

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question Paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question Paper is divided into **five** Sections – Section **A, B, C, D** and **E**.
- (iii) In **Section A** – Questions no. **1 to 18** are Multiple Choice Questions (MCQs) and Questions no. **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B** – Questions no. **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C** – Questions no. **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D** – Questions no. **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In **Section E** – Questions no. **36 to 38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in **Section B**, **3** questions in **Section C**, **3** questions in **Section D** and **2** questions in **Section E**.
- (ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section consists of 20 multiple choice questions of 1 mark each.

1. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$, then the angle between $2\vec{a}$ and $-\vec{b}$ is: [1 mark]

- (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{3}$
- (C) $\frac{5\pi}{6}$
- (D) $\frac{11\pi}{6}$

Correct Answer: (C) $\frac{5\pi}{6}$

Solution:

Concept:

- For non-zero vectors, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$, where $0 \leq \theta \leq \pi$ is their angle.
- Multiplication by a positive scalar preserves direction; multiplication by a negative scalar reverses direction.

Step 1: Calculate the cosine for the requested vectors Put $\vec{u} = 2\vec{a}$ and $\vec{v} = -\vec{b}$. Their dot product is $-2\vec{a} \cdot \vec{b} = -2\sqrt{3}$, while their magnitudes are $2|\vec{a}| = 2$ and $|\vec{b}| = 2$. Thus

$$\cos \theta = \frac{-2\sqrt{3}}{2 \times 2} = -\frac{\sqrt{3}}{2}.$$

Step 2: Use the angle range and check the choices The angle in $[0, \pi]$ with this cosine is $5\pi/6$. Options (A) and (B) are acute and have positive cosines. Option (D), $11\pi/6$, lies outside the angle range and also has positive cosine. Reversing $\vec{b}$ changes the original angle $\pi/6$ to its supplement.

Step 3: Conclusion The required angle is $5\pi/6$. Hence, the correct option is (C).

Quick Tip to Solve: Keep the minus sign in the dot product: reversing one vector replaces its angle by its supplement.

2. The vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = -3\hat{i} + 4\hat{j} + 4\hat{k}$ represent the sides of: [1 mark]

- (A) an equilateral triangle
- (B) an obtuse-angled triangle
- (C) an isosceles triangle
- (D) a right-angled triangle

Correct Answer: (D) a right-angled triangle

Solution:

Concept:

- Three directed side vectors of a triangle satisfy $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.
- A zero dot product means perpendicular vectors; alternatively, a triangle is right-angled if the largest squared side equals the sum of the other two squared sides.

Step 1: Check closure and compute the side lengths Adding components gives

$$\vec{a} + \vec{b} + \vec{c} = (2 + 1 - 3)\hat{i} + (-1 - 3 + 4)\hat{j} + (1 - 5 + 4)\hat{k} = \vec{0}.$$

The squared lengths are $|\vec{a}|^2 = 4 + 1 + 1 = 6$, $|\vec{b}|^2 = 1 + 9 + 25 = 35$ and $|\vec{c}|^2 = 9 + 16 + 16 = 41$.

Step 2: Classify the triangle and reject the other types Since $41 = 6 + 35$, the sides satisfy the Pythagorean relation. Also, $\vec{a} \cdot \vec{b} = 2 + 3 - 5 = 0$, confirming perpendicular side directions. The three lengths are unequal, so the triangle is neither equilateral nor isosceles. Equality in the Pythagorean relation gives a right angle, not an obtuse angle.

Step 3: Conclusion The vectors represent a right-angled triangle. Hence, the correct option is (D).

Quick Tip to Solve: Check the dot product first: here $2 + 3 - 5 = 0$ identifies perpendicular sides immediately.

3. Let $\vec{a}$ be any vector such that $|\vec{a}| = a$. The value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$ is: [1 mark]

- (A) a^2
- (B) $2a^2$
- (C) $3a^2$
- (D) 0

Correct Answer: (B) $2a^2$

Solution:

Concept:

- Write $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, so $|\vec{a}|^2 = x^2 + y^2 + z^2 = a^2$.
- The squared magnitude of a cross product with a coordinate unit vector is the sum of the squares of the other two components.

Step 1: Compute each squared cross-product magnitude Using the orthonormal coordinate unit

Quick Tip to Solve: In the three cross products, each component disappears once and occurs twice; hence the total is twice the squared magnitude.

vectors,

$$|\vec{a} \times \hat{i}|^2 = y^2 + z^2,$$

$$|\vec{a} \times \hat{j}|^2 = z^2 + x^2,$$

$$|\vec{a} \times \hat{k}|^2 = x^2 + y^2.$$

For instance, $\vec{a} \times \hat{i} = z\hat{j} - y\hat{k}$.

Step 2: Add the three expressions and compare options Each component square occurs twice. Therefore the requested sum is

$$2x^2 + 2y^2 + 2z^2 = 2(x^2 + y^2 + z^2) = 2a^2.$$

Options (A) and (C) count each component once and three times respectively. Option (D) holds only for the special zero vector and is not the identity for an arbitrary vector.

Step 3: Conclusion The value is $2a^2$. Hence, the correct option is (B).

Quick Tip to Solve: In the three cross products, each component disappears once and occurs twice; hence the total is twice the squared magnitude.

4. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and $A^2 + 7I = kA$, then the value of k is:

[1 mark]

- (A) 1
- (B) 2
- (C) 5
- (D) 7

Correct Answer: (C) 5

Solution:

Concept:

- Matrix multiplication uses row-by-column products: $(A^2)_{ij} = \sum_r A_{ir}A_{rj}$.
- The identity matrix I adds only to diagonal entries, while scalar multiplication multiplies every entry.

Step 1: Square the given matrix and add the identity term

$$A^2 = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}.$$

Hence

$$A^2 + 7I = \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix}.$$

Step 2: Compare corresponding entries The right side is $kA = \begin{pmatrix} 3k & k \\ -k & 2k \end{pmatrix}$. The upper-right entry requires $k = 5$; the other entries then give 15, -5 and 10, as required. The proposed values 1, 2 and 7

Quick Tip to Solve: After forming A^2 , compare an off-diagonal entry: $7I$ does not affect it, so it gives k directly.

fail already in that upper-right entry, so only option (C) satisfies the matrix equation.

Step 3: Conclusion $k = 5$. Hence, the correct option is (C).

Quick Tip to Solve: After forming A^2 , compare an off-diagonal entry: $7I$ does not affect it, so it gives k directly.

5. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$ and $B = \frac{1}{3} \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & \lambda \end{pmatrix}$. If $AB = I$, then the value of λ is: [1 mark]

- (A) $-\frac{9}{4}$
- (B) -2
- (C) $-\frac{3}{2}$
- (D) 0

Correct Answer: No printed option satisfies $AB = I$. Option (B), $\lambda = -2$, instead gives $AB = I/3$.

Solution:

Concept:

- For $AB = I$, every diagonal entry must be one and every off-diagonal entry must be zero.
- The printed factor $1/3$ outside B scales every entry of the product; it must be retained when checking the identity.

Step 1: Check a diagonal entry independent of the unknown The first row of A is $(1, -1, 2)$, and the first column of the unscaled matrix in B is $(-2, 9, 6)^T$. Therefore

$$(AB)_{11} = \frac{-2 - 9 + 12}{3} = \frac{1}{3}.$$

This entry is independent of λ , and it is not the required diagonal value 1. Thus no choice of λ can make the printed product equal to I .

Step 2: Check the apparent candidate against the whole product The complete product is

$$AB = \frac{1}{3} \begin{pmatrix} 1 & 0 & 4 + 2\lambda \\ 0 & 1 & -6 - 3\lambda \\ 0 & 0 & 9 + 4\lambda \end{pmatrix}.$$

Making the upper-right entry zero gives $\lambda = -2$, option (B). At this value the unscaled matrix becomes I , so the actual product is $I/3$, not I . Options (A), (C) and (D) also fail the upper-right zero condition. Hence even the apparent candidate does not satisfy the stated equation with the printed prefactor.

Step 3: Conclusion With the printed matrices, $\boxed{\text{no value of } \lambda \text{ satisfies } AB = I}$. None of the printed options is valid; option (B) only gives $AB = I/3$.

Quick Tip to Solve: Before solving for an unknown in an inverse matrix, test one diagonal product and retain every scalar prefactor.

6. Derivative of x^2 with respect to x^3 , is:

[1 mark]

- (A) $\frac{2}{3x}$
- (B) $\frac{3x}{2}$
- (C) $\frac{2x}{3}$
- (D) $6x^5$

Correct Answer: (A) $\frac{2}{3x}$

Solution:

Concept:

- For $y = y(x)$ and $u = u(x)$, $dy/du = (dy/dx)/(du/dx)$ wherever $du/dx \neq 0$.
- The power rule is $d(x^n)/dx = nx^{n-1}$. Here the independent expression is x^3 , not x .

Step 1: Differentiate both expressions with respect to x Let $y = x^2$ and $u = x^3$. Applying the power rule separately gives

$$\frac{dy}{dx} = 2x, \quad \frac{du}{dx} = 3x^2.$$

We now divide these derivatives because the question asks for the rate of change of y with respect to u .

Step 2: Form the ratio and compare the options For $x \neq 0$,

$$\frac{d(x^2)}{d(x^3)} = \frac{2x}{3x^2} = \frac{2}{3x}.$$

Option (B) reverses the ratio, option (C) cancels powers incorrectly, and option (D) multiplies expressions instead of using the chain rule. At $x = 0$ the derivative with respect to u is not finite; this does not change the required formula where it exists.

Step 3: Conclusion The derivative is $\boxed{2/(3x)}$, for $x \neq 0$. Hence, the correct option is (A).

Quick Tip to Solve: Differentiate both powers with respect to x , then divide the derivative of x^2 by that of x^3 .

7. The function $f(x) = |x| + |x - 2|$ is:

[1 mark]

- (A) continuous, but not differentiable at $x = 0$ and $x = 2$.
- (B) differentiable but not continuous at $x = 0$ and $x = 2$.
- (C) continuous but not differentiable at $x = 0$ only.
- (D) neither continuous nor differentiable at $x = 0$ and $x = 2$.

Correct Answer: (A) continuous, but not differentiable at $x = 0$ and $x = 2$.

Solution:

Concept:

- Absolute-value functions are continuous everywhere, and the sum of continuous functions is continuous.
- A function is differentiable at a point only when its left-hand and right-hand derivatives are equal.

Step 1: Write the function without absolute values The signs change at 0 and 2, giving

$$f(x) = \begin{cases} 2 - 2x, & x < 0, \\ 2, & 0 \leq x \leq 2, \\ 2x - 2, & x > 2. \end{cases}$$

At 0 and 2, the neighbouring formulae agree with the function value 2. Thus the function is continuous at both points, as also follows from continuity of absolute values.

Step 2: Compare the one-sided derivatives at each corner At $x = 0$, the left derivative is -2 and the right derivative is 0 ; they differ. At $x = 2$, the left derivative is 0 and the right derivative is 2 ; they also differ. Therefore the function is not differentiable at either point. Options (B) and (D) incorrectly deny continuity, while (C) misses the second corner at 2.

Step 3: Conclusion The function is continuous but not differentiable at $x = 0$ and $x = 2$. Hence, the correct option is (A).

Quick Tip to Solve: Split at both zeros of the absolute-value arguments, 0 and 2, and compare the slopes on their two sides.

8. The value of $\int_0^\pi \tan^2\left(\frac{\theta}{3}\right) d\theta$ is:

[1 mark]

- (A) $\pi + \sqrt{3}$
- (B) $3\sqrt{3} - \pi$
- (C) $\sqrt{3} - \pi$
- (D) $\pi - \sqrt{3}$

Correct Answer: (B) $3\sqrt{3} - \pi$

Solution:

Concept:

- The identity $\tan^2 u = \sec^2 u - 1$ converts the integrand into terms with elementary antiderivatives.

Quick Tip to Solve: Use $\tan^2 u = \sec^2 u - 1$ and remember the factor 3 from the inner angle $\theta/3$.

- Because $d \tan(\theta/3)/d\theta = (1/3) \sec^2(\theta/3)$, its integral contains the factor 3.

Step 1: Find an antiderivative with the correct scale factor Using the identity,

$$\begin{aligned}\int \tan^2(\theta/3) d\theta &= \int [\sec^2(\theta/3) - 1] d\theta \\ &= 3 \tan(\theta/3) - \theta + C.\end{aligned}$$

The angle $\theta/3$ ranges from 0 to $\pi/3$, so the integrand has no singularity on the interval.

Step 2: Apply the limits and inspect the choices

$$\begin{aligned}\int_0^\pi \tan^2(\theta/3) d\theta &= [3 \tan(\theta/3) - \theta]_0^\pi \\ &= 3 \tan(\pi/3) - \pi \\ &= 3\sqrt{3} - \pi.\end{aligned}$$

Option (A) has the wrong sign and coefficient; (C) omits the factor 3; (D) has the wrong coefficient and sign combination. Only option (B) equals the endpoint evaluation.

Step 3: Conclusion The value is $3\sqrt{3} - \pi$. Hence, the correct option is (B).

Quick Tip to Solve: Use $\tan^2 u = \sec^2 u - 1$ and remember the factor 3 from the inner angle $\theta/3$.

9. The integrating factor of the differential equation $\frac{dy}{dx} + \frac{2}{x}y = 0$, $x \neq 0$ is:

[1 mark]

- (A) $\frac{2}{x}$
- (B) x^2
- (C) $e^{2/x}$
- (D) $e^{\log(2x)}$

Correct Answer: (B) x^2

Solution:

Concept:

- For a linear equation $dy/dx + P(x)y = Q(x)$, an integrating factor is $I(x) = e^{\int P(x) dx}$.
- A nonzero constant multiple of an integrating factor works equally well, on any interval excluding a singular point.

Step 1: Identify and integrate the coefficient Here $P(x) = 2/x$, not the right-hand side, which is zero. Since $x \neq 0$, work on an interval entirely on one side of zero. Thus

$$\int P(x) dx = 2 \log |x|.$$

Step 2: Form the factor and check the options Therefore $I(x) = e^{2 \log |x|} = x^2$. Multiplication gives

Quick Tip to Solve: Integrate the coefficient of y before exponentiating; the right-hand side does not determine the integrating factor.

$x^2y' + 2xy = (x^2y)'$, confirming the factor. Option (A) merely repeats the coefficient. Option (C) exponentiates it without integration. Option (D) equals $2x$ wherever its logarithm is defined, so it is not a constant multiple of x^2 .

Step 3: Conclusion The integrating factor is x^2 . Hence, the correct option is (B).

Quick Tip to Solve: Integrate the coefficient of y before exponentiating; the right-hand side does not determine the integrating factor.

10. The lines $\frac{1-x}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{2x-3}{2p} = \frac{y}{-1} = \frac{z-4}{7}$ are perpendicular to each other for p equal to: [1 mark]

- (A) $-\frac{1}{2}$
- (B) $\frac{1}{2}$
- (C) 2
- (D) 3

Correct Answer: (C) 2

Solution:

Concept:

- Two lines are perpendicular when the dot product of their direction vectors is zero.
- Obtain direction ratios from the coefficients of a common line parameter; reverse signs and numerator factors must be handled first.

Step 1: Extract the correct direction vectors Set the common ratio of the first line equal to t . Then $x = 1 - 2t$, $y = 1 + 3t$, and $z = t$, giving $(-2, 3, 1)$. For the second line, put the common ratio equal to s . From $2x - 3 = 2ps$, we obtain $x = 3/2 + ps$, with $y = -s$, $z = 4 + 7s$. Its direction vector is therefore $(p, -1, 7)$.

Step 2: Apply perpendicularity and test the choices The required condition is

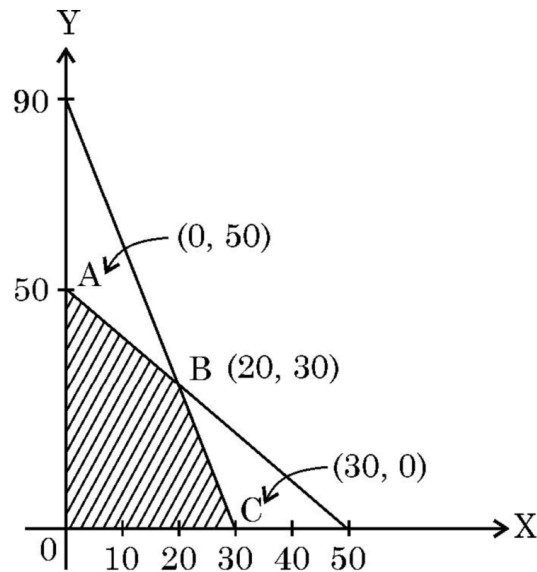
$$(-2)p + 3(-1) + 1(7) = 0, \quad -2p + 4 = 0.$$

It gives $p = 2$. The dot product $4 - 2p$ equals 5, 3, 0, -2 for options (A), (B), (C), (D), respectively; only (C) makes it zero.

Step 3: Conclusion $p = 2$. Hence, the correct option is (C).

Quick Tip to Solve: In the second line divide $2x-3=2ps$ by 2: the x direction ratio is p , not $2p$.

11. The maximum value of $Z = 4x + y$ for a L.P.P. whose feasible region is given below is: [1 mark]



- (A) 50
- (B) 110
- (C) 120
- (D) 170

Correct Answer: (C) 120

Solution:

Concept:

- A linear objective function achieves its maximum on a bounded polygonal feasible region at a vertex.
- Use only vertices of the shaded feasible region, and evaluate the objective at every such vertex.

Step 1: Read the feasible vertices The shaded region in the given figure is the quadrilateral with vertices $O(0,0)$, $A(0,50)$, $B(20,30)$, and $C(30,0)$. The other intercepts $(0,90)$ and $(50,0)$ lie outside this shaded region and are not candidates.

Step 2: Compare the objective values Evaluating $Z = 4x + y$ at each feasible vertex gives

Vertex	Z
$O(0,0)$	0
$A(0,50)$	50
$B(20,30)$	110
$C(30,0)$	120

Thus the greatest feasible value occurs at C . Option (A) is the value at A , and (B) is the value at B ; neither is the maximum. Option (D), 170, exceeds the largest vertex value and cannot be attained in this bounded region.

Step 3: Conclusion The maximum is 120, attained at $(30,0)$. Hence, the correct option is (C).

Quick Tip to Solve: List the shaded polygon vertices first; intercepts of extended boundary lines outside the shading must be excluded.

12. The probability distribution of a random variable X is:

X	0	1	2	3	4
$P(X)$	0.1	k	$2k$	k	0.1

where k is some unknown constant.

The probability that the random variable X takes the value 2 is:

[1 mark]

- (A) $\frac{1}{5}$
(B) $\frac{2}{5}$
(C) $\frac{4}{5}$
(D) 1

Correct Answer: (B) $\frac{2}{5}$

Solution:

Concept:

- For a discrete probability distribution, every probability is nonnegative and the sum over all possible values is one.
- The probability of a specified value is the entry in its column; here $P(X = 2) = 2k$.

Step 1: Use normalization to find the constant The five mutually exclusive values exhaust the distribution. Therefore

$$0.1 + k + 2k + k + 0.1 = 1, \quad 4k = 0.8, \quad k = 0.2 = \frac{1}{5}.$$

The resulting probabilities are all nonnegative, so this is a valid distribution.

Step 2: Read the requested probability The column headed $X = 2$ contains $2k$, not k . Hence

$$P(X = 2) = 2 \left(\frac{1}{5} \right) = \frac{2}{5}.$$

Option (A) is the value of k itself. Option (C) is the sum of the three middle probabilities, while option (D) is the total probability of all possible values; neither answers the question.

Step 3: Conclusion $P(X = 2) = \frac{2}{5}$. Hence, the correct option is (B).

Quick Tip to Solve: After finding k from the total probability, return to the requested column: the entry for $X=2$ is $2k$.

13. The function $f(x) = kx - \sin x$ is strictly increasing for:

[1 mark]

- (A) $k > 1$
- (B) $k < 1$
- (C) $k > -1$
- (D) $k < -1$

Correct Answer: (A) $k > 1$

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- For every real x , $-1 \leq \cos x \leq 1$; this bound controls the sign of $k - \cos x$.

Step 1: Differentiate and bound the derivative For $f(x) = kx - \sin x$,

$$f'(x) = k - \cos x \geq k - 1.$$

If $k > 1$, the lower bound is positive; hence the derivative is positive for every real x , ensuring strict increase.

Step 2: Check the other parameter conditions For any $k < 1$, $f'(0) = k - 1 < 0$, and continuity makes the function decrease near zero. Thus (B) and (D) cannot ensure increase. Condition (C), $k > -1$, permits $k = 0$, giving the same failure. At the boundary $k = 1$, the function is also strictly increasing: its derivative $1 - \cos x$ is nonnegative and positive except at isolated points, so its integral over any nonempty interval is positive. This boundary does not affect the valid sufficient condition printed in (A).

Step 3: Conclusion Among the printed choices, $k > 1$ ensures strict increase. Hence, the correct option is (A).

Quick Tip to Solve: Use the largest possible cosine, 1, to find the smallest derivative; distinguish a sufficient option from the exact boundary condition.

14. The Cartesian equation of a line passing through the point with position vector $\vec{a} = \hat{i} - \hat{j}$ and parallel to the line $\vec{r} = \hat{i} + \hat{k} + \mu(2\hat{i} - \hat{j})$, is: [1 mark]

(A) $\frac{x-2}{1} = \frac{y+1}{0} = \frac{z}{1}$

(B) $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{0}$

(C) $\frac{x+1}{2} = \frac{y+1}{-1} = \frac{z}{0}$

(D) $\frac{x-1}{2} = \frac{y}{-1} = \frac{z-1}{0}$

Correct Answer: (B) $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{0}$

Solution:

Concept:

- A line through (x_0, y_0, z_0) with direction ratios (l, m, n) has parametric form $(x, y, z) = (x_0, y_0, z_0) + t(l, m, n)$.
- Parallel lines share proportional direction vectors; a zero direction component means the corresponding coordinate stays constant.

Step 1: Identify the point and direction The position vector $\hat{i} - \hat{j}$ gives the point $(1, -1, 0)$. The coefficient of μ in the given line is $2\hat{i} - \hat{j}$, so a parallel direction vector is $(2, -1, 0)$. Its fixed part $\hat{i} + \hat{k}$ does not give the required point.

Step 2: Write the equation and compare the options The required parametric equations are

$$x = 1 + 2t, \quad y = -1 - t, \quad z = 0.$$

In the conventional symmetric notation used by the choices, they are

$$(x-1)/2 = (y+1)/(-1) = z/0.$$

The notation $z/0$ encodes $z = 0$, not an actual division. Option (A) has a different direction. Option (C) has the correct direction but passes through $(-1, -1, 0)$, which is not on the required line. Option (D) lies in $z = 1$ and misses the required point.

Step 3: Conclusion $\frac{x-1}{2} = \frac{y+1}{-1}, z = 0$, represented by option (B).

Quick Tip to Solve: Take direction ratios from the parameter coefficient, and treat the zero denominator as a constant-coordinate condition.

15. If $\begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a scalar matrix, then the value of $a + 2b + 3c + 4d$ is:

[1 mark]

- (A) 0
- (B) 5
- (C) 10
- (D) 25

Correct Answer: (D) 25

Solution:

Concept:

- A scalar matrix has the form λI : all diagonal entries equal λ , and all off-diagonal entries vanish.
- Identify diagonal and off-diagonal positions before substituting their values into the requested expression.

Step 1: Apply the scalar-matrix conditions The given diagonal entries are $a, d, 5$. Their equality requires $a = d = 5$. The entries b and c lie off the diagonal, so $b = c = 0$. Thus the complete matrix is $5I_3$; the value 5 is fixed by its bottom-right entry.

Step 2: Calculate the requested combination Substitute all four values:

$$a + 2b + 3c + 4d = 5 + 2(0) + 3(0) + 4(5) = 25.$$

Option (A) incorrectly makes the diagonal entries zero; (B) retains only a ; (C) adds the diagonal values without the required coefficient of d . The weighted expression gives the value in option (D).

Step 3: Conclusion $a + 2b + 3c + 4d = 25$. Hence, the correct option is (D).

Quick Tip to Solve: For a scalar matrix use the known diagonal value 5 for both a and d , and set only off-diagonal entries to zero.

16. Given that $A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$, matrix A is:

[1 mark]

- (A) $7 \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(C) $\frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$
(D) $\frac{1}{49} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

Solution:

Concept:

- For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, $M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- For a nonzero scalar s , $(sM)^{-1} = s^{-1}M^{-1}$.

Step 1: Invert the matrix inside the scalar factor Put $M = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$. Its determinant is $2(2) - 1(-3) = 7$. Therefore

$$M^{-1} = \frac{1}{7} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

The inverse exists because the determinant is nonzero.

Step 2: Account for the prefactor and verify Since $A^{-1} = M/7$,

$$A = 7M^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

Indeed, multiplying this matrix by $M/7$ gives I_2 . Relative to this verified inverse, options (A), (C), and (D) have extra factors 7, $1/7$, and $1/49$; their products with A^{-1} are those multiples of the identity, not the identity itself.

Step 3: Conclusion $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Invert the scalar factor as well as the matrix: the external 7 cancels the determinant factor $1/7$.

17. If $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$, then the value of $I - A + A^2 - A^3 + \dots$ is:

[1 mark]

(A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$

Solution:

Concept:

- For matrix powers, multiply rows by columns. A matrix with $A^2 = O$, where O is the zero matrix, has $A^n = O$ for every integer $n \geq 2$.
- The identity matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; subtraction is performed entry by entry.

Step 1: Calculate the square

$$A^2 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 4-4 & 2-2 \\ -8+8 & -4+4 \end{pmatrix} = O.$$

Thus $A^3 = A^2 A = O$, and every subsequent power vanishes.

Step 2: Simplify the series and check the options The series therefore reduces exactly to its first two terms:

$$I - A = \begin{pmatrix} 1-2 & 0-1 \\ 0-(-4) & 1-(-2) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}.$$

Option (B) equals $I + A$, using the wrong sign. Option (C) confuses the vanishing square with the full series, while option (D) omits the nonzero term $-A$.

Step 3: Conclusion The value is $\boxed{\begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}}$. Hence, the correct option is (A).

Quick Tip to Solve: Compute A^2 before treating this as an infinite series; a zero square makes every later term vanish.

18. The integrating factor of the differential equation $(x + 2y^2)\frac{dy}{dx} = y$ ($y > 0$) is:

[1 mark]

- (A) $\frac{1}{x}$
- (B) x
- (C) y
- (D) $\frac{1}{y}$

Correct Answer: (D) $\frac{1}{y}$

Solution:

Concept:

- A linear equation with x as the dependent variable has the form $\frac{dx}{dy} + P(y)x = Q(y)$. Its integrating factor is $e^{\int P(y) dy}$.
- For $y > 0$, $\int \frac{dy}{y} = \log y$. A nonzero constant multiple of an integrating factor is equally valid.

Step 1: Use y as the independent variable The given equation is not linear in y as a function of x . Invert the derivative to obtain

$$\frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y, \quad \frac{dx}{dy} - \frac{x}{y} = 2y.$$

Hence $P(y) = -1/y$.

Step 2: Calculate and compare the integrating factors

$$\text{I.F.} = e^{\int (-1/y) dy} = e^{-\log y} = \frac{1}{y}.$$

The options $1/x$ and x incorrectly treat x as the independent variable. Option y would correspond to $P(y) = +1/y$, whereas this equation has the negative coefficient. Multiplication by $1/y$ converts its left side into $\frac{d}{dy}(x/y)$.

Step 3: Conclusion The integrating factor is $\boxed{1/y}$. Hence, the correct option is (D).

Quick Tip to Solve: Rewrite the equation in dx/dy form first; the coefficient of x , including its minus sign, determines the integrating factor.

19. Assertion (A): The relation $R = \{(x, y) : (x + y) \text{ is a prime number and } x, y \in \mathbb{N}\}$ is not a reflexive relation.

Reason (R): The number ' $2n$ ' is composite for all natural numbers n .

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A relation on $\mathbb{N}$ is reflexive only if (n, n) belongs to it for every $n \in \mathbb{N}$. One missing diagonal pair disproves reflexivity.
- A prime has exactly two positive divisors. The even number 2 is prime, whereas an even number greater than 2 is composite.

Step 1: Judge the assertion For a diagonal pair (n, n) , the required sum is $n + n = 2n$. Take $n = 2$: then $2 + 2 = 4$ is not prime, so $(2, 2) \notin R$. Since reflexivity requires every diagonal pair, R is not reflexive. Assertion (A) is true.

Step 2: Judge the reason The reason claims that $2n$ is composite for all natural numbers. At $n = 1$, however, $2n = 2$, which is prime. This single counterexample disproves the universal claim. Reason (R) is false.

Step 3: Check whether the reason explains the assertion A false universal statement cannot be the required correct explanation. The assertion follows from the missing pair $(2, 2)$; it does not require every $2n$ to be composite. Thus options (A) and (B), which call the reason true, and option (D), which calls the assertion false, are excluded.

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Test $n = 1$ in the reason and $n = 2$ for reflexivity: the two statements require different counterexample checks.

Section B (10 marks)

In this section there are 5 very short answer type questions of 2 marks each.

21(a). If $y = \cos^3(\sec^2 2t)$, find $\frac{dy}{dt}$.

[2 marks]

Solution:

Concept:

- The chain rule gives $\frac{d}{dt}f(g(t)) = f'(g(t))g'(t)$. Apply it once for each nested function.
- The standard derivatives are $\frac{d}{du} \cos u = -\sin u$ and $\frac{d}{dv} \sec v = \sec v \tan v$.

Step 1: Differentiate the outer cube and cosine Let $u = \sec^2(2t)$, so $y = (\cos u)^3$. Differentiating the cube first and then the cosine gives

$$\frac{dy}{dt} = 3 \cos^2 u (-\sin u) \frac{du}{dt}.$$

The factor $-\sin u$ is necessary because the cube is applied to the cosine, rather than directly to u .

Step 2: Differentiate the squared secant and its argument Write $u = (\sec v)^2$, where $v = 2t$. Then

$$\frac{du}{dt} = 2 \sec v (\sec v \tan v) \frac{dv}{dt} = 4 \sec^2(2t) \tan(2t).$$

This includes the factor 2 from dv/dt , as well as the factor 2 from differentiating the square.

Step 3: Combine the chain-rule factors Substituting the inner derivative into the outer one yields

$$\frac{dy}{dt} = -12 \cos^2(\sec^2 2t) \sin(\sec^2 2t) \sec^2(2t) \tan(2t).$$

The formula applies wherever the given function is defined, namely where $\cos(2t) \neq 0$.

Step 4: Conclusion

$$\boxed{\frac{dy}{dt} = -12 \cos^2(\sec^2 2t) \sin(\sec^2 2t) \sec^2(2t) \tan(2t)}.$$

Quick Tip to Solve: Track the four layers: cube, cosine, squared secant and $2t$; their numerical factors give $3 \times (-1) \times 2 \times 2 = -12$.

21(b). If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

[2 marks]

Solution:

Concept:

- Taking natural logarithms of $x^y = e^{x-y}$ gives $y \log x = x - y$, for $x > 0$. Here $\log$ denotes the natural logarithm.
- The product rule gives $\frac{d}{dx}(y \log x) = y' \log x + y/x$, since y depends on x .

Step 1: Take logarithms and express y From the given relation,

$$y \log x = x - y, \quad y(1 + \log x) = x.$$

Hence $y = \frac{x}{1 + \log x}$. The denominator cannot be zero on the relation: if $x = e^{-1}$, the previous equation would read $0 = x$, a contradiction.

Step 2: Differentiate the logarithmic relation Differentiating both sides of $y \log x = x - y$ gives

$$\frac{dy}{dx} \log x + \frac{y}{x} = 1 - \frac{dy}{dx}.$$

Collecting derivative terms,

$$\frac{dy}{dx}(1 + \log x) = 1 - \frac{y}{x}.$$

Both appearances of y must be differentiated because y is a function of x .

Step 3: Substitute and simplify Using $y/x = 1/(1 + \log x)$, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 - 1/(1 + \log x)}{1 + \log x} \\ &= \frac{\log x}{(1 + \log x)^2}. \end{aligned}$$

The domain is $x > 0$, with $x \neq e^{-1}$, consistent with the original relation.

Step 4: Conclusion Thus $\boxed{\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}}$, as required.

Quick Tip to Solve: After taking logs, first write $y(1 + \log x) = x$; this supplies the substitution y/x needed to eliminate y from the derivative.

22. The volume of a cube is increasing at the rate of $6 \text{ cm}^3/\text{s}$. How fast is the surface area of cube increasing, when the length of an edge is 8 cm ? [2 marks]

Solution:

Concept:

- For a cube with edge length a , its volume and total surface area are $V = a^3$ and $S = 6a^2$, respectively.
- The chain rule relates rates with respect to time t : $\frac{dV}{dt} = 3a^2 \frac{da}{dt}$ and $\frac{dS}{dt} = 12a \frac{da}{dt}$.

Step 1: Find the instantaneous rate of change of the edge Let a be the edge length in centimetres and t the time in seconds. Differentiate $V = a^3$ with respect to time, treating a as a function of t . At the stated instant, $a = 8$ and $\frac{dV}{dt} = 6$, so

$$6 = 3(8)^2 \frac{da}{dt} \implies \frac{da}{dt} = \frac{1}{32} \text{ cm/s}.$$

The given volume rate cannot be used directly as the edge rate, since volume varies with the cube of the edge length.

Step 2: Use the edge rate to calculate the surface-area rate The total surface area consists of six square faces, so $S = 6a^2$. Differentiating and substituting the edge length and edge rate at the same instant gives

$$\frac{dS}{dt} = 12a \frac{da}{dt} = 12(8) \left(\frac{1}{32} \right) = 3 \text{ cm}^2/\text{s}.$$

The rate is positive, as expected because every face becomes larger when the edge increases.

Step 3: Conclusion The surface area is increasing at $3 \text{ cm}^2/\text{s}$ when the edge length is 8 cm .

Quick Tip to Solve: Eliminate the edge rate first: $\frac{dS}{dt} = \frac{4}{a} \frac{dV}{dt}$, then insert the edge length at the specified instant.

23. Show that the function f given by $f(x) = \sin x + \cos x$, is strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. [2 marks]

Solution:

Concept:

- A differentiable function is strictly decreasing on an interval when its derivative is negative throughout that interval.
- The identity $\cos x - \sin x = \sqrt{2} \cos\left(x + \frac{\pi}{4}\right)$ converts the derivative into a single trigonometric function whose sign is easy to determine.

Step 1: Differentiate the given function For $f(x) = \sin x + \cos x$, differentiate term by term to obtain

$$f'(x) = \cos x - \sin x = \sqrt{2} \cos\left(x + \frac{\pi}{4}\right).$$

Quick Tip to Solve: Combine $\cos x - \sin x$ into $\sqrt{2} \cos(x + \pi/4)$; shifting the interval makes its negative sign immediate.

Here the factor $\sqrt{2}$ is positive, so the sign of the derivative is exactly the sign of the cosine factor.

Step 2: Determine the derivative sign on the entire interval If $\frac{\pi}{4} < x < \frac{5\pi}{4}$, adding $\frac{\pi}{4}$ yields

$$\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{3\pi}{2}.$$

Cosine is negative throughout this open interval, hence $f'(x) < 0$ for every allowed x . In particular, for any two points $u < v$ in the given interval, the mean value theorem gives $f(v) - f(u) = f'(c)(v - u) < 0$ for some $c \in (u, v)$. This establishes strict decrease throughout the interval, not merely at selected points.

Step 3: Conclusion Therefore, $f(x) = \sin x + \cos x$ is strictly decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

Quick Tip to Solve: Combine $\cos x - \sin x$ into $\sqrt{2}\cos(x + \pi/4)$; shifting the interval makes its negative sign immediate.

24(a). Express $\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the simplest form.

[2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus $\tan^{-1}(\tan \theta) = \theta$ only when θ lies in this range.
- The half-angle identity $\frac{1 + \sin x}{\cos x} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ is valid here because $\cos x > 0$.

Step 1: Transform the argument using a conjugate Since $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we have $\cos x > 0$ and $1 - \sin x > 0$; hence all the following divisions are valid. Multiplying numerator and denominator by $1 + \sin x$ gives

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x(1 + \sin x)}{1 - \sin^2 x} = \frac{1 + \sin x}{\cos x} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

The final equality follows from the half-angle formula for tangent.

Step 2: Check the principal branch before cancelling tangent Dividing the stated interval by two and then adding $\frac{\pi}{4}$ gives

$$0 < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}.$$

Thus this angle belongs to the principal range of $\tan^{-1}$. We can therefore apply $\tan^{-1}$ to the transformed argument without adding or subtracting π . The interval check is essential: inverse tangent does not return an arbitrary angle with the same tangent.

Step 3: Conclusion

$$\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right) = \frac{\pi}{4} + \frac{x}{2}$$

Quick Tip to Solve: After rationalising the denominator, check that $\pi/4 + x/2$ is between 0 and $\pi/2$ before cancelling $\tan^{-1}$ with $\tan$.

24(b). Find the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

[2 marks]

Solution:**Concept:**

- The principal ranges are $\tan^{-1} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos^{-1} : [0, \pi]$, and $\sin^{-1} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- Each inverse trigonometric term must first be evaluated in its own principal range; the resulting real angles are then added.

Step 1: Evaluate the three principal inverse values Because $\tan(\pi/4) = 1$ and $\pi/4$ lies in the inverse-tangent range,

$$\tan^{-1}(1) = \frac{\pi}{4}.$$

Cosine is $-1/2$ at $2\pi/3$, which lies in $[0, \pi]$, so

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

Finally, $\sin(-\pi/4) = -1/\sqrt{2}$, and this negative angle lies in the inverse-sine range. Therefore

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

Choosing $5\pi/4$ for the last term would give the same sine but would violate its principal range.

Step 2: Add the angles after selecting their principal values Substitute these values into the expression:

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{4}.$$

The tangent and sine contributions cancel exactly. No further inverse trigonometric operation is applied to the sum.

Step 3: Conclusion The required value is $\boxed{\frac{2\pi}{3}}$.

Quick Tip to Solve: Use the negative angle $-\pi/4$ for the inverse-sine term; it then cancels the inverse-tangent term immediately.

25. Find: $\int \frac{2x}{(x^2 + 1)(x^2 - 4)} dx$.

[2 marks]

Solution:**Concept:**

- A substitution $u = x^2$ gives $du = 2x dx$, reducing the integral to a rational function in u .

Quick Tip to Solve: The numerator is the derivative of x^2 ; substitute first, then use the difference of the two denominator roots to obtain the factor $1/5$.

- For a nonzero variable expression v , $\int \frac{dv}{v} = \ln |v| + C$; absolute values preserve validity wherever the original integrand is defined.

Step 1: Substitute and decompose into partial fractions Put $u = x^2$, so $du = 2x dx$. The integral becomes

$$I = \int \frac{du}{(u+1)(u-4)}.$$

Write

$$\frac{1}{(u+1)(u-4)} = \frac{A}{u+1} + \frac{B}{u-4}.$$

Clearing denominators gives $1 = A(u-4) + B(u+1)$. Set $u = -1$ to obtain $A = -1/5$, and set $u = 4$ to obtain $B = 1/5$. These substitutions determine the coefficients of the polynomial identity; they do not substitute singular values into the original integrand.

Step 2: Integrate and restore the original variable Using the coefficients,

$$\begin{aligned} I &= \frac{1}{5} \int \left(\frac{1}{u-4} - \frac{1}{u+1} \right) du \\ &= \frac{1}{5} (\ln |u-4| - \ln |u+1|) + C. \end{aligned}$$

Replace u by x^2 . Since $x^2 + 1 > 0$, its logarithm needs no absolute-value sign. The antiderivative applies on any interval avoiding $x = \pm 2$, where the original denominator vanishes. Differentiation of the logarithmic difference gives $\frac{2x}{5} \left(\frac{1}{x^2-4} - \frac{1}{x^2+1} \right)$, which simplifies to the given integrand.

Step 3: Conclusion

$$\int \frac{2x dx}{(x^2+1)(x^2-4)} = \frac{1}{5} \ln \left| \frac{x^2-4}{x^2+1} \right| + C$$

Quick Tip to Solve: The numerator is the derivative of x^2 ; substitute first, then use the difference of the two denominator roots to obtain the factor $1/5$.

Section C (18 marks)

In this section there are 6 short answer type questions of 3 marks each.

26. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + \cos^{-1} \sqrt{x}$ is given.

[3 marks]

Solution:

Concept:

- Logarithmic differentiation handles both a variable base and exponent: if $u = (\cos x)^x$, then $\ln u = x \ln(\cos x)$.
- The chain rule gives $\frac{d}{dx} \cos^{-1} v = -\frac{v'}{\sqrt{1-v^2}}$, where $v = v(x)$. Here $\cos^{-1}$ denotes inverse cosine.

Step 1: Differentiate the variable power Put $u = (\cos x)^x$. For $0 < x < 1$, the base $\cos x$ is positive, so

Quick Tip to Solve: Take logarithms for a variable raised to a variable power, and retain the minus sign in the inverse cosine derivative.

taking natural logarithms is valid. Use the product rule on the right:

$$\begin{aligned}\ln u &= x \ln(\cos x), \\ \frac{1}{u} \frac{du}{dx} &= \ln(\cos x) + x \frac{-\sin x}{\cos x} \\ &= \ln(\cos x) - x \tan x.\end{aligned}$$

Thus,

$$\frac{du}{dx} = (\cos x)^x [\ln(\cos x) - x \tan x].$$

Treating the exponent as constant would omit the logarithmic term; both the base and exponent vary with x .

Step 2: Differentiate the inverse cosine term Let $v = \sqrt{x}$. Then $v' = 1/(2\sqrt{x})$ and $1 - v^2 = 1 - x$. The chain rule therefore gives

$$\frac{d}{dx} \cos^{-1} \sqrt{x} = -\frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x(1-x)}}.$$

Add this to the derivative of u , since y is the sum of the two terms. The inverse cosine expression is real on $[0, 1]$; the derivative formula holds in the interior $0 < x < 1$, where both factors under the square root are positive. The endpoint derivatives are singular.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = (\cos x)^x [\ln(\cos x) - x \tan x] - \frac{1}{2\sqrt{x(1-x)}}}, \quad 0 < x < 1.$$

Quick Tip to Solve: Take logarithms for a variable raised to a variable power, and retain the minus sign in the inverse cosine derivative.

27(a). Find the particular solution of the differential equation $\frac{dy}{dx} = y \cot 2x$, given that $y\left(\frac{\pi}{4}\right) = 2$. [3 marks]

Solution:

Concept:

- A separable differential equation can be integrated after placing all factors involving y with dy , and all factors involving x with dx .
- Since $\frac{d}{dx} \ln |\sin 2x| = 2 \cot 2x$, the integral of $\cot 2x$ is $\frac{1}{2} \ln |\sin 2x| + C$. The initial condition fixes the integration constant.

Step 1: Separate variables and integrate The initial value $y(\pi/4) = 2$ is nonzero, so divide by y near that point:

$$\frac{dy}{y} = \cot 2x \, dx.$$

Quick Tip to Solve: In integrating $\cot 2x$, include the factor $1/2$, then use $\sin(\pi/2) = 1$ to fix the constant immediately.

Integrating both sides yields

$$\ln |y| = \frac{1}{2} \ln |\sin 2x| + C.$$

The factor $1/2$ comes from the inner argument $2x$. On the interval $0 < x < \pi/2$ containing $\pi/4$, we have $\sin 2x > 0$, and the solution satisfying the positive initial value can be written

$$y = A\sqrt{\sin 2x}.$$

The constant A absorbs the exponential of the integration constant. The zero solution discarded during division cannot satisfy the given initial condition.

Step 2: Determine the constant from the given value Substitute $x = \pi/4$ and $y = 2$:

$$2 = A\sqrt{\sin(\pi/2)} = A.$$

To check the resulting expression, differentiate $2\sqrt{\sin 2x}$:

$$\frac{dy}{dx} = \frac{2 \cos 2x}{\sqrt{\sin 2x}}.$$

Its right-hand side in the original equation is

$$y \cot 2x = 2\sqrt{\sin 2x} \frac{\cos 2x}{\sin 2x} = \frac{2 \cos 2x}{\sqrt{\sin 2x}}.$$

These agree throughout the stated interval, and the initial value is also satisfied.

Step 3: Conclusion The required particular solution is $y = 2\sqrt{\sin 2x}$, on the interval $0 < x < \pi/2$ containing the initial point.

Quick Tip to Solve: In integrating $\cot 2x$, include the factor $1/2$, then use $\sin(\pi/2) = 1$ to fix the constant immediately.

OR

27(b). Find the particular solution of the differential equation $(xe^{y/x} + y) dx = x dy$, given that $y = 1$ when $x = 1$. [3 marks]

Solution:

Concept:

- An equation whose right-hand side depends only on y/x is homogeneous. Substitute $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$.
- After substitution the variables separate. The integral $\int e^{-v} dv = -e^{-v}$ is used, and $\int dx/x = \ln |x|$.

Step 1: Reduce the homogeneous differential equation Divide the printed equation by $x dx$, where $x \neq 0$:

$$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}.$$

Quick Tip to Solve: Set $y = vx$: the extra v in dy/dx cancels the printed y/x , leaving a separable equation.

Set $v = y/x$. Substituting $dy/dx = v + x dv/dx$ gives

$$v + x \frac{dv}{dx} = e^v + v, \quad x \frac{dv}{dx} = e^v.$$

Cancel the two v terms, then separate and integrate:

$$e^{-v} dv = \frac{dx}{x}, \quad -e^{-v} = \ln|x| + C.$$

Equivalently, with a renamed constant $K = -C$,

$$e^{-y/x} + \ln|x| = K.$$

Step 2: Apply the initial condition and verify At $x = 1$ and $y = 1$, we have $v = 1$, hence

$$K = e^{-1} + \ln 1 = \frac{1}{e}.$$

Near the initial point $x > 0$, so $\ln|x| = \ln x$. The explicit form follows by taking logarithms:

$$y = -x \ln \left(\frac{1}{e} - \ln x \right).$$

Its logarithm requires $\ln x < 1/e$, giving $0 < x < e^{1/e}$. For an exact check, differentiate the implicit expression:

$$-e^{-v} \frac{dv}{dx} + \frac{1}{x} = 0 \implies x \frac{dv}{dx} = e^v.$$

Since $y' = v + xv'$, this recovers $y' = y/x + e^{y/x}$; substituting $x = 1$ also gives $y = 1$.

Step 3: Conclusion The required particular solution is $e^{-y/x} + \ln x = 1/e$. Equivalently, $y = -x \ln(1/e - \ln x)$, for $0 < x < e^{1/e}$.

Quick Tip to Solve: Set $y = vx$: the extra v in dy/dx cancels the printed y/x , leaving a separable equation.

28. Find: $\int \sec^3 \theta d\theta$

[3 marks]

Solution:

Concept:

- Integration by parts uses $\int u dv = uv - \int v du$. For $\sec^3 \theta$, split the integrand as $\sec \theta \cdot \sec^2 \theta$.
- The identity $\tan^2 \theta = \sec^2 \theta - 1$ makes the original integral reappear. Also, $\int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$.

Step 1: Apply integration by parts Write $I = \int \sec^3 \theta d\theta$, and choose

$$u = \sec \theta, \quad dv = \sec^2 \theta d\theta, \quad du = \sec \theta \tan \theta d\theta, \quad v = \tan \theta.$$

Quick Tip to Solve: Split $\sec^3 \theta$ into $\sec \theta \sec^2 \theta$, then move the repeated integral to the left before dividing by two.

Integration by parts then gives

$$I = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta.$$

Using $\tan^2 \theta = \sec^2 \theta - 1$, expand the remaining integrand:

$$I = \sec \theta \tan \theta - I + \int \sec \theta d\theta.$$

Move the repeated integral to the left to obtain

$$2I = \sec \theta \tan \theta + \int \sec \theta d\theta.$$

Step 2: Evaluate the remaining integral and check Multiply numerator and denominator by $\sec \theta + \tan \theta$:

$$\int \sec \theta d\theta = \int \frac{\sec^2 \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta = \ln |\sec \theta + \tan \theta| + C.$$

The numerator is the derivative of the denominator. Substituting this in the expression for $2I$ and dividing by two yields the required antiderivative, with the constant renamed C . As a check, its derivative is

$$\frac{1}{2}(\sec \theta \tan^2 \theta + \sec^3 \theta + \sec \theta) = \sec^3 \theta,$$

because $\tan^2 \theta + 1 = \sec^2 \theta$. These expressions apply on any interval where $\cos \theta \neq 0$.

Step 3: Conclusion

$$\int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C.$$

Quick Tip to Solve: Split $\sec^3 \theta$ into $\sec \theta \sec^2 \theta$, then move the repeated integral to the left before dividing by two.

29(a). A card from a well shuffled deck of 52 playing cards is lost. From the remaining cards of the pack, a card is drawn at random and is found to be a King. Find the probability of the lost card being a King. [3 marks]

Solution:

Concept:

- Conditional probability updates an event probability after new information is known. Here the observed information is that the drawn card is a King.
- Bayes' theorem is $P(L | K) = \frac{P(L)P(K | L)}{P(L)P(K | L) + P(L^c)P(K | L^c)}$, where L denotes a lost King and K a drawn King.

Step 1: Define the mutually exclusive cases Let L be the event that the lost card is a King, and let

Quick Tip to Solve: Condition on the King actually observed: among the other fifty-one cards, only three are Kings.

K be the event that the card drawn from the remaining pack is a King. There are four Kings and forty-eight non-Kings in the original deck. Thus

$$P(L) = \frac{4}{52}, \quad P(L^c) = \frac{48}{52}.$$

These two cases exhaust the possibilities for the lost card; the card subsequently drawn is selected from fifty-one remaining cards.

Step 2: Find the likelihood of drawing a King in each case If a King was lost, three Kings remain, whereas if a non-King was lost, all four Kings remain. Therefore

$$P(K | L) = \frac{3}{51}, \quad P(K | L^c) = \frac{4}{51}.$$

By the law of total probability,

$$P(K) = \frac{4}{52} \cdot \frac{3}{51} + \frac{48}{52} \cdot \frac{4}{51} = \frac{204}{52 \cdot 51}.$$

The observation of a King is slightly less likely when the lost card was itself a King.

Step 3: Apply Bayes' theorem The required probability is conditional on the observed King, so the original prior probability $4/52$ must be updated:

$$P(L | K) = \frac{(4/52)(3/51)}{(4/52)(3/51) + (48/52)(4/51)} = \frac{12}{204} = \frac{1}{17}.$$

Equivalently, after setting aside the observed King, the lost card is equally likely to be any of the other fifty-one original cards, of which three are Kings.

Step 4: Conclusion The probability that the lost card is a King is $\boxed{\frac{1}{17}}$.

Quick Tip to Solve: Condition on the King actually observed: among the other fifty-one cards, only three are Kings.

OR

29(b). A biased die is twice as likely to show an even number as an odd number. If such a die is thrown twice, find the probability distribution of the number of sixes. Also, find the mean of the distribution. [3 marks]

Solution:

Concept:

- For the stated biased die, assign probability k to each odd face and $2k$ to each even face; normalise the six face probabilities to obtain k .
- For two independent throws with probability p of a six, the number X of sixes has $P(X = x) = \binom{2}{x} p^x (1 - p)^{2-x}$, for $x = 0, 1, 2$.
- The mean of a discrete distribution is $E(X) = \sum_x xP(X = x)$.

Quick Tip to Solve: First normalise the six face probabilities; exactly one six can occur in either of two orders.

Step 1: Find the probability of a six on one throw Let each odd face, namely 1, 3 and 5, have probability k . Each even face, namely 2, 4 and 6, then has probability $2k$. Since all six face probabilities add to one,

$$3k + 3(2k) = 1 \implies k = \frac{1}{9}.$$

In particular, $P(6) = 2/9$, and the probability of a result other than six is $1 - 2/9 = 7/9$. The event of an even result is broader than the event of a six, so its probability must not be used as the success probability.

Step 2: Form the probability distribution Let X denote the number of sixes in the two throws. Its possible values are 0, 1 and 2. Independence gives

$$\begin{aligned} P(X = 0) &= \left(\frac{7}{9}\right)^2 = \frac{49}{81}, \\ P(X = 1) &= 2 \left(\frac{2}{9}\right) \left(\frac{7}{9}\right) = \frac{28}{81}, \\ P(X = 2) &= \left(\frac{2}{9}\right)^2 = \frac{4}{81}. \end{aligned}$$

The factor 2 for exactly one six accounts for the orders six followed by non-six and non-six followed by six. The probabilities add to $(49 + 28 + 4)/81 = 1$, confirming a valid distribution.

Step 3: Calculate the mean Using the definition of expectation,

$$E(X) = 0 \cdot \frac{49}{81} + 1 \cdot \frac{28}{81} + 2 \cdot \frac{4}{81} = \frac{36}{81} = \frac{4}{9}.$$

This agrees with the binomial mean $np = 2(2/9)$.

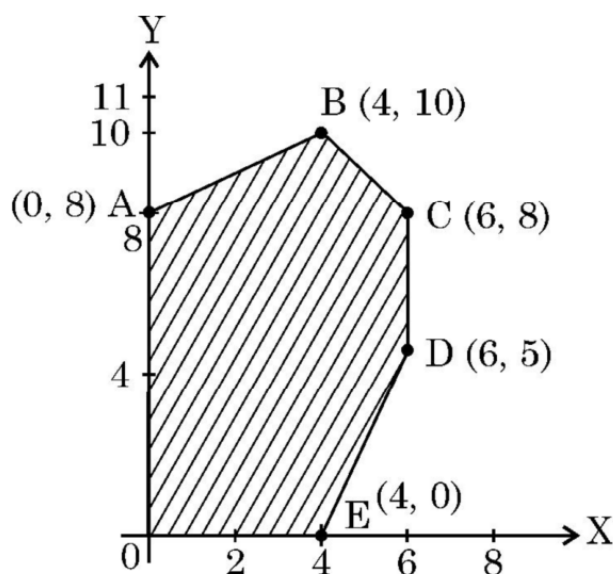
Step 4: Conclusion The distribution is $P(X = 0) = 49/81$, $P(X = 1) = 28/81$, $P(X = 2) = 4/81$, and its mean is $\boxed{4/9}$.

Quick Tip to Solve: First normalise the six face probabilities; exactly one six can occur in either of two orders.

30. The corner points of the feasible region determined by the system of linear constraints are as shown in the following figure:

(i) If $Z = 3x - 4y$ be the objective function, then find the maximum value of Z .

(ii) If $Z = px + qy$ where $p, q > 0$ be the objective function. Find the condition on p and q so that maximum value of Z occurs at $B(4, 10)$ and $C(6, 8)$. [3 marks]



Solution:

Concept:

- A linear objective function on a bounded polygonal feasible region attains its maximum at a vertex; evaluate it at every corner, including the origin.
- If two adjacent vertices give the same maximum, every point of the line segment joining them also gives that maximum, since the objective is linear.

Step 1: List the vertices and evaluate the first objective The shaded feasible polygon has vertices $O(0, 0)$, $A(0, 8)$, $B(4, 10)$, $C(6, 8)$, $D(6, 5)$, and $E(4, 0)$. For $Z = 3x - 4y$, their objective values are

Vertex	$Z = 3x - 4y$
$O(0, 0)$	0
$A(0, 8)$	-32
$B(4, 10)$	-28
$C(6, 8)$	-14
$D(6, 5)$	-2
$E(4, 0)$	12

Since the polygon is closed and bounded, comparing these corner values suffices. The greatest entry is 12 at $E(4, 0)$.

Step 2: Require equal values at B and C For $Z = px + qy$, both specified corners can be maximisers only if their objective values agree:

$$4p + 10q = 6p + 8q \implies 2q = 2p \implies p = q.$$

Quick Tip to Solve: Include the origin in the corner test, then equate the objective values at the two requested maximisers.

Thus equality of the two positive coefficients is necessary. Geometrically, the objective lines then have slope $-p/q = -1$, the same slope as the edge BC .

Step 3: Check that the common value really is maximal Set $p = q = r$, where $r > 0$. Then $Z = r(x+y)$. The values at O, A, B, C, D, E , respectively, are

$$0, \quad 8r, \quad 14r, \quad 14r, \quad 11r, \quad 4r.$$

Because r is positive, the shared value $14r$ exceeds every other corner value. Hence $p = q > 0$ is sufficient as well as necessary. Every point of the segment BC , where $x + y = 14$, also has this maximum.

Step 4: Conclusion (i) $Z_{\max} = 12$ at $E(4, 0)$. (ii) The required condition is $p = q > 0$; both B and C , and their joining segment, maximise Z .

Quick Tip to Solve: Include the origin in the corner test, then equate the objective values at the two requested maximisers.

31(a). Evaluate: $\int_0^{\pi/4} \frac{x \, dx}{1 + \cos 2x + \sin 2x}$

[3 marks]

Solution:

Concept:

- The substitution $x = a - t$ gives $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$. If a denominator is symmetric about $a/2$, adding the two forms removes a numerator x .
- Use $\cos 2x = 2 \cos^2 x - 1$ and $\sin 2x = 2 \sin x \cos x$, followed by $t = \tan x$ with $dt = \sec^2 x \, dx$.

Step 1: Exploit the symmetry of the denominator Write

$$I = \int_0^{\pi/4} \frac{x \, dx}{1 + \cos 2x + \sin 2x}.$$

Under $x = \pi/4 - t$, the terms $\cos 2x$ and $\sin 2x$ interchange. Reversing the transformed limits and renaming the variable gives

$$I = \int_0^{\pi/4} \frac{\pi/4 - x}{1 + \cos 2x + \sin 2x} \, dx.$$

Add this to the original form:

$$2I = \frac{\pi}{4} \int_0^{\pi/4} \frac{dx}{1 + \cos 2x + \sin 2x}.$$

Therefore $I = (\pi/8)J$, where J denotes the last integral without its constant prefactor.

Step 2: Evaluate the remaining integral On the given interval, $\cos x > 0$, and the denominator factors as

$$1 + \cos 2x + \sin 2x = 2 \cos x (\cos x + \sin x) = 2 \cos^2 x (1 + \tan x).$$

Quick Tip to Solve: Replace x by $\pi/4$ minus x before using $\tan x$; symmetry eliminates the troublesome factor x .

Consequently,

$$J = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 x}{1 + \tan x} dx.$$

Set $t = \tan x$, so $dt = \sec^2 x dx$. The limits become $t = 0$ and $t = 1$. Hence

$$J = \frac{1}{2} \int_0^1 \frac{dt}{1+t} = \frac{1}{2} [\ln(1+t)]_0^1 = \frac{1}{2} \ln 2.$$

The logarithm is well defined throughout these limits because $1+t > 0$.

Step 3: Combine the two results Substituting this value into the symmetry relation,

$$I = \frac{\pi}{8} \left(\frac{1}{2} \ln 2 \right) = \frac{\pi \ln 2}{16}.$$

The answer is positive, consistent with the nonnegative integrand on the interval.

Step 4: Conclusion $\int_0^{\pi/4} \frac{x dx}{1 + \cos 2x + \sin 2x} = \frac{\pi \ln 2}{16}.$

Quick Tip to Solve: Replace x by $\pi/4$ minus x before using $\tan x$; symmetry eliminates the troublesome factor x .

OR

31(b). Find: $\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right] dx$

[3 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Thus an integrand of this form can be integrated by recognising the product being differentiated.
- The power rule and chain rule give $\frac{d}{dx}(1+x^2)^{-1/2} = -x(1+x^2)^{-3/2}$. Here $1+x^2 > 0$ for every real x , so all expressions used are defined on $\mathbb{R}$.

Step 1: Identify the function inside the product The second term in the bracket suggests choosing

$$f(x) = \frac{x}{\sqrt{1+x^2}} = x(1+x^2)^{-1/2}.$$

Differentiate this expression using the product rule, taking care to differentiate the inner function $1+x^2$ as well:

$$\begin{aligned} f'(x) &= (1+x^2)^{-1/2} - x^2(1+x^2)^{-3/2} \\ &= \frac{1+x^2 - x^2}{(1+x^2)^{3/2}} \\ &= \frac{1}{(1+x^2)^{3/2}}. \end{aligned}$$

Quick Tip to Solve: For an integrand containing e^x , differentiate the simpler term in the bracket first. Here the derivative of $x/\sqrt{1+x^2}$ is exactly the other term, revealing the reverse product rule.

Therefore the first term in the printed bracket is exactly $f'(x)$, and the second is $f(x)$.

Step 2: Apply the reverse product rule Since the exponential and its derivative are both e^x , the entire printed integrand becomes

$$e^x[f'(x) + f(x)] = \frac{d}{dx} \left(e^x \frac{x}{\sqrt{1+x^2}} \right).$$

Integrating a derivative recovers its differentiable function up to an arbitrary constant C . This identification also verifies the antiderivative: differentiating the product reproduces both printed terms, with their positive signs and their exact denominator powers.

Step 3: Conclusion

$$\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}} \right] dx = \frac{xe^x}{\sqrt{1+x^2}} + C.$$

Quick Tip to Solve: For an integrand containing e^x , differentiate the simpler term in the bracket first. Here the derivative of $x/\sqrt{1+x^2}$ is exactly the other term, revealing the reverse product rule.

Section D (20 marks)

In the section there are 4 long answer type questions of 5 marks each.

32(a). Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \rightarrow B$, defined by $f(x) = \frac{x-3}{x-5}$. Show that f is one-one and onto. [5 marks]

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1, x_2 in its domain.
- A function $f : A \rightarrow B$ is onto if every $y \in B$ has a preimage $x \in A$ satisfying $f(x) = y$. The preimage must belong to the stated domain.
- Before proving these properties, check that the formula is defined throughout A and that its values lie in B ; excluded points must be respected during cross-multiplication.

Step 1: Check the domain and codomain For $x \in A = \mathbb{R} \setminus \{5\}$, the denominator $x - 5$ is nonzero. Also write

$$f(x) = \frac{x-3}{x-5} = 1 + \frac{2}{x-5}.$$

The fraction $2/(x-5)$ cannot be zero. Hence $f(x) \neq 1$ and every value of the formula belongs to $B = \mathbb{R} \setminus \{1\}$. This establishes that the stated mapping is well defined.

Step 2: Prove that the function is one-one Let $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Since neither

Quick Tip to Solve: Rewrite the formula as $1 + 2/(x-5)$ to see why 1 is excluded from the range. For onto, solve for x and explicitly check that the resulting preimage is not 5.

denominator is zero, cross-multiplication is valid:

$$\begin{aligned}\frac{x_1 - 3}{x_1 - 5} &= \frac{x_2 - 3}{x_2 - 5}, \\ (x_1 - 3)(x_2 - 5) &= (x_2 - 3)(x_1 - 5), \\ x_1x_2 - 5x_1 - 3x_2 + 15 &= x_1x_2 - 5x_2 - 3x_1 + 15, \\ -2x_1 + 2x_2 &= 0.\end{aligned}$$

Thus $x_1 = x_2$, as required by the definition of a one-one function. Distinct elements of A therefore cannot have the same image.

Step 3: Construct a preimage for every element of the codomain Take an arbitrary $y \in B$, so $y \neq 1$. Solve the equation $y = f(x)$ for x :

$$\begin{aligned}y(x - 5) &= x - 3, \\ x(y - 1) &= 5y - 3, \\ x &= \frac{5y - 3}{y - 1} = 5 + \frac{2}{y - 1}.\end{aligned}$$

This is a real number because $y - 1 \neq 0$. Moreover,

$$x - 5 = \frac{2}{y - 1} \neq 0,$$

so this preimage is in A , rather than being the excluded value 5.

Step 4: Verify the constructed preimage For this choice of x , we have

$$x - 3 = \frac{2y}{y - 1}, \quad x - 5 = \frac{2}{y - 1}.$$

Substitution in the original formula gives

$$f(x) = \frac{2y/(y - 1)}{2/(y - 1)} = y.$$

Thus every arbitrary element y of B is actually attained by an element of A , which proves that the function is onto.

Step 5: Conclusion The function $f : A \rightarrow B$, given by $f(x) = (x - 3)/(x - 5)$, is both one-one and onto, and is therefore bijective.

Quick Tip to Solve: Rewrite the formula as $1 + 2/(x - 5)$ to see why 1 is excluded from the range. For onto, solve for x and explicitly check that the resulting preimage is not 5.

32(b). Check whether the relation S in the set of real numbers $\mathbb{R}$ defined by $S = \{(a, b) : \text{where } a - b + \sqrt{2} \text{ is an irrational number}\}$ is reflexive, symmetric or transitive. [5 marks]

Solution:

Concept:

- A relation on $\mathbb{R}$ is reflexive when (a, a) belongs to it for every real number a .
- Symmetry requires $(a, b) \in S$ to imply $(b, a) \in S$. Transitivity requires $(a, b), (b, c) \in S$ to imply $(a, c) \in S$.
- To disprove a universal property, one counterexample is sufficient. Every nonzero rational multiple of $\sqrt{2}$ is irrational, whereas 0 is rational.

Step 1: Test reflexivity for an arbitrary real number Let $a \in \mathbb{R}$. Substituting the pair (a, a) into the defining condition gives

$$a - a + \sqrt{2} = \sqrt{2}.$$

This is irrational, independently of the value of a . Consequently, $(a, a) \in S$ for every real number a , and S is reflexive. The cancellation of the two occurrences of a ensures that this argument covers all real numbers, not merely a few examples.

Step 2: Test symmetry using an ordered pair Choose $a = \sqrt{2}$ and $b = 0$, both of which belong to the underlying set $\mathbb{R}$. In the forward order,

$$a - b + \sqrt{2} = 2\sqrt{2},$$

which is irrational. Hence $(\sqrt{2}, 0) \in S$. In the reverse order, however,

$$b - a + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0,$$

which is rational. Thus $(0, \sqrt{2}) \notin S$. This single pair satisfies the hypothesis of symmetry but violates its required consequence, so S is not symmetric. The fixed added term $\sqrt{2}$ prevents reversing the pair from merely negating the entire defining expression.

Step 3: Test transitivity with three real numbers Take $a = 0$, $b = \frac{\sqrt{2}}{2}$ and $c = \sqrt{2}$. For the first two ordered pairs we obtain

$$\begin{aligned} a - b + \sqrt{2} &= \frac{\sqrt{2}}{2}, \\ b - c + \sqrt{2} &= \frac{\sqrt{2}}{2}. \end{aligned}$$

Both expressions are irrational, so $(a, b) \in S$ and $(b, c) \in S$. But for the pair demanded by transitivity,

$$a - c + \sqrt{2} = 0 - \sqrt{2} + \sqrt{2} = 0,$$

which is rational. Therefore $(a, c) \notin S$, even though both premises hold. This disproves transitivity; sums or differences involving irrational numbers need not stay irrational.

Step 4: Conclusion The relation S is reflexive, but neither symmetric nor transitive.

Quick Tip to Solve: Make the failed condition equal to zero: use $(\sqrt{2}, 0)$ for symmetry and the chain $0, \sqrt{2}/2, \sqrt{2}$ for transitivity.

33(a). Find the distance between the line $\frac{x}{2} = \frac{2y-6}{4} = \frac{1-z}{-1}$ and another line parallel to it passing through the point $(4, 0, -5)$. [5 marks]

Solution:

Concept:

- A line written as $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ passes through the point with position vector $\mathbf{a}$ and has direction vector $\mathbf{d} \neq \mathbf{0}$.
- The distance between parallel lines equals the perpendicular distance from any point of one line to the other: $D = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$.
- The cross-product magnitude is the area of the parallelogram with adjacent vectors $\mathbf{p} - \mathbf{a}$ and $\mathbf{d}$. Dividing by the base length $|\mathbf{d}|$ gives its perpendicular height.

Step 1: Extract a point and the correct direction vector Set the common ratio equal to t :

$$\frac{x}{2} = \frac{2y-6}{4} = \frac{1-z}{-1} = t.$$

Solving each equality for its coordinate gives

$$x = 2t, \quad y = 3 + 2t, \quad z = 1 + t.$$

Thus the given line has vector form

$$\mathbf{r} = (0, 3, 1) + t(2, 2, 1).$$

We may take $A = (0, 3, 1)$ and $\mathbf{d} = (2, 2, 1)$. Notice that the coefficient of t in y is 2, not 4, and the coefficient in z is positive. These follow from solving the printed expressions, rather than reading their denominators directly.

Step 2: Describe the second parallel line and its separation The parallel line through $P = (4, 0, -5)$ has the same direction vector, so its equation is

$$\mathbf{r} = (4, 0, -5) + s(2, 2, 1).$$

The vector joining the selected points on the two lines is

$$\overrightarrow{AP} = (4, -3, -6).$$

Its length alone is not the line separation because it may contain a component parallel to the lines. We need its perpendicular component. All points on the second line have the same perpendicular distance from the first line, since translating along their common direction changes only the parallel component.

Step 3: Compute the cross product and the perpendicular distance Using the point-to-line distance

Quick Tip to Solve: First convert each printed ratio to a coordinate in terms of the common parameter; this prevents the misleading denominators from giving the wrong direction vector.

formula,

$$\begin{aligned}\vec{AP} \times \mathbf{d} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -6 \\ 2 & 2 & 1 \end{vmatrix} \\ &= 9\mathbf{i} - 16\mathbf{j} + 14\mathbf{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\vec{AP} \times \mathbf{d}| &= \sqrt{9^2 + (-16)^2 + 14^2} = \sqrt{533}, \\ |\mathbf{d}| &= \sqrt{2^2 + 2^2 + 1^2} = 3, \\ D &= \frac{\sqrt{533}}{3}.\end{aligned}$$

As an arithmetic check, $|\vec{AP}|^2 = 61$ and $\vec{AP} \cdot \mathbf{d} = -4$. The squared perpendicular component is therefore $61 - \frac{16}{9} = \frac{533}{9}$, agreeing with the cross-product result.

Step 4: Conclusion The distance between the two parallel lines is $\boxed{\frac{\sqrt{533}}{3} \text{ units}}.$

Quick Tip to Solve: First convert each printed ratio to a coordinate in terms of the common parameter; this prevents the misleading denominators from giving the wrong direction vector.

OR

33(b). If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-7}$ are perpendicular to each other, find the value of k and hence write the vector equation of a line perpendicular to these two lines and passing through the point $(3, -4, 7)$. [5 marks]

Solution:

Concept:

- In a symmetric line equation, the three denominators give direction ratios. Two nonzero direction vectors are perpendicular when their scalar product is zero.
- The vector product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. A line through position vector $\vec{p}$, with nonzero direction vector $\vec{d}$, is $\vec{r} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$.

Step 1: Determine the parameter from perpendicularity Read the direction vectors directly from the two symmetric equations:

$$\vec{a} = -3\hat{i} + 2k\hat{j} + 2\hat{k}, \quad \vec{b} = 3k\hat{i} + \hat{j} - 7\hat{k}.$$

Their initial points do not enter the condition for perpendicular directions. Since the two lines are

Quick Tip to Solve: First substitute the value of k into both direction vectors; then use their cross product, checking each dot product to catch a middle-component sign error.

perpendicular,

$$\begin{aligned}\vec{a} \cdot \vec{b} &= 0, \\ (-3)(3k) + (2k)(1) + (2)(-7) &= 0, \\ -9k + 2k - 14 &= 0, \\ -7k &= 14, \\ k &= -2.\end{aligned}$$

On substitution, the direction vectors become $\vec{a} = (-3, -4, 2)$ and $\vec{b} = (-6, 1, -7)$.

Step 2: Find a direction perpendicular to both lines Take their vector product, keeping the negative sign in the middle component:

$$\begin{aligned}\vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -4 & 2 \\ -6 & 1 & -7 \end{vmatrix} \\ &= \{28 - 2\}\hat{i} - \{21 + 12\}\hat{j} + \{-3 - 24\}\hat{k} \\ &= 26\hat{i} - 33\hat{j} - 27\hat{k}.\end{aligned}$$

This vector is nonzero. Check both required scalar products:

$$\begin{aligned}(26, -33, -27) \cdot (-3, -4, 2) &= -78 + 132 - 54 = 0, \\ (26, -33, -27) \cdot (-6, 1, -7) &= -156 - 33 + 189 = 0.\end{aligned}$$

Thus it has precisely the required perpendicular direction; any nonzero scalar multiple would give the same line.

Step 3: Use the specified point The point $(3, -4, 7)$ has position vector $\vec{p} = 3\hat{i} - 4\hat{j} + 7\hat{k}$. Insert this point and the direction $\vec{d}$ into $\vec{r} = \vec{p} + t\vec{d}$. At $t = 0$ the line passes through the required point, and its direction has already been checked against both original lines.

Step 4: Conclusion

$$\boxed{k = -2}, \quad \boxed{\vec{r} = (3\hat{i} - 4\hat{j} + 7\hat{k}) + t(26\hat{i} - 33\hat{j} - 27\hat{k}), \quad t \in \mathbb{R}.}$$

Quick Tip to Solve: First substitute the value of k into both direction vectors; then use their cross product, checking each dot product to catch a middle-component sign error.

34. Find A^{-1} , if $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. Hence, solve the following system of equations:

$$x + 2y + z = 5$$

$$2x + 3y = 1$$

$$x - y + z = 8$$

[5 marks]

Solution:

Concept:

- A square matrix is invertible when its determinant is nonzero. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- For a column vector $X = (x, y, z)^T$, multiplication by A forms the linear expressions specified by its rows. Express the printed equations in these row expressions before multiplying by A^{-1} .

Step 1: Evaluate the determinant Expanding along the first row gives

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \\ &= 3 - 2(2 + 1) + (0 - 3) = -6. \end{aligned}$$

Since $|A| \neq 0$, the inverse exists.

Step 2: Form the cofactor matrix and inverse Using the alternating cofactor signs, the nine cofactors are

$$\begin{aligned} C_{11} &= 3, & C_{12} &= -3, & C_{13} &= -3, \\ C_{21} &= -2, & C_{22} &= 0, & C_{23} &= 2, \\ C_{31} &= -5, & C_{32} &= 3, & C_{33} &= -1. \end{aligned}$$

Transpose this array and divide by the determinant:

$$\begin{aligned} \text{adj } A &= \begin{pmatrix} 3 & -2 & -5 \\ -3 & 0 & 3 \\ -3 & 2 & -1 \end{pmatrix}, \\ A^{-1} &= \frac{1}{-6} \begin{pmatrix} -3 & 2 & 5 \\ 3 & 0 & -3 \\ 3 & -2 & 1 \end{pmatrix}. \end{aligned}$$

Step 3: Express the printed equations using A The coefficient matrix of the printed system has second row (2, 3, 0) and third row (1, -1, 1), while A has rows (2, 3, -1) and (1, 0, 1) there. Thus, to use the inverse just obtained, write the given equations as

$$AX = \begin{pmatrix} x + 2y + z \\ 2x + 3y - z \\ x + z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 - z \\ 8 + y \end{pmatrix}.$$

Quick Tip to Solve: Compare each printed equation with the corresponding row of A . Here use the adjusted column (5, 1-z, 8+y) before multiplying by the inverse.

Multiplication by A^{-1} is valid even though the column on the right contains unknowns. It produces equivalent equations:

$$6x = -3(5) + 2(1 - z) + 5(8 + y) = 27 + 5y - 2z,$$

$$6y = 3(5) - 3(8 + y) = -9 - 3y,$$

$$6z = 3(5) - 2(1 - z) + (8 + y) = 21 + y + 2z.$$

Step 4: Solve and verify the values The second equation gives $9y = -9$, hence $y = -1$. The third then gives $4z = 21 + y = 20$, hence $z = 5$. Finally, the first gives $6x = 27 - 5 - 10 = 12$, hence $x = 2$. All three values satisfy the actual printed system:

$$x + 2y + z = 2 - 2 + 5 = 5,$$

$$2x + 3y = 4 - 3 = 1,$$

$$x - y + z = 2 + 1 + 5 = 8.$$

Step 5: Conclusion

$$A^{-1} = \frac{1}{6} \begin{pmatrix} -3 & 2 & 5 \\ 3 & 0 & -3 \\ 3 & -2 & 1 \end{pmatrix}, \quad \boxed{x = 2, y = -1, z = 5}.$$

Quick Tip to Solve: Compare each printed equation with the corresponding row of A . Here use the adjusted column $(5, 1-z, 8+y)$ before multiplying by the inverse.

35(a). Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, X -axis and the ordinates $x = -2$ and $x = 2$, using integration. [5 marks]

Solution:

Concept:

- The absolute value satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$. Thus the graph must be drawn using separate formulas on the two sides of the origin.
- Geometrical area between a curve $y = f(x)$ and the X -axis is $\int |f(x)| dx$. A portion below the axis contributes positive area even though its signed integral is negative.
- An odd function satisfies $f(-x) = -f(x)$; its graph is symmetric about the origin. Corresponding regions on opposite sides of the origin have equal geometrical areas.

Step 1: Express the curve in piecewise form Using the definition of absolute value,

$$y = x|x| = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

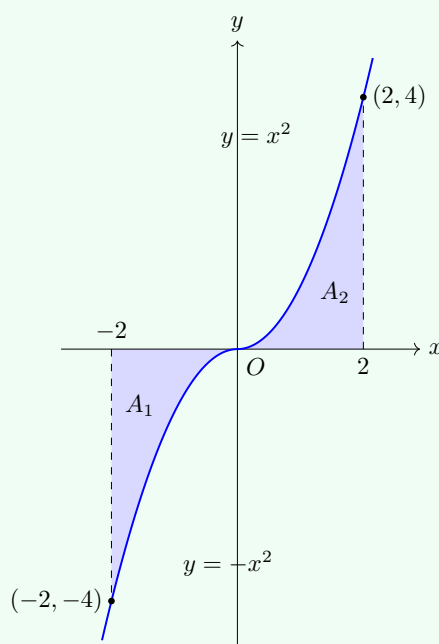
Quick Tip to Solve: Split at the zero $x = 0$ before integrating: turn the negative branch's signed integral positive, or use symmetry to compute $2 \int_0^2 x^2 dx$.

Therefore the negative branch is the left half of $y = -x^2$, lying in the third quadrant, and the non-negative branch is the right half of $y = x^2$, lying in the first quadrant. Both branches meet at the origin $(0, 0)$, which is the only intersection with the X -axis. In particular, the ordinates $x = -2$ and $x = 2$ meet the curve at $(-2, -4)$ and $(2, 4)$, respectively.

Step 2: Sketch the bounded regions Plot the points

$$(-2, -4), \quad (-1, -1), \quad (0, 0), \quad (1, 1), \quad (2, 4).$$

Join them by the appropriate smooth parabolic branches. Draw the vertical bounding ordinates at $x = -2$ and $x = 2$. The shaded region on the left lies below the X -axis, whereas the shaded region on the right lies above it. Since $x|x|$ is odd, the two shaded regions are images of each other under a rotation through 180° about the origin.



The curve $y = x|x|$ and the two bounded regions

Step 3: Evaluate the area below the axis For $-2 \leq x \leq 0$, the upper boundary is $y = 0$ and the lower boundary is $y = -x^2$. Hence

$$A_1 = \int_{-2}^0 (0 - (-x^2)) dx = \left[\frac{x^3}{3} \right]_{-2}^0 = 0 - \left(-\frac{8}{3} \right) = \frac{8}{3}.$$

Subtracting the lower boundary from the upper boundary ensures that the area is positive.

Step 4: Evaluate the area above the axis and combine For $0 \leq x \leq 2$, the upper boundary is $y = x^2$ and the lower boundary is $y = 0$. Thus

$$A_2 = \int_0^2 (x^2 - 0) dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}.$$

Quick Tip to Solve: Split at the zero $x = 0$ before integrating: turn the negative branch's signed integral positive, or use symmetry to compute $2 \int_0^2 x^2 dx$.

Adding the two disjoint regions gives

$$A = A_1 + A_2 = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.$$

Integrating $x|x|$ directly from -2 to 2 would give zero because the signed contributions cancel; that cancellation does not represent the geometrical area.

Step 5: Conclusion The graph consists of $y = -x^2$ for $x < 0$ and $y = x^2$ for $x \geq 0$, meeting at the origin. The required bounded area is $\boxed{\frac{16}{3} \text{ square units}}$.

Quick Tip to Solve: Split at the zero $x = 0$ before integrating: turn the negative branch's signed integral positive, or use symmetry to compute $2 \int_0^2 x^2 dx$.

OR

35(b). Using integration, find the area bounded by the ellipse $9x^2 + 25y^2 = 225$, the lines $x = -2$, $x = 2$, and the X -axis. [5 marks]

Solution:

Concept:

- The area between a nonnegative curve $y = f(x)$, the X -axis and the ordinates $x = a$ and $x = b$ is $A = \int_a^b f(x) dx$.
- For an ellipse $x^2/a^2 + y^2/b^2 = 1$, its upper branch is $y = (b/a)\sqrt{a^2 - x^2}$; reflection in either coordinate axis preserves area.
- A trigonometric substitution $x = a \sin \theta$ evaluates $\int \sqrt{a^2 - x^2} dx$, using $\cos^2 \theta = (1 + \cos 2\theta)/2$.

Step 1: Express the upper boundary of the region Dividing the given ellipse by 225 gives

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

The semiaxes are 5 and 3. Its upper branch is

$$y = 3\sqrt{1 - \frac{x^2}{25}} = \frac{3}{5}\sqrt{25 - x^2}.$$

On $[-2, 2]$, a vertical strip between this curve and $y = 0$ has area $\frac{3}{5}\sqrt{25 - x^2} dx$.

Step 2: Set up the definite integral The integrand is even, so left-right symmetry gives:

$$A = \frac{3}{5} \int_{-2}^2 \sqrt{25 - x^2} dx = \frac{6}{5} \int_0^2 \sqrt{25 - x^2} dx.$$

Step 3: Evaluate by trigonometric substitution Put $x = 5 \sin \theta$. Then $dx = 5 \cos \theta d\theta$ and $\sqrt{25 - x^2} =$

Quick Tip to Solve: Write the ellipse in standard form first, then use left-right symmetry to integrate its upper ordinate only from zero to two.

$5 \cos \theta$ on the integration interval. Let $\alpha = \sin^{-1}(2/5)$; the limits become 0 and α . Therefore,

$$\begin{aligned} A &= 30 \int_0^\alpha \cos^2 \theta \, d\theta \\ &= 15 \int_0^\alpha (1 + \cos 2\theta) \, d\theta \\ &= 15 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^\alpha. \end{aligned}$$

Here $\sin \alpha = 2/5$ and $\cos \alpha = \sqrt{21}/5$, since α lies in the first quadrant. Thus $\sin 2\alpha = 4\sqrt{21}/25$, and

$$A = 15 \sin^{-1} \left(\frac{2}{5} \right) + \frac{6\sqrt{21}}{5}.$$

Step 4: Conclusion The area bounded with the X -axis is $15 \sin^{-1}(2/5) + 6\sqrt{21}/5$ square units, with the inverse sine in radians. The lower bounded region has the same area.

Quick Tip to Solve: Write the ellipse in standard form first, then use left-right symmetry to integrate its upper ordinate only from zero to two.

Section E (12 marks)

In this section, there are 3 case study based question of 4 marks each.

36. Case Study - 3: Rohit, Jaspreet and Alia appeared for an interview for three vacancies in the same post. The probability of Rohit's selection is $\frac{1}{5}$, Jaspreet's selection is $\frac{1}{3}$ and Alia's selection is $\frac{1}{4}$. The event of selection is independent of each other. [4 marks]

36(i). What is the probability that at least one of them is selected? [1 mark]

Solution:

Concept:

- For an event E , $P(E^c) = 1 - P(E)$. The complement of at least one selection is no selection.
- For independent selection events, multiply the relevant selection or nonselection probabilities to find a joint outcome.

Step 1: Find the probability that none is selected Let R, J, A denote selection of Rohit, Jaspreet and Alia respectively. Their nonselection probabilities are $4/5$, $2/3$ and $3/4$. Independence also holds for their complements, so

$$P(R^c \cap J^c \cap A^c) = \frac{4}{5} \times \frac{2}{3} \times \frac{3}{4} = \frac{2}{5}.$$

Step 2: Apply the complement rule At least one selected includes exactly one, exactly two and all three selected. These possibilities together are complementary to none selected. Hence

$$P(\text{at least one selected}) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Quick Tip to Solve: For at least one independent success, subtract the product of all failure probabilities from one.

Adding the three individual selection probabilities directly would count outcomes involving multiple selections more than once.

Step 3: Conclusion The required probability is $\boxed{3/5}$.

Quick Tip to Solve: For at least one independent success, subtract the product of all failure probabilities from one.

36(ii). Find $P(G | \overline{H})$ where G is the event of Jaspreet's selection and $\overline{H}$ denotes the event that Rohit is not selected. [1 mark]

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, provided $P(F) > 0$.
- Independence gives $P(E \cap F) = P(E)P(F)$; independence of two events also implies independence of one event and the complement of the other.

Step 1: Evaluate the joint probability Let H be Rohit's selection event, so the question's event $\overline{H}$ means his nonselection. Then $P(\overline{H}) = 1 - 1/5 = 4/5$. Jaspreet's selection event G has probability $1/3$. Independence yields

$$P(G \cap \overline{H}) = \frac{1}{3} \times \frac{4}{5} = \frac{4}{15}.$$

Step 2: Divide by the conditioning probability Since $P(\overline{H}) > 0$, the conditional probability is defined. Thus

$$P(G | \overline{H}) = \frac{4/15}{4/5} = \frac{1}{3}.$$

Knowing that Rohit is not selected does not change Jaspreet's selection probability because their outcomes are independent.

Step 3: Conclusion The required conditional probability is $\boxed{P(G | \overline{H}) = 1/3}$.

Quick Tip to Solve: When the conditioning event concerns another independent candidate, the original selection probability stays unchanged.

36(iii)(a). Find the probability that exactly one of them is selected. [2 marks]

Solution:

Concept:

- Exactly one of three selections is the union of three mutually exclusive outcomes, each specifying one selected candidate and two nonselected candidates.
- For independent candidates, multiply the three probabilities within each outcome; then add the probabilities of mutually exclusive outcomes.

Step 1: List the three disjoint possibilities Let R, J, A be selection of Rohit, Jaspreet and Alia.

Quick Tip to Solve: For exactly one selection, write one success factor and two failure factors in each of the three terms.

Their selection probabilities are $1/5, 1/3, 1/4$, and their complementary probabilities are $4/5, 2/3, 3/4$. Exactly one selected means one of

$$R \cap J^c \cap A^c, \quad R^c \cap J \cap A^c, \quad R^c \cap J^c \cap A.$$

Each outcome specifies all three candidates, so it excludes any second selection. No two of these outcomes can occur together.

Step 2: Calculate and add their probabilities Using the given independence,

$$\begin{aligned} P(\text{Rohit only}) &= \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{10}, \\ P(\text{Jaspreet only}) &= \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{5}, \\ P(\text{Alia only}) &= \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{2}{15}. \end{aligned}$$

By the addition rule for disjoint events, the required probability is

$$\frac{1}{10} + \frac{1}{5} + \frac{2}{15} = \frac{3}{30} + \frac{6}{30} + \frac{4}{30} = \frac{13}{30}.$$

Step 3: Conclusion The probability that exactly one candidate is selected is $\boxed{13/30}$.

Quick Tip to Solve: For exactly one selection, write one success factor and two failure factors in each of the three terms.

OR

36(iii)(b). Find the probability that exactly two of them are selected.

[2 marks]

Solution:

Concept:

- Exactly two selections out of three occur when one of the three possible pairs is selected and the remaining candidate is not selected.
- Independence lets us multiply the probabilities within a specified outcome. The three exact-pair outcomes are disjoint, so their probabilities are added.

Step 1: Identify outcomes with exactly two selections Let R, J, A denote Rohit's, Jaspreet's and Alia's selection events. Their probabilities are $1/5, 1/3, 1/4$, with complementary probabilities $4/5, 2/3, 3/4$. The required event is the disjoint union of

$$R \cap J \cap A^c, \quad R \cap J^c \cap A, \quad R^c \cap J \cap A.$$

The complement factor in each term is necessary: omitting it would also include the outcome in which all three are selected, which is excluded by exactly two.

Quick Tip to Solve: For exactly two independent selections, sum the three products containing two success factors and one failure factor.

Step 2: Multiply for each pair and add The independence given in the case study gives

$$P(R \cap J \cap A^c) = \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{20},$$

$$P(R \cap J^c \cap A) = \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{30},$$

$$P(R^c \cap J \cap A) = \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{15}.$$

Because these outcomes are mutually exclusive,

$$P(\text{exactly two selected}) = \frac{1}{20} + \frac{1}{30} + \frac{1}{15} = \frac{3}{60} + \frac{2}{60} + \frac{4}{60} = \frac{3}{20}.$$

Step 3: Conclusion The probability that exactly two candidates are selected is $\boxed{3/20}$.

Quick Tip to Solve: For exactly two independent selections, sum the three products containing two success factors and one failure factor.

37. Case Study - 1: A store has been selling calculators at ₹350 each. A market survey indicates that a reduction in price (p) of calculator increases the number of units (x) sold. The relation between the price and quantity sold is given by the demand function $p = 450 - \frac{1}{2}x$. [4 marks]

37(i). Determine the number of units (x) that should be sold to maximise the revenue $R(x) = xp(x)$. Also, verify the result. [2 marks]

Solution:

Concept:

- Revenue is the total amount received from sales: $R(x) = xp(x)$, where x is the number of calculators sold and $p(x)$ is the price per calculator.
- At an interior maximum of a differentiable revenue function, $R'(x) = 0$. The condition $R''(x) < 0$ verifies a strict maximum.

Step 1: Form the revenue function Using the given demand relation, $p(x) = 450 - \frac{x}{2}$, we obtain

$$R(x) = x \left(450 - \frac{x}{2} \right) = 450x - \frac{x^2}{2}.$$

The economically meaningful domain has $x \geq 0$ and $p(x) \geq 0$, so $0 \leq x \leq 900$.

Step 2: Find and verify the maximum Differentiating with respect to the quantity sold gives

$$R'(x) = 450 - x.$$

The stationary-point condition is $450 - x = 0$, giving $x = 450$. Further,

$$R''(x) = -1 < 0.$$

Quick Tip to Solve: Form revenue by multiplying demand price by quantity before differentiating; maximising price alone does not maximise sales revenue.

Thus revenue is strictly concave throughout the domain, and this stationary point is the unique maximum. Indeed, $R'(x) > 0$ for $x < 450$ and $R'(x) < 0$ for $x > 450$, confirming that revenue first increases and then decreases.

Step 3: Conclusion The store should sell 450 calculators. The second derivative $R''(450) = -1 < 0$ verifies maximum revenue.

Quick Tip to Solve: Form revenue by multiplying demand price by quantity before differentiating; maximising price alone does not maximise sales revenue.

37(ii). What rebate in price of calculator should the store give to maximise the revenue? [2 marks]

Solution:

Concept:

- The revenue corresponding to quantity x and price $p(x)$ is $R(x) = xp(x)$. Its maximum is found from $R'(x) = 0$ and verified by a negative second derivative.
- The rebate per calculator is the original selling price minus the price that maximises total revenue.

Step 1: Determine the revenue-maximising quantity From $p(x) = 450 - \frac{x}{2}$, the revenue is

$$R(x) = 450x - \frac{x^2}{2}.$$

Therefore $R'(x) = 450 - x$, and setting this equal to zero gives $x = 450$. Since $R''(x) = -1 < 0$, this quantity maximises revenue. The quantity is admissible because it is positive and yields a positive selling price.

Step 2: Find the price and calculate the rebate Substitute the optimal quantity into the demand function:

$$p(450) = 450 - \frac{450}{2} = 225.$$

Hence the calculator should be sold for ₹225 each. The original selling price is ₹350, so the required reduction is

$$350 - 225 = 125.$$

This is the rebate on each calculator; it is not the total reduction in the store's revenue.

Step 3: Conclusion The store should give a rebate of ₹125 per calculator, making the selling price ₹225.

Quick Tip to Solve: After maximising revenue, substitute the optimal quantity into demand, then subtract that price from the original price of 350.

38. Case Study - 2: An instructor at the astronomical centre shows three among the brightest stars in a particular constellation. Assume that the telescope is located at $O(0, 0, 0)$ and the three stars have their locations at the points D , A and V having position vectors $2\hat{i} + 3\hat{j} + 4\hat{k}$, $7\hat{i} + 5\hat{j} + 8\hat{k}$ and $-3\hat{i} + 7\hat{j} + 11\hat{k}$ respectively. [4 marks]

38(i). How far is the star V from star A ? [1 mark]

Solution:

Concept:

- The displacement from A to V is $\overrightarrow{AV} = \vec{OV} - \vec{OA}$.
- The distance between two points is the magnitude of their displacement: $|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$.

Step 1: Find the displacement vector The position vectors give $A = (7, 5, 8)$ and $V = (-3, 7, 11)$. Subtract the coordinates of the initial point A from those of the final point V :

$$\overrightarrow{AV} = (-3 - 7)\hat{i} + (7 - 5)\hat{j} + (11 - 8)\hat{k} = -10\hat{i} + 2\hat{j} + 3\hat{k}.$$

Step 2: Calculate the separation The distance is

$$AV = |\overrightarrow{AV}| = \sqrt{(-10)^2 + 2^2 + 3^2} = \sqrt{100 + 4 + 9} = \sqrt{113}.$$

Using the reverse displacement would change all three signs but leave this magnitude unchanged. The magnitude of $\vec{OV}$ alone would give the distance from the telescope, not from star A .

Step 3: Conclusion The distance of star V from star A is $\boxed{\sqrt{113} \text{ units}}$.

Quick Tip to Solve: Subtract the two position vectors first; taking the magnitude of a single position vector measures distance from the origin.

38(ii). Find a unit vector in the direction of $\vec{DA}$. [1 mark]

Solution:

Concept:

- A vector directed from D to A is $\vec{DA} = \vec{OA} - \vec{OD}$.
- The unit vector along a nonzero vector $\vec{a}$ is $\hat{a} = \vec{a}/|\vec{a}|$, with magnitude one.

Step 1: Obtain the directed vector Using $D = (2, 3, 4)$ and $A = (7, 5, 8)$,

$$\vec{DA} = (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} = 5\hat{i} + 2\hat{j} + 4\hat{k}.$$

The subtraction order matters because the required direction starts at D and ends at A .

Step 2: Normalise the vector Its magnitude is

$$|\vec{DA}| = \sqrt{5^2 + 2^2 + 4^2} = \sqrt{45} = 3\sqrt{5}.$$

Quick Tip to Solve: For a vector named DA , subtract D from A , then divide all components by the same magnitude.

Divide every component by this positive magnitude. The squared magnitude of the resulting vector is $(25 + 4 + 16)/45 = 1$, verifying that it is a unit vector; division by a positive scalar preserves the direction.

Step 3: Conclusion A unit vector in the direction of $\overrightarrow{DA}$ is $\frac{5\hat{i} + 2\hat{j} + 4\hat{k}}{3\sqrt{5}}$.

Quick Tip to Solve: For a vector named DA, subtract D from A, then divide all components by the same magnitude.

38(iii)(a). Find the measure of $\angle VDA$.

[2 marks]

Solution:

Concept:

- The angle $\angle VDA$ has vertex D , so use the two vectors $\overrightarrow{DV}$ and $\overrightarrow{DA}$ directed away from D .
- For nonzero vectors $\vec{a}$ and $\vec{b}$, $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$, with $0 \leq \theta \leq \pi$.

Step 1: Form the vectors at the vertex The coordinates are $D = (2, 3, 4)$, $A = (7, 5, 8)$ and $V = (-3, 7, 11)$. Therefore

$$\begin{aligned}\overrightarrow{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \overrightarrow{DA} &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Using vectors from the telescope instead would calculate an angle at O , which is not the angle asked for.

Step 2: Apply the dot-product formula Calculate

$$\overrightarrow{DV} \cdot \overrightarrow{DA} = (-5)(5) + (4)(2) + (7)(4) = 11.$$

Also,

$$|\overrightarrow{DV}| = \sqrt{25 + 16 + 49} = \sqrt{90}, \quad |\overrightarrow{DA}| = \sqrt{45}.$$

Thus

$$\cos \angle VDA = \frac{11}{\sqrt{90}\sqrt{45}} = \frac{11}{45\sqrt{2}}.$$

The dot product is positive, so the required angle is acute, consistently with this positive cosine.

Step 3: Conclusion $\angle VDA = \cos^{-1} \left(\frac{11}{45\sqrt{2}} \right)$.

Quick Tip to Solve: The middle letter names the vertex: for angle VDA, use DV and DA, both starting at D.

38(iii)(b). What is the projection of vector $\overrightarrow{DV}$ on vector $\overrightarrow{DA}$?

[2 marks]

Solution:

Concept:

- The projection of $\vec{a}$ on $\vec{b}$, in the standard scalar-projection convention, is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.
- This is the signed component of $\vec{a}$ in the direction of $\vec{b}$; the denominator is the magnitude of the vector onto which the projection is taken.

Step 1: Find the two displacement vectors Using the stated position vectors, subtract the coordinates of D from the destination coordinates:

$$\begin{aligned}\overrightarrow{DV} &= (-3 - 2)\hat{i} + (7 - 3)\hat{j} + (11 - 4)\hat{k} \\ &= -5\hat{i} + 4\hat{j} + 7\hat{k}, \\ \overrightarrow{DA} &= (7 - 2)\hat{i} + (5 - 3)\hat{j} + (8 - 4)\hat{k} \\ &= 5\hat{i} + 2\hat{j} + 4\hat{k}.\end{aligned}$$

Step 2: Evaluate the projection on DA The dot product and the required magnitude are

$$\begin{aligned}\overrightarrow{DV} \cdot \overrightarrow{DA} &= (-5)(5) + 4(2) + 7(4) = 11, \\ |\overrightarrow{DA}| &= \sqrt{25 + 4 + 16} = 3\sqrt{5}.\end{aligned}$$

Consequently, the scalar projection is

$$\frac{\overrightarrow{DV} \cdot \overrightarrow{DA}}{|\overrightarrow{DA}|} = \frac{11}{3\sqrt{5}}.$$

Its positive sign shows that the component lies along the direction from D to A . Dividing by $|\overrightarrow{DV}|$ instead would project $\overrightarrow{DA}$ onto $\overrightarrow{DV}$, reversing the requested roles.

Step 3: Conclusion The projection of $\overrightarrow{DV}$ on $\overrightarrow{DA}$ is $\boxed{\frac{11}{3\sqrt{5}} \text{ units}}.$

Quick Tip to Solve: In projection of DV on DA , place the magnitude of DA in the denominator of the dot-product formula.