CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-5-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This Question paper contains **38** questions. **All** questions are **compulsory**.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Questions Number **1** to **18** are Multiple Choice Questions (MCQs) type and Questions Number **19 & 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Questions Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C** – Questions Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Questions Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.

(vii) In **Section E** – Questions Number **36** to **38** are case study based questions, carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

*This section has **20** multiple choice questions of **1** mark each.*

1. Derivative of $e^{\sin^2 x}$ with respect to $\cos x$ is: [1 mark]

(A) $\sin x\, e^{\sin^2 x}$

(B) $\cos x\, e^{\sin^2 x}$

(C) $-2\cos x\, e^{\sin^2 x}$

(D) $-2\sin^2 x \cos x\, e^{\sin^2 x}$

Correct Answer: (C) $-2\cos x\, e^{\sin^2 x}$

Solution:

Concept:

- To differentiate with respect to another expression, first express the dependent quantity in terms of that expression.
- Use $\sin^2 x = 1 - \cos^2 x$ and the chain rule $\dfrac{d}{du}e^{g(u)} = e^{g(u)}g'(u)$.

Step 1: Change the independent variable Put $u = \cos x$. The quantity to be differentiated is then

$$y = e^{\sin^2 x} = e^{1-u^2}.$$

This substitution expresses y directly in terms of the requested variable.

Step 2: Apply the chain rule Differentiate the exponential and multiply by the derivative of its exponent:

$$\frac{dy}{du} = e^{1-u^2}(-2u) = -2\cos x\, e^{\sin^2 x}.$$

Options (A) and (B) miss the factor from differentiating $-u^2$. Option (D) contains an extra factor $\sin^2 x$, which the chain rule does not produce.

Step 3: Conclusion The required derivative is $\boxed{-2\cos x\, e^{\sin^2 x}}$. Hence, the correct option is (C).

Quick Tip to Solve: For differentiation with respect to $\cos x$, replace $\sin^2 x$ by $1 - \cos^2 x$ before differentiating.

2. If A is a square matrix of order 2 and $|A| = -2$, then the value of $|5A'|$ is: [1 mark]

(A) -50

(B) -10

(C) 10

(D) 50

Correct Answer: (A) -50

Solution:

Concept:

- Transposing a square matrix does not change its determinant: $|A'| = |A|$.
- For a square matrix of order n, scalar multiplication gives $|kA| = k^n|A|$, since each of its n rows contributes a factor k.

Step 1: Use the transpose property The symbol A' denotes the transpose of A. Thus its determinant is still

$$|A'| = |A| = -2.$$

Transposition interchanges rows and columns; it does not reverse the determinant's sign.

Step 2: Account for the order of the matrix Since A' is of order 2, multiplying the matrix by 5 multiplies both rows by 5. Therefore

$$|5A'| = 5^2|A'| = 25(-2) = -50.$$

Option (B) uses just one factor of 5. Options (C) and (D) incorrectly make the determinant positive; a positive scalar factor cannot change its sign here.

Step 3: Conclusion $\boxed{|5A'| = -50}$. Hence, the correct option is (A).

Quick Tip to Solve: Check the matrix order before taking a scalar outside a determinant: order 2 means the scalar is squared.

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3. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minima at x equal to: [1 mark]

(A) 2

(B) 1

(C) 0

(D) -2

Correct Answer: (A) 2

Solution:

Concept:

- An interior stationary point of a differentiable function satisfies $f'(x) = 0$.
- At a stationary point $x = a$, $f''(a) > 0$ implies a local minimum, whereas $f''(a) < 0$ implies a local

Quick Tip to Solve: When solving $f'(x) = 0$ produces both signs of a root, classify both with f'' before choosing a minimum.

maximum.

Step 1: Find the stationary points The domain excludes $x = 0$. For $x \neq 0$, differentiate the given function:

$$f'(x) = \frac{1}{2} - \frac{2}{x^2}.$$

Setting this equal to zero gives $x^2 = 4$, so the possible stationary points are $x = 2$ and $x = -2$. At $x = 1$, the derivative is $-3/2$, so option (B) is not a stationary point.

Step 2: Apply the second derivative test Differentiate once more and evaluate at the two candidates:

$$f''(x) = \frac{4}{x^3}, \qquad f''(2) = \frac{1}{2} > 0, \qquad f''(-2) = -\frac{1}{2} < 0.$$

Thus the positive candidate gives a local minimum, while option (D) gives a local maximum. Option (C) is outside the function's domain.

Step 3: Conclusion The function has a local minimum at $\boxed{x = 2}$. Hence, the correct option is (A).

Quick Tip to Solve: When solving $f'(x) = 0$ produces both signs of a root, classify both with f'' before choosing a minimum.

4. Given a curve $y = 7x - x^3$ and x increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when $x = 5$ is: [1 mark]

(A) -60 units/sec

(B) 60 units/sec

(C) -70 units/sec

(D) -140 units/sec

Correct Answer: (A) -60 units/sec

Solution:

Concept:

- The slope of a curve $y = f(x)$ is $m = \frac{dy}{dx}$.
- If x varies with time t, the slope changes at the rate $\frac{dm}{dt} = \frac{dm}{dx}\frac{dx}{dt}$, by the chain rule.

Step 1: Express the slope as a function of position Differentiate $y = 7x - x^3$ with respect to x:

$$m = \frac{dy}{dx} = 7 - 3x^2, \qquad \frac{dm}{dx} = -6x.$$

The problem asks for the time rate of change of this slope, not the rate of change of y itself.

Step 2: Use the given rate of increase of x Since $\frac{dx}{dt} = 2$, substitute into the chain rule at $x = 5$:

$$\frac{dm}{dt} = (-6x)(2) = -12x, \qquad \left.\frac{dm}{dt}\right|_{x=5} = -60.$$

Quick Tip to Solve: For the time rate of change of slope, differentiate the curve twice with respect to x, then multiply by dx/dt.

The sign is negative because the slope becomes smaller as positive x increases, ruling out (B). The magnitudes in (C) and (D) do not follow from differentiating the slope and multiplying by the given rate.

Step 3: Conclusion The slope changes at $\boxed{-60\,\text{units/sec}}$. Hence, the correct option is (A).

Quick Tip to Solve: For the time rate of change of slope, differentiate the curve twice with respect to x, then multiply by dx/dt.

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5. The product of matrix P and Q is equal to a diagonal matrix. If the order of matrix Q is 3×2, then order of matrix P is: [1 mark]

(A) 2×2

(B) 3×3

(C) 2×3

(D) 3×2

Correct Answer: (C) 2×3

Solution:

Concept:

- A product of matrices of orders $m \times n$ and $n \times p$ is defined and has order $m \times p$.
- A diagonal matrix is square, so its number of rows equals its number of columns.

Step 1: Make the product possible The given matrix Q has order 3×2. For the product PQ to exist, P must have exactly 3 columns. Write its order as $m \times 3$. Then

$$(m \times 3)(3 \times 2) \quad \text{gives a product of order } m \times 2.$$

Step 2: Use the diagonal condition Since PQ is diagonal, it must be square. Hence $m = 2$. Options (A) and (D) give only 2 columns in P, so the product with Q cannot be formed. Option (B) gives a 3×2 product, which is not square and therefore cannot be a diagonal matrix.

Step 3: Conclusion The order of P is $\boxed{2 \times 3}$. Hence, the correct option is (C).

Quick Tip to Solve: First match the inner dimensions for multiplication; then make the outer dimensions equal because the product is diagonal.

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6. A function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2 - 4x + 5$ is: [1 mark]

(A) injective but not surjective.

(B) surjective but not injective.

(C) both injective and surjective.

(D) neither injective nor surjective.

Correct Answer: (D) neither injective nor surjective.

Solution:

Concept:

- A function is injective if distinct inputs always have distinct outputs. A single pair of distinct inputs with the same image disproves injectivity.
- A function is surjective if its range equals its stated codomain. Here both the domain and codomain are $\mathbb{R}$.

Step 1: Test injectivity using equal images Complete the square:

$$f(x) = x^2 - 4x + 5 = (x - 2)^2 + 1.$$

Inputs equally far from 2 have equal images. In particular, $f(1) = 2 = f(3)$, although $1 \neq 3$. Hence the function is not injective, eliminating options (A) and (C).

Step 2: Compare the range with the codomain Since $(x - 2)^2 \geq 0$, every output satisfies $f(x) \geq 1$. Conversely, every $y \geq 1$ is attained by $x = 2 + \sqrt{y - 1}$. Thus the range is $[1, \infty)$, which is smaller than the codomain $\mathbb{R}$. For example, 0 has no preimage. This disproves surjectivity and rules out option (B).

Step 3: Conclusion The function is neither injective nor surjective. Hence, the correct option is (D).

Quick Tip to Solve: Rewrite a real quadratic in completed-square form: its symmetry tests injectivity and its minimum or maximum reveals its range.

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7. If $\sin(xy) = 1$, then $\frac{dy}{dx}$ is equal to: [1 mark]

(A) $\frac{x}{y}$

(B) $-\frac{x}{y}$

(C) $\frac{y}{x}$

(D) $-\frac{y}{x}$

Correct Answer: (D) $-\frac{y}{x}$

Solution:

Concept:

- For real u, $\sin u = 1$ implies $u = \frac{\pi}{2} + 2k\pi$, where k is an integer.

Quick Tip to Solve: When a sine equals 1, first express its argument as a constant branch; never cancel its cosine, which is zero.

- On a differentiable branch, the product rule gives $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$.

Step 1: Express the trigonometric constraint The given equation requires

$$xy = \frac{\pi}{2} + 2k\pi.$$

On any differentiable branch, xy is continuous and cannot jump between these separated values; hence k is constant locally. Also $x \neq 0$, because $x = 0$ would make $\sin(xy) = 0$.

Step 2: Differentiate the constant product Differentiate the branch equation with respect to x:

$$y + x\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Options (A) and (B) interchange the numerator and denominator. Option (C) misses the negative sign from moving y to the other side. Direct differentiation of the sine gives $\cos(xy)(y + xy') = 0$, but $\cos(xy) = 0$ here, so cancelling that factor would be invalid.

Step 3: Conclusion $\boxed{\frac{dy}{dx} = -\frac{y}{x}}$. Hence, the correct option is (D).

Quick Tip to Solve: When a sine equals 1, first express its argument as a constant branch; never cancel its cosine, which is zero.

8. If inverse of matrix $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ **is the matrix** $\begin{pmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{pmatrix}$**, then value of** λ **is:** [1 mark]

(A) -4

(B) 1

(C) 3

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- If $B = A^{-1}$, then $AB = I$, where I is the identity matrix of the same order.
- An entry of a matrix product is the scalar product of a row of the first matrix and a column of the second. Diagonal entries of I are 1, and off-diagonal entries are 0.

Step 1: Choose an entry containing the unknown Let the first displayed matrix be A and its stated inverse be B. The second row of A is $(-1, 1, 0)$, while the second column of B is $(3, \lambda, 3)^T$. Thus

$$(AB)_{22} = (-1)(3) + (1)(\lambda) + (0)(3) = \lambda - 3.$$

Since this is a diagonal entry of I, it must equal 1.

Quick Tip to Solve: To find one unknown in a stated inverse, use a convenient entry of $AB = I$ instead of calculating the entire inverse.

Step 2: Solve and check the condition We obtain $\lambda - 3 = 1$, so $\lambda = 4$. As an independent check, the off-diagonal entry in row 1, column 2 becomes

$$(AB)_{12} = 7(3) - 3\lambda - 3(3) = 12 - 3\lambda = 0$$

at this value. For options (A), (B) and (C), the diagonal expression $\lambda - 3$ equals -7, -2 and 0, respectively, instead of 1.

Step 3: Conclusion The required value is $\boxed{\lambda = 4}$. Hence, the correct option is (D).

Quick Tip to Solve: To find one unknown in a stated inverse, use a convenient entry of $AB = I$ instead of calculating the entire inverse.

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9. Find the matrix A^2, where $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by $a_{ij} =$ maximum$(i, j) -$ minimum(i, j): [1 mark]

(A) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Correct Answer: (C) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

Concept:

- The entry $a_{ij} = \max(i, j) - \min(i, j) = |i - j|$ is zero on the diagonal.
- The square $A^2 = A \cdot A$ uses row-by-column multiplication, not the separate squaring of entries.

Step 1: Form the matrix For $i, j \in \{1, 2\}$, the four entries are $a_{11} = 0$, $a_{12} = 1$, $a_{21} = 1$ and $a_{22} = 0$. Thus

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Step 2: Multiply the matrix by itself Using the rows of the first factor and columns of the second gives

$$A^2 = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Option (B) is the original matrix rather than its square; (A) has incorrect diagonal entries and (D) has incorrect off-diagonal entries.

Quick Tip to Solve: This matrix swaps two coordinates; applying the swap twice returns each coordinate to its original place.

Step 3: Conclusion $A^2 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the correct option is (C).

Quick Tip to Solve: This matrix swaps two coordinates; applying the swap twice returns each coordinate to its original place.

10. If A is a square matrix of order 3 and $|\text{adj} \cdot A| = 8$, then the value of $|A^T|$ is: [1 mark]

(A) $\sqrt{2}$

(B) $-\sqrt{2}$

(C) 8

(D) $2\sqrt{2}$

Correct Answer: (D) $2\sqrt{2}$

Solution:

Concept:

- For a square matrix of order n, $|\text{adj } A| = |A|^{n-1}$.
- Transposition preserves a determinant: $|A^T| = |A|$.

Step 1: Apply the adjugate identity Put $d = |A|$. Since the order is 3, the given condition gives

$$d^2 = 8, \qquad d = \pm\sqrt{8} = \pm 2\sqrt{2}.$$

The determinant itself is not assumed positive; both signs satisfy the printed condition.

Step 2: Compare the listed values The determinant of A^T has the same value as d. Of the listed options, only $2\sqrt{2}$ has square 8. The squares of $\sqrt{2}$ and $-\sqrt{2}$ are 2, and the square of 8 is 64. The other possible value $-2\sqrt{2}$ is not listed. Thus (D) is the sole compatible listed value, although the data do not uniquely fix its sign.

Step 3: Conclusion $|A^T| = \pm 2\sqrt{2}$; the compatible listed answer is $2\sqrt{2}$, option (D).

Quick Tip to Solve: Use the exponent n-1 for the adjugate determinant, and retain both signs when solving a squared determinant.

11. $\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta \, d\theta$ **is:** [1 mark]

(A) $\frac{1}{2}$

(B) $-\frac{1}{2}$

(C) 0

(D) $-\frac{\pi}{8}$

Correct Answer: (A) $\frac{1}{2}$

Solution:

Concept:

- The derivative $\frac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta$ gives a direct substitution.
- A definite integral can be evaluated by substituting both its variable and its endpoint values.

Step 1: Substitute the cotangent Let $u = \cot\theta$; then $du = -\operatorname{cosec}^2\theta \, d\theta$. At $\theta = \pi/4$, $u = 1$; at $\theta = \pi/2$, $u = 0$. The new limits are therefore in decreasing order.

Step 2: Evaluate and check the sign

$$I = -\int_1^0 u \, du = -\left[\frac{u^2}{2}\right]_1^0 = \frac{1}{2}.$$

The original integrand is positive in the interior of the interval, so the integral must be positive. This rules out the negative values in (B) and (D), as well as zero in (C).

Step 3: Conclusion The value of the integral is $\frac{1}{2}$. Hence, the correct option is (A).

Quick Tip to Solve: When substituting u = cot(theta), change the limits immediately so its negative derivative cannot cause a sign error.

12. The integral $\int \frac{dx}{\sqrt{9-4x^2}}$ is equal to: [1 mark]

(A) $\frac{1}{6}\sin^{-1}\left(\frac{2x}{3}\right)+c$

(B) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

(C) $\sin^{-1}\left(\frac{2x}{3}\right)+c$

(D) $\frac{3}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

Correct Answer: (B) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$

Solution:

Concept:

- The standard identity is $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}(u/a)+c$, for $a>0$ and $|u|<a$.
- A substitution must transform the differential as well as the expression under the radical.

Step 1: Reduce to the standard form Take $u=2x$, so $du=2\,dx$ and $dx=du/2$. Thus

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{3^2-u^2}}.$$

The real integrand is defined for $-3/2<x<3/2$.

Step 2: Integrate and verify the coefficient The antiderivative is $\frac{1}{2}\sin^{-1}(2x/3)+c$. Indeed,

$$\frac{d}{dx}\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right] = \frac{1/3}{\sqrt{1-4x^2/9}} = \frac{1}{\sqrt{9-4x^2}}.$$

The coefficients in (A), (C) and (D) instead produce respectively one-third, twice and three times the required integrand.

Step 3: Conclusion $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$. Hence, the correct option is (B).

Quick Tip to Solve: Set u = 2x first; the factor one-half comes from dx and must remain outside the standard inverse-sine integral.

13. The area of the region bounded by the curve $y^2 = 4x$ and $x = 1$ is: [1 mark]

(A) $\frac{4}{3}$

(B) $\frac{8}{3}$

(C) $\frac{64}{3}$

(D) $\frac{32}{3}$

Correct Answer: (B) $\frac{8}{3}$

Solution:

Concept:

- For vertical strips, area is $\int (\text{upper curve} - \text{lower curve})\, dx$.
- The parabola $y^2 = 4x$ has branches $y = 2\sqrt{x}$ and $y = -2\sqrt{x}$, symmetric about the x-axis.

Step 1: Identify the bounded region The line $x = 1$ meets the parabola where $y^2 = 4$, namely at $(1, 2)$ and $(1, -2)$. The enclosed region extends from the vertex $x = 0$ to $x = 1$, between the two branches.

Step 2: Integrate the full vertical height The height of a strip is $2\sqrt{x} - (-2\sqrt{x}) = 4\sqrt{x}$. Hence

$$\mathcal{A} = \int_0^1 4x^{1/2}\, dx = 4\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{8}{3}.$$

Option (A) counts only the upper half of the region; the much larger values in (C) and (D) do not agree with the stated limits and strip height.

Step 3: Conclusion The bounded area is $\frac{8}{3}$ square units. Hence, the correct option is (B).

Quick Tip to Solve: Include both branches of y squared = 4x: using only the positive square root gives exactly half the enclosed area.

14. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is: [1 mark]

(A) $e^x + e^{-y} = c$

(B) $e^{-x} + e^{-y} = c$

(C) $e^{x+y} = c$

(D) $2e^{x+y} = c$

Correct Answer: (A) $e^x + e^{-y} = c$

Solution:

Concept:

- An equation $dy/dx = f(x)g(y)$ is separable by moving the y-dependent factor beside dy.
- The exponential is nonzero, and $\int e^{-y}\, dy = -e^{-y}$.

Quick Tip to Solve: Remember the minus sign in the integral of e to the power -y, then differentiate the proposed implicit answer to check it.

Step 1: Separate the variables Since $e^{x+y} = e^x e^y$, multiply by $e^{-y}\,dx$ to obtain

$$e^{-y}\,dy = e^x\,dx.$$

This division loses no solutions because $e^y \neq 0$ for every real y.

Step 2: Integrate and compare the options Integration gives $-e^{-y} = e^x + C$. Moving terms and renaming the arbitrary constant yields $e^x + e^{-y} = c$. Differentiating this relation gives $e^x - e^{-y}y\prime = 0$, or $y\prime = e^{x+y}$. Option (B) gives $y\prime = -e^{y-x}$, whereas (C) and (D) give $y\prime = -1$; none matches the equation.

Step 3: Conclusion The general solution is $e^x + e^{-y} = c$, on intervals where $c - e^x > 0$. Hence, the correct option is (A).

Quick Tip to Solve: Remember the minus sign in the integral of e to the power -y, then differentiate the proposed implicit answer to check it.

15. The angle which the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$ makes with the positive direction of Y-axis is: [1 mark]

(A) $\frac{5\pi}{6}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{4}$

(D) $\frac{7\pi}{4}$

Correct Answer: (B) $\frac{3\pi}{4}$

Solution:

Concept:

- The denominators in the symmetric equation give direction ratios (a, b, c).
- The direction cosine with the positive Y-axis is $\cos\beta = b/\sqrt{a^2 + b^2 + c^2}$, with $0 \leq \beta \leq \pi$.

Step 1: Read and normalize the direction ratios From $x/1 = y/(-1) = z/0$, take the direction ratios $(1, -1, 0)$. The zero third ratio means that the z-coordinate stays zero along this line. Their magnitude is

$$\sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}.$$

Step 2: Find the direction angle The Y-direction cosine is

$$\cos\beta = \frac{-1}{\sqrt{2}}.$$

The angle in the direction-angle range $[0, \pi]$ with this cosine is $3\pi/4$. Option (A) has cosine $-\sqrt{3}/2$, while (C) and (D) lie outside this range. A negative Y-direction cosine also confirms that the required angle is obtuse.

Quick Tip to Solve: Keep the negative sign of the Y-direction ratio; taking its absolute value would incorrectly give an acute direction angle.

Step 3: Conclusion The angle with the positive Y-axis is $\frac{3\pi}{4}$. Hence, the correct option is (B).

Quick Tip to Solve: Keep the negative sign of the Y-direction ratio; taking its absolute value would incorrectly give an acute direction angle.

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16. The Cartesian equation of the line passing through the point $(1, -3, 2)$ and parallel to the line $\vec{r} = (2+\lambda)\hat{i} + \lambda\hat{j} + (2\lambda - 1)\hat{k}$ is: [1 mark]

(A) $\frac{x-1}{2} = \frac{y+3}{0} = \frac{z-2}{-1}$

(B) $\frac{x+1}{1} = \frac{y-3}{1} = \frac{z+2}{2}$

(C) $\frac{x+1}{2} = \frac{y-3}{0} = \frac{z+2}{-1}$

(D) $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Correct Answer: (D) $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Solution:

Concept:

- In $\vec{r} = \vec{a} + \lambda\vec{b}$, the vector $\vec{b}$ gives the direction ratios of the line.
- A line through (x_0, y_0, z_0) with direction ratios (a, b, c) has equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.

Step 1: Extract the direction vector Rearrange the given equation as

$$\vec{r} = (2\hat{i} - \hat{k}) + \lambda(\hat{i} + \hat{j} + 2\hat{k}).$$

Thus its direction ratios are $(1, 1, 2)$. The constants $(2, 0, -1)$ locate a point on the original line and are not its direction ratios.

Step 2: Use the required point The parallel line through $(1, -3, 2)$ is parametrically

$$x = 1 + t, \qquad y = -3 + t, \qquad z = 2 + 2t.$$

Eliminating t gives $(x-1)/1 = (y+3)/1 = (z-2)/2$. Options (A) and (C) use the wrong direction ratios. Option (B) has the right direction but incorrect coordinate offsets and does not pass through the required point.

Step 3: Conclusion The Cartesian equation is $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$. Hence, the correct option is (D).

Quick Tip to Solve: Read direction ratios from the coefficient of lambda, then subtract each coordinate of the specified point in the numerators.

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17. If A and B are two events such that $P(A/B) = P(B/A) \neq 0$, then: [1 mark]

(A) $A \subset B$, but $A \neq B$

(B) $A = B$

(C) $A \cap B = \phi$

(D) $P(A) = P(B)$

Correct Answer: (D) $P(A) = P(B)$

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$, where $P(B) > 0$.
- A nonzero conditional probability implies that the intersection has positive probability, so its factor may be cancelled.

Step 1: Expand the conditional probabilities The stated condition means both conditioning events have positive probability. Hence

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \neq 0.$$

In particular, $P(A \cap B) > 0$, so cancellation gives $1/P(B) = 1/P(A)$.

Step 2: Check what the equality implies Taking reciprocals yields equal probabilities. This does not require the events themselves to be identical or one to be a proper subset of the other, so (A) and (B) need not hold. Option (C) would make the intersection probability zero and contradict the hypothesis.

Step 3: Conclusion The two events have equal probabilities. Hence, the correct option is (D).

Quick Tip to Solve: Expand both conditional probabilities first; the nonzero condition is precisely what permits cancelling the common intersection probability.

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18. The position vectors of points P and Q are $\vec{p}$ and $\vec{q}$ respectively. The point R divides line segment PQ in the ratio $3:1$ and S is the mid-point of line segment PR. The position vector of S is: [1 mark]

(A) $\dfrac{\vec{p} + 3\vec{q}}{4}$

(B) $\dfrac{\vec{p} + 3\vec{q}}{8}$

(C) $\dfrac{5\vec{p} + 3\vec{q}}{4}$

(D) $\dfrac{5\vec{p} + 3\vec{q}}{8}$

Correct Answer: (D) $\dfrac{5\vec{p} + 3\vec{q}}{8}$

Solution:

Concept:

- For internal division $PR : RQ = m : n$, the position vector of R is $(n\vec{p} + m\vec{q})/(m + n)$.

Quick Tip to Solve: First find the dividing point, then average it with the specified endpoint; the coefficients of a position-vector average must add to 1.

- The midpoint of two points has position vector equal to the average of their position vectors.

Step 1: Apply the section formula Since $PR : RQ = 3 : 1$, the point R is closer to Q. Its vector is

$$\vec{r} = \frac{\vec{p} + 3\vec{q}}{4}.$$

This is option (A), but the question asks for S, not R.

Step 2: Take the midpoint of the correct segment Average $\vec{p}$ and $\vec{r}$ because S bisects PR:

$$\vec{s} = \frac{\vec{p} + \vec{r}}{2} = \frac{4\vec{p} + \vec{p} + 3\vec{q}}{8} = \frac{5\vec{p} + 3\vec{q}}{8}.$$

Option (B) omits the contribution of P when averaging. Option (C) omits division by 2 in the midpoint formula.

Step 3: Conclusion The position vector is $\vec{s} = (5\vec{p} + 3\vec{q})/8$. Hence, the correct option is (D).

Quick Tip to Solve: First find the dividing point, then average it with the specified endpoint; the coefficients of a position-vector average must add to 1.

19. Assertion (A): The vectors $\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$, $\vec{b} = 10\hat{i} - 2\hat{j} - 6\hat{k}$ and $\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$ represent the sides of a right angled triangle.

Reason (R): Three non-zero vectors of which no two are collinear, form a triangle, if their resultant is zero vector or sum of any two vectors is equal to the third. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- Non-collinear vectors can represent triangle sides when their directed sum is zero, allowing reversal of a side's direction.
- Two nonzero vectors are perpendicular exactly when their scalar product is zero.

Step 1: Test the assertion The vectors satisfy $\vec{a} + \vec{c} = (10, -2, -6) = \vec{b}$, so they form a triangle. Moreover,

$$\vec{a} \cdot \vec{c} = 6(4) + 2(-4) + (-8)(2) = 24 - 8 - 16 = 0.$$

These two nonzero vectors are perpendicular, so the triangle is right angled. The assertion is true.

Quick Tip to Solve: Separate the triangle closure test from the right-angle test: a vector sum proves closure, while a zero dot product proves perpendicularity.

Step 2: Test the reason The reason states the triangle closure condition. A zero resultant gives a closed head-to-tail triangle. If two vectors sum to the third, reversing that third vector gives zero resultant. Non-collinearity prevents a degenerate triangle. Thus the reason is true.

Step 3: Decide whether the reason explains the assertion Closure alone proves that a triangle forms; it does not prove a right angle. The extra scalar-product calculation is essential. Thus option (A) claims too much, while options (C) and (D) incorrectly label one of the true statements as false.

Step 4: Conclusion Both statements are true, but the reason does not establish the right angle. Hence, the correct option is (B).

Quick Tip to Solve: Separate the triangle closure test from the right-angle test: a vector sum proves closure, while a zero dot product proves perpendicularity.

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20. Assertion (A): Domain of $y = \cos^{-1}(x)$ is $[-1, 1]$.

Reason (R): The range of the principal value branch of $y = \cos^{-1}(x)$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- Cosine restricted to $[0, \pi]$ is one-one and maps that interval onto $[-1, 1]$.
- Its inverse $\cos^{-1} x$ consequently has domain $[-1, 1]$ and principal range $[0, \pi]$.

Step 1: Judge the assertion A real angle has cosine between -1 and 1, inclusive. Every number in that interval is attained on $[0, \pi]$, so the domain of the real inverse cosine is exactly $[-1, 1]$. The assertion is true.

Step 2: Judge the reason The actual principal range includes every angle from 0 to π. In particular, $\cos^{-1}(0) = \pi/2$. Removing $\pi/2$, as the reason does, is incorrect, so the reason is false.

Step 3: Select the matching truth combination A false reason cannot explain the true assertion. Options (A) and (B) incorrectly call the reason true; option (D) also incorrectly calls the assertion false. The required combination is true assertion and false reason.

Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Test the excluded angle directly: since cosine of pi/2 is zero, inverse cosine at zero must include pi/2 in its range.

Section B (10 marks)

*This section has **5** Very Short Answer questions of **2** marks each.*

21. If $a = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ **and** $b = \tan^{-1}(\sqrt{3}) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$**, then find the value of** $a + b$**.**

[2 marks]

Solution:

Concept:

- Use principal values: sine inverse lies in $[-\pi/2, \pi/2]$, cosine inverse in $[0, \pi]$, and tangent inverse in $(-\pi/2, \pi/2)$.
- The principal range of cotangent inverse is $(0, \pi)$; a negative argument therefore gives a second-quadrant angle.

Step 1: Evaluate the two terms in a Since $\sin(\pi/4) = \sqrt{2}/2$ and $\cos(2\pi/3) = -1/2$, with both angles in their required principal ranges,

$$a = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}.$$

Step 2: Evaluate b with the correct cotangent branch We have $\tan^{-1}(\sqrt{3}) = \pi/3$. The angle $2\pi/3$ lies in $(0, \pi)$ and has cotangent $-1/\sqrt{3}$, so

$$b = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$

A negative angle with the same cotangent is outside the principal range and cannot be used.

Step 3: Add the principal-value expressions Express the two quantities with denominator 12:

$$a + b = \frac{11\pi}{12} - \frac{4\pi}{12} = \frac{7\pi}{12}.$$

Step 4: Conclusion The required value is $\boxed{a + b = 7\pi/12}$.

Quick Tip to Solve: For a negative cotangent-inverse argument, choose the second-quadrant angle in (0, pi) before combining the fractions.

22(a). Find: $\int \cos^3 x \, e^{\log \sin x} \, dx$**.**

[2 marks]

Solution:

Concept:

- The exponential and natural logarithm are inverse functions: $e^{\log t} = t$ for $t > 0$.
- The substitution $u = \cos x$, with $du = -\sin x \, dx$, reduces a product of $\sin x$ and a power of $\cos x$ to a power integral.

Quick Tip to Solve: Cancel the exponential and logarithm first; then pair the remaining sine factor with dx so that differentiating cosine supplies the substitution.

Step 1: Simplify the exponential factor The real logarithm requires $\sin x > 0$. On each such interval, $e^{\log \sin x} = \sin x$. Hence the given integral becomes

$$I = \int \cos^3 x \sin x \, dx.$$

Keeping this domain in mind ensures that the simplification uses the real logarithm correctly.

Step 2: Substitute and integrate the power Let $u = \cos x$. Then $du = -\sin x \, dx$, so

$$I = -\int u^3 \, du = -\frac{u^4}{4} + C.$$

The negative sign comes from differentiating cosine. The constant C is arbitrary on each interval of definition.

Step 3: Return to x and verify Substituting $u = \cos x$ gives $-\cos^4 x/4 + C$. Differentiation confirms the result:

$$\frac{d}{dx}\left(-\frac{\cos^4 x}{4}\right) = \cos^3 x \sin x = \cos^3 x \, e^{\log \sin x}.$$

Step 4: Conclusion $\int \cos^3 x \, e^{\log \sin x} \, dx = -\frac{\cos^4 x}{4} + C$, on intervals where $\sin x > 0$.

Quick Tip to Solve: Cancel the exponential and logarithm first; then pair the remaining sine factor with dx so that differentiating cosine supplies the substitution.

OR

22(b). Find: $\int \frac{1}{5 + 4x - x^2} \, dx$. [2 marks]

Solution:

Concept:

- Completing the square converts a quadratic denominator into a standard form: here $5 + 4x - x^2 = 9 - (x - 2)^2$.
- For a positive constant a, $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \log \left|\frac{a + u}{a - u}\right| + C$, on intervals avoiding the denominator's zeros.

Step 1: Complete the square and substitute Denote the required integral by I. Rearranging the denominator gives

$$5 + 4x - x^2 = 9 - (x^2 - 4x + 4) = 9 - (x - 2)^2.$$

Put $u = x - 2$, so that $du = dx$. Thus the quadratic becomes a difference of squares with $a = 3$:

$$I = \int \frac{du}{3^2 - u^2}.$$

Step 2: Apply the standard integral and restore the variable Using the logarithmic formula with

Quick Tip to Solve: Complete the square as $9 - (x - 2)^2$; a difference of squares uses the logarithmic formula, whereas a sum of squares uses inverse tangent.

$a = 3$, we obtain

$$I = \frac{1}{6}\log\left|\frac{3+u}{3-u}\right| + C$$
$$= \frac{1}{6}\log\left|\frac{x+1}{5-x}\right| + C.$$

The absolute value makes the logarithmic expression valid on each interval where the original integrand is defined; $x = -1, 5$ must be excluded.

Step 3: Conclusion $\int \frac{dx}{5+4x-x^2} = \frac{1}{6}\log\left|\frac{x+1}{5-x}\right| + C$, on any interval avoiding $x = -1$ and $x = 5$.

Quick Tip to Solve: Complete the square as $9-(x-2)^2$; a difference of squares uses the logarithmic formula, whereas a sum of squares uses inverse tangent.

23. Sand is pouring from a pipe at the rate of 15 cm^3/minute. The falling sand forms a cone on the ground such that the height of the cone is always one-third of the radius of the base. How fast is the height of the sand cone increasing at the instant when the height is 4 cm? [2 marks]

Solution:

Concept:

- The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is its base radius and h its height.
- In a related-rates problem, express the volume in terms of the required variable before differentiating with respect to time t.

Step 1: Express the volume using the height Let r and h be measured in centimetres and t in minutes. The condition holds throughout the filling process, so

$$h = \frac{r}{3} \quad \Longrightarrow \quad r = 3h.$$

Substitution into the volume formula gives

$$V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3.$$

Both radius and height change; eliminating r accounts for their linked change.

Step 2: Differentiate and evaluate the rate Applying the chain rule with respect to time, we have

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}.$$

Sand arrives at $\frac{dV}{dt} = 15\,\text{cm}^3/\text{minute}$. At the required instant $h = 4\,\text{cm}$, and hence

$$15 = 9\pi(4)^2\frac{dh}{dt}, \qquad \frac{dh}{dt} = \frac{15}{144\pi} = \frac{5}{48\pi}\ \text{cm/minute}.$$

Step 3: Conclusion The height of the cone is increasing at $\boxed{\frac{5}{48\pi}\ \text{cm/minute}}$ when its height is 4 cm.

Quick Tip to Solve: Translate the height-radius condition into $r = 3h$, then substitute before differentiating; using $r = h/3$ reverses the given relationship.

24. Find the vector equation of the line passing through the point $(2, 3, -5)$ and making equal angles with the co-ordinate axes. [2 marks]

Solution:

Concept:

- If a directed line makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, satisfying $l^2 + m^2 + n^2 = 1$.
- A line through a point with position vector $\vec{a}$ and parallel to a nonzero vector $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.

Step 1: Determine a direction vector The angles with the coordinate axes are equal. Their cosines are therefore equal: $l = m = n$. The direction-cosine identity becomes

$$3l^2 = 1 \quad \Longrightarrow \quad l = m = n = \pm\frac{1}{\sqrt{3}}.$$

The common sign is the same for all three cosines. Reversing all signs merely reverses the direction along the same line. Thus we may use the direction ratios $1, 1, 1$, giving $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

Step 2: Use the given point The position vector of $P(2, 3, -5)$ is

$$\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}.$$

Substitute this vector and the direction vector into the point-direction form. The parameter $\lambda = 0$ gives the specified point; varying λ moves along the required direction.

Step 3: Conclusion The required vector equation is

$$\boxed{\vec{r} = (2\hat{i} + 3\hat{j} - 5\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.}$$

Quick Tip to Solve: Equal angles imply equal direction cosines, so use $1, 1, 1$ as direction ratios; a unit direction vector is unnecessary in a vector line equation.

25(a). Verify whether the function f defined by $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$ or not. [2 marks]

Solution:

Concept:

- A function is continuous at $x = a$ when its two-sided limit exists and equals the defined value $f(a)$.
- The sine function satisfies $|\sin t| \leq 1$ for every real t. A quantity whose absolute value is bounded by a quantity tending to zero also tends to zero.

Quick Tip to Solve: Bound the absolute value of x sin(1/x) by $|x|$; do not try to evaluate the oscillating sine factor separately.

Step 1: Identify the value and bound the nearby expression The definition gives $f(0) = 0$. For every $x \neq 0$, the argument $1/x$ is defined and the sine bound gives

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|, \qquad -|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|.$$

Using $|x|$ makes the inequality valid on both sides of zero, including negative values of x.

Step 2: Evaluate the two-sided limit Both bounding functions $-|x|$ and $|x|$ tend to zero as $x \to 0$. The squeeze theorem therefore gives

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = f(0).$$

Although $\sin(1/x)$ oscillates near zero, multiplication by x forces the magnitude of the product to approach zero. Thus the left and right limits coincide with the assigned function value.

Step 3: Conclusion Since $\lim_{x \to 0} f(x) = f(0) = 0$, the function f is continuous at $x = 0$.

Quick Tip to Solve: Bound the absolute value of x sin(1/x) by |x|; do not try to evaluate the oscillating sine factor separately.

OR

25(b). Check for differentiability of the function f defined by $f(x) = |x - 5|$, at the point $x = 5$.

[2 marks]

Solution:

Concept:

- A function is differentiable at $x = a$ if the left and right limits of $\big(f(a + h) - f(a)\big)/h$ are finite and equal.
- For a real number h, $|h| = h$ when $h > 0$, whereas $|h| = -h$ when $h < 0$. This change determines the two slopes at the vertex of an absolute-value function.

Step 1: Form the difference quotient at the given point Here $f(5) = |5 - 5| = 0$. For a nonzero increment h, $f(5 + h) = |h|$, so the quotient needed for the derivative is

$$\frac{f(5 + h) - f(5)}{h} = \frac{|h|}{h}.$$

We must examine positive and negative increments separately because the expression defining $|h|$ changes with the sign of h.

Step 2: Compare the derivatives from the two sides For $h < 0$, $|h| = -h$, and for $h > 0$, $|h| = h$.

Quick Tip to Solve: At x=5, replace x by 5+h and compare |h|/h for positive and negative h to test the corner directly.

Hence

$$f'_-(5) = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} (-1) = -1,$$

$$f'_+(5) = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} 1 = 1.$$

Both one-sided derivatives are finite, but their values differ. Consequently the two-sided limit of the difference quotient does not exist.

Step 3: Conclusion The function $f(x) = |x-5|$ is not differentiable at $x = 5$, since $f'_-(5) = -1 \neq 1 = f'_+(5)$.

Quick Tip to Solve: At x=5, replace x by 5+h and compare |h|/h for positive and negative h to test the corner directly.

Section C (18 marks)

*There are **6** short answer questions in this section. Each is of **3** marks.*

26(a). Find the particular solution of the differential equation $\frac{dy}{dx} - 2xy = 3x^2e^{x^2}$; $y(0) = 5$. [3 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x)\,dx}$.
- Multiplying by the integrating factor makes the left side an exact product derivative: $\frac{d}{dx}(Iy) = IQ$. An initial condition fixes the arbitrary constant after integration.

Step 1: Identify the linear equation and integrating factor Comparing the given equation with the standard linear form gives

$$P(x) = -2x, \qquad Q(x) = 3x^2e^{x^2}.$$

Therefore choose the integrating factor

$$I(x) = e^{\int -2x\,dx} = e^{-x^2}.$$

A nonzero constant multiplier in the integrating factor is unnecessary, so this simple choice suffices.

Step 2: Multiply and integrate the product derivative Multiplying every term by e^{-x^2} cancels the exponential on the right:

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 3x^2.$$

Since $\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$, the product rule yields

$$\frac{d}{dx}\left(ye^{-x^2}\right) = 3x^2.$$

Quick Tip to Solve: Keep the negative sign in $P(x) = -2x$: the factor e^{-x^2} cancels the right-side exponential and leaves the easy integral of $3x^2$.

Integrating with respect to x,

$$ye^{-x^2} = x^3 + C.$$

Step 3: Apply the initial condition and check At $x = 0$, the condition $y(0) = 5$ gives $5 = C$. Hence $y = (x^3 + 5)e^{x^2}$. Differentiating this expression gives

$$y' = 3x^2e^{x^2} + 2x(x^3 + 5)e^{x^2}.$$

Subtracting $2xy$ leaves exactly $3x^2e^{x^2}$, and substitution of $x = 0$ gives $y = 5$, verifying both requirements.

Step 4: Conclusion The required particular solution is $\boxed{y = (x^3 + 5)e^{x^2}}$, valid for all real x.

Quick Tip to Solve: Keep the negative sign in $P(x) = -2x$: the factor e^{-x^2} cancels the right-side exponential and leaves the easy integral of $3x^2$.

OR

26(b). Solve the following differential equation: $x^2\,dy + y(x + y)\,dx = 0$. [3 marks]

Solution:

Concept:

- A first-order equation is homogeneous when dy/dx is a function of y/x. The substitution $y = vx$ gives $dy/dx = v + x\,dv/dx$.
- Separate the variables and integrate using partial fractions. Any constant value of v lost through division must be checked in the original equation.

Step 1: Reduce the homogeneous equation On an interval where $x \neq 0$, divide the given equation by $x^2\,dx$:

$$\frac{dy}{dx} = -\frac{y}{x} - \left(\frac{y}{x}\right)^2.$$

Put $y = vx$. Substitution gives

$$v + x\frac{dv}{dx} = -v - v^2, \qquad x\frac{dv}{dx} = -v(v + 2).$$

Thus the equation in v and x is separable. The constant possibilities $v = 0$ and $v = -2$ will be checked separately.

Step 2: Integrate the separated equation For $v \neq 0, -2$,

$$\frac{dv}{v(v + 2)} = -\frac{dx}{x}, \qquad \frac{1}{v(v + 2)} = \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v + 2}\right).$$

Integrating both sides gives

$$\frac{1}{2}\ln\left|\frac{v}{v + 2}\right| = -\ln|x| + C_1.$$

Quick Tip to Solve: After using $y = vx$, record $v = 0, -2$ before dividing by $v(v + 2)$, so no straight-line solution is lost.

Multiply by two and combine logarithms. Absorbing the sign and exponential of the integration constant into a real constant C, we obtain

$$x^2\frac{v}{v+2} = C.$$

On replacing v by y/x and clearing the denominator, this becomes $x^2y = C(y+2x)$.

Step 3: Restore the solutions excluded by division The value $v = 0$ yields $y = 0$, which satisfies the original equation and is already included by taking $C = 0$. The value $v = -2$ yields $y = -2x$. For this function,

$$x^2\frac{dy}{dx} + y(x+y) = -2x^2 + (-2x)(-x) = 0.$$

Hence this second straight-line solution must be included separately; it is not represented by any finite C in the implicit family above.

Step 4: Conclusion The solution is $\boxed{x^2y = C(y+2x)}$, together with $\boxed{y = -2x}$. The family includes $y = 0$, and the derivation applies on intervals with $x \neq 0$.

Quick Tip to Solve: After using $y = vx$, record $v = 0, -2$ before dividing by $v(v+2)$, so no straight-line solution is lost.

27. Find the values of a and b so that the following function is differentiable for all values of x:

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

[3 marks]

Solution:

Concept:

- A differentiable function is necessarily continuous. At the joining point of a piecewise function, first match its limiting values with its actual value.
- Differentiability at a joining point also requires equality of the left-hand and right-hand derivatives. Polynomial pieces are differentiable at all interior points of their respective intervals.

Step 1: Impose continuity at the joining point Both expressions are polynomials, so the only point that needs checking is $x = -1$. Since the second branch includes equality,

$$f(-1) = b(-1)^2 - 3 = b - 3.$$

The left-hand limit equals $b - 3$, while the right-hand limit is $-a + b$. Therefore continuity demands

$$-a + b = b - 3, \qquad a = 3.$$

This condition must be used before comparing one-sided derivatives; equality of slopes alone would not rule out a jump.

Step 2: Match the one-sided derivatives For the quadratic branch $f'(x) = 2bx$, and for the linear

Quick Tip to Solve: At a piecewise join, match the function values first and the slopes second; the two conditions determine the two constants.

branch $f'(x) = a$. With continuity established, their limiting slopes give

$$f'_-(-1) = -2b, \qquad f'_+(-1) = a.$$

For example, from the left the difference quotient is

$$\frac{b(-1+h)^2 - 3 - (b-3)}{h} = -2b + bh,$$

which tends to $-2b$ as $h \to 0^-$. From the right it tends to a. Thus $-2b = a = 3$, giving $b = -3/2$.

Step 3: Verify differentiability everywhere With these constants, both branch values at the join approach $-9/2$, so the function is continuous there. The derivative from the quadratic side is $2(-3/2)(-1) = 3$, equal to the derivative of the linear side. Consequently the derivative exists at $x = -1$. At every other real point, the function agrees locally with a polynomial and is differentiable automatically.

Step 4: Conclusion The required values are $\boxed{a = 3, \quad b = -\frac{3}{2}}$.

Quick Tip to Solve: At a piecewise join, match the function values first and the slopes second; the two conditions determine the two constants.

28(a). Find $\frac{dy}{dx}$, if $(\cos x)^y = (\cos y)^x$. [3 marks]

Solution:

Concept:

- When both the base and exponent vary, logarithmic differentiation changes powers into products: $\ln(u^v) = v \ln u$, for $u > 0$.
- In implicit differentiation, y is a function of x. Hence $d(\ln \cos y)/dx = -\tan y \, dy/dx$, by the chain rule.

Step 1: Take logarithms of the given relation On a real branch where $\cos x > 0$ and $\cos y > 0$, take natural logarithms of both sides of

$$(\cos x)^y = (\cos y)^x.$$

Using the power rule for logarithms gives

$$y \ln(\cos x) = x \ln(\cos y).$$

This form is easier to differentiate because the variable exponents have become factors. Both factors in each product must be differentiated.

Step 2: Apply the product rule and chain rule Differentiate both sides with respect to x, remembering that y depends on x:

$$\frac{dy}{dx} \ln(\cos x) - y \tan x = \ln(\cos y) - x \tan y \frac{dy}{dx}.$$

Quick Tip to Solve: Take logarithms before differentiating variable powers, and attach dy/dx whenever a function of y is differentiated.

In particular, the term $-x\tan y\,dy/dx$ on the right contains the chain-rule factor dy/dx; omitting this factor would treat y incorrectly as independent of x.

Step 3: Collect the derivative terms Move both terms containing dy/dx to the left and the remaining terms to the right:

$$\frac{dy}{dx}\left[\ln(\cos x)+x\tan y\right]=\ln(\cos y)+y\tan x.$$

Factoring the derivative leaves a single algebraic equation for the slope. At a point where the bracket on the left is nonzero, divide by that bracket to isolate the derivative. This nonzero-denominator condition is the usual condition for the displayed implicit derivative formula.

Step 4: Conclusion $\boxed{\dfrac{dy}{dx}=\dfrac{\ln(\cos y)+y\tan x}{\ln(\cos x)+x\tan y}}$, wherever the logarithms are defined and the denominator is nonzero.

Quick Tip to Solve: Take logarithms before differentiating variable powers, and attach dy/dx whenever a function of y is differentiated.

OR

28(b). If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that $\dfrac{dy}{dx}=\sqrt{\dfrac{1-y^2}{1-x^2}}$. [3 marks]

Solution:

Concept:

- Differentiate an implicit relation using the chain rule: $d\sqrt{1-y^2}/dx=-y(1-y^2)^{-1/2}\,dy/dx$.
- The parameter a is constant. After differentiation, use the original relation to eliminate it and simplify the resulting expression.

Step 1: Introduce notation and differentiate Write $A=\sqrt{1-x^2}$ and $B=\sqrt{1-y^2}$. For $|x|<1$, $|y|<1$, both are positive, and

$$A+B=a(x-y).$$

Differentiating with respect to x, with a constant, gives

$$-\frac{x}{A}-\frac{y}{B}\frac{dy}{dx}=a\left(1-\frac{dy}{dx}\right).$$

Collecting the terms containing the derivative,

$$\left(a-\frac{y}{B}\right)\frac{dy}{dx}=a+\frac{x}{A}.$$

Step 2: Eliminate the constant using the original relation Since $A+B>0$, the original equation implies $x\neq y$, so

$$a=\frac{A+B}{x-y}.$$

Quick Tip to Solve: Name the two square roots A and B; substituting $a=(A+B)/(x-y)$ exposes the identical factor in numerator and denominator.

Substitution gives

$$a - \frac{y}{B} = \frac{B(A+B) - y(x-y)}{B(x-y)} = \frac{1 + AB - xy}{B(x-y)},$$

where $B^2 = 1 - y^2$. Similarly,

$$a + \frac{x}{A} = \frac{A(A+B) + x(x-y)}{A(x-y)} = \frac{1 + AB - xy}{A(x-y)},$$

using $A^2 = 1 - x^2$.

Step 3: Cancel the common factor Taking the ratio of the two expressions gives

$$\frac{dy}{dx} = \frac{a + x/A}{a - y/B} = \frac{B}{A}.$$

This cancellation is valid: $|xy| < 1$ and $AB > 0$, so the common factor $1 + AB - xy$ is positive. Finally replace A and B by their square roots.

Step 4: Conclusion Hence $\boxed{\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}}$, at the interior points where the relation defines the derivative.

Quick Tip to Solve: Name the two square roots A and B; substituting $a = (A+B)/(x-y)$ exposes the identical factor in numerator and denominator.

29(a). Evaluate: $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$. [3 marks]

Solution:

Concept:

- For a continuous function on $[0, a]$, reflection of the interval gives $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$.
- The identity $\cos(\pi - x) = -\cos x$ interchanges the two positive terms in the denominator. Adding the original and reflected integrands can therefore simplify the integral.

Step 1: Write the integral and check its domain Let

$$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx.$$

Both exponential terms are strictly positive for every real x, so the denominator never vanishes. The integrand is continuous throughout $[0, \pi]$, and the definite-integral reflection property applies without any singular endpoint.

Step 2: Reflect the interval Put $x = \pi - t$, so $dx = -dt$, and the limits $x = 0, \pi$ become $t = \pi, 0$. Using $\cos(\pi - t) = -\cos t$, obtain

$$I = \int_0^{\pi} \frac{e^{-\cos t}}{e^{-\cos t} + e^{\cos t}}\, dt.$$

The integration variable is a dummy variable, so it may now be renamed x.

Quick Tip to Solve: For limits 0 to π, try $x \mapsto \pi - x$: here the reflected integrand and the original add to 1.

Step 3: Add the two equivalent integrals The denominators in the two forms are identical. Thus

$$2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$$
$$= \int_0^{\pi} 1\, dx = [x]_0^{\pi} = \pi.$$

The numerator becomes exactly the denominator; no antiderivative of an exponential of a trigonometric function is needed.

Step 4: Conclusion The value of the given definite integral is $\boxed{\frac{\pi}{2}}$.

Quick Tip to Solve: For limits 0 to π, try $x \mapsto \pi - x$: here the reflected integrand and the original add to 1.

OR

29(b). Find: $\int \frac{2x+1}{(x+1)^2(x-1)}\, dx$. [3 marks]

Solution:

Concept:

- A proper rational function with denominator $(x+1)^2(x-1)$ requires terms for both powers of the repeated factor: $A/(x-1) + B/(x+1) + C/(x+1)^2$.
- Use $\int dx/(x-a) = \ln|x-a| + K$ and the power rule $\int (x+1)^{-2} dx = -(x+1)^{-1} + K$. The formula holds on intervals excluding the poles.

Step 1: Decompose the rational function For $x \neq \pm 1$, write

$$\frac{2x+1}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplication by the common denominator gives the polynomial identity

$$2x + 1 = A(x+1)^2 + B(x+1)(x-1) + C(x-1).$$

Step 2: Determine the constants In the polynomial identity, set $x = 1$ to obtain $3 = 4A$, hence $A = 3/4$. Set $x = -1$ to get $-1 = -2C$, hence $C = 1/2$. Comparing coefficients of x^2 gives $A + B = 0$, so $B = -3/4$. These values also give the coefficient of x as $2A + C = 2$, and the constant as $A - B - C = 1$, checking the decomposition.

Step 3: Integrate each term Substitute the constants and use linearity:

$$\int \frac{2x+1}{(x+1)^2(x-1)}\, dx = \frac{3}{4}\int \frac{dx}{x-1} - \frac{3}{4}\int \frac{dx}{x+1} + \frac{1}{2}\int (x+1)^{-2}\, dx.$$

The first two terms give logarithms of absolute values, while the last term gives $-1/[2(x+1)]$.

Quick Tip to Solve: For the repeated factor $(x+1)^2$, include both $1/(x+1)$ and $1/(x+1)^2$; the squared term integrates to a negative reciprocal.

Differentiating these three terms recovers the partial fractions above, and hence the printed integrand.

Step 4: Conclusion

$$\boxed{\frac{3}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2(x+1)} + K}, \qquad x \neq \pm 1.$$

Here K is an arbitrary constant on any interval avoiding -1 and 1.

Quick Tip to Solve: For the repeated factor $(x+1)^2$, include both $1/(x+1)$ and $1/(x+1)^2$; the squared term integrates to a negative reciprocal.

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30. Given $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} - \hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 3$. [3 marks]

Solution:

Concept:

- For non-parallel vectors $\vec{a}$ and $\vec{b}$, their cross product $\vec{a} \times \vec{b}$ is perpendicular to both. Every common perpendicular vector is a scalar multiple of this cross product.
- The dot product is the sum of products of corresponding components. The condition $\vec{c} \cdot \vec{d} = 3$ determines the scalar multiplier and hence the magnitude and direction of $\vec{d}$.

Step 1: Find a common perpendicular direction Expand the determinant for the cross product, remembering the negative sign in the middle cofactor:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{vmatrix} \\ &= \hat{i}\{(-1)(-1) - 1(0)\} - \hat{j}\{2(-1) - 1(3)\} + \hat{k}\{2(0) - (-1)(3)\} \\ &= \hat{i} + 5\hat{j} + 3\hat{k}.\end{aligned}$$

This vector is nonzero, so the two given vectors are not parallel. We can therefore write the required vector as

$$\vec{d} = \lambda(\hat{i} + 5\hat{j} + 3\hat{k}),$$

where λ is a real scalar.

Step 2: Use the prescribed dot product Substitute $\vec{c} = 2\hat{i} + \hat{j} - 2\hat{k}$ and the above expression for $\vec{d}$:

$$\begin{aligned}\vec{c} \cdot \vec{d} &= \lambda\{2(1) + 1(5) + (-2)(3)\} \\ &= \lambda(2 + 5 - 6) = \lambda.\end{aligned}$$

Since this dot product must be 3, the multiplier is $\lambda = 3$.

Step 3: Verify the conditions The resulting components are $(3, 15, 9)$. Check them directly against all

Quick Tip to Solve: Take $\vec{a} \times \vec{b}$ first, then divide 3 by its dot product with $\vec{c}$ to get the required multiplier.

the given conditions:

$$\vec{a}\cdot\vec{d}=2(3)-15+9=0,$$
$$\vec{b}\cdot\vec{d}=3(3)-9=0,$$
$$\vec{c}\cdot\vec{d}=2(3)+15-2(9)=3.$$

The first two equalities establish perpendicularity, and the third establishes the required scaling.

Step 4: Conclusion The required vector is $\boxed{\vec{d}=3\hat{i}+15\hat{j}+9\hat{k}}$.

Quick Tip to Solve: Take $\vec{a}\times\vec{b}$ first, then divide 3 by its dot product with $\vec{c}$ to get the required multiplier.

31. Bag I contains 3 red and 4 black balls, Bag II contains 5 red and 2 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn at random from Bag II. Find the probability that the drawn ball is red in colour. [3 marks]

Solution:

Concept:

- When two balls are transferred without replacement, the possible colour combinations are two red, one red and one black, or two black. These events are mutually exclusive and exhaustive.
- By the law of total probability, $P(R)=\sum_i P(E_i)P(R\mid E_i)$, where R is the event that the final ball is red and E_i specifies the transferred pair.

Step 1: Calculate the probabilities of the transferred pairs Bag I contains 3 red and 4 black balls. There are $\binom{7}{2}=21$ equally likely ways of choosing the two balls. Let E_1, E_2, E_3 denote transferring two red, one of each colour, and two black, respectively. Then

$$P(E_1)=\frac{\binom{3}{2}}{\binom{7}{2}}=\frac{3}{21}=\frac{1}{7},$$
$$P(E_2)=\frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}}=\frac{12}{21}=\frac{4}{7},$$
$$P(E_3)=\frac{\binom{4}{2}}{\binom{7}{2}}=\frac{6}{21}=\frac{2}{7}.$$

Their sum is 1, confirming that all possible transferred colour combinations have been included. The middle count already includes every unordered mixed pair; no additional factor of 2 is needed.

Step 2: Find the conditional probabilities of drawing red Bag II initially contains 5 red and 2 black balls. After either transfer it contains $7+2=9$ balls. Depending on the transferred pair, the red-ball counts are 7, 6 and 5. Hence

$$P(R\mid E_1)=\frac{7}{9},\qquad P(R\mid E_2)=\frac{6}{9},\qquad P(R\mid E_3)=\frac{5}{9}.$$

The denominator must be 9 because the final draw is made after adding both transferred balls.

Step 3: Apply the law of total probability Weight each conditional probability by the probability of its

Quick Tip to Solve: List the transferred pairs as RR, RB and BB, and use 9 as the final bag total in every conditional probability.

transfer event:

$$\begin{aligned} P(R) &= P(E_1)P(R \mid E_1) + P(E_2)P(R \mid E_2) + P(E_3)P(R \mid E_3) \\ &= \frac{1}{7}\cdot\frac{7}{9} + \frac{4}{7}\cdot\frac{6}{9} + \frac{2}{7}\cdot\frac{5}{9} \\ &= \frac{7+24+10}{63}. \end{aligned}$$

Each term represents one disjoint route by which the final red ball can be drawn.

Step 4: Conclusion The probability that the ball drawn from Bag II is red is $\boxed{\frac{41}{63}}$.

Quick Tip to Solve: List the transferred pairs as RR, RB and BB, and use 9 as the final bag total in every conditional probability.

Section D (20 marks)

*There are **4** long answer questions in this section. Each question is of **5** marks.*

32(a). Find the co-ordinates of the foot of the perpendicular drawn from the point $(2, 3, -8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance of the given point from the line. [5 marks]

Solution:

Concept:

- A line in parametric form is $\vec{r} = \vec{a} + t\vec{d}$, where $\vec{a}$ locates a point on the line and $\vec{d}$ is its direction vector.
- If H is the perpendicular foot from P, then H lies on the line and $\overrightarrow{PH} \cdot \vec{d} = 0$. The required distance is the length PH.

Step 1: Express a general point of the line Let the common value of the three printed ratios be t. Keep the minus signs in the numerators when rearranging:

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3} = t \quad \Longrightarrow \quad x = 4-2t, \quad y = 6t, \quad z = 1-3t.$$

Thus a point on the line is $A = (4, 0, 1)$, and a direction vector is $\vec{d} = (-2, 6, -3)$. In particular, the direction ratios are not $(2, 6, 3)$, because the first and third numerators have reversed signs.

Step 2: Impose the perpendicularity condition Write the foot as $H = (4-2t, 6t, 1-3t)$. From the given point $P = (2, 3, -8)$, the joining vector is

$$\overrightarrow{PH} = (2-2t,\ 6t-3,\ 9-3t).$$

Quick Tip to Solve: First rewrite $4-x = 2t$ and $1-z = 3t$; this avoids sign errors in the direction vector. Then use one dot-product equation to locate the foot.

Its scalar product with the line's direction must vanish:

$$\begin{aligned}(2-2t)(-2)+(6t-3)6+(9-3t)(-3)&=0,\\ -4+4t+36t-18-27+9t&=0,\\ 49t-49&=0.\end{aligned}$$

Hence the parameter corresponding to the perpendicular foot is $t=1$. The nonzero coefficient $49=|\vec{d}|^2$ also gives a unique foot.

Step 3: Obtain the foot and measure the perpendicular Substitute $t=1$ into the coordinates of H:

$$H=(4-2,\ 6,\ 1-3)=(2,6,-2).$$

The distance formula then gives

$$PH=\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}=\sqrt{0+9+36}=3\sqrt{5}.$$

As a check, all three ratios at H equal 1, and $\overrightarrow{PH}\cdot\vec{d}=(0,3,6)\cdot(-2,6,-3)=18-18=0$. Thus this point lies on the line and the joining segment is indeed perpendicular to it, so its length is the shortest distance from P to the line.

Step 4: Conclusion The foot of the perpendicular is $\boxed{(2,6,-2)}$, and the perpendicular distance is $\boxed{3\sqrt{5}\text{ units}}$.

Quick Tip to Solve: First rewrite $4-x=2t$ and $1-z=3t$; this avoids sign errors in the direction vector. Then use one dot-product equation to locate the foot.

OR

32(b). Find the shortest distance between the lines L_1 and L_2 given below:
L_1: the line passing through $(2,-1,1)$ and parallel to $\frac{x}{1}=\frac{y}{1}=\frac{z}{3}$, and
L_2: $\vec{r}=\hat{i}+(2\mu+1)\hat{j}-(\mu+2)\hat{k}$. [5 marks]

Solution:

Concept:

- For lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, $\vec{a}_1,\vec{a}_2$ locate points and $\vec{b}_1,\vec{b}_2$ give directions.
- If $\vec{n}=\vec{b}_1\times\vec{b}_2\neq\vec{0}$, the shortest distance is $d=\dfrac{|(\vec{a}_2-\vec{a}_1)\cdot\vec{n}|}{|\vec{n}|}$. This is the magnitude of the connecting vector's projection onto the common perpendicular.

Step 1: Write both lines in vector form The reference line $x/1=y/1=z/3$ has direction ratios $(1,1,3)$. Thus the parallel line through $(2,-1,1)$, and the given second line, can be written as

$$\begin{aligned}L_1:&\quad \vec{r}=(2\hat{i}-\hat{j}+\hat{k})+\lambda(\hat{i}+\hat{j}+3\hat{k}),\\ L_2:&\quad \vec{r}=(\hat{i}+\hat{j}-2\hat{k})+\mu(2\hat{j}-\hat{k}).\end{aligned}$$

Quick Tip to Solve: Expand $-(\mu+2)$ before choosing the point and direction on L_2, then divide the absolute scalar triple product by the cross-product magnitude.

Hence $\vec{a}_1 = (2, -1, 1)$, $\vec{b}_1 = (1, 1, 3)$, $\vec{a}_2 = (1, 1, -2)$, and $\vec{b}_2 = (0, 2, -1)$. The minus sign in $-(\mu + 2)$ affects both the constant and the parameter coefficient.

Step 2: Find the common perpendicular direction Compute the cross product of the two direction vectors:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 0 & 2 & -1 \end{vmatrix} = (-1 - 6)\hat{i} - (-1)\hat{j} + 2\hat{k} = -7\hat{i} + \hat{j} + 2\hat{k}.$$

Its magnitude is $|\vec{n}| = \sqrt{49 + 1 + 4} = 3\sqrt{6}$. Since this is nonzero, the lines are nonparallel and the distance formula applies. Also, $(-7, 1, 2) \cdot (1, 1, 3) = 0$ and $(-7, 1, 2) \cdot (0, 2, -1) = 0$, verifying that this direction is perpendicular to both lines.

Step 3: Calculate the perpendicular projection The connecting vector is $\vec{a}_2 - \vec{a}_1 = (1-2, 1+1, -2-1) = (-1, 2, -3)$. Consequently,

$$(\vec{a}_2 - \vec{a}_1) \cdot \vec{n} = (-1)(-7) + 2(1) + (-3)(2) = 3,$$
$$d = \frac{|3|}{3\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

The nonzero scalar triple product confirms that the nonparallel lines are skew. Dividing by $|\vec{n}|$ converts the scalar product into a projection length; taking its absolute value makes the result a positive distance, independent of the orientation chosen for the normal.

Step 4: Conclusion The shortest distance between the two lines is $\boxed{\frac{1}{\sqrt{6}} \text{ units}}$.

Quick Tip to Solve: Expand $-(\mu + 2)$ before choosing the point and direction on L_2, then divide the absolute scalar triple product by the cross-product magnitude.

33(a). If $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{pmatrix}$, then find A^{-1} and hence solve the following system of equations:

$x + 2y - 3z = 1$

$2x - 3z = 2$

$x + 2y = 3$ [5 marks]

Solution:

Concept:

- A square matrix A has an inverse exactly when $|A| \neq 0$. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor of entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant left after deleting row i and column j. The adjoint is the transpose of the cofactor matrix.
- A linear system with coefficient matrix A, unknown column X, and constant column B is $AX = B$. If A is invertible, its unique solution is $X = A^{-1}B$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by -12. For a quick check on this system, subtract the first equation from the third to obtain $3z = 2$.

Step 1: Check that the inverse exists Expand the determinant along the first row, retaining the alternating cofactor signs:

$$\begin{aligned}|A| &= 1(0+6) - 2(0+3) - 3(4-0)\\ &= 6 - 6 - 12 = -12.\end{aligned}$$

Since this is nonzero, the inverse exists and the given system has exactly one solution.

Step 2: Calculate the cofactors and transpose them Evaluating the nine signed two-by-two determinants gives

$$\begin{aligned} C_{11} &= 6, & C_{12} &= -3, & C_{13} &= 4,\\ C_{21} &= -6, & C_{22} &= 3, & C_{23} &= 0,\\ C_{31} &= -6, & C_{32} &= -3, & C_{33} &= -4. \end{aligned}$$

For instance, $C_{12} = -(2\cdot 0 - (-3)\cdot 1) = -3$, whereas $C_{23} = -(1\cdot 2 - 2\cdot 1) = 0$. Thus, transposing the cofactor matrix,

$$\operatorname{adj} A = \begin{pmatrix} 6 & -6 & -6\\ -3 & 3 & -3\\ 4 & 0 & -4 \end{pmatrix}.$$

Divide every entry by the determinant:

$$A^{-1} = -\frac{1}{12}\begin{pmatrix} 6 & -6 & -6\\ -3 & 3 & -3\\ 4 & 0 & -4 \end{pmatrix}.$$

Step 3: Use the inverse to solve the system In the order x, y, z, the coefficients in the equations are precisely the rows of A. Therefore put

$$X = \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \qquad B = \begin{pmatrix} 1\\ 2\\ 3 \end{pmatrix}, \qquad AX = B.$$

Premultiply by A^{-1} and use row-by-column multiplication:

$$\begin{aligned} X &= -\frac{1}{12}\begin{pmatrix} 6 & -6 & -6\\ -3 & 3 & -3\\ 4 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1\\ 2\\ 3 \end{pmatrix}\\ &= -\frac{1}{12}\begin{pmatrix} 6-12-18\\ -3+6-9\\ 4+0-12 \end{pmatrix} = -\frac{1}{12}\begin{pmatrix} -24\\ -6\\ -8 \end{pmatrix}. \end{aligned}$$

As a check, substitution gives $2 + 1 - 2 = 1$, $4 - 2 = 2$, and $2 + 1 = 3$, the three required right-hand sides.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by -12. For a quick check on this system, subtract the first equation from the third to obtain $3z = 2$.

Step 4: Conclusion

$$\boxed{A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}}, \qquad \boxed{x = 2, \quad y = \frac{1}{2}, \quad z = \frac{2}{3}}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by -12. For a quick check on this system, subtract the first equation from the third to obtain $3z = 2$.

OR

33(b). Find the product of the matrices $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$ **and hence solve the following system of linear equations:**

$x + 2y - 3z = -4$

$2x + 3y + 2z = 2$

$3x - 3y - 4z = 11$ [5 marks]

Solution:

Concept:

- The (i, j) entry of a matrix product AB is the dot product of row i of A with column j of B. The order of the factors must be preserved.
- If square matrices satisfy $AB = kI$ with $k \neq 0$, then $A^{-1} = \frac{1}{k}B$, where I is the identity matrix.
- Write a linear system as $AX = D$, where X is the column of unknowns and D is the column of constants. Premultiplying by the inverse gives $X = A^{-1}D$.

Step 1: Name the matrices and calculate each product entry Let the first printed matrix be A and the second be B:

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}.$$

For example, the first entry is $1(-6) + 2(14) + (-3)(-15) = 67$. Calculate the remaining entries in exactly the same row-by-column order:

$$AB = \begin{pmatrix} -6 + 28 + 45 & 17 + 10 - 27 & 13 - 16 + 3 \\ -12 + 42 - 30 & 34 + 15 + 18 & 26 - 24 - 2 \\ -18 - 42 + 60 & 51 - 15 - 36 & 39 + 24 + 4 \end{pmatrix}.$$

Quick Tip to Solve: Once the product is $67I$, divide the second matrix by 67 to obtain the inverse. Multiply it directly by $(-4, 2, 11)^T$ instead of calculating cofactors again.

Hence

$$AB = \begin{pmatrix} 67 & 0 & 0 \\ 0 & 67 & 0 \\ 0 & 0 & 67 \end{pmatrix} = 67I.$$

Both negative factors in $(-3)(-15)$ and $(-4)(-1)$ produce positive contributions; retaining these signs is essential.

Step 2: Identify the inverse and form the matrix equation Since $A(B/67) = I$ and A is square, it is invertible and

$$A^{-1} = \frac{1}{67}B.$$

The coefficients of x, y, z in the printed system agree, in order, with the rows of A. Consequently the three equations can be written as

$$AX = D, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad D = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}.$$

Premultiplication by A^{-1} gives $X = BD/67$; the matrix of constants stays on the right.

Step 3: Multiply by the constant column and verify Substitute the column of constants and evaluate its product with B:

$$X = \frac{1}{67}\begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}$$

$$= \frac{1}{67}\begin{pmatrix} 24 + 34 + 143 \\ -56 + 10 - 88 \\ 60 + 18 - 11 \end{pmatrix} = \frac{1}{67}\begin{pmatrix} 201 \\ -134 \\ 67 \end{pmatrix}.$$

Substituting the resulting entries in the original equations confirms all three constants:

$$\begin{aligned} 3 + 2(-2) - 3(1) &= -4, \\ 2(3) + 3(-2) + 2(1) &= 2, \\ 3(3) - 3(-2) - 4(1) &= 11. \end{aligned}$$

Step 4: Conclusion The required product is $\boxed{67I}$, and the unique solution of the given system is

$$\boxed{x = 3, \qquad y = -2, \qquad z = 1}.$$

Quick Tip to Solve: Once the product is $67I$, divide the second matrix by 67 to obtain the inverse. Multiply it directly by $(-4, 2, 11)^T$ instead of calculating cofactors again.

34. Find the area of the region bounded by the curve $4x^2 + y^2 = 36$ using integration. [5 marks]

Solution:

Concept:

- The area under a nonnegative curve $y = f(x)$, between $x = a$ and $x = b$, is $\int_a^b f(x)\,dx$.
- If a closed curve is symmetric about both coordinate axes, its enclosed area is four times the area enclosed in the first quadrant.
- For a square root of the form $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ gives $dx = a\cos\theta\,d\theta$. On $0 \le \theta \le \pi/2$, the square root equals $a\cos\theta$.

Step 1: Identify the intercepts and symmetry Dividing the equation by 36 gives

$$\frac{x^2}{9} + \frac{y^2}{36} = 1.$$

Thus the curve is an ellipse centred at the origin. Its intercepts on the coordinate axes are $(\pm 3, 0)$ and $(0, \pm 6)$. Since replacing either x by $-x$ or y by $-y$ leaves the equation unchanged, the four quadrants enclose equal areas.

Step 2: Set up the area integral In the first quadrant, x runs from 0 to 3, and the upper boundary is the positive root

$$y = \sqrt{36 - 4x^2} = 2\sqrt{9 - x^2}.$$

A vertical strip extends from the x-axis to this boundary. Hence, if A denotes the entire area enclosed by the ellipse,

$$A = 4\int_0^3 y\,dx = 8\int_0^3 \sqrt{9 - x^2}\,dx.$$

Step 3: Evaluate by trigonometric substitution Put $x = 3\sin\theta$, so $dx = 3\cos\theta\,d\theta$. The limits $x = 0$ and $x = 3$ correspond to $\theta = 0$ and $\theta = \pi/2$, respectively. Since $\cos\theta \ge 0$ on this interval,

$$\sqrt{9 - x^2} = 3\cos\theta.$$

Substitution and the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ yield

$$\begin{aligned} A &= 72\int_0^{\pi/2} \cos^2\theta\,d\theta \\ &= 36\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta \\ &= 36\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\ &= 36\left(\frac{\pi}{2} + 0 - 0\right). \end{aligned}$$

Step 4: Conclusion The area of the region bounded by $4x^2 + y^2 = 36$, obtained using integration, is $\boxed{18\pi \text{ square units}}$.

Quick Tip to Solve: Use the positive root in the first quadrant and multiply its integral by four. Check the result against πab, with semi-axes $a = 3$ and $b = 6$.

35. Solve the following Linear Programming problem graphically:

Maximise $Z = 300x + 600y$

Subject to $x + 2y \leq 12$

$2x + y \leq 12$

$x + \frac{5}{4}y \geq 5$

$x \geq 0,\ y \geq 0$**.** [5 marks]

Solution:

Concept:

- Each linear inequality describes a half-plane. The common part of these half-planes in the first quadrant is the feasible region.
- For a bounded feasible polygon, a linear objective attains its maximum at a vertex. If adjacent vertices have the same maximum value, every point of their joining edge is also optimal.

Step 1: Plot the three boundary lines Replace each inequality by equality. For the third constraint, multiply by the positive number 4, obtaining $4x + 5y \geq 20$. The intercepts used to draw the lines are

Boundary	x-intercept	y-intercept
$x + 2y = 12$	$(12, 0)$	$(0, 6)$
$2x + y = 12$	$(6, 0)$	$(0, 12)$
$4x + 5y = 20$	$(5, 0)$	$(0, 4)$

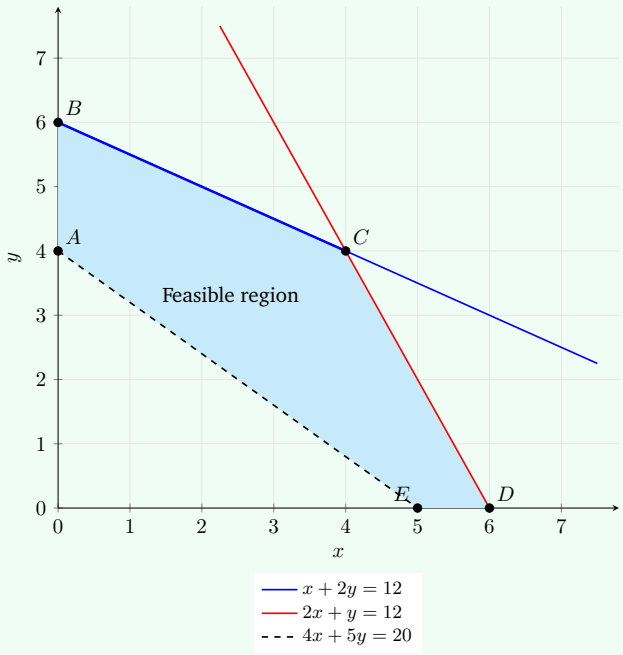
The graph shows the portions of these lines near the feasible region. The origin satisfies the first two inequalities, so take their sides containing the origin. It fails the third inequality, so take the side of $4x + 5y = 20$ away from the origin. Also retain only $x \geq 0$, $y \geq 0$.

Step 2: Identify the common feasible polygon The shaded intersection is the closed, bounded pentagon $ABCDE$. On the y-axis its endpoints are $A = (0, 4)$ and $B = (0, 6)$. On the x-axis its endpoints are $D = (6, 0)$ and $E = (5, 0)$. The remaining vertex C is the intersection of the first two lines:

$$x + 2y = 12, \quad 2x + y = 12,$$
$$x - y = 0, \qquad 3x = 12.$$

Thus $C = (4, 4)$, which also satisfies $4x + 5y = 36 \geq 20$. The entire lower edge EA lies below both upper constraint lines, as can also be checked at its endpoints; no additional intersection cuts the polygon.

Quick Tip to Solve: Notice that $Z = 300(x + 2y)$: the first constraint gives the maximum bound immediately. Use the graph and the other inequalities to identify the whole feasible edge on which equality holds.



Shaded feasible pentagon; the heavy edge BC gives all maximum points.

Step 3: Evaluate the objective at every vertex Substitute the coordinates of all five corners into $Z = 300x + 600y$:

Vertex	Z
$A(0, 4)$	2400
$B(0, 6)$	3600
$C(4, 4)$	3600
$D(6, 0)$	1800
$E(5, 0)$	1500

Adjacent vertices B and C both give the largest value.

Step 4: Determine every maximising point The objective is $Z = 300(x + 2y)$, so the constraint $x + 2y \leq 12$ gives the upper bound $Z \leq 3600$. Equality requires $y = 6 - x/2$. On this line the other upper constraint becomes

$$2x + 6 - \frac{x}{2} \leq 12 \quad \Longrightarrow \quad x \leq 4.$$

For $0 \leq x \leq 4$, non-negativity holds and

$$x + \frac{5}{4}y = \frac{15}{2} + \frac{3}{8}x \geq \frac{15}{2} > 5.$$

Thus all of BC is feasible and attains the upper bound.

Step 5: Conclusion The maximum value is $\boxed{Z_{\max} = 3600}$. It is attained at every point on the segment joining $(0, 6)$ and $(4, 4)$, namely

$$\boxed{(x, y) = (t, 6 - t/2), \quad 0 \leq t \leq 4}.$$

Quick Tip to Solve: Notice that $Z = 300(x + 2y)$: the first constraint gives the maximum bound immediately. Use the graph and the other inequalities to identify the whole feasible edge on which equality holds.

Section E (12 marks)

*In this section there are **3** case study questions of **4** marks each.*

36. A departmental store sends bills to charge its customers once a month. Past experience shows that 70% of its customers pay their first month bill in time. The store also found that the customer who pays the bill in time has the probability of 0.8 of paying in time next month and the customer who doesn't pay in time has the probability of 0.4 of paying in time the next month.

Based on the above information, answer the following questions: [4 marks]

36(i). Let E_1 and E_2 respectively denote the event of customer paying or not paying the first month bill in time. Find $P(E_1)$, $P(E_2)$. [1 mark]

Solution:

Concept:

- A percentage becomes a probability on division by 100.
- Complementary events cover all possibilities and cannot occur together, so their probabilities add to 1.

Step 1: Read the first-month payment probability The event E_1 means that the customer pays the first month's bill in time. The given first-month percentage is 70%, hence

$$P(E_1) = \frac{70}{100} = 0.7.$$

Step 2: Use the complementary event The customer either pays the first month's bill in time or does not; thus $E_2 = E_1^c$. Therefore

$$P(E_2) = 1 - P(E_1) = 1 - 0.7 = 0.3.$$

The next-month conditional probabilities are not needed to calculate these first-month probabilities.

Step 3: Conclusion $\boxed{P(E_1) = 0.7, \qquad P(E_2) = 0.3}$.

Quick Tip to Solve: Match the month before substituting: use the initial 70% for the first month and its complement for late payment.

36(ii). Let A denotes the event of customer paying second month's bill in time, then find $P(A|E_1)$ and $P(A|E_2)$. [1 mark]

Solution:

Concept:

- $P(A \mid E)$ means the probability of event A given that event E has occurred.
- Here A refers to an on-time second-month payment, whereas E_1 and E_2 describe the two first-month outcomes.

Quick Tip to Solve: Read the condition to the right of the vertical bar first, then select the next-month probability stated for that group.

Step 1: Condition on an on-time first payment Given that a customer paid the first month's bill in time, the stated chance of paying the next bill in time is 0.8. In event notation this is

$$P(A \mid E_1) = 0.8.$$

Step 2: Condition on a late first payment Given that the customer did not pay the first bill in time, the chance of paying the second bill in time is 0.4. Therefore

$$P(A \mid E_2) = 0.4.$$

These are probabilities within different first-month groups; they are not complementary probabilities and need not add to 1.

Step 3: Conclusion $\boxed{P(A \mid E_1) = 0.8, \qquad P(A \mid E_2) = 0.4}$.

Quick Tip to Solve: Read the condition to the right of the vertical bar first, then select the next-month probability stated for that group.

36(iii)(a). Find the probability of customer paying second month's bill in time. [2 marks]

Solution:

Concept:

- Let A denote an on-time second-month payment. The first-month events E_1 (on time) and E_2 (not on time) form a mutually exclusive, exhaustive partition.
- The theorem of total probability gives $P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$.

Step 1: Record the probabilities for both groups The first-month probabilities are $P(E_1) = 0.7$ and $P(E_2) = 1 - 0.7 = 0.3$. The corresponding probabilities of an on-time second payment are $P(A \mid E_1) = 0.8$ and $P(A \mid E_2) = 0.4$. These two conditional probabilities must be weighted by the sizes of the first-month groups.

Step 2: Add the two disjoint ways to pay on time An on-time second payment can follow either an on-time or a late first payment. These possibilities cannot overlap, so

$$\begin{aligned} P(A) &= P(A \cap E_1) + P(A \cap E_2) \\ &= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) \\ &= (0.7)(0.8) + (0.3)(0.4) \\ &= 0.56 + 0.12 = 0.68. \end{aligned}$$

The weighted probability lies between 0.4 and 0.8, as expected, and is closer to 0.8 because most customers belong to E_1.

Step 3: Conclusion The probability that the customer pays the second month's bill in time is $\boxed{0.68 = \frac{17}{25}}$.

Quick Tip to Solve: Multiply along each first-month-to-second-month path, then add the two products; do not take an unweighted average of 0.8 and 0.4.

OR

36(iii)(b). Find the probability of customer paying first month's bill in time if it is found that customer has paid the second month's bill in time. [2 marks]

Solution:

Concept:

- Let E_1 denote paying the first month's bill in time, $E_2 = E_1^c$, and A denote paying the second month's bill in time.
- Bayes' theorem reverses the conditioning: $P(E_1 \mid A) = P(E_1)P(A \mid E_1)/P(A)$, where total probability gives $P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$.

Step 1: Identify the two possible first-month outcomes From the given payment history, $P(E_1) = 0.7$, so $P(E_2) = 1 - 0.7 = 0.3$. The second-month payment probabilities are $P(A \mid E_1) = 0.8$ and $P(A \mid E_2) = 0.4$. These are conditional probabilities and must be weighted by their first-month probabilities.

Step 2: Compute the probability of the observed second-month payment The events E_1 and E_2 are mutually exclusive and exhaustive. Thus the two routes to paying the second bill in time give

$$P(A) = (0.7)(0.8) + (0.3)(0.4) = 0.56 + 0.12 = 0.68.$$

Step 3: Reverse the condition using Bayes' theorem Among customers in event A, the fraction also in E_1 is the joint probability divided by $P(A)$:

$$P(E_1 \mid A) = \frac{P(E_1 \cap A)}{P(A)} = \frac{(0.7)(0.8)}{0.68} = \frac{56}{68} = \frac{14}{17}.$$

Step 4: Conclusion The required probability is $\boxed{\frac{14}{17}}$.

Quick Tip to Solve: For a reversed conditional probability, put the desired route 0.7×0.8 in the numerator and both routes to the observed payment in the denominator.

37. Students of a school are taken to a railway museum to learn about railways heritage and its history.

An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by

$$R = \{(l_1, l_2) : l_1 \text{ is parallel to } l_2\}.$$

On the basis of the above information, answer the following questions: [4 marks]

37(a)(i). Find whether the relation R is symmetric or not. [1 mark]

Solution:

Concept:

- A relation R on L is symmetric if $(l_1, l_2) \in R$ implies $(l_2, l_1) \in R$ for every pair in L.
- Parallelism does not depend on the order of the two lines: $l_1 \parallel l_2$ implies $l_2 \parallel l_1$.

Step 1: Take an arbitrary related pair Let $l_1, l_2 \in L$ and assume $(l_1, l_2) \in R$. By the definition of the given relation, the rail line l_1 is parallel to the rail line l_2.

Step 2: Reverse the order The same parallelism says that l_2 is parallel to l_1. Therefore $(l_2, l_1) \in R$. This holds for every related pair, exactly satisfying the definition of symmetry; it does not require every possible pair of rail lines to be parallel.

Step 3: Conclusion The relation $\boxed{R \text{ is symmetric}}$.

Quick Tip to Solve: To test symmetry, reverse an arbitrary related pair and check that the defining geometric condition still holds.

37(a)(ii). Find whether the relation R is transitive or not. [1 mark]

Solution:

Concept:

- Transitivity means that $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$ together imply $(l_1, l_3) \in R$.
- Lines parallel to the same line have the same direction and are parallel to one another.

Step 1: Take two linked pairs Let $l_1, l_2, l_3 \in L$ with $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$. Translating these statements gives $l_1 \parallel l_2$ and $l_2 \parallel l_3$.

Step 2: Compare the first and third directions Both l_1 and l_3 have the direction of l_2, so $l_1 \parallel l_3$. Equivalently, nonvertical lines have equal slopes in both given pairs, and equality of slopes is transitive; if the middle line is vertical, all three are vertical. Hence $(l_1, l_3) \in R$.

Step 3: Conclusion The relation $\boxed{R \text{ is transitive}}$.

Quick Tip to Solve: For transitivity of parallelism, use the shared middle line to show that the first and third lines have the same direction.

37(a)(iii). If one of the rail lines on the railway track is represented by the equation $y = 3x + 2$, then find the set of rail lines in R related to it. [2 marks]

Solution:

Concept:

- The line $y = mx + c$ has slope m, where c is its intercept on the y-axis.
- All lines parallel to a nonvertical line have the same slope. For a relation on L, retain only those members of the family that belong to L.

Step 1: Identify the direction of the given line Write the given rail line as $l_0 : y = 3x + 2$. Comparing with $y = mx + c$ shows that its slope is $m = 3$ and its intercept is 2. A rail line related to l_0 under R must be parallel to it, so its slope must also be 3.

Step 2: Describe the complete related family A line of slope 3 has equation $y = 3x + c$, where c is real. Conversely, each such line has the same direction as l_0, so the parallel family is precisely this family of equations. Because the given relation is on the set of rail lines L, take the intersection of this family with L. The value $c = 2$ recovers the given line itself; all distinct parallel companions have $c \neq 2$.

Step 3: Conclusion The related rail lines form $\boxed{\{\, l \in L : l \text{ has equation } y = 3x + c,\ c \in \mathbb{R} \,\}}$, the family of rail lines of slope 3.

Quick Tip to Solve: Keep the slope 3 fixed and vary the intercept: parallelism changes the position of a line, not its direction.

OR

37(b). Let S be the relation defined by

$$S = \{(l_1, l_2) : l_1 \text{ is perpendicular to } l_2\},$$

check whether the relation S is symmetric and transitive. [4 marks]

Solution:

Concept:

- A relation S is symmetric if $(l_1, l_2) \in S$ implies $(l_2, l_1) \in S$ for all lines under consideration.
- Transitivity requires $(l_1, l_2) \in S$ and $(l_2, l_3) \in S$ to imply $(l_1, l_3) \in S$. One counterexample disproves this universal implication.
- Perpendicular lines meet at a right angle; two lines in a plane perpendicular to the same line are parallel to each other.

Step 1: Test symmetry using an arbitrary pair Assume $(l_1, l_2) \in S$. By definition, $l_1 \perp l_2$, so the angle between these lines is $90°$. Reversing their order leaves the right angle unchanged: $l_2 \perp l_1$. Consequently $(l_2, l_1) \in S$. This argument applies to every related pair and establishes the symmetry implication.

Step 2: Test the transitivity implication If $l_1 \perp l_2$ and $l_2 \perp l_3$, the first and third lines have the same direction. They are parallel, rather than perpendicular. Thus the two hypotheses of transitivity do not

Quick Tip to Solve: Use a horizontal–vertical–horizontal chain: each consecutive pair is perpendicular, but the first and third lines are parallel.

provide its required conclusion. We can disprove the implication explicitly using ordinary coordinate lines.

Step 3: Give a concrete counterexample Take

$$l_1 : y = 0, \qquad l_2 : x = 0, \qquad l_3 : y = 1.$$

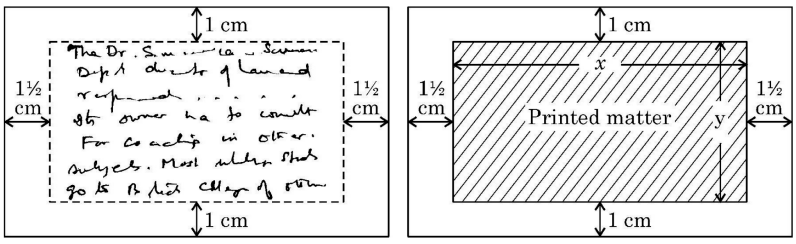
The horizontal line l_1 is perpendicular to the vertical line l_2, and l_2 is perpendicular to the horizontal line l_3. Thus $(l_1, l_2) \in S$ and $(l_2, l_3) \in S$. But l_1 and l_3 are distinct parallel horizontal lines, so $(l_1, l_3) \notin S$. This supplies both true hypotheses and a false conclusion, which is enough to rule out transitivity.

Step 4: Conclusion The perpendicularity relation $\boxed{S \text{ is symmetric but not transitive}}$.

Quick Tip to Solve: Use a horizontal–vertical–horizontal chain: each consecutive pair is perpendicular, but the first and third lines are parallel.

38. A rectangular visiting card is to contain 24 sq.cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be $1\frac{1}{2}$ cm as shown below:

On the basis of the above information, answer the following questions: [4 marks]



38(i). Write the expression for the area of the visiting card in terms of x. [2 marks]

Solution:

Concept:

- The area of a rectangle is length times breadth. In the printed diagram, x and y are the horizontal and vertical dimensions, in centimetres, of the printed matter.
- A fixed printed area gives the constraint $xy = 24$. The card's outer dimensions include the margins on both sides of each printed dimension.

Step 1: Express the printed height in terms of its width Since the printed matter occupies $24\,\text{cm}^2$, its dimensions satisfy

$$xy = 24 \quad \Longrightarrow \quad y = \frac{24}{x}, \qquad x > 0.$$

The positivity restriction is necessary because x is a physical length; it also makes y positive.

Step 2: Add the margins to obtain the card dimensions The left and right margins each contribute $1\frac{1}{2}$ cm, making the card's horizontal dimension $x + 3$. The top and bottom margins each contribute

Quick Tip to Solve: Use $xy = 24$ for the printed region first, then add 3 to the horizontal length and 2 to the vertical length before multiplying.

1 cm, making its vertical dimension $y+2$. Hence its area is

$$\begin{aligned} A &= (x+3)(y+2) \\ &= (x+3)\left(\frac{24}{x}+2\right) \\ &= 24+2x+\frac{72}{x}+6. \end{aligned}$$

The fixed 24 accounts for the printed rectangle; the remaining terms account for the margins, including their corner regions.

Step 3: Conclusion The area of the visiting card is $\boxed{A(x)=2x+\frac{72}{x}+30\,\text{cm}^2}$, where $x>0$ is measured in centimetres.

Quick Tip to Solve: Use $xy=24$ for the printed region first, then add 3 to the horizontal length and 2 to the vertical length before multiplying.

38(ii). Obtain the dimensions of the card of minimum area. [2 marks]

Solution:

Concept:

- For a fixed printed area, if the printed width is x cm and height is y cm, then $xy=24$, with $x>0$ and $y>0$.
- Include both opposite margins to obtain the card dimensions. A stationary point of its area $A(x)$ satisfies $A'(x)=0$; if $A''(x)>0$ throughout the domain, it gives the minimum.

Step 1: Express the total card area The left and right margins together add 3 cm to the printed width, while the top and bottom margins together add 2 cm to the printed height. Since $y=24/x$, the area of the complete card is

$$A(x)=(x+3)\left(\frac{24}{x}+2\right)=30+2x+\frac{72}{x}, \qquad x>0.$$

Step 2: Minimise the area and restore the margins Differentiating with respect to x,

$$A'(x)=2-\frac{72}{x^2}, \qquad A''(x)=\frac{144}{x^3}>0 \quad (x>0).$$

The stationary condition gives $2-72/x^2=0$, so $x^2=36$. Rejecting the negative root because a length is positive, $x=6$ cm. The positive second derivative throughout the domain confirms the unique minimum. The printed height is $y=24/6=4$ cm. Thus the card width is $x+3=6+3=9$ cm, and its height is $y+2=4+2=6$ cm.

Step 3: Conclusion The dimensions of the card of minimum area are $\boxed{9\,\text{cm}\times 6\,\text{cm}}$, with minimum area $54\,\text{cm}^2$.

Quick Tip to Solve: Optimise using the printed width first, then add 3 cm horizontally and 2 cm vertically to report the card dimensions.