pi/4$ belongs to the principal range of inverse tangent,

$$\tan^{-1}(1) = \frac{\pi}{4}.$$

Other angles having tangent 1 are excluded by that range.

Step 2: Evaluate the negative arguments Cosine is negative in the second quadrant, so the angle in $[0, \pi]$ with cosine $-1/2$ is $2\pi/3$. Sine is negative at $-\pi/4$, which lies in $[-\pi/2, \pi/2]$. Thus

$$\cos^{-1}(-1/2) = \frac{2\pi}{3}, \quad \sin^{-1}(-1/\sqrt{2}) = -\frac{\pi}{4}.$$

Step 3: Add the principal values The inverse tangent and inverse sine contributions cancel. Therefore the requested sum is

$$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{4} = \frac{2\pi}{3}.$$

The question is a sum of inverse-function values, so no further inverse trigonometric operation is applied.

Step 4: Conclusion The required principal-value sum is $\boxed{2\pi/3}$.

Quick Tip to Solve: For negative arguments, check the principal range of each inverse function before choosing its angle.

23. Show that $f(x) = \frac{4 \sin x}{2 + \cos x} - x$ is an increasing function of x in $\left[0, \frac{\pi}{2}\right]$.

[2 marks]

Solution:**Concept:**

- A continuous function with positive derivative throughout the interior of an interval is strictly increasing on that interval.
- Use the quotient rule and $\sin^2 x + \cos^2 x = 1$ to simplify the derivative.

Step 1: Differentiate the function The denominator $2 + \cos x$ is positive on the given interval, so f is

Quick Tip to Solve: After differentiating, factor the numerator as $\cos x(4 - \cos x)$ to read its sign immediately.

continuous there and differentiable in its interior. By the quotient rule,

$$\begin{aligned} f'(x) &= 4 \frac{(2 + \cos x) \cos x + \sin^2 x}{(2 + \cos x)^2} - 1 \\ &= \frac{8 \cos x + 4 - (2 + \cos x)^2}{(2 + \cos x)^2} \\ &= \frac{4 \cos x - \cos^2 x}{(2 + \cos x)^2}. \end{aligned}$$

Step 2: Determine the sign Factor the numerator to obtain

$$f'(x) = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}.$$

For $0 < x < \pi/2$, $0 < \cos x < 1$, hence both numerator factors are positive and the denominator is positive. Thus $f'(x) > 0$ in the interior. At $x = \pi/2$ the derivative is zero; this endpoint does not prevent increase on the interval. For any $a < b$ in the interval, the mean value theorem gives $f(b) - f(a) = f'(c)(b - a) > 0$ for some $a < c < b$.

Step 3: Conclusion Hence $f(x)$ is strictly increasing on $[0, \pi/2]$.

Quick Tip to Solve: After differentiating, factor the numerator as $\cos x(4 - \cos x)$ to read its sign immediately.

24(a). If $y = \cos^3(\sec^2 2t)$, find $\frac{dy}{dt}$.

[2 marks]

Solution:

Concept:

- The chain rule differentiates a composition from its outermost function to its innermost function.
- Use $d(\cos u)/du = -\sin u$ and $d(\sec v)/dv = \sec v \tan v$.

Step 1: Differentiate the outer power and cosine Put $u = \sec^2(2t)$, so $y = (\cos u)^3$. Applying the power rule followed by the cosine derivative,

$$\frac{dy}{dt} = 3 \cos^2 u (-\sin u) \frac{du}{dt}.$$

The square on the cosine comes from differentiating its cube; the argument u stays unchanged in these two factors.

Step 2: Differentiate the inner secant square Now differentiate $u = (\sec(2t))^2$. The derivative of the square contributes $2 \sec(2t)$, and differentiating $\sec(2t)$ contributes $\sec(2t) \tan(2t) \cdot 2$. Consequently,

$$\frac{du}{dt} = 4 \sec^2(2t) \tan(2t).$$

Substituting this in the preceding expression accounts for all four layers: cube, cosine, secant square and the argument $2t$. The derivative is defined wherever $\cos(2t) \neq 0$, as required for the original expression.

Quick Tip to Solve: Keep $\sec^2(2t)$ unchanged inside sine and cosine, and include the extra factor 2 from $2t$.

Step 3: Conclusion

$$\frac{dy}{dt} = -12 \cos^2(\sec^2(2t)) \sin(\sec^2(2t)) \sec^2(2t) \tan(2t).$$

Quick Tip to Solve: Keep $\sec^2(2t)$ unchanged inside sine and cosine, and include the extra factor 2 from $2t$.

OR

24(b). If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

[2 marks]

Solution:**Concept:**

- For the real logarithmic method, take $x > 0$ and use $\log(x^y) = y \log x$. Here $\log$ denotes the natural logarithm.
- The product rule gives $d(y \log x)/dx = y' \log x + y/x$, because y depends on x .

Step 1: Take logarithms and rearrange Taking natural logarithms of $x^y = e^{x-y}$ gives

$$y \log x = x - y, \quad y(1 + \log x) = x.$$

At $x = e^{-1}$, this equality would read $0 = x$, which is impossible. Thus the real solution curve has $x > 0$, $x \neq e^{-1}$, and

$$y = \frac{x}{1 + \log x}.$$

Step 2: Differentiate and substitute for y Differentiate the logarithmic relation implicitly:

$$\begin{aligned} y' \log x + \frac{y}{x} &= 1 - y', \\ y'(1 + \log x) &= 1 - \frac{y}{x}. \end{aligned}$$

Since $y/x = 1/(1 + \log x)$, the right side becomes

$$1 - \frac{1}{1 + \log x} = \frac{\log x}{1 + \log x}.$$

Dividing by $1 + \log x$, which is nonzero on the solution curve, proves the required expression.

Step 3: Conclusion Hence $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$, for $x > 0$, $x \neq e^{-1}$.

Quick Tip to Solve: First obtain $y/x = 1/(1 + \log x)$; substituting it into the implicit derivative makes the proof immediate.

25. Evaluate: $\int_{-1/2}^{1/2} \cos x \cdot \log \left(\frac{1+x}{1-x} \right) dx$

[2 marks]

Solution:

Concept:

- If $g(-x) = -g(x)$, then g is odd and its integral over $[-a, a]$ is zero, provided it is integrable.
- Cosine is even, while taking the reciprocal inside a logarithm changes the sign: $\log(1/u) = -\log u$ for $u > 0$.

Step 1: Check the integrand and its domain Let $g(x) = \cos x \log((1+x)/(1-x))$. For $-1/2 \leq x \leq 1/2$, both $1+x$ and $1-x$ are positive, so the logarithm is defined and continuous. Thus g is continuous on the whole interval and its definite integral exists.

Step 2: Establish oddness Replacing x by $-x$ gives

$$\begin{aligned} g(-x) &= \cos(-x) \log \left(\frac{1-x}{1+x} \right) \\ &= \cos x \left[-\log \left(\frac{1+x}{1-x} \right) \right] \\ &= -g(x). \end{aligned}$$

Therefore the integrand is odd. The negative and positive halves of the interval contribute equal magnitudes with opposite signs. More explicitly, substituting $x = -u$ in the negative half yields

$$\int_{-1/2}^0 g(x) dx = - \int_0^{1/2} g(u) du.$$

Adding the positive half cancels these contributions.

Step 3: Conclusion

$$\boxed{\int_{-1/2}^{1/2} \cos x \log \left(\frac{1+x}{1-x} \right) dx = 0}.$$

Quick Tip to Solve: For symmetric limits, replace x with $-x$ before attempting any antiderivative.

Section C (18 marks)

In this section there are 6 short answer type questions of 3 marks each.

26. Given that $x^y + y^x = a^b$, where a and b are positive constants, find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x . The quantity a^b is constant because both a and b are constants.

Quick Tip to Solve: Write each variable power as an exponential first; this keeps both contributions to its derivative visible.

- For positive bases, write $x^y = e^{y \ln x}$ and $y^x = e^{x \ln y}$. The chain rule differentiates the exponent as well as the exponential.

Step 1: Differentiate each variable power Work on a real differentiable branch with $x > 0$ and $y > 0$, so that both logarithms exist. Put $y' = dy/dx$. Using the product rule inside each exponent gives

$$\begin{aligned}\frac{d}{dx}(x^y) &= x^y \left(y' \ln x + \frac{y}{x} \right), \\ \frac{d}{dx}(y^x) &= y^x \left(\ln y + \frac{x}{y} y' \right).\end{aligned}$$

In particular, neither power can be differentiated by treating its exponent and its base as simultaneously constant.

Step 2: Collect the derivative terms Differentiate the given identity and use $d(a^b)/dx = 0$:

$$x^y \left(y' \ln x + \frac{y}{x} \right) + y^x \left(\ln y + \frac{x}{y} y' \right) = 0.$$

Hence

$$y' (x^y \ln x + x y^{x-1}) = - (y x^{y-1} + y^x \ln y).$$

Division is valid wherever the coefficient of y' is nonzero; at other points the implicit equation must be examined separately before claiming a finite derivative.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = -\frac{y x^{y-1} + y^x \ln y}{x^y \ln x + x y^{x-1}}}$$

This formula holds where the displayed denominator is nonzero on the positive real branch.

Quick Tip to Solve: Write each variable power as an exponential first; this keeps both contributions to its derivative visible.

27(a). Find the particular solution of the differential equation $\frac{dy}{dx} = y \cot 2x$, given that $y\left(\frac{\pi}{4}\right) = 2$. [3 marks]

Solution:

Concept:

- A separable differential equation can be integrated after putting all factors involving y with dy , and all factors involving x with dx .
- Use $\int \cot u \, du = \ln |\sin u| + C$. Substitution $u = 2x$ supplies the factor $1/2$.

Step 1: Separate and integrate The initial value $y(\pi/4) = 2$ is nonzero, so separation is valid on a neighbourhood of the initial point. From the given equation,

$$\frac{dy}{y} = \cot 2x \, dx.$$

Quick Tip to Solve: Keep the factor one-half when integrating cotangent of $2x$, and choose the square-root sign from the initial value.

Integrating both sides gives

$$\ln |y| = \frac{1}{2} \ln |\sin 2x| + C.$$

Exponentiation yields $y = K\sqrt{|\sin 2x|}$, where K is a nonzero signed constant. The zero solution lost on division by y cannot satisfy the given initial value.

Step 2: Use the initial value and check The interval containing $\pi/4$ on which the coefficient $\cot 2x$ is continuous is $0 < x < \pi/2$; there $\sin 2x > 0$. Thus $y = K\sqrt{\sin 2x}$. Substituting the condition gives

$$2 = K\sqrt{\sin(\pi/2)} = K.$$

For this value,

$$\frac{dy}{dx} = \frac{2 \cos 2x}{\sqrt{\sin 2x}}, \quad y \cot 2x = \frac{2 \cos 2x}{\sqrt{\sin 2x}}.$$

Thus the candidate satisfies both the differential equation and the initial condition; the positive square-root branch is required by $y(\pi/4) = 2$.

Step 3: Conclusion

$$y = 2\sqrt{\sin 2x}, \quad 0 < x < \frac{\pi}{2}.$$

Quick Tip to Solve: Keep the factor one-half when integrating cotangent of $2x$, and choose the square-root sign from the initial value.

OR

27(b). Find the particular solution of the differential equation $(xe^{y/x} + y) dx = x dy$, given that $y = 1$ when $x = 1$. [3 marks]

Solution:

Concept:

- A first-order homogeneous equation $dy/dx = F(y/x)$ is reduced by $v = y/x$, so that $y = vx$ and $dy/dx = v + x dv/dx$.
- After substitution, separate the variables. The initial point fixes the integration constant and the real branch of the logarithm.

Step 1: Reduce the homogeneous equation For $x \neq 0$, divide the printed equation by $x dx$:

$$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}.$$

Put $v = y/x$, so $y = vx$. Differentiating by the product rule gives $dy/dx = v + x dv/dx$. Substitution therefore yields

$$v + x \frac{dv}{dx} = e^v + v, \quad e^{-v} dv = \frac{dx}{x}.$$

The v terms cancel; this is what makes the substitution useful.

Quick Tip to Solve: After setting $y=vx$, cancel the two v terms before separating; remember that the integral of $\exp(-v)$ has a minus sign.

Step 2: Integrate and impose the condition Integrate the separated equation:

$$-e^{-v} = \ln|x| + C.$$

Equivalently, $e^{-y/x} + \ln|x| = C_1$. The initial condition $x = 1, y = 1$ gives

$$C_1 = e^{-1} + \ln 1 = e^{-1}.$$

On the connected interval through $x = 1$, we have $x > 0$, and hence

$$e^{-y/x} = e^{-1} - \ln x.$$

The right side must be positive, since an exponential is positive. Therefore $0 < x < e^{1/e}$, and taking logarithms gives $y = -x \ln(e^{-1} - \ln x)$. Differentiating the preceding implicit relation gives $e^{-y/x}(y' - y/x) = 1$, recovering the original equation and checking the sign of the integrated exponential.

Step 3: Conclusion

$$e^{-y/x} + \ln x = e^{-1}$$

$$y = -x \ln(e^{-1} - \ln x), \quad 0 < x < e^{1/e}.$$

Quick Tip to Solve: After setting $y=vx$, cancel the two v terms before separating; remember that the integral of $\exp(-v)$ has a minus sign.

28. Find: $\int \frac{2x+3}{x^2(x+3)} dx$

[3 marks]

Solution:

Concept:

- A rational function with denominator $x^2(x+3)$ requires terms $A/x + B/x^2 + C/(x+3)$; both powers of the repeated factor must be included.
- Integrate the simple fractions using $\int dx/x = \ln|x| + C$ and $\int x^{-2} dx = -x^{-1} + C$.

Step 1: Determine the partial fractions Write

$$\frac{2x+3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}.$$

Multiplication by $x^2(x+3)$ gives

$$2x+3 = Ax(x+3) + B(x+3) + Cx^2.$$

Comparing coefficients,

$$A + C = 0, \quad 3A + B = 2, \quad 3B = 3.$$

Quick Tip to Solve: For the repeated factor x squared, include both A/x and B/x^2 before comparing coefficients.

Thus $B = 1$, $A = 1/3$, and $C = -1/3$. Consequently, the integrand is $1/(3x) + 1/x^2 - 1/[3(x+3)]$. As a check, recombining the fractions gives numerator $(x(x+3) + 3(x+3) - x^2)/3 = 2x + 3$.

Step 2: Integrate term by term and verify Therefore

$$\begin{aligned}\int \frac{2x+3}{x^2(x+3)} dx &= \frac{1}{3} \int \frac{dx}{x} + \int x^{-2} dx - \frac{1}{3} \int \frac{dx}{x+3} \\ &= \frac{1}{3} \ln|x| - \frac{1}{x} - \frac{1}{3} \ln|x+3| + C.\end{aligned}$$

Differentiating this expression gives $1/(3x) + 1/x^2 - 1/[3(x+3)]$, exactly the verified partial-fraction expression. The antiderivative applies on any interval that excludes $x = 0$ and $x = -3$, where the original integrand is undefined. The integration constant may differ between such intervals.

Step 3: Conclusion

$$\int \frac{2x+3}{x^2(x+3)} dx = \frac{1}{3} \ln \left| \frac{x}{x+3} \right| - \frac{1}{x} + C.$$

Quick Tip to Solve: For the repeated factor x squared, include both A/x and B/x squared before comparing coefficients.

29(a). A card from a well shuffled deck of 52 playing cards is lost. From the remaining cards of the pack, a card is drawn at random and is found to be a King. Find the probability of the lost card being a King. [3 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: $P(A | B) = P(A)P(B | A)/P(B)$.
- Total probability gives $P(B) = P(A)P(B | A) + P(A^c)P(B | A^c)$, where A^c is the complement of A .

Step 1: Define the events and prior probabilities Let A be the event that the lost card is a King and B the event that the drawn card is a King. There are four Kings and forty-eight other cards in the original pack. Each card is equally likely to be lost, so

$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(A^c) = \frac{48}{52} = \frac{12}{13}.$$

We require the conditional probability $P(A | B)$, because a King has actually been observed in the subsequent draw.

Step 2: Find the probabilities of the observed draw If the lost card is a King, three Kings remain among fifty-one cards; if it is a non-King, four Kings remain among fifty-one cards. Hence

$$P(B | A) = \frac{3}{51}, \quad P(B | A^c) = \frac{4}{51}.$$

The denominators are fifty-one in both cases because exactly one card has been removed. Total

Quick Tip to Solve: After setting aside the observed King, the lost card is equally likely to be any of the other fifty-one cards; only three are Kings.

probability gives

$$P(B) = \frac{1}{13} \cdot \frac{3}{51} + \frac{12}{13} \cdot \frac{4}{51} = \frac{51}{13 \cdot 51} = \frac{1}{13}.$$

Step 3: Apply Bayes theorem Substituting the probabilities,

$$P(A | B) = \frac{(1/13)(3/51)}{(1/13)(3/51) + (12/13)(4/51)} = \frac{3}{3 + 48} = \frac{1}{17}.$$

Observing a King reduces the chance of a lost King: a lost King leaves only three possible Kings for the observation, whereas a lost non-King leaves four. This explains why the conditional probability is smaller than the prior probability $1/13$.

Step 4: Conclusion The required probability that the lost card is a King is $\boxed{1/17}$.

Quick Tip to Solve: After setting aside the observed King, the lost card is equally likely to be any of the other fifty-one cards; only three are Kings.

OR

29(b). A biased die is twice as likely to show an even number as an odd number. If such a die is thrown twice, find the probability distribution of the number of sixes. Also, find the mean of the distribution. [3 marks]

Solution:

Concept:

- For the stated biased die, assign probability p to each odd face and $2p$ to each even face; the six face probabilities sum to one.
- For independent throws, the number of sixes X has binomial probabilities $P(X = r) = \binom{2}{r} s^r (1 - s)^{2-r}$, where $s = P(6)$.

Step 1: Determine the probability of a six The odd faces are 1, 3, 5 and the even faces are 2, 4, 6. Each even face is twice as likely as each odd face under the stated bias. Normalising the probabilities gives

$$3p + 3(2p) = 1, \quad p = \frac{1}{9}.$$

Thus the probability of a six on a single throw is $s = 2/9$, and the probability of a result other than six is $1 - s = 7/9$. A six is one particular even face, so its probability is not the probability $2/3$ of getting any even number.

Step 2: Construct the probability distribution Let X denote the number of sixes in two independent throws. Its possible values are 0, 1, 2. No six requires both throws to be non-sixes; exactly one six can

Quick Tip to Solve: Find the probability of one particular even face first; the factor two for exactly one six counts the two possible orders.

occur in either order; two sixes require a six on each throw. Therefore

$$P(X = 0) = \left(\frac{7}{9}\right)^2 = \frac{49}{81},$$

$$P(X = 1) = 2 \left(\frac{2}{9}\right) \left(\frac{7}{9}\right) = \frac{28}{81},$$

$$P(X = 2) = \left(\frac{2}{9}\right)^2 = \frac{4}{81}.$$

The distribution can be written as

x	0	1	2
$P(X = x)$	49/81	28/81	4/81

Its probabilities sum to $(49 + 28 + 4)/81 = 1$, as required.

Step 3: Calculate the mean The mean of a discrete distribution is the sum of each value multiplied by its probability. Thus

$$E(X) = 0 \frac{49}{81} + 1 \frac{28}{81} + 2 \frac{4}{81} = \frac{36}{81} = \frac{4}{9}.$$

Equivalently, the binomial mean is $ns = 2(2/9) = 4/9$, which verifies the calculation.

Step 4: Conclusion The probabilities for $X = 0, 1, 2$ are respectively $49/81, 28/81, 4/81$, and the mean is $4/9$.

Quick Tip to Solve: Find the probability of one particular even face first; the factor two for exactly one six counts the two possible orders.

30. Solve the following L.P.P. graphically:

Maximise $Z = x + 3y$

subject to the constraints:

$$x + 2y \leq 200$$

$$x + y \leq 150$$

$$y \leq 75$$

$$x, y \geq 0$$

[3 marks]

Solution:

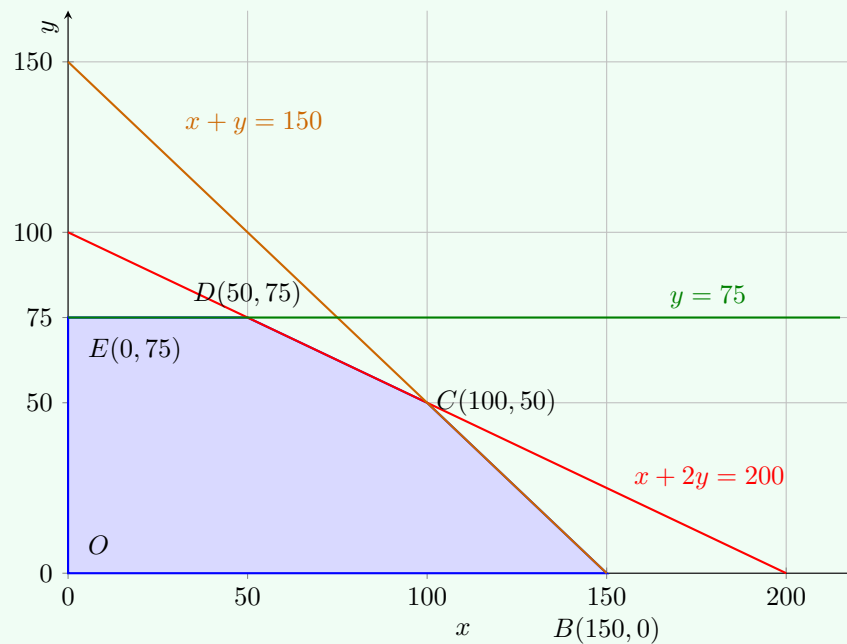
Concept:

- Each linear inequality determines a half-plane; their intersection with $x, y \geq 0$ is the feasible region.
- A linear objective function attains its maximum at a vertex of a nonempty bounded feasible polygon. Evaluate $Z = x + 3y$ at every vertex.

Step 1: Draw the boundary lines and identify the feasible side For $x + 2y = 200$, the intercepts are $(200, 0)$ and $(0, 100)$. For $x + y = 150$, they are $(150, 0)$ and $(0, 150)$. The third boundary is the

Quick Tip to Solve: Check each intersection against every inequality before including it as a corner; the intersection $(75, 75)$ lies outside the feasible region.

horizontal line $y = 75$. The origin satisfies both sloping inequalities, so choose the sides containing the origin, and choose the region below $y = 75$. Restrict the graph to the first quadrant. The shaded common region in the diagram satisfies all five inequalities.



Shaded feasible polygon OBCDE in the first quadrant

Step 2: Find the vertices of the feasible polygon On the x -axis the tighter bound is $x \leq 150$, giving $B = (150, 0)$. Solving $x + 2y = 200$ and $x + y = 150$ by subtraction gives $y = 50$, then $x = 100$, so $C = (100, 50)$. At $y = 75$, the first line gives $x = 50$, so $D = (50, 75)$; it satisfies $x + y = 125 \leq 150$. The remaining vertices are $O = (0, 0)$ and $E = (0, 75)$. The point $(75, 75)$, from $x + y = 150$ and $y = 75$, is excluded because $75 + 2(75) = 225 > 200$.

Step 3: Evaluate the objective at all feasible vertices The region is closed and bounded, so the corner-point method applies. The objective values are

Vertex	$Z = x + 3y$
$(0, 0)$	0
$(150, 0)$	150
$(100, 50)$	250
$(50, 75)$	275
$(0, 75)$	225

The greatest value in this table occurs at $(50, 75)$. This point obeys $x + 2y = 200$, $x + y = 125 \leq 150$, $y = 75$, and nonnegativity, confirming that it is feasible.

Step 4: Conclusion The maximum value is $Z_{\max} = 275$, attained at $x = 50, y = 75$.

Quick Tip to Solve: Check each intersection against every inequality before including it as a corner; the intersection $(75, 75)$ lies outside the feasible region.

31(a). Evaluate: $\int_0^{\pi/4} \frac{x \, dx}{1 + \cos 2x + \sin 2x}$

[3 marks]

Solution:

Concept:

- For a continuous function g on $[0, a]$, reflection of the interval gives $\int_0^a xg(x) \, dx = \int_0^a (a-x)g(a-x) \, dx$. Here $a = \pi/4$.
- Use $1 + \cos 2x = 2 \cos^2 x$ and $\sin 2x = 2 \sin x \cos x$. The substitution $t = \tan x$ has derivative $dt = \sec^2 x \, dx$.

Step 1: Use symmetry of the denominator Put $D(x) = 1 + \cos 2x + \sin 2x$ and denote the required integral by I . On the interval of integration, $D(x) > 0$, so the integral exists. Under $x = \pi/4 - u$, the two trigonometric terms interchange: $\cos(\pi/2 - 2u) = \sin 2u$ and $\sin(\pi/2 - 2u) = \cos 2u$. Reversing the transformed limits and renaming the variable therefore gives

$$I = \int_0^{\pi/4} \frac{(\pi/4 - x) \, dx}{D(x)}.$$

Adding this to the original expression cancels the variable numerator and yields

$$2I = \frac{\pi}{4} \int_0^{\pi/4} \frac{dx}{D(x)}.$$

Step 2: Evaluate the remaining integral Let $J = \int_0^{\pi/4} dx/D(x)$. Factor the denominator before substituting:

$$D(x) = 2 \cos x (\cos x + \sin x) = 2 \cos^2 x (1 + \tan x).$$

Thus

$$J = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 x}{1 + \tan x} \, dx.$$

With $t = \tan x$, the limits become 0 and 1. Hence

$$J = \frac{1}{2} \int_0^1 \frac{dt}{1+t} = \frac{1}{2} [\ln(1+t)]_0^1 = \frac{1}{2} \ln 2.$$

The earlier symmetry relation now determines the required value as $I = (\pi/8)J$.

Step 3: Conclusion Therefore, $\int_0^{\pi/4} \frac{x \, dx}{1 + \cos 2x + \sin 2x} = \boxed{\frac{\pi \ln 2}{16}}.$

Quick Tip to Solve: Reflect about $\pi/8$ first: the sine and cosine terms interchange, removing the numerator x when the two integrals are added.

31(b). Find: $\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right] dx$

[3 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Recognising this form avoids unnecessary integration by parts.
- The chain rule gives $\frac{d}{dx}(1+x^2)^{-1/2} = -x(1+x^2)^{-3/2}$. All the expressions here are defined for every real x , since $1+x^2 > 0$.

Step 1: Identify a function and its derivative The term $x/\sqrt{1+x^2}$ suggests choosing

$$f(x) = \frac{x}{\sqrt{1+x^2}} = x(1+x^2)^{-1/2}.$$

Differentiate by the product and chain rules:

$$\begin{aligned} f'(x) &= (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2} \\ &= \frac{1+x^2 - x^2}{(1+x^2)^{3/2}} \\ &= \frac{1}{(1+x^2)^{3/2}}. \end{aligned}$$

This is exactly the other term in the printed bracket; the exponent $3/2$ is essential for the cancellation.

Step 2: Apply and verify the product rule The given integrand is consequently $e^x[f'(x) + f(x)]$, which equals the derivative of $e^x f(x)$. Therefore its antiderivative is obtained directly from that product, with an arbitrary real constant C . As an explicit check,

$$\begin{aligned} \frac{d}{dx} \left(\frac{xe^x}{\sqrt{1+x^2}} \right) &= e^x \frac{x}{\sqrt{1+x^2}} + e^x \frac{1}{(1+x^2)^{3/2}} \\ &= e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right]. \end{aligned}$$

Thus the derivative reproduces both terms, with their positive signs, for all real x .

Step 3: Conclusion The required indefinite integral is $\boxed{\frac{xe^x}{\sqrt{1+x^2}} + C}$, where C is an arbitrary constant.

Quick Tip to Solve: Differentiate $x/\sqrt{1+x^2}$ first; if it produces the remaining bracket term, use the derivative of its product with e^x .

Section D (20 marks)

In the section there are 4 long answer type questions of 5 marks each.

32(a). Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \rightarrow B$, defined by $f(x) = \frac{x-3}{x-5}$. Show that f is one-one and onto. [5 marks]

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all inputs in its domain. It is onto if every element of its specified codomain has a preimage in its domain.
- For $f : A \rightarrow B$, well-definedness requires the denominator to be nonzero for every $x \in A$, and the value $f(x)$ to belong to B . Here $A = \mathbb{R} \setminus \{5\}$ and $B = \mathbb{R} \setminus \{1\}$.

Step 1: Check that the function maps into its codomain For each $x \in A$, we have $x \neq 5$, so $f(x) = (x-3)/(x-5)$ is a real number. Furthermore,

$$f(x) = \frac{x-5+2}{x-5} = 1 + \frac{2}{x-5}.$$

The second term is nonzero for every permitted input. Hence $f(x) \neq 1$, and thus $f(x) \in B$.

Step 2: Prove that the function is one-one Take any $x_1, x_2 \in A$ and suppose that their images are equal. Then

$$\frac{x_1-3}{x_1-5} = \frac{x_2-3}{x_2-5}.$$

Both denominators are nonzero, so cross multiplication is valid:

$$\begin{aligned}(x_1-3)(x_2-5) &= (x_2-3)(x_1-5), \\ x_1x_2 - 5x_1 - 3x_2 + 15 &= x_1x_2 - 5x_2 - 3x_1 + 15, \\ -2x_1 + 2x_2 &= 0, \\ x_1 &= x_2.\end{aligned}$$

Thus equal outputs force equal inputs, proving injectivity.

Step 3: Construct a preimage for every element of the codomain Let y be an arbitrary element of B ; in particular $y \neq 1$. Solving $y = (x-3)/(x-5)$ for the candidate preimage gives

$$\begin{aligned}y(x-5) &= x-3, \\ yx - x &= 5y - 3, \\ x &= \frac{5y-3}{y-1} = 5 + \frac{2}{y-1}.\end{aligned}$$

Since $y-1 \neq 0$, this x is real and differs from 5; hence it lies in A . Verify its image explicitly:

$$x-3 = \frac{2y}{y-1}, \quad x-5 = \frac{2}{y-1}, \quad f(x) = \frac{2y/(y-1)}{2/(y-1)} = y.$$

Quick Tip to Solve: For onto proofs, solve $y = f(x)$ and check the resulting input avoids the excluded domain value; here write $x = 5 + 2/(y-1)$.

Thus every $y \in B$ is actually attained by an allowed input, proving surjectivity.

Step 4: Conclusion The function $f : A \rightarrow B$ is both one-one and onto, and therefore is a bijection.

Quick Tip to Solve: For onto proofs, solve $y = f(x)$ and check the resulting input avoids the excluded domain value; here write $x = 5 + 2/(y - 1)$.

OR

32(b). Check whether the relation S in the set of real numbers $\mathbb{R}$ defined by $S = \{(a, b) : \text{where } a - b + \sqrt{2} \text{ is an irrational number}\}$ is reflexive, symmetric or transitive. [5 marks]

Solution:

Concept:

- A relation on $\mathbb{R}$ is reflexive if (a, a) belongs to it for every real number a .
- Symmetry requires that $(a, b) \in S$ imply $(b, a) \in S$ for every pair. One counterexample disproves this universal implication.
- Transitivity requires that $(a, b) \in S$ and $(b, c) \in S$ together imply $(a, c) \in S$. Here membership is decided by whether the specified expression is irrational.

Step 1: Test reflexivity for an arbitrary real number Let $a \in \mathbb{R}$. For the pair (a, a) , the expression in the definition becomes

$$a - a + \sqrt{2} = \sqrt{2}.$$

Since $\sqrt{2}$ is irrational, $(a, a) \in S$. This holds for every real a , so S is reflexive. No restriction on the value of a is needed because its two occurrences cancel.

Step 2: Disprove symmetry with a pair and its reverse Choose $a = \sqrt{2}$ and $b = 0$, which both belong to the underlying set $\mathbb{R}$. Then

$$a - b + \sqrt{2} = 2\sqrt{2}$$

is irrational, so $(\sqrt{2}, 0) \in S$. However, for the reversed pair,

$$b - a + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0,$$

which is rational. Thus $(0, \sqrt{2}) \notin S$, and the implication required for symmetry fails. Therefore S is not symmetric.

Step 3: Disprove transitivity with three real numbers Take $a = 0$, $b = \sqrt{2}/2$ and $c = \sqrt{2}$. The two expressions for the consecutive pairs are

$$\begin{aligned} a - b + \sqrt{2} &= 0 - \frac{\sqrt{2}}{2} + \sqrt{2} = \frac{\sqrt{2}}{2}, \\ b - c + \sqrt{2} &= \frac{\sqrt{2}}{2} - \sqrt{2} + \sqrt{2} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Both are irrational: a nonzero rational multiple of $\sqrt{2}$ is irrational. Hence $(a, b) \in S$ and $(b, c) \in S$.

Quick Tip to Solve: To disprove symmetry or transitivity, choose real numbers that make the unwanted membership expression zero; then check that the required premise expressions remain irrational.

But

$$a - c + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0$$

is rational, so $(a, c) \notin S$. The two required premises hold while the required conclusion fails. Therefore S is not transitive.

Step 4: Conclusion The relation S is reflexive, but neither symmetric nor transitive. In particular, it is not an equivalence relation.

Quick Tip to Solve: To disprove symmetry or transitivity, choose real numbers that make the unwanted membership expression zero; then check that the required premise expressions remain irrational.

33(a). Find the distance between the line $\frac{x}{2} = \frac{2y-6}{4} = \frac{1-z}{-1}$ and another line parallel to it passing through the point $(4, 0, -5)$. [5 marks]

Solution:

Concept:

- The line $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ passes through the point with position vector $\mathbf{a}$ and has nonzero direction vector $\mathbf{d}$.
- For two parallel lines with common direction $\mathbf{d}$, their distance is $D = \frac{|(\mathbf{b} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$, where $\mathbf{a}$ and $\mathbf{b}$ represent points on the respective lines.
- The cross-product magnitude is the area of the parallelogram formed by the joining vector and the direction vector. Dividing by the direction-vector length gives the perpendicular separation.

Step 1: Rewrite the given line and identify its direction Put each of the three printed ratios equal to t . Then

$$\begin{aligned}x &= 2t, \\2y - 6 &= 4t \Rightarrow y = 3 + 2t, \\1 - z &= -t \Rightarrow z = 1 + t.\end{aligned}$$

In particular, the negative denominator in the last ratio gives a positive coefficient of t in z . Thus the line is

$$\mathbf{r} = (0, 3, 1) + t(2, 2, 1).$$

We may choose $A = (0, 3, 1)$ on it and take $\mathbf{d} = (2, 2, 1)$ as its direction vector.

Step 2: Express the parallel line and the joining vector The other line passes through $B = (4, 0, -5)$ and has the same direction vector because it is parallel to the given line. Therefore its vector form is

$$\mathbf{r} = (4, 0, -5) + s(2, 2, 1).$$

The vector joining the selected points on the two lines is

$$\overrightarrow{AB} = B - A = (4, -3, -6).$$

This joining vector need not itself be perpendicular to the lines. The distance formula extracts its

Quick Tip to Solve: First set the printed ratios equal to a parameter: the expression $(1-z)/(-1)$ gives $z = 1+t$, so the third direction component is positive.

component perpendicular to their common direction.

Step 3: Calculate the perpendicular separation Compute the cross product with the common direction:

$$\begin{aligned}\vec{AB} \times \mathbf{d} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -6 \\ 2 & 2 & 1 \end{vmatrix} \\ &= 9\mathbf{i} - 16\mathbf{j} + 14\mathbf{k}.\end{aligned}$$

Its magnitude and the direction-vector magnitude are

$$|\vec{AB} \times \mathbf{d}| = \sqrt{9^2 + (-16)^2 + 14^2} = \sqrt{533}, \quad |\mathbf{d}| = \sqrt{4 + 4 + 1} = 3.$$

Hence the formula for parallel-line distance gives

$$D = \frac{\sqrt{533}}{3}.$$

As a check, the squared perpendicular component of $\vec{AB}$ is

$$|\vec{AB}|^2 - \frac{(\vec{AB} \cdot \mathbf{d})^2}{|\mathbf{d}|^2} = 61 - \frac{(-4)^2}{9} = \frac{533}{9},$$

consistent with the calculated distance.

Step 4: Conclusion The distance between the two parallel lines is $\boxed{\frac{\sqrt{533}}{3} \text{ units}}.$

Quick Tip to Solve: First set the printed ratios equal to a parameter: the expression $(1-z)/(-1)$ gives $z = 1+t$, so the third direction component is positive.

OR

33(b). If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-7}$ are perpendicular to each other, find the value of k and hence write the vector equation of a line perpendicular to these two lines and passing through the point $(3, -4, 7)$. [5 marks]

Solution:

Concept:

- The denominators in the symmetric equation of a line give its direction ratios. Two perpendicular directions have zero scalar product.
- For two nonparallel direction vectors $\vec{a}$ and $\vec{b}$, the cross product $\vec{a} \times \vec{b}$ is perpendicular to both. A line through position vector $\vec{p}$ with direction $\vec{d}$ has equation $\vec{r} = \vec{p} + t\vec{d}$, where $t \in \mathbb{R}$.

Step 1: Apply the perpendicularity condition From the printed symmetric equations, the two direction vectors are

$$\vec{a} = -3\hat{i} + 2k\hat{j} + 2\hat{k}, \quad \vec{b} = 3k\hat{i} + \hat{j} - 7\hat{k}.$$

Quick Tip to Solve: First use the dot product to determine the parameter; then cross the resulting direction vectors and attach the given point.

Their scalar product must vanish:

$$(-3)(3k) + (2k)(1) + (2)(-7) = 0.$$

Thus $-9k + 2k - 14 = 0$, so $-7k = 14$ and $k = -2$. Notice that the numerators specify points on the original lines; they do not enter the calculation of the angle between their directions.

Step 2: Find a direction perpendicular to both given directions Substituting $k = -2$ gives $\vec{a} = (-3, -4, 2)$ and $\vec{b} = (-6, 1, -7)$. Therefore

$$\begin{aligned}\vec{d} = \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -4 & 2 \\ -6 & 1 & -7 \end{vmatrix} \\ &= (28 - 2)\hat{i} - (21 + 12)\hat{j} + (-3 - 24)\hat{k} \\ &= 26\hat{i} - 33\hat{j} - 27\hat{k}.\end{aligned}$$

This vector is nonzero, so it defines a valid direction for the required line. Any nonzero scalar multiple would describe the same line.

Step 3: Form the line through the specified point and check it The position vector of $(3, -4, 7)$ is $\vec{p} = 3\hat{i} - 4\hat{j} + 7\hat{k}$. Using the point-direction form gives

$$\vec{r} = (3\hat{i} - 4\hat{j} + 7\hat{k}) + t(26\hat{i} - 33\hat{j} - 27\hat{k}), \quad t \in \mathbb{R}.$$

At $t = 0$, this line passes through the specified point. Its direction satisfies

$$\begin{aligned}\vec{d} \cdot \vec{a} &= -78 + 132 - 54 = 0, \\ \vec{d} \cdot \vec{b} &= -156 - 33 + 189 = 0.\end{aligned}$$

Both required right-angle conditions are therefore satisfied.

Step 4: Conclusion $k = -2$, and the required vector equation is

$$\vec{r} = (3\hat{i} - 4\hat{j} + 7\hat{k}) + t(26\hat{i} - 33\hat{j} - 27\hat{k}), \quad t \in \mathbb{R}.$$

Quick Tip to Solve: First use the dot product to determine the parameter; then cross the resulting direction vectors and attach the given point.

34. Use the product of matrices $\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{pmatrix}$ **to solve the following system of equations:**

$$x + 2y - 3z = 6$$

$$3x + 2y - 2z = 3$$

$$2x - y + z = 2$$

[5 marks]

Solution:

Concept:

- A system of linear equations can be written as $AX = C$, where A contains the coefficients, X contains the unknowns and C contains the constants.
- If the given matrix B satisfies $AB = \lambda I$ with $\lambda \neq 0$, then $A^{-1} = \lambda^{-1}B$. Thus $X = A^{-1}C$ gives the unique solution.

Step 1: Multiply the two given matrices Let

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{pmatrix}.$$

Multiply each row of A by each column of B :

$$\begin{aligned} AB &= \begin{pmatrix} 0 - 14 + 21 & 1 + 14 - 15 & 2 - 14 + 12 \\ 0 - 14 + 14 & 3 + 14 - 10 & 6 - 14 + 8 \\ 0 + 7 - 7 & 2 - 7 + 5 & 4 + 7 - 4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} = 7I. \end{aligned}$$

All off-diagonal entries vanish, and all diagonal entries equal 7. Hence A is invertible and its inverse is $A^{-1} = \frac{1}{7}B$. The factor $\frac{1}{7}$ must be retained because the product is $7I$, rather than I .

Step 2: Express the equations as a matrix equation Put $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$. The three printed equations become

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}, \quad \text{that is, } AX = C.$$

Multiplying both sides on the left by A^{-1} gives $X = A^{-1}C$. The order of multiplication matters: the unknown column vector is to the right of the coefficient matrix.

Step 3: Calculate the unknowns and verify the solution Using the inverse obtained from the prescribed

Quick Tip to Solve: When a supplied matrix product is a scalar multiple of the identity, divide the second matrix by that scalar before applying it to the constants column.

product,

$$\begin{aligned} X &= \frac{1}{7} \begin{pmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \\ &= \frac{1}{7} \begin{pmatrix} 0 + 3 + 4 \\ -42 + 21 - 14 \\ -42 + 15 - 8 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 7 \\ -35 \\ -35 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix}. \end{aligned}$$

Substitution into the original equations confirms each constant:

$$\begin{aligned} 1 + 2(-5) - 3(-5) &= 6, \\ 3(1) + 2(-5) - 2(-5) &= 3, \\ 2(1) - (-5) + (-5) &= 2. \end{aligned}$$

Since A has an inverse, these values give the unique solution.

Step 4: Conclusion The given product is $AB = 7I$, and the system has the solution

$$\boxed{x = 1, y = -5, z = -5}.$$

Quick Tip to Solve: When a supplied matrix product is a scalar multiple of the identity, divide the second matrix by that scalar before applying it to the constants column.

35(a). Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, X -axis and the ordinates $x = -2$ and $x = 2$, using integration. [5 marks]

Solution:

Concept:

- The absolute value satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$. Therefore the curve $y = x|x|$ has the two branches $y = -x^2$ on the negative side and $y = x^2$ on the nonnegative side.
- Geometrical area between a curve and the X -axis is the integral of the nonnegative vertical distance $|y|$. A portion below the axis contributes positive area even though its signed integral is negative.
- For a power function, $\int x^n dx = x^{n+1}/(n+1) + C$, where $n \neq -1$. Here the required antiderivative is $x^3/3$.

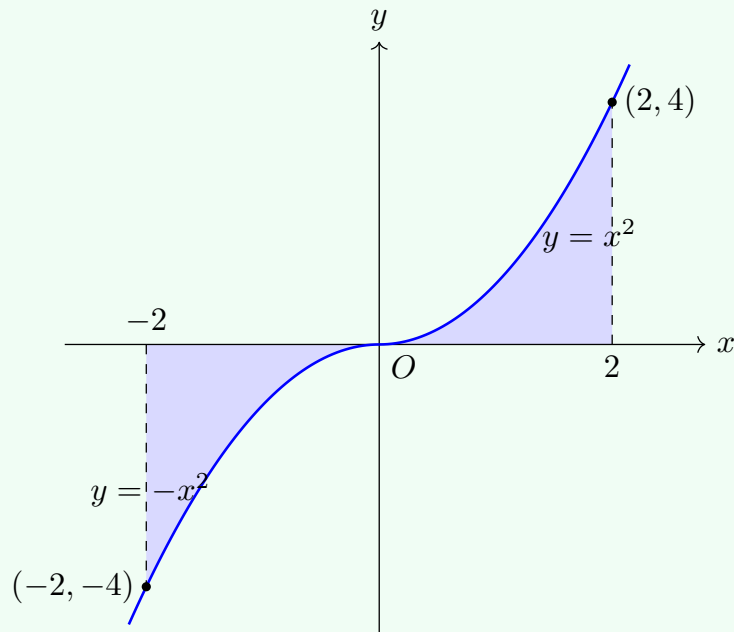
Step 1: Resolve the absolute value and sketch the two branches Write the function explicitly as

$$y = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

The left branch lies in the third quadrant and passes through $(-2, -4)$, $(-1, -1)$ and $(0, 0)$. The right branch lies in the first quadrant and passes through $(0, 0)$, $(1, 1)$ and $(2, 4)$. Both branches meet at the origin with a horizontal tangent. The graph is symmetric about the origin because $f(-x) = -f(x)$.

Quick Tip to Solve: Split at the zero of the curve, $x = 0$, before integrating for area; use $2 \int_0^2 x^2 dx$ once the equal shaded regions are identified.

Draw the ordinates at $x = -2$ and $x = 2$, and shade the two regions between the curve and the axis.



Graph of $y = x|x|$ with the two bounded regions shaded

Step 2: Find the area to the left of the origin On $[-2, 0]$, the upper boundary is the X -axis, whose ordinate is zero, and the lower boundary is $y = -x^2$. Thus the vertical distance is $0 - (-x^2) = x^2$, so

$$\begin{aligned} A_1 &= \int_{-2}^0 [0 - (-x^2)] dx \\ &= \left[\frac{x^3}{3} \right]_{-2}^0 = 0 - \left(-\frac{8}{3} \right) = \frac{8}{3}. \end{aligned}$$

Step 3: Find the area to the right of the origin and combine On $[0, 2]$, the curve $y = x^2$ lies above the axis. Hence

$$\begin{aligned} A_2 &= \int_0^2 (x^2 - 0) dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}, \\ A &= A_1 + A_2 = \frac{8}{3} + \frac{8}{3}. \end{aligned}$$

The two areas agree, as also follows from the symmetry of the sketch. Integrating $x|x|$ directly from -2 to 2 would give zero because the signed contributions cancel; that signed integral does not represent the total bounded area.

Step 4: Conclusion The required area bounded by $y = x|x|$, the X -axis and the ordinates $x = -2$ and $x = 2$ is $\boxed{\frac{16}{3} \text{ square units}}$.

Quick Tip to Solve: Split at the zero of the curve, $x = 0$, before integrating for area; use $2 \int_0^2 x^2 dx$ once the equal shaded regions are identified.

35(b). Using integration, find the area bounded by the ellipse $9x^2 + 25y^2 = 225$, the lines $x = -2$, $x = 2$, and the X -axis. [5 marks]

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$, the X -axis and ordinates $x = a, b$ is $\int_a^b f(x) dx$.
- For the ellipse $x^2/25 + y^2/9 = 1$, the upper branch is $y = (3/5)\sqrt{25 - x^2}$. The upper and lower regions have equal areas by symmetry.
- Use $x = 5 \sin \theta$ to integrate the square root; the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ gives its elementary antiderivative.

Step 1: Express the upper boundary From the printed equation,

$$25y^2 = 225 - 9x^2 = 9(25 - x^2), \quad y = \frac{3}{5}\sqrt{25 - x^2}$$

on the upper half of the ellipse. The ordinates $x = -2$ and $x = 2$ lie within its horizontal extent $[-5, 5]$. The region between this branch and the X -axis is bounded by precisely the four stated boundaries. The corresponding region below the axis has the same area.

Step 2: Set up the area integral The upper boundary is an even function of x , so the strips to the left and right of the Y -axis contribute equally. Thus

$$\mathcal{A} = \int_{-2}^2 \frac{3}{5}\sqrt{25 - x^2} dx = \frac{6}{5} \int_0^2 \sqrt{25 - x^2} dx.$$

The factor two here accounts for the two sides of the Y -axis within one region bounded by the X -axis.

Step 3: Evaluate by trigonometric substitution Put $x = 5 \sin \theta$, where $0 \leq \theta \leq \alpha = \sin^{-1}(2/5)$. Then $dx = 5 \cos \theta d\theta$ and $\sqrt{25 - x^2} = 5 \cos \theta$. Hence

$$\begin{aligned} \mathcal{A} &= 30 \int_0^\alpha \cos^2 \theta d\theta \\ &= 15 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^\alpha. \end{aligned}$$

All angles in this integral are in radians.

Step 4: Substitute the endpoint values Since $\sin \alpha = 2/5$ and $\cos \alpha = \sqrt{21}/5$, we obtain $\sin 2\alpha = 4\sqrt{21}/25$. Therefore

$$\mathcal{A} = 15 \left(\sin^{-1} \frac{2}{5} + \frac{2\sqrt{21}}{25} \right).$$

The same computation with the absolute value of the lower branch gives an equal positive area; the X -axis is a boundary of either chosen region.

Step 5: Conclusion The required area is $\boxed{15 \sin^{-1}(2/5) + \frac{6\sqrt{21}}{5}}$ square units.

Quick Tip to Solve: First isolate the upper branch and use evenness to integrate from 0 to 2; keep the inverse sine in radians.

Section E (12 marks)

In this section, there are 3 case study based questions of 4 marks each.

36. Case Study - 2: An instructor at the astronomical centre shows three among the brightest stars in a particular constellation. Assume that the telescope is located at $O(0, 0, 0)$ and the three stars have their locations at the points D , A and V having position vectors $2\hat{i} + 3\hat{j} + 4\hat{k}$, $7\hat{i} + 5\hat{j} + 8\hat{k}$ and $-3\hat{i} + 7\hat{j} + 11\hat{k}$ respectively. [4 marks]

36(i). How far is the star V from star A ? [1 mark]

Solution:

Concept:

- The displacement from A to V is $\overrightarrow{AV} = \vec{OV} - \vec{OA}$.
- The distance between the points is the magnitude $\sqrt{u_x^2 + u_y^2 + u_z^2}$ of their displacement vector.

Step 1: Form the displacement The given coordinates are $A(7, 5, 8)$ and $V(-3, 7, 11)$. Subtract the coordinates of the starting point from those of the endpoint:

$$\overrightarrow{AV} = (-3 - 7)\hat{i} + (7 - 5)\hat{j} + (11 - 8)\hat{k} = -10\hat{i} + 2\hat{j} + 3\hat{k}.$$

The telescope location is not needed for the distance between these two stars.

Step 2: Take its magnitude The three components are perpendicular Cartesian components. Therefore

$$AV = \sqrt{(-10)^2 + 2^2 + 3^2} = \sqrt{100 + 4 + 9} = \sqrt{113}.$$

Reversing the displacement changes every sign but leaves this length unchanged.

Step 3: Conclusion The star V is $\boxed{\sqrt{113}}$ units from star A .

Quick Tip to Solve: Subtract the two position vectors before taking a magnitude; the distance is not the difference of their magnitudes.

36(ii). Find a unit vector in the direction of $\overrightarrow{DA}$. [1 mark]

Solution:

Concept:

- The directed vector $\overrightarrow{DA} = \vec{OA} - \vec{OD}$ points from D towards A .
- A unit vector along a nonzero vector $\vec{u}$ is $\vec{u}/|\vec{u}|$.

Step 1: Find the directed vector Using $D(2, 3, 4)$ and $A(7, 5, 8)$,

$$\overrightarrow{DA} = (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} = 5\hat{i} + 2\hat{j} + 4\hat{k}.$$

Subtract the position vector of the starting point from that of the endpoint: endpoint minus start.

Quick Tip to Solve: For DA , calculate A minus D , then divide all three components by the same positive magnitude.

Step 2: Normalize the vector Its magnitude is

$$|\overrightarrow{DA}| = \sqrt{5^2 + 2^2 + 4^2} = \sqrt{45} = 3\sqrt{5}.$$

Divide every component by this positive magnitude. The resulting vector retains the direction towards A, and its squared magnitude is $(25 + 4 + 16)/45 = 1$, as required.

Step 3: Conclusion The required unit vector is $\boxed{\frac{5\hat{i} + 2\hat{j} + 4\hat{k}}{3\sqrt{5}}}.$

Quick Tip to Solve: For DA, calculate A minus D, then divide all three components by the same positive magnitude.

36(iii)(a). Find the measure of $\angle VDA$.

[2 marks]

Solution:

Concept:

- The angle $\angle VDA$ is the angle between $\overrightarrow{DV}$ and $\overrightarrow{DA}$, both starting at the vertex D .
- For nonzero vectors, $\cos \theta = (\vec{u} \cdot \vec{v}) / (|\vec{u}||\vec{v}|)$, with $0 \leq \theta \leq \pi$.

Step 1: Form both vectors at D The position vectors give

$$\begin{aligned}\overrightarrow{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \overrightarrow{DA} &= (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} \\ &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Using a vector towards D for just one ray would instead produce the supplementary angle.

Step 2: Compute the dot product and lengths We have

$$\overrightarrow{DV} \cdot \overrightarrow{DA} = (-5)(5) + (4)(2) + (7)(4) = 11.$$

Also,

$$|\overrightarrow{DV}| = \sqrt{25 + 16 + 49} = 3\sqrt{10}, \quad |\overrightarrow{DA}| = \sqrt{25 + 4 + 16} = 3\sqrt{5}.$$

Their product is $45\sqrt{2}$.

Step 3: Apply the angle formula Therefore

$$\cos \angle VDA = \frac{11}{45\sqrt{2}}.$$

The positive dot product confirms an acute angle. The principal inverse cosine selects the geometric angle between the two rays, so no supplementary value is required.

Step 4: Conclusion $\boxed{\angle VDA = \cos^{-1}\left(\frac{11}{45\sqrt{2}}\right)}.$

Quick Tip to Solve: Place both vectors at the middle letter D before applying the dot-product angle formula.

36(iii)(b). What is the projection of vector $\vec{DV}$ on vector $\vec{DA}$?

[2 marks]

Solution:

Concept:

- The scalar projection of $\vec{u}$ on $\vec{v}$ is $\vec{u} \cdot \hat{v} = (\vec{u} \cdot \vec{v})/|\vec{v}|$, where $\hat{v} = \vec{v}/|\vec{v}|$.
- The signed projection measures the component along the direction of the second vector; use its magnitude in the denominator.

Step 1: Find the vectors from D Using the given coordinates,

$$\begin{aligned}\vec{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \vec{DA} &= (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} \\ &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Both vectors are based at D; $\vec{DA}$ is the direction onto which the projection is taken.

Step 2: Evaluate the component along DA The dot product is

$$\vec{DV} \cdot \vec{DA} = -25 + 8 + 28 = 11,$$

while

$$|\vec{DA}| = \sqrt{25 + 4 + 16} = 3\sqrt{5}.$$

Thus $\vec{DV} \cdot \hat{DA} = 11/(3\sqrt{5})$. The positive value means that this component points towards A. If expressed as a vector component, it is $(11/45)(5\hat{i} + 2\hat{j} + 4\hat{k})$; its magnitude equals the positive scalar projection found above.

Step 3: Conclusion The projection of $\vec{DV}$ on $\vec{DA}$ is $\boxed{\frac{11}{3\sqrt{5}}}$ units.

Quick Tip to Solve: For the scalar projection of DV on DA, divide their dot product by the magnitude of DA only.

37. Case Study - 3: Rohit, Jaspreet and Alia appeared for an interview for three vacancies in the same post. The probability of Rohit's selection is $\frac{1}{5}$, Jaspreet's selection is $\frac{1}{3}$ and Alia's selection is $\frac{1}{4}$. The event of selection is independent of each other.

[4 marks]

37(i). What is the probability that at least one of them is selected?

[1 mark]

Solution:

Concept:

- For independent events, the probability of their joint occurrence is the product of their probabilities. Complements remain independent.
- Let R, J, A denote selection of Rohit, Jaspreet and Alia. Thus $P(R) = 1/5$, $P(J) = 1/3$, $P(A) = 1/4$;

Quick Tip to Solve: For at least one independent success, multiply all failure probabilities and subtract from 1.

the respective non-selection probabilities are $4/5, 2/3, 3/4$.

Step 1: Use the complementary event The complement of at least one selection is no selection at all. This includes all three candidates being rejected, so independence gives

$$P(R^c \cap J^c \cap A^c) = \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{2}{5}.$$

Step 2: Subtract from unity The events no selection and at least one selection partition all possible outcomes. Therefore

$$P(R \cup J \cup A) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Adding the three individual selection probabilities would count overlapping selections more than once.

Step 3: Conclusion The probability that at least one is selected is $\boxed{\frac{3}{5}}$.

Quick Tip to Solve: For at least one independent success, multiply all failure probabilities and subtract from 1.

37(ii). Find $P(G \mid \overline{H})$, where G is the event of Jaspreet's selection and $\overline{H}$ denotes the event that Rohit is not selected. [1 mark]

Solution:

Concept:

- Conditional probability satisfies $P(G \mid \overline{H}) = P(G \cap \overline{H}) / P(\overline{H})$, provided $P(\overline{H}) > 0$.
- Independence of selection events also makes one selection independent of the other candidate not being selected.

Step 1: Find the joint probability Here $P(G) = 1/3$ and $P(\overline{H}) = 1 - 1/5 = 4/5$. Hence

$$P(G \cap \overline{H}) = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}.$$

Alia is unrestricted in this event, so no factor for her selection or rejection is needed.

Step 2: Apply the conditional probability formula As $4/5 > 0$, the requested probability is well-defined:

$$P(G \mid \overline{H}) = \frac{4/15}{4/5} = \frac{1}{3}.$$

Thus knowing that Rohit was not selected does not alter Jaspreet's selection probability, because the events are independent.

Step 3: Conclusion $\boxed{P(G \mid \overline{H}) = \frac{1}{3}}$.

Quick Tip to Solve: When independent candidates are involved, conditioning on another candidate failing leaves the success probability unchanged.

37(iii)(a). Find the probability that exactly one of them is selected.

[2 marks]

Solution:

Concept:

- For independent events, the probability of their joint occurrence is the product of their probabilities. Complements remain independent.
- Let R, J, A denote selection of Rohit, Jaspreet and Alia. Thus $P(R) = 1/5, P(J) = 1/3, P(A) = 1/4$; the respective non-selection probabilities are $4/5, 2/3, 3/4$.

Step 1: List the mutually exclusive cases Exactly one selection means Rohit alone, Jaspreet alone, or Alia alone. These cases cannot occur together. In each case, one candidate succeeds and both others fail. Thus the event is

$$(R \cap J^c \cap A^c) \cup (R^c \cap J \cap A^c) \cup (R^c \cap J^c \cap A).$$

Step 2: Compute and add the probabilities By independence within each case and additivity between disjoint cases,

$$\begin{aligned} P(\text{exactly one}) &= \frac{1}{5} \frac{2}{3} \frac{3}{4} + \frac{4}{5} \frac{1}{3} \frac{3}{4} + \frac{4}{5} \frac{2}{3} \frac{1}{4} \\ &= \frac{1}{10} + \frac{1}{5} + \frac{2}{15} \\ &= \frac{3+6+4}{30} = \frac{13}{30}. \end{aligned}$$

The failure factors are essential: the sum of selection probabilities alone also counts cases in which two or three candidates succeed.

Step 3: Conclusion The probability of exactly one selection is $\boxed{\frac{13}{30}}$.

Quick Tip to Solve: Write one term for each possible sole success, including failure factors for both other candidates.

OR

37(iii)(b). Find the probability that exactly two of them are selected.

[2 marks]

Solution:

Concept:

- For independent events, the probability of their joint occurrence is the product of their probabilities. Complements remain independent.
- Let R, J, A denote selection of Rohit, Jaspreet and Alia. Thus $P(R) = 1/5, P(J) = 1/3, P(A) = 1/4$; the respective non-selection probabilities are $4/5, 2/3, 3/4$.

Step 1: Separate the possible selected pairs Exactly two selections can occur through Rohit and Jaspreet with Alia rejected, Rohit and Alia with Jaspreet rejected, or Jaspreet and Alia with Rohit rejected. These three events are mutually exclusive, and together exhaust the requested event. The third candidate must be rejected in every term; otherwise all-three selection would be included.

Quick Tip to Solve: For exactly two successes among three candidates, sum the three pair products with the remaining failure factor.

Step 2: Multiply within cases and add between them Independence gives

$$\begin{aligned}P(\text{exactly two}) &= \frac{1}{5} \frac{1}{3} \frac{3}{4} + \frac{1}{5} \frac{2}{3} \frac{1}{4} + \frac{4}{5} \frac{1}{3} \frac{1}{4} \\&= \frac{1}{20} + \frac{1}{30} + \frac{1}{15} \\&= \frac{3 + 2 + 4}{60} = \frac{3}{20}.\end{aligned}$$

As a check, all-three selection has probability $1/60$; together with probabilities $2/5$ for none and $13/30$ for exactly one, the four counts sum to unity.

Step 3: Conclusion The probability of exactly two selections is $\boxed{\frac{3}{20}}$.

Quick Tip to Solve: For exactly two successes among three candidates, sum the three pair products with the remaining failure factor.

38. Case Study - 1: A store has been selling calculators at ₹350 each. A market survey indicates that a reduction in price (p) of calculator increases the number of units (x) sold. The relation between the price and quantity sold is given by the demand function $p = 450 - \frac{1}{2}x$. [4 marks]

38(i). Determine the number of units (x) that should be sold to maximise the revenue $R(x) = xp(x)$. Also, verify the result. [2 marks]

Solution:

Concept:

- Revenue is quantity times price: $R(x) = xp(x)$, where x is the number of calculators sold and $p(x)$ is price per calculator.
- An interior maximum satisfies $R'(x) = 0$. The second derivative test confirms a strict local maximum when $R''(x) < 0$.

Step 1: Form revenue and find the stationary point Using the demand function,

$$R(x) = x \left(450 - \frac{x}{2} \right) = 450x - \frac{x^2}{2}.$$

Nonnegative price and quantity give $0 \leq x \leq 900$. Differentiating with respect to quantity,

$$R'(x) = 450 - x.$$

The stationary-point equation $450 - x = 0$ gives $x = 450$, an admissible integer quantity.

Step 2: Verify that revenue is maximised The second derivative is $R''(x) = -1 < 0$, so the stationary point is a maximum. In fact,

$$R(x) = 101250 - \frac{1}{2}(x - 450)^2 \leq 101250.$$

Quick Tip to Solve: For a linear demand function, first multiply price by quantity; optimise this quadratic revenue, not the price alone.

Equality holds only at $x = 450$, proving a global maximum over the feasible interval. The corresponding price is $450 - 450/2 = 225$ rupees, and revenue is $450 \times 225 = 101250$ rupees.

Step 3: Conclusion The store should sell 450 calculators. Since $R''(450) = -1 < 0$, the revenue is maximised there.

Quick Tip to Solve: For a linear demand function, first multiply price by quantity; optimise this quadratic revenue, not the price alone.

38(ii). What rebate in price of calculator should the store give to maximise the revenue? [2 marks]

Solution:

Concept:

- Revenue is quantity times price: $R(x) = xp(x)$, where x is the number of calculators sold and $p(x)$ is price per calculator.
- An interior maximum satisfies $R'(x) = 0$. The second derivative test confirms a strict local maximum when $R''(x) < 0$.

Step 1: Find the price at maximum revenue The demand function gives

$$R(x) = 450x - \frac{x^2}{2}, \quad R'(x) = 450 - x.$$

Thus maximum revenue occurs at $x = 450$, because $R''(x) = -1 < 0$. Substituting this quantity into the demand function gives

$$p(450) = 450 - \frac{1}{2}(450) = 225 \text{ rupees.}$$

This is the selling price per calculator that corresponds to maximum revenue.

Step 2: Compare with the current selling price The current selling price stated in the question is 350 rupees per calculator. Rebate means the reduction from that actual price to the optimal price, so

$$\text{Rebate} = 350 - 225 = 125 \text{ rupees.}$$

As a check, giving this rebate leaves a price of $350 - 125 = 225$, which gives demand $x = 2(450 - 225) = 450$, exactly the revenue-maximising quantity. The intercept 450 in the demand equation is not the current selling price.

Step 3: Conclusion The store should give a rebate of 125 rupees per calculator.

Quick Tip to Solve: Subtract the optimal price from the stated current price 350, rather than from the demand intercept 450.