

CBSE Class 12 Mathematics Question Paper 2024 with Solution PDF-Set 65-5-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In **Section A** – Questions Number **1** to **18** are Multiple Choice Questions (MCQs) type and Questions Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B** – Questions Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C** – Questions Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D** – Questions Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In **Section E** – Questions Number **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.
- (ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

There are 20 multiple choice questions of 1 mark each in this section.

1. The position vectors of points P and Q are $\vec{p}$ and $\vec{q}$ respectively. The point R divides line segment PQ in the ratio $3 : 1$ and S is the mid-point of line segment PR . The position vector of S is: [1 mark]

- (A) $\frac{\vec{p} + 3\vec{q}}{4}$
(B) $\frac{\vec{p} + 3\vec{q}}{8}$
(C) $\frac{5\vec{p} + 3\vec{q}}{4}$
(D) $\frac{5\vec{p} + 3\vec{q}}{8}$

Correct Answer: (D) $\frac{5\vec{p} + 3\vec{q}}{8}$

Solution:

Concept:

- For internal division of PQ in the ratio $m : n$, the position vector is $\frac{n\vec{p} + m\vec{q}}{m + n}$.
- The midpoint of two points has the average of their position vectors.

Step 1: Locate the dividing point Here $PR : RQ = 3 : 1$. Thus the coefficient of the position vector of Q is 3, and

$$\vec{r} = \frac{\vec{p} + 3\vec{q}}{4}.$$

This gives the point three quarters of the way from P to Q .

Step 2: Take the midpoint of PR The endpoints of PR have position vectors $\vec{p}$ and $\vec{r}$. Hence

$$\vec{s} = \frac{\vec{p} + \vec{r}}{2} = \frac{\vec{p} + (\vec{p} + 3\vec{q})/4}{2} = \frac{5\vec{p} + 3\vec{q}}{8}.$$

Option (A) is the vector of R , while (B) omits the contribution of P in the midpoint calculation. Option (C) has the correct numerator but fails to divide by 2 when taking the midpoint.

Step 3: Conclusion $\vec{s} = \frac{5\vec{p} + 3\vec{q}}{8}$. The correct option is (D).

Quick Tip to Solve: Apply the section formula first; average that vector with $\vec{p}$, not with $\vec{q}$.

2. For the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ \lambda & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$ to be invertible, the value of λ is:

[1 mark]

- (A) 0
- (B) 10
- (C) $\mathbb{R} - \{10\}$
- (D) $\mathbb{R} - \{-10\}$

Correct Answer: (D) $\mathbb{R} - \{-10\}$

Solution:

Concept:

- A square matrix is invertible if and only if its determinant is nonzero.
- Expand a determinant along a row using alternating cofactor signs $+$, $-$, $+$.

Step 1: Evaluate the determinant Expanding along the first row gives

$$\begin{aligned} |A| &= 2(2 \cdot 3 - 0 \cdot (-2)) \\ &\quad - (-1)(\lambda \cdot 3 - 0 \cdot 1) \\ &\quad + 1(\lambda \cdot (-2) - 2 \cdot 1) \\ &= 12 + 3\lambda - 2\lambda - 2 = \lambda + 10. \end{aligned}$$

The middle contribution is positive because both the cofactor sign and the matrix entry are negative.

Step 2: Impose invertibility and compare options The condition is $\lambda + 10 \neq 0$, so $\lambda \neq -10$. Values 0 and 10 in options (A) and (B) do make the matrix invertible, but neither describes all allowed real values. Option (C) incorrectly excludes 10 and includes the singular value -10 . The set in option (D) excludes precisely the singular value.

Step 3: Conclusion $\lambda \in \mathbb{R} \setminus \{-10\}$. The correct option is (D).

Quick Tip to Solve: Find the single determinant-zero value, then exclude it from the real numbers.

3. The angle which the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$ makes with the positive direction of Y-axis is: [1 mark]

- (A) $\frac{5\pi}{6}$
- (B) $\frac{3\pi}{4}$
- (C) $\frac{5\pi}{4}$
- (D) $\frac{7\pi}{4}$

Correct Answer: (B) $\frac{3\pi}{4}$

Solution:

Concept:

- The denominators in a symmetric line equation give direction ratios a, b, c .
- The direction cosine with the positive Y-axis is $\cos \beta = b/\sqrt{a^2 + b^2 + c^2}$, where $0 \leq \beta \leq \pi$.

Step 1: Identify the direction ratios The displayed line has direction ratios $(1, -1, 0)$. Its zero third direction ratio means that z remains constant; it does not require an arithmetic division by zero. The length of the direction vector is $\sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$.

Step 2: Evaluate the angle and test the options Thus

$$\cos \beta = \frac{-1}{\sqrt{2}}, \quad \beta = \frac{3\pi}{4}.$$

Option (A), $5\pi/6$, has cosine $-\sqrt{3}/2$, which does not agree. Option (C), $5\pi/4$, is outside the range of an angle with a positive coordinate axis. Option (D), $7\pi/4$, is also outside that range and has positive cosine.

Step 3: Conclusion The angle is $\frac{3\pi}{4}$; the correct option is (B).

Quick Tip to Solve: Keep the sign of the Y direction ratio: a negative value gives an obtuse axis angle.

4. The Cartesian equation of the line passing through the point $(1, -3, 2)$ and parallel to the line

$$\vec{r} = (2 + \lambda)\hat{i} + \lambda\hat{j} + (2\lambda - 1)\hat{k} \text{ is:}$$

[1 mark]

(A) $\frac{x-1}{2} = \frac{y+3}{0} = \frac{z-2}{-1}$

(B) $\frac{x+1}{1} = \frac{y-3}{1} = \frac{z+2}{2}$

(C) $\frac{x+1}{2} = \frac{y-3}{0} = \frac{z+2}{-1}$

(D) $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Correct Answer: (D) $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$

Solution:

Concept:

- Parallel lines have proportional direction vectors.
- A line through (x_0, y_0, z_0) with direction ratios (a, b, c) has equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.

Step 1: Read the direction vector Rearrange the given vector equation as

$$\vec{r} = (2\hat{i} - \hat{k}) + \lambda(\hat{i} + \hat{j} + 2\hat{k}).$$

Therefore the direction ratios are $(1, 1, 2)$. The constant vector describes a point on the given line; its components $(2, 0, -1)$ are not direction ratios.

Step 2: Use the specified point The required parallel line passes through $(1, -3, 2)$. Consequently

$$\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}.$$

Option (B) has the right direction ratios but passes through $(-1, 3, -2)$. Options (A) and (C) use the constant vector as their direction ratios, so they cannot be parallel to the given line. Only (D) satisfies both requirements.

Step 3: Conclusion The required equation is $\frac{x-1}{1} = \frac{y+3}{1} = \frac{z-2}{2}$. The correct option is (D).

Quick Tip to Solve: Take coefficients of the line parameter for direction ratios and subtract the new point coordinates.

5. If $A = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix}$, then value of x for which $A^2 = B$ is:

[1 mark]

- (A) -2
- (B) 2
- (C) 2 or -2
- (D) 4

Correct Answer: (A) -2

Solution:

Concept:

- Matrix squaring means matrix multiplication $A^2 = A \cdot A$, not squaring each entry.
- Equal matrices have equal corresponding entries, so every resulting equation must hold.

Step 1: Multiply the matrix by itself Using row-by-column multiplication,

$$A^2 = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ x+1 & 1 \end{pmatrix}.$$

In particular, the lower-left entry is $1 \cdot x + 1 \cdot 1 = x + 1$; it cannot be ignored.

Step 2: Compare all entries Equality with B requires both $x^2 = 4$ and $x + 1 = -1$. The first equation permits $x = 2$ or -2 , but the second requires $x = -2$, which satisfies both. Thus (B) fails the lower-left entry, (C) incorrectly retains 2, and (D) fails both nonconstant entries. The zero and lower-right entries already agree.

Step 3: Conclusion $x = -2$. The correct option is (A).

Quick Tip to Solve: Use the off-diagonal entry as well as $x^2 = 4$ to select the correct sign.

6. Given a curve $y = 7x - x^3$ and x increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when $x = 5$ is:

[1 mark]

- (A) -60 units/sec
- (B) 60 units/sec
- (C) -70 units/sec
- (D) -140 units/sec

Correct Answer: (A) -60 units/sec

Solution:

Concept:

- The slope of the curve is $m = dy/dx$.
- If x depends on time t , the slope changes at the rate $dm/dt = (dm/dx)(dx/dt)$.

Quick Tip to Solve: Differentiate the slope once more with respect to x , then multiply by the given dx/dt .

Step 1: Differentiate to find the slope For $y = 7x - x^3$,

$$m = \frac{dy}{dx} = 7 - 3x^2, \quad \frac{dm}{dx} = -6x.$$

The question asks how the slope changes with time, so neither the slope itself nor dy/dt is the requested rate.

Step 2: Apply the time rate and evaluate Using $dx/dt = 2$,

$$\frac{dm}{dt} = -6x \cdot 2 = -12x.$$

At $x = 5$, this gives $-12 \cdot 5 = -60$ units/sec. The negative sign indicates that the slope decreases as x increases. Option (B) reverses this sign; (C) and (D) have incorrect magnitudes. The factor 2 from the given time rate is essential.

Step 3: Conclusion The slope changes at -60 units/sec. The correct option is (A).

Quick Tip to Solve: Differentiate the slope once more with respect to x , then multiply by the given dx/dt .

7. Let $f(x) = \begin{vmatrix} x^2 & \sin x \\ p & -1 \end{vmatrix}$, where p is a constant. The value of p for which $f'(0) = 1$ is: [1 mark]

- (A) $\mathbb{R}$
- (B) 1
- (C) 0
- (D) -1

Correct Answer: (D) -1

Solution:

Concept:

- The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ equals $ad - bc$.
- When differentiating, p remains constant, and $d(\sin x)/dx = \cos x$.

Step 1: Expand before differentiating The determinant defines the scalar function

$$f(x) = x^2(-1) - p \sin x = -x^2 - p \sin x.$$

The minus sign on the second term follows from the determinant formula; replacing it by a plus would change the required value of p .

Step 2: Impose the derivative condition Differentiate to obtain

$$f'(x) = -2x - p \cos x, \quad f'(0) = -p,$$

because $\cos 0 = 1$. The condition $f'(0) = 1$ therefore gives $-p = 1$. Option (B) would give derivative

Quick Tip to Solve: Expand the determinant first and retain its subtraction sign when differentiating the sine term.

-1 , and (C) would give 0. Option (A) cannot hold because the prescribed derivative fixes one unique constant.

Step 3: Conclusion $p = -1$. The correct option is (D).

Quick Tip to Solve: Expand the determinant first and retain its subtraction sign when differentiating the sine term.

8. If A and B are two events such that $P(A/B) = P(B/A) \neq 0$, then:

[1 mark]

(A) $A \subset B$, but $A \neq B$

(B) $A = B$

(C) $A \cap B = \phi$

(D) $P(A) = P(B)$

Correct Answer: (D) $P(A) = P(B)$

Solution:

Concept:

- For events of positive probability, $P(A/B) = P(A \cap B)/P(B)$ and $P(B/A) = P(A \cap B)/P(A)$.
- The common conditional probability is nonzero, so $P(A \cap B) > 0$, allowing cancellation.

Step 1: Write both conditional probabilities The given equality becomes

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \neq 0.$$

Both denominators are positive because the conditional probabilities are defined. The numerator is also positive by the nonzero condition, so cross-multiplication and cancellation are valid.

Step 2: Determine what must follow Cancelling the common positive numerator gives $1/P(B) = 1/P(A)$, hence equal marginal probabilities. Equal probabilities do not require equal events, so (B) need not follow. For example, two events with an overlapping outcome can have equal probability and still differ. Nor must one event be a proper subset of the other as in (A). Option (C) contradicts the nonzero intersection probability.

Step 3: Conclusion The two marginal probabilities are equal. The correct option is (D).

Quick Tip to Solve: Use the nonzero condition to justify cancelling $P(A \cap B)$ from the two conditional probabilities.

9. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = x^2 - 4x + 5$ is:

[1 mark]

- (A) injective but not surjective.
- (B) surjective but not injective.
- (C) both injective and surjective.
- (D) neither injective nor surjective.

Correct Answer: (D) neither injective nor surjective.

Solution:

Concept:

- A function is injective if distinct inputs always give distinct outputs; it is surjective if its range equals its stated codomain.
- Completing the square gives $f(x) = (x - 2)^2 + 1$. A real square is non-negative.

Step 1: Test injectivity Take the distinct inputs 1 and 3. Then $f(1) = 1 - 4 + 5 = 2$ and $f(3) = 9 - 12 + 5 = 2$. Equal outputs for distinct inputs show that the function is not injective.

Step 2: Test surjectivity and the options Since $(x - 2)^2 \geq 0$, every output satisfies $f(x) \geq 1$. The value 0 in the codomain $\mathbb{R}$ has no preimage, so the function is not surjective. In fact its range is $[1, \infty)$. Options (A) and (C) incorrectly assert injectivity, while (B) incorrectly asserts surjectivity.

Step 3: Conclusion The function is neither injective nor surjective. Hence, the correct option is (D).

Quick Tip to Solve: Complete the square to find the range, and test two inputs equally spaced from the vertex for injectivity.

10. If A is a square matrix of order 3 and $|\text{adj} \cdot A| = 8$, then the value of $|A^T|$ is:

[1 mark]

- (A) $\sqrt{2}$
- (B) $-\sqrt{2}$
- (C) 8
- (D) $2\sqrt{2}$

Correct Answer: (D) $2\sqrt{2}$

Solution:

Concept:

- For a square matrix A of order n , $|\text{adj} A| = |A|^{n-1}$. Here $n = 3$.
- Transposition leaves a determinant unchanged: $|A^T| = |A|$.

Step 1: Apply the adjugate determinant identity For the matrix of order 3, the printed condition implies

$$|A|^2 = |\text{adj} A| = 8.$$

Consequently $|A| = \pm\sqrt{8} = \pm 2\sqrt{2}$. The vertical bars denote a determinant, so this calculation alone does not force its sign to be positive.

Step 2: Compare all printed options We have $|A^T| = \pm 2\sqrt{2}$. The proposed values $\sqrt{2}$ and $-\sqrt{2}$ in

Quick Tip to Solve: Use the exponent $n-1$ in the adjugate identity, then check which listed determinant squares to 8.

(A) and (B) square to 2, and the value 8 in (C) squares to 64, so each contradicts the given condition. The value $2\sqrt{2}$ in (D) squares to 8 and is the only compatible listed value. The negative possibility $-2\sqrt{2}$ is not listed; the printed condition itself admits that sign as well.

Step 3: Conclusion Among the printed options, $|A^T| = 2\sqrt{2}$ is the compatible value, so select (D). In general the condition allows $|A^T| = \pm 2\sqrt{2}$.

Quick Tip to Solve: Use the exponent $n-1$ in the adjugate identity, then check which listed determinant squares to 8.

11. If $\int_{-2}^3 x^2 dx = k \int_0^2 x^2 dx + \int_2^3 x^2 dx$, then the value of k is:

[1 mark]

- (A) 2
- (B) 1
- (C) 0
- (D) $\frac{1}{2}$

Correct Answer: (A) 2

Solution:

Concept:

- A definite integral may be split at an interior point: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.
- The function x^2 is even, so $\int_{-a}^a x^2 dx = 2 \int_0^a x^2 dx$.

Step 1: Split the integral using symmetry Divide the interval at 2:

$$\int_{-2}^3 x^2 dx = \int_{-2}^2 x^2 dx + \int_2^3 x^2 dx = 2 \int_0^2 x^2 dx + \int_2^3 x^2 dx.$$

The interval from -2 to 2 contributes twice the integral from 0 to 2 .

Step 2: Identify the coefficient Comparing this expression with the given equation gives $(k - 2) \int_0^2 x^2 dx = 0$. Since $\int_0^2 x^2 dx = 8/3 \neq 0$, the coefficient must satisfy $k - 2 = 0$. Values 1, 0 and $1/2$ would not supply the contribution from both halves of the symmetric interval.

Step 3: Conclusion $k = 2$. Hence, the correct option is (A).

Quick Tip to Solve: Cancel the common integral from 2 to 3 and use evenness on the remaining interval from -2 to 2 .

12. The value of $\int_1^e \log x \, dx$ is:

[1 mark]

- (A) 0
- (B) 1
- (C) e
- (D) $e \log e$

Correct Answer: (B) 1

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. Choose $u = \log x$ and $dv = dx$.
- Here $\log x$ is the natural logarithm, with $\log e = 1$ and $\log 1 = 0$.

Step 1: Find the antiderivative With $du = dx/x$ and $v = x$, integration by parts yields

$$\int \log x \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

Differentiation checks the formula: $d(x \log x - x)/dx = \log x + 1 - 1 = \log x$.

Step 2: Evaluate both endpoints Therefore

$$\int_1^e \log x \, dx = [x \log x - x]_1^e = (e - e) - (0 - 1) = 1.$$

The value is positive, so (A) is false. Options (C) and (D) both equal e , rather than the difference of the two endpoint values. Keeping the lower endpoint contribution is essential.

Step 3: Conclusion The integral equals 1. Hence, the correct option is (B).

Quick Tip to Solve: Use $x \log x - x$ and subtract its value at 1; the upper endpoint alone gives zero.

13. The area bounded by the curve $y = \sqrt{x}$, Y-axis and between the lines $y = 0$ and $y = 3$ is: [1 mark]

- (A) $2\sqrt{3}$
- (B) 27
- (C) 9
- (D) 3

Correct Answer: (C) 9

Solution:

Concept:

- For horizontal strips, area is $\int_c^d (x_{\text{right}} - x_{\text{left}}) \, dy$.
- On the curve $y = \sqrt{x}$, we have $x = y^2$ and $y \geq 0$; the Y-axis has equation $x = 0$.

Step 1: Express the width of a horizontal strip For every height y between 0 and 3, the left boundary is the Y-axis $x = 0$, and the right boundary is the curve $x = y^2$. Thus the strip has width $y^2 - 0 = y^2$.

Quick Tip to Solve: When bounds are y-values and one boundary is the Y-axis, rewrite the curve as $x = y^2$ and integrate horizontally.

Because the bounds are heights, integration with respect to y directly describes the enclosed region.

Step 2: Integrate the strip widths The area is

$$\mathcal{A} = \int_0^3 y^2 dy = \left[\frac{y^3}{3} \right]_0^3 = \frac{27}{3} = 9.$$

Option (B) omits the division by 3; (D) gives only the vertical height. The value $2\sqrt{3}$ in (A) also differs from the evaluated area. The area is measured in square units.

Step 3: Conclusion The bounded area is 9 square units. Hence, the correct option is (C).

Quick Tip to Solve: When bounds are y -values and one boundary is the Y -axis, rewrite the curve as $x = y$ squared and integrate horizontally.

14. The order of the following differential equation $\frac{d^3y}{dx^3} + x \left(\frac{dy}{dx} \right)^5 = 4 \log \left(\frac{d^4y}{dx^4} \right)$ is: [1 mark]

- (A) not defined
- (B) 3
- (C) 4
- (D) 5

Correct Answer: (C) 4

Solution:

Concept:

- The order of a differential equation is the order of the highest derivative occurring in it.
- The exponent on a derivative and the function applied to it do not change its derivative order. Order and degree are different concepts.

Step 1: List the derivative orders The printed equation contains d^3y/dx^3 , dy/dx raised to the fifth power, and d^4y/dx^4 inside the logarithm. Their derivative orders are respectively 3, 1, and 4. The factor x multiplying the fifth power does not introduce any additional derivative.

Step 2: Identify the highest derivative The fourth derivative is present even though it is the argument of $\log$. Thus the highest derivative order is 4. Option (A) confuses the issue of degree with order; (B) ignores the derivative inside the logarithm, and (D) treats the exponent 5 on the first derivative as an order. None of these interpretations follows the definition of order.

Step 3: Conclusion The order is 4. Hence, the correct option is (C).

Quick Tip to Solve: Inspect derivatives inside logarithms as well as outside them; the exponent 5 on dy/dx does not mean order 5.

15. If inverse of matrix $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{pmatrix}$, then value of λ is: [1 mark]

- (A) -4
(B) 1
(C) 3
(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- If $B = A^{-1}$, then $AB = I$, where I is the identity matrix.
- An entry of a matrix product is the dot product of the corresponding row of the first matrix and column of the second.

Step 1: Use the entry containing the unknown Let the proposed inverse be B . The second row of A is $(-1, 1, 0)$, and the second column of B is $(3, \lambda, 3)^T$. Their dot product must be the second diagonal entry of the identity matrix:

$$-3 + \lambda = 1.$$

Step 2: Check the resulting inverse This condition requires $\lambda = 4$. With this value, the product is

$$AB = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The alternative values -4 , 1 , and 3 make the second diagonal entry -7 , -2 , and 0 , respectively, instead of 1 .

Step 3: Conclusion $\lambda = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Use row 2 times column 2: its dot product is $\lambda - 3$ and must equal the identity entry 1 .

16. Find the matrix A^2 , where $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by $a_{ij} = \text{maximum}(i, j) - \text{minimum}(i, j)$: [1 mark]

(A) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Correct Answer: (C) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:

Concept:

- For $i, j \in \{1, 2\}$, $\max(i, j) - \min(i, j) = |i - j|$.
- The square A^2 means the matrix product AA , computed by row-column multiplication.

Step 1: Construct the given matrix The diagonal entries satisfy $a_{11} = 1 - 1 = 0$ and $a_{22} = 2 - 2 = 0$. The off-diagonal entries are $a_{12} = 2 - 1 = 1$ and $a_{21} = 2 - 1 = 1$. Therefore

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Step 2: Multiply the matrix by itself Row-column multiplication gives

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Option (B) is A itself, not its square. Option (A) loses the nonzero diagonal products, and (D) incorrectly gives nonzero off-diagonal entries.

Step 3: Conclusion $A^2 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the correct option is (C).

Quick Tip to Solve: The matrix swaps two coordinates; applying the swap twice returns them, so its square is the identity.

17. Derivative of $e^{\sin^2 x}$ with respect to $\cos x$ is:

[1 mark]

- (A) $\sin x e^{\sin^2 x}$
- (B) $\cos x e^{\sin^2 x}$
- (C) $-2 \cos x e^{\sin^2 x}$
- (D) $-2 \sin^2 x \cos x e^{\sin^2 x}$

Correct Answer: (C) $-2 \cos x e^{\sin^2 x}$

Solution:

Concept:

- For functions $y = y(x)$ and $u = u(x)$, differentiation with respect to u gives $\frac{dy}{du} = \frac{dy/dx}{du/dx}$, wherever the denominator is non-zero.
- The chain rule gives $\frac{d}{dx} e^{v(x)} = e^{v(x)} v'(x)$. Here $v = \sin^2 x$.

Step 1: Differentiate the two functions Put $y = e^{\sin^2 x}$ and $u = \cos x$. Then

$$\frac{dy}{dx} = 2 \sin x \cos x e^{\sin^2 x}, \quad \frac{du}{dx} = -\sin x.$$

The factor $2 \sin x \cos x$ comes from differentiating the square of $\sin x$, so it must be retained before taking the ratio.

Step 2: Take the ratio and compare the options For $\sin x \neq 0$,

$$\frac{dy}{du} = \frac{2 \sin x \cos x e^{\sin^2 x}}{-\sin x} = -2 \cos x e^{\sin^2 x}.$$

Equivalently, $y = e^{1-u^2}$ gives $dy/du = -2ue^{1-u^2}$ directly. Options (A) and (B) miss the negative factor 2; option (D) incorrectly retains $\sin^2 x$ after simplification.

Step 3: Conclusion $-2 \cos x e^{\sin^2 x}$. Hence, the correct option is (C).

Quick Tip to Solve: Rewrite $\sin^2 x = 1 - \cos^2 x$ first to differentiate directly with respect to $\cos x$.

18. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minima at x equal to:

[1 mark]

- (A) 2
- (B) 1
- (C) 0
- (D) -2

Correct Answer: (A) 2

Solution:

Concept:

- A stationary point satisfies $f'(x) = 0$ within the domain of f . The given function is defined only for $x \neq 0$.

Quick Tip to Solve: After finding $x = \pm 2$, check the sign of $4/x^3$; the positive root gives the minimum.

- The second derivative test gives a strict local minimum at $x = a$ when $f'(a) = 0$ and $f''(a) > 0$.

Step 1: Find the stationary points For $f(x) = x/2 + 2/x$,

$$f'(x) = \frac{1}{2} - \frac{2}{x^2}.$$

Setting the derivative equal to zero gives $x^2 = 4$, so the stationary points occur at $x = 2$ and $x = -2$. Both belong to the domain.

Step 2: Classify the points and reject the other options The second derivative is $f''(x) = 4/x^3$. Thus $f''(2) = 1/2 > 0$, whereas $f''(-2) = -1/2 < 0$. The point -2 is a local maximum. Also, $f'(1) = -3/2 \neq 0$, and $x = 0$ is excluded from the domain. Therefore neither (B), (C), nor (D) can give the required local minimum.

Step 3: Conclusion The local minimum occurs at $x = 2$. Hence, the correct option is (A).

Quick Tip to Solve: After finding $x = \pm 2$, check the sign of $4/x^3$; the positive root gives the minimum.

19. Assertion (A): Domain of $y = \cos^{-1}(x)$ is $[-1, 1]$.

Reason (R): The range of the principal value branch of $y = \cos^{-1}(x)$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
 (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
 (C) Assertion (A) is true, but Reason (R) is false.
 (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- The real inverse cosine has domain $[-1, 1]$ and principal range $[0, \pi]$.
- In an Assertion–Reason question, test the two statements separately before checking whether the reason explains the assertion.

Step 1: Judge the Assertion The cosine function restricted to $[0, \pi]$ takes every value in $[-1, 1]$ exactly once. Its inverse $y = \cos^{-1} x$ is consequently defined for all $x \in [-1, 1]$, including the endpoints. Thus Assertion (A) is true.

Step 2: Judge the Reason The complete principal range of $\cos^{-1} x$ is $[0, \pi]$. In particular, $\cos^{-1} 0 = \pi/2$, so $\pi/2$ is included in the range. The printed Reason removes this valid value and is therefore false.

Step 3: Check the explanation and select the code A false Reason cannot explain the true Assertion. Options (A) and (B) require both statements to be true, while option (D) requires the Assertion to be false. The truth values found above therefore match only option (C).

Step 4: Conclusion Assertion is true, but Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Test the disputed range value using $x = 0$: $\cos^{-1} 0 = \pi/2$ immediately disproves the Reason.

20. Assertion (A): The vectors $\vec{a} = 6\hat{i} + 2\hat{j} - 8\hat{k}$, $\vec{b} = 10\hat{i} - 2\hat{j} - 6\hat{k}$ and $\vec{c} = 4\hat{i} - 4\hat{j} + 2\hat{k}$ represent the sides of a right angled triangle.

Reason (R): Three non-zero vectors of which no two are collinear, form a triangle, if their resultant is zero vector or sum of any two vectors is equal to the third. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- Non-collinear non-zero vectors can represent triangle sides when they close head-to-tail; equivalently, two consistently directed side vectors can add to the third.
- For non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \cdot \vec{v} = 0$ proves that their directions are perpendicular.

Step 1: Judge the Assertion by checking closure and perpendicularity The given vectors satisfy

$$\vec{a} + \vec{c} = (6 + 4)\hat{i} + (2 - 4)\hat{j} + (-8 + 2)\hat{k} = \vec{b}.$$

Moreover,

$$\vec{a} \cdot \vec{c} = 6(4) + 2(-4) + (-8)(2) = 24 - 8 - 16 = 0.$$

Both $\vec{a}$ and $\vec{c}$ are non-zero, so they are perpendicular and non-collinear. The vectors $\vec{a}, \vec{c}, -\vec{b}$ close to form a triangle with a right angle. Assertion (A) is true.

Step 2: Judge the Reason The Reason states the closure condition for three non-zero vectors, with non-collinearity excluding a degenerate triangle. A zero resultant gives a closed triangle. If two vectors sum to the third, reversing the third vector gives this same closure. Thus Reason (R) is true.

Step 3: Decide whether the Reason explains the Assertion The Reason establishes that a triangle can be formed, but it gives no condition for a right angle. A general closed triangle need not have perpendicular sides. Here the additional calculation $\vec{a} \cdot \vec{c} = 0$ is essential to establish the right angle. Hence the true Reason does not fully explain the Assertion; options (A), (C), and (D) do not match this judgement.

Step 4: Conclusion Both Assertion and Reason are true, but Reason is not the correct explanation of Assertion. Hence, the correct option is (B).

Quick Tip to Solve: Check both closure and a zero dot product: closure proves a triangle, while perpendicularity proves its right angle.

Section B (10 marks)

This section has 5 Very Short Answer questions of 2 marks each.

21. Simplify: $\cos^{-1} x + \cos^{-1} \left(\frac{x + \sqrt{3 - 3x^2}}{2} \right)$, $\frac{1}{2} \leq x \leq 1$. [2 marks]

Solution:

Concept:

- For $x = \cos \theta$ with $\theta \in [0, \pi]$, $\cos^{-1} x = \theta$ and $\sqrt{1 - x^2} = \sin \theta$.
- Use $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, then respect the principal range $[0, \pi]$ of inverse cosine.

Step 1: Choose a substitution consistent with the given interval Let $x = \cos \theta$. Since $1/2 \leq x \leq 1$, cosine is decreasing on its principal interval and gives $0 \leq \theta \leq \pi/3$. Thus $\cos^{-1} x = \theta$, and the non-negative square root is $\sqrt{3 - 3x^2} = \sqrt{3} \sin \theta$.

Step 2: Convert the second inverse-cosine argument The argument becomes

$$\frac{x + \sqrt{3 - 3x^2}}{2} = \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = \cos \left(\frac{\pi}{3} - \theta \right).$$

The angle $\pi/3 - \theta$ lies in $[0, \pi/3]$, which is contained in the principal range. Therefore

$$\cos^{-1} \left(\frac{x + \sqrt{3 - 3x^2}}{2} \right) = \frac{\pi}{3} - \theta.$$

The range check is necessary: inverse cosine does not cancel cosine for every real angle.

Step 3: Add the two principal values The required sum is $\theta + (\pi/3 - \theta) = \pi/3$. At $x = 1$, the two terms are 0 and $\pi/3$; at $x = 1/2$, they are $\pi/3$ and 0, so the stated endpoints are covered as well.

Step 4: Conclusion The expression simplifies to $\boxed{\frac{\pi}{3}}$ for every $x \in [1/2, 1]$.

Quick Tip to Solve: Use $x = \cos \theta$ and keep $0 \leq \theta \leq \pi/3$; this fixes the principal inverse-cosine branch.

22(a). Find: $\int \cos^3 x e^{\log \sin x} dx$. [2 marks]

Solution:

Concept:

- For a positive real number t , $e^{\log t} = t$. Here the real logarithm requires $\sin x > 0$.
- If $u = \cos x$, then $du = -\sin x dx$; the power rule gives $\int u^3 du = u^4/4 + C$, where C is the constant of integration.

Step 1: Simplify the exponential factor On every interval where the printed integrand is real, $\sin x > 0$,

Quick Tip to Solve: Simplify $e^{\log \sin x}$ first, then use $u = \cos x$ so the remaining sine factor becomes $-du$.

so $e^{\log \sin x} = \sin x$. The integral consequently reduces to

$$I = \int \sin x \cos^3 x \, dx.$$

This simplification removes the logarithm and exposes the derivative of $\cos x$ as a factor.

Step 2: Substitute and integrate Put $u = \cos x$, so $du = -\sin x \, dx$. Then

$$I = - \int u^3 \, du = -\frac{u^4}{4} + C = -\frac{\cos^4 x}{4} + C.$$

The negative sign is essential because the derivative of cosine is minus sine.

Step 3: Verify by differentiation Differentiating the proposed antiderivative gives

$$\frac{d}{dx} \left(-\frac{\cos^4 x}{4} + C \right) = -\cos^3 x (-\sin x) = \cos^3 x \sin x.$$

This equals $\cos^3 x e^{\log \sin x}$ throughout each interval in the original real domain.

Step 4: Conclusion $\int \cos^3 x e^{\log \sin x} \, dx = -\frac{\cos^4 x}{4} + C.$

Quick Tip to Solve: Simplify $e^{\log \sin x}$ first, then use $u = \cos x$ so the remaining sine factor becomes $-du$.

OR

22(b). Find: $\int \frac{1}{5 + 4x - x^2} \, dx.$

[2 marks]

Solution:

Concept:

- Completing the square converts a quadratic denominator into a standard form: here $5 + 4x - x^2 = 9 - (x - 2)^2$.
- Partial fractions give $\frac{1}{a^2 - u^2} = \frac{1}{2a} \left(\frac{1}{a + u} + \frac{1}{a - u} \right)$, and integrating reciprocal linear factors produces logarithms of absolute values.

Step 1: Complete the square and substitute Let the required antiderivative be I . Rewrite its denominator as

$$5 + 4x - x^2 = 9 - (x^2 - 4x + 4) = 9 - (x - 2)^2.$$

Put $u = x - 2$, so $du = dx$. Then

$$I = \int \frac{du}{9 - u^2} = \frac{1}{6} \int \left(\frac{1}{3 + u} + \frac{1}{3 - u} \right) du.$$

The factor $1/6$ follows because the sum inside the brackets is $6/(9 - u^2)$.

Step 2: Integrate the two linear factors Since the derivative of $3 - u$ is -1 , its logarithmic term has a

Quick Tip to Solve: Complete the square first: the minus sign on x^2 gives a difference of squares and hence a logarithm, not an arctangent.

minus sign:

$$I = \frac{1}{6} (\log |3 + u| - \log |3 - u|) + C = \frac{1}{6} \log \left| \frac{x+1}{5-x} \right| + C.$$

The original integrand is undefined at $x = -1, 5$; the formula holds on any interval avoiding these points, with C an arbitrary constant.

Step 3: Conclusion $\int \frac{dx}{5+4x-x^2} = \frac{1}{6} \log \left| \frac{x+1}{5-x} \right| + C.$

Quick Tip to Solve: Complete the square first: the minus sign on x^2 gives a difference of squares and hence a logarithm, not an arctangent.

23. The surface area of a cube increases at the rate of $72 \text{ cm}^2/\text{sec}$. Find the rate of change of its volume, when the edge of the cube measures 3 cm . [2 marks]

Solution:

Concept:

- For a cube with edge a , its total surface area is $S = 6a^2$ and its volume is $V = a^3$.
- When a varies with time t , the chain rule gives $\frac{dS}{dt} = 12a \frac{da}{dt}$ and $\frac{dV}{dt} = 3a^2 \frac{da}{dt}$.

Step 1: Find the rate at which the edge increases Let a be measured in centimetres and t in seconds. The given rate is $dS/dt = 72 \text{ cm}^2/\text{s}$. Differentiating $S = 6a^2$ with respect to time,

$$\frac{dS}{dt} = 12a \frac{da}{dt}.$$

At the instant when $a = 3 \text{ cm}$,

$$72 = 12(3) \frac{da}{dt}, \quad \frac{da}{dt} = 2 \text{ cm/s}.$$

This is the instantaneous edge growth rate corresponding to the given surface-area growth rate.

Step 2: Use the volume formula and the chain rule Differentiating $V = a^3$ with respect to the same time variable and substituting the edge length and its rate gives

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt} = 3(3)^2(2) = 54 \text{ cm}^3/\text{s}.$$

The positive sign agrees with the increasing surface area: the edge and therefore the volume both increase. Notice that the volume rate has cubic units, whereas the given area rate has square units.

Step 3: Conclusion The volume increases at $54 \text{ cm}^3/\text{s}$ when the cube's edge is 3 cm .

Quick Tip to Solve: Eliminate the edge rate to get $dV/dt = (a/4)(dS/dt)$; substituting $a = 3$ immediately gives 54.

24. Find the vector equation of the line passing through the point $(2, 3, -5)$ and making equal angles with the co-ordinate axes. [2 marks]

Solution:

Concept:

- If a line has equal angles $\alpha = \beta = \gamma$ with the coordinate axes, its direction cosines satisfy $l = m = n$, where $l = \cos \alpha$, $m = \cos \beta$, and $n = \cos \gamma$.
- Direction cosines obey $l^2 + m^2 + n^2 = 1$. A line through a point with position vector $\vec{a}$ and nonzero direction vector $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda \vec{b}$, $\lambda \in \mathbb{R}$.

Step 1: Determine a direction vector Equal angles with all three axes imply equal direction cosines. Write $l = m = n = k$. Then

$$3k^2 = 1, \quad k = \pm \frac{1}{\sqrt{3}}.$$

Thus the direction cosines are either all positive or all negative. They give opposite orientations of the same line. We may choose direction ratios 1, 1, 1, so a suitable direction vector is

$$\vec{b} = \hat{i} + \hat{j} + \hat{k}.$$

Step 2: Insert the given point in the vector form The point $(2, 3, -5)$ has position vector $\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}$. Using the point-direction form,

$$\vec{r} = (2\hat{i} + 3\hat{j} - 5\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

For $\lambda = 0$, the equation gives the specified point. Its direction vector has equal components, and normalising it gives equal direction cosines $1/\sqrt{3}$, verifying both conditions.

Step 3: Conclusion The required line is $\boxed{\vec{r} = 2\hat{i} + 3\hat{j} - 5\hat{k} + \lambda(\hat{i} + \hat{j} + \hat{k}), \lambda \in \mathbb{R}}.$

Quick Tip to Solve: Equal angles mean equal direction cosines, so use direction ratios 1 : 1 : 1; changing all three signs only reverses the same line.

25(a). Verify whether the function f defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$ or not.

[2 marks]

Solution:

Concept:

- A function is continuous at $x = a$ when $\lim_{x \rightarrow a} f(x) = f(a)$; the limit must agree from both sides.
- Since $|\sin t| \leq 1$ for every real t , the squeeze theorem can evaluate a product with a bounded oscillating factor.

Quick Tip to Solve: For a vanishing factor multiplying an oscillating sine, bound the absolute value by the vanishing factor and use the squeeze theorem.

Step 1: Identify the value and bound the function The second branch of the definition gives $f(0) = 0$. For every $x \neq 0$, the sine factor is bounded, although $\sin(1/x)$ itself has no limit at zero. Therefore

$$|f(x)| = \left| x \sin \left(\frac{1}{x} \right) \right| \leq |x|, \quad -|x| \leq f(x) \leq |x|.$$

This bound holds for positive as well as negative x , so it deals with the two-sided limit.

Step 2: Evaluate the limit and apply continuity As $x \rightarrow 0$, both bounds $-|x|$ and $|x|$ tend to zero. By the squeeze theorem,

$$\lim_{x \rightarrow 0} x \sin \left(\frac{1}{x} \right) = 0 = f(0).$$

Thus the left-hand and right-hand limits both exist and equal the assigned value. It is the vanishing factor x that controls the oscillations; a separate limit for the sine factor is unnecessary.

Step 3: Conclusion Hence f is continuous at $x = 0$, because $\lim_{x \rightarrow 0} f(x) = f(0) = 0$.

Quick Tip to Solve: For a vanishing factor multiplying an oscillating sine, bound the absolute value by the vanishing factor and use the squeeze theorem.

OR

25(b). Check for differentiability of the function f defined by $f(x) = |x - 5|$, at the point $x = 5$.

[2 marks]

Solution:

Concept:

- The derivative at $x = a$ is the limit $\lim_{h \rightarrow 0} [f(a + h) - f(a)]/h$, provided the same finite value is obtained from both sides.
- For real h , $|h| = h$ when $h > 0$, and $|h| = -h$ when $h < 0$. This gives the two branches of the difference quotient.

Step 1: Set up the difference quotient at the given point With $f(x) = |x - 5|$, we have $f(5) = 0$ and $f(5 + h) = |h|$. For $h \neq 0$,

$$\frac{f(5 + h) - f(5)}{h} = \frac{|h|}{h}.$$

We must examine the signs of h separately, because the absolute-value expression changes its algebraic form at the point being tested.

Step 2: Compare the left and right derivatives For $h > 0$, $|h|/h = 1$, whereas for $h < 0$, $|h|/h = -1$. Hence

$$f'_+(5) = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1, \quad f'_-(5) = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1.$$

Both one-sided derivatives exist, but their values differ. Consequently the two-sided difference-quotient limit does not exist. Although $\lim_{x \rightarrow 5} |x - 5| = 0 = f(5)$, continuity alone does not establish differentiability.

Quick Tip to Solve: At the zero of an absolute-value expression, calculate both one-sided difference quotients; unequal slopes rule out differentiability.

Step 3: Conclusion Since $f'_-(5) = -1 \neq 1 = f'_+(5)$, f is not differentiable at $x = 5$.

Quick Tip to Solve: At the zero of an absolute-value expression, calculate both one-sided difference quotients; unequal slopes rule out differentiability.

Section C (18 marks)

There are 6 short answer questions in this section. Each is of 3 marks.

26(a). Evaluate: $\int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$.

[3 marks]

Solution:

Concept:

- The reflection property of definite integrals is $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$. Here the interval is $[0, \pi]$, so reflection replaces x by $\pi - x$.
- The identity $\cos(\pi - x) = -\cos x$ interchanges the positive exponential terms $e^{\cos x}$ and $e^{-\cos x}$. Their sum in the denominator is always positive.

Step 1: Apply reflection about the midpoint of the interval Denote the required integral by

$$I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$$

The integrand is continuous throughout $[0, \pi]$ because its denominator is a sum of two positive exponentials. Using the substitution $u = \pi - x$, the limits $0, \pi$ become $\pi, 0$, while $dx = -du$. Reversing the limits removes the minus sign:

$$I = \int_\pi^0 \frac{e^{\cos(\pi-u)}}{e^{\cos(\pi-u)} + e^{-\cos(\pi-u)}} du = \int_0^\pi \frac{e^{-\cos u}}{e^{-\cos u} + e^{\cos u}} du.$$

Since the variable of integration is a dummy variable, rename u as x to compare this form directly with the original.

Step 2: Add the complementary forms of the integral Adding the original expression and the reflected expression gives

$$\begin{aligned} 2I &= \int_0^\pi \left(\frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} + \frac{e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} \right) dx \\ &= \int_0^\pi 1 dx = [x]_0^\pi = \pi. \end{aligned}$$

The two numerators together are exactly the common denominator. Hence no antiderivative of the original exponential quotient is needed; the definite-integral property evaluates it directly.

Step 3: Conclusion Therefore, $\int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx = \boxed{\frac{\pi}{2}}$.

Quick Tip to Solve: On $[0, \pi]$, test $x \mapsto \pi - x$: if the original and reflected integrands sum to 1, the integral is half the interval length.

26(b). Find: $\int \frac{2x+1}{(x+1)^2(x-1)} dx.$

[3 marks]

Solution:

Concept:

- A proper rational function with a repeated linear factor is resolved into partial fractions, using one term for each power of the repeated factor.
- The standard integrals are $\int dx/(x-a) = \ln|x-a| + C$ and $\int (x+1)^{-2} dx = -(x+1)^{-1} + C$, where C is an arbitrary constant.

Step 1: Set up the partial fractions For $x \neq -1, 1$, write

$$\frac{2x+1}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{D}{(x+1)^2}.$$

Multiplication by the denominator gives the polynomial identity

$$2x+1 = A(x+1)^2 + B(x+1)(x-1) + D(x-1).$$

The term with denominator $(x+1)^2$ is essential because $x+1$ is a repeated factor.

Step 2: Determine the coefficients Substituting $x = 1$ in the identity gives $3 = 4A$, so $A = 3/4$. Substituting $x = -1$ gives $-1 = -2D$, so $D = 1/2$. Comparing coefficients of x^2 gives $A + B = 0$, hence $B = -3/4$. Therefore,

$$\frac{2x+1}{(x+1)^2(x-1)} = \frac{3}{4(x-1)} - \frac{3}{4(x+1)} + \frac{1}{2(x+1)^2}.$$

The coefficients of x and the constant term also agree: $2A + D = 2$ and $A - B - D = 1$.

Step 3: Integrate the three terms Integrating term by term,

$$\begin{aligned} I &= \frac{3}{4} \int \frac{dx}{x-1} - \frac{3}{4} \int \frac{dx}{x+1} + \frac{1}{2} \int (x+1)^{-2} dx \\ &= \frac{3}{4} \ln|x-1| - \frac{3}{4} \ln|x+1| - \frac{1}{2(x+1)} + C. \end{aligned}$$

This antiderivative is valid on any interval avoiding -1 and 1 . Differentiating the last rational term produces the positive repeated-factor term required above.

Step 4: Conclusion

$$\int \frac{2x+1}{(x+1)^2(x-1)} dx = \frac{3}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2(x+1)} + C.$$

Quick Tip to Solve: Substitute the two roots of the denominator first; then compare the highest power to find the remaining partial-fraction coefficient.

27. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2$.

[3 marks]

Solution:

Concept:

- The chain rule gives $d(u^2)/dx = 2u du/dx$, where u is a differentiable function of x .
- The derivative of the inverse tangent is $d(\tan^{-1} x)/dx = 1/(1 + x^2)$. Product differentiation gives $d(fg)/dx = f'g + fg'$.

Step 1: Differentiate the given function once Let $u = \tan^{-1} x$, so that $y = u^2$. Applying the chain rule,

$$\frac{dy}{dx} = 2u \frac{du}{dx} = \frac{2 \tan^{-1} x}{1 + x^2}.$$

Multiplying by $1 + x^2$, which is positive for every real x , gives the useful identity

$$(1 + x^2) \frac{dy}{dx} = 2 \tan^{-1} x.$$

Keeping this identity in product form makes the next differentiation simpler than expanding the second derivative at once.

Step 2: Differentiate the identity a second time Differentiate both sides with respect to x . On the left, both $1 + x^2$ and dy/dx vary with x , so the product rule is required:

$$2x \frac{dy}{dx} + (1 + x^2) \frac{d^2 y}{dx^2} = \frac{2}{1 + x^2}.$$

The factor $2x$ is the derivative of $1 + x^2$, whereas $d^2 y/dx^2$ is the derivative of dy/dx . No constant of integration occurs, since this is a differentiation calculation.

Step 3: Multiply to obtain the required expression Multiplication of the last identity by $1 + x^2$ yields

$$2x(1 + x^2) \frac{dy}{dx} + (1 + x^2)^2 \frac{d^2 y}{dx^2} = 2.$$

As a direct check,

$$\frac{d^2 y}{dx^2} = \frac{2 - 4x \tan^{-1} x}{(1 + x^2)^2}.$$

Substitution of the first and second derivatives cancels the terms $-4x \tan^{-1} x$ and $+4x \tan^{-1} x$, leaving exactly 2.

Step 4: Conclusion Thus, for every real x ,

$$(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} = 2.$$

Hence the required identity is proved.

Quick Tip to Solve: After the first differentiation, clear the factor $1 + x^2$ before differentiating again; the product rule then gives the target identity directly.

28(a). Find the particular solution of the differential equation $\frac{dy}{dx} - 2xy = 3x^2e^{x^2}$; $y(0) = 5$. [3 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $dy/dx + P(x)y = Q(x)$. Its integrating factor is $IF = e^{\int P(x) dx}$.
- Multiplication by the integrating factor converts the left side into $d(y IF)/dx$. An initial condition determines the arbitrary constant in the general solution.

Step 1: Identify the equation and its integrating factor The given equation is

$$\frac{dy}{dx} - 2xy = 3x^2e^{x^2}.$$

Comparing it with $dy/dx + P(x)y = Q(x)$, we have $P(x) = -2x$ and $Q(x) = 3x^2e^{x^2}$. Therefore,

$$IF = e^{\int -2x dx} = e^{-x^2}.$$

A constant multiplier in the integrating factor is unnecessary and can be taken as one.

Step 2: Multiply and integrate Multiplying every term by e^{-x^2} gives

$$e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2}y = 3x^2.$$

Since $d(e^{-x^2})/dx = -2xe^{-x^2}$, the product rule identifies the left side as

$$\frac{d}{dx} (ye^{-x^2}) = 3x^2.$$

Integrating with respect to x ,

$$ye^{-x^2} = x^3 + C, \quad y = (x^3 + C)e^{x^2}.$$

Here C is the arbitrary constant of the general solution.

Step 3: Apply the initial condition and check The condition $y(0) = 5$ gives $5 = (0 + C)e^0$, so $C = 5$. For this value, differentiation gives

$$y' = 3x^2e^{x^2} + 2x(x^3 + 5)e^{x^2}.$$

Consequently $y' - 2xy = 3x^2e^{x^2}$, exactly the prescribed right side. Also, at $x = 0$, the expression for y gives 5. Thus it satisfies both the differential equation and the initial condition.

Step 4: Conclusion The required particular solution is

$$y = (x^3 + 5)e^{x^2}.$$

Quick Tip to Solve: The integrating factor e^{-x^2} cancels the exponential on the right, leaving the elementary integral of $3x^2$.

28(b). Solve the following differential equation: $x^2 dy + y(x + y) dx = 0$.

[3 marks]

Solution:

Concept:

- A homogeneous differential equation has a right side depending only on y/x . Put $y = vx$, where v is a function of x , so that $dy/dx = v + x dv/dx$.
- After separation, integrate each side. Whenever division by an expression involving v is used, check its zero values separately for solutions that division may omit.

Step 1: Reduce the equation to homogeneous form The equation $x^2 dy + y(x + y) dx = 0$, on an interval with $x \neq 0$, becomes

$$\frac{dy}{dx} = -\frac{y(x+y)}{x^2} = -\frac{y}{x} - \left(\frac{y}{x}\right)^2.$$

Set $y = vx$. Then

$$v + x \frac{dv}{dx} = -v - v^2, \quad x \frac{dv}{dx} = -v(v+2).$$

Thus the transformed equation is separable.

Step 2: Separate variables and integrate For $v \neq 0, -2$, separation gives

$$\frac{dv}{v(v+2)} = -\frac{dx}{x}.$$

Resolve the left side using

$$\frac{1}{v(v+2)} = \frac{1}{2} \left(\frac{1}{v} - \frac{1}{v+2} \right).$$

Integration therefore yields

$$\frac{1}{2} \ln \left| \frac{v}{v+2} \right| = -\ln |x| + C_1.$$

Multiplying by two and exponentiating, with the constant sign absorbed into a real constant C , gives

$$x^2 \frac{v}{v+2} = C.$$

Step 3: Return to the original variables and check omitted solutions Since $v = y/x$, the integrated relation becomes

$$\frac{x^2 y}{y+2x} = C, \quad x^2 y = C(y+2x).$$

Division by $v(v+2)$ excluded $v = 0$ and $v = -2$. Both solve $xv' = -v(v+2)$: they give $y = 0$ and $y = -2x$. The first is already represented by the family with $C = 0$. The second must be listed separately. Indeed, substituting $y = -2x$, $dy = -2 dx$ into the original equation gives $-2x^2 dx + 2x^2 dx = 0$.

Step 4: Conclusion Hence $\boxed{x^2 y = C(y+2x)}$, together with $\boxed{y = -2x}$; the family includes $y = 0$ at $C = 0$.

Quick Tip to Solve: In the substitution $y = vx$, remember the extra v in $y' = v + xv'$, and check $v = 0, -2$ before dividing.

29(a). If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- Logarithmic differentiation converts variable powers into products. On a real differentiable branch where $\cos x > 0$ and $\cos y > 0$, use $\log(u^v) = v \log u$.
- When $y = y(x)$, implicit differentiation uses the product and chain rules: $\frac{d}{dx} \log(\cos x) = -\tan x$ and $\frac{d}{dx} \log(\cos y) = -\tan y \frac{dy}{dx}$.

Step 1: Take logarithms of both sides The given relation is $(\cos x)^y = (\cos y)^x$. Taking natural logarithms on both sides gives

$$y \log(\cos x) = x \log(\cos y).$$

This form is suitable for differentiation because each side is now a product. Both x and y occur in the bases and the exponents, so treating either power as a constant-exponent power would miss a term.

Step 2: Differentiate implicitly and collect derivative terms Write $y' = \frac{dy}{dx}$. Applying the product rule on each side and the chain rule to the logarithms,

$$y' \log(\cos x) - y \tan x = \log(\cos y) - x \tan y y'.$$

Move the term containing y' on the right to the left and move $-y \tan x$ to the right. Then

$$y' [\log(\cos x) + x \tan y] = \log(\cos y) + y \tan x.$$

Divide by the coefficient of y' wherever it is nonzero. At a point where that coefficient vanishes, this division does not supply a derivative; the undivided implicit relation must instead be examined.

Step 3: Conclusion

$$\frac{dy}{dx} = \frac{\log(\cos y) + y \tan x}{\log(\cos x) + x \tan y}$$

on the real differentiable branch, wherever the denominator is nonzero.

Quick Tip to Solve: For a relation with variable powers on both sides, take logarithms first and retain both product-rule terms.

29(b). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[3 marks]

Solution:

Concept:

- For $A = \sqrt{1-x^2}$ and $B = \sqrt{1-y^2}$, the chain rule gives $A' = -x/A$ and $B' = -(y/B) dy/dx$. Work where $|x| < 1$ and $|y| < 1$, so these denominators are positive.
- The parameter a is constant. Use the original relation $A + B = a(x-y)$ to eliminate a after implicit differentiation.

Step 1: Differentiate the given relation Put $A = \sqrt{1-x^2}$, $B = \sqrt{1-y^2}$ and $y' = dy/dx$. Since a is constant, differentiating $A + B = a(x-y)$ gives

$$-\frac{x}{A} - \frac{y}{B} y' = a(1 - y').$$

Collecting the terms that contain y' ,

$$\left(a - \frac{y}{B}\right) y' = a + \frac{x}{A}.$$

Step 2: Eliminate the constant and simplify the coefficients As $A + B > 0$ in the interior domain, the original relation implies $x \neq y$, and

$$a = \frac{A+B}{x-y}.$$

Using $A^2 = 1-x^2$ and $B^2 = 1-y^2$, obtain

$$\begin{aligned} a + \frac{x}{A} &= \frac{A(A+B) + x(x-y)}{A(x-y)} \\ &= \frac{1+AB-xy}{A(x-y)}, \\ a - \frac{y}{B} &= \frac{B(A+B) - y(x-y)}{B(x-y)} \\ &= \frac{1+AB-xy}{B(x-y)}. \end{aligned}$$

Here $1+AB-xy > 0$, because $AB > 0$ and $xy < 1$. The common numerator therefore cancels legitimately, yielding

$$y' = \frac{(1+AB-xy)/[A(x-y)]}{(1+AB-xy)/[B(x-y)]} = \frac{B}{A}.$$

Step 3: Conclusion

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Hence the required relation is proved wherever the implicit derivative is defined in the interior domain.

Quick Tip to Solve: After differentiating, replace the constant by its value from the original equation; use the squared-root identities to expose a common numerator.

30. Find the projection of vector $(\vec{b} + \vec{c})$ on vector $\vec{a}$, where $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{k}$. [3 marks]

Solution:

Concept:

- The scalar projection of a vector $\vec{v}$ on a nonzero vector $\vec{a}$ is $\vec{v} \cdot \hat{a} = \frac{\vec{v} \cdot \vec{a}}{|\vec{a}|}$, where $\hat{a} = \vec{a}/|\vec{a}|$ is the unit vector along $\vec{a}$.
- In an orthonormal Cartesian basis, add corresponding components to find a sum, take the dot product by summing component products, and use $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Step 1: Find the vector to be projected The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{k}$. The $\hat{j}$ component of $\vec{c}$ is zero. Thus, adding components gives

$$\begin{aligned}\vec{v} = \vec{b} + \vec{c} &= (1+1)\hat{i} + (3+0)\hat{j} + (1+1)\hat{k} \\ &= 2\hat{i} + 3\hat{j} + 2\hat{k}.\end{aligned}$$

We project this sum along $\vec{a}$, so the direction used for normalization is $\vec{a}$.

Step 2: Compute the dot product and normalize the direction The dot product of the sum with the direction vector is

$$\vec{v} \cdot \vec{a} = (2)(2) + (3)(2) + (2)(1) = 4 + 6 + 2 = 12.$$

The magnitude of the direction vector is

$$|\vec{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3.$$

Therefore its unit vector is $\hat{a} = (2\hat{i} + 2\hat{j} + \hat{k})/3$, and the signed scalar component along this unit direction is

$$\text{proj}_{\vec{a}}(\vec{b} + \vec{c}) = \frac{12}{3} = 4.$$

The positive dot product shows that the component points along $\vec{a}$. For distinction, the vector component itself is $4\hat{a} = \frac{8}{3}\hat{i} + \frac{8}{3}\hat{j} + \frac{4}{3}\hat{k}$; its magnitude is also 4.

Step 3: Conclusion The required scalar projection of $\vec{b} + \vec{c}$ on $\vec{a}$ is $\boxed{4}$.

Quick Tip to Solve: Normalize the vector on which you project: divide the dot product by its magnitude, not by the magnitude of the vector being projected.

31. An urn contains 3 red and 2 white marbles. Two marbles are drawn one by one with replacement from the urn. Find the probability distribution of the number of white balls. Also, find the mean of the number of white balls drawn. [3 marks]

Solution:

Concept:

- Let X be the number of white marbles in two draws. Since each marble is replaced, each draw has $P(W) = 2/5$ and $P(R) = 3/5$, and the two draws are independent.

Quick Tip to Solve: For exactly one white marble, include both orders WR and RW; replacement makes every draw use the same white probability of $2/5$.

- The possible values are $X = 0, 1, 2$. For independent draws, multiply probabilities along a sequence and add probabilities of mutually exclusive sequences. The mean is $E(X) = \sum_x xP(X = x)$.

Step 1: Find the probability for each possible count For no white marbles, both draws must be red:

$$P(X = 0) = P(RR) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}.$$

For exactly one white marble there are two distinct, mutually exclusive orders, white then red and red then white:

$$\begin{aligned} P(X = 1) &= P(WR) + P(RW) \\ &= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{25}. \end{aligned}$$

For two white marbles both draws must be white:

$$P(X = 2) = P(WW) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}.$$

Step 2: State the distribution and calculate its mean The probability distribution is

x	0	1	2
$P(X = x)$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{4}{25}$

It is valid because each probability is nonnegative and

$$\frac{9}{25} + \frac{12}{25} + \frac{4}{25} = 1.$$

Using the weighted average of the possible counts,

$$\begin{aligned} E(X) &= 0 \left(\frac{9}{25} \right) + 1 \left(\frac{12}{25} \right) + 2 \left(\frac{4}{25} \right) \\ &= \frac{12 + 8}{25} = \frac{20}{25} = \frac{4}{5}. \end{aligned}$$

As a check, X has a binomial distribution with $n = 2$ and $p = 2/5$; its mean $np = 2(2/5) = 4/5$ agrees with the direct calculation.

Step 3: Conclusion The distribution is $P(X = 0) = 9/25$, $P(X = 1) = 12/25$, $P(X = 2) = 4/25$, and the mean number of white marbles drawn is $\boxed{4/5}$.

Quick Tip to Solve: For exactly one white marble, include both orders WR and RW; replacement makes every draw use the same white probability of $2/5$.

Section D (20 marks)

There are 4 long answer questions in this section. Each question is of 5 marks.

32. Find the area of the region bounded by the curve $4x^2 + y^2 = 36$ using integration.

[5 marks]

Solution:

Concept:

- The area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$, from $x = a$ to $x = b$, is $\int_a^b [u(x) - l(x)] dx$.
- A curve containing only x^2 and y^2 is symmetric about both coordinate axes. Its enclosed area is four times its first-quadrant area.
- For $\sqrt{a^2 - x^2}$, use $x = a \sin \theta$, with $dx = a \cos \theta d\theta$. For $0 \leq x \leq a$, take $0 \leq \theta \leq \pi/2$.

Step 1: Identify the boundary and its intercepts Dividing the given equation by 36 gives

$$\frac{x^2}{9} + \frac{y^2}{36} = 1.$$

This is an ellipse centred at the origin. Setting $y = 0$ gives $x = \pm 3$, and setting $x = 0$ gives $y = \pm 6$. In the first quadrant its boundary is

$$y = \sqrt{36 - 4x^2} = 2\sqrt{9 - x^2}, \quad 0 \leq x \leq 3.$$

Replacing either coordinate by its negative leaves the equation unchanged, so the four quadrants contribute equal areas.

Step 2: Set up the required definite integral In the first quadrant the enclosed region lies between the positive branch of the ellipse and the x -axis. A vertical strip has height $2\sqrt{9 - x^2}$ and width dx . If A denotes the full enclosed area,

$$A = 4 \int_0^3 2\sqrt{9 - x^2} dx = 8 \int_0^3 \sqrt{9 - x^2} dx.$$

Step 3: Evaluate using trigonometric substitution Put $x = 3 \sin \theta$, so $dx = 3 \cos \theta d\theta$. The limits $x = 0, 3$ become $\theta = 0, \pi/2$. Since $\cos \theta \geq 0$ on this interval, $\sqrt{9 - x^2} = 3 \cos \theta$. Therefore,

$$\begin{aligned} A &= 72 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 36 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= 36 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}. \end{aligned}$$

The sine term vanishes at both endpoints, and the lower endpoint also makes the first term zero.

Quick Tip to Solve: Integrate the positive branch from zero to the positive x -intercept and multiply by four to include all quadrants.

Step 4: Conclusion The area bounded by $4x^2 + y^2 = 36$ is

$$A = 18\pi \text{ square units.}$$

Quick Tip to Solve: Integrate the positive branch from zero to the positive x-intercept and multiply by four to include all quadrants.

33(a). Find the co-ordinates of the foot of the perpendicular drawn from the point $(2, 3, -8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance of the given point from the line. [5 marks]

Solution:

Concept:

- A line through A with direction vector $\mathbf{d}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$, where t is a real parameter.
- If H is the foot of the perpendicular from P to a line, then H lies on the line and $\overrightarrow{PH} \cdot \mathbf{d} = 0$.
- The perpendicular distance is the length $PH = \sqrt{(x_H - x_P)^2 + (y_H - y_P)^2 + (z_H - z_P)^2}$.

Step 1: Write the given line in parametric form Let the common value of the three ratios be t :

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3} = t.$$

Solving each relation for the corresponding coordinate gives

$$x = 4 - 2t, \quad y = 6t, \quad z = 1 - 3t.$$

Thus the line passes through $A = (4, 0, 1)$, and its direction vector is $\mathbf{d} = (-2, 6, -3)$.

Step 2: Apply the perpendicularity condition Since the foot H lies on this line, write

$$H = (4 - 2t, 6t, 1 - 3t).$$

With $P = (2, 3, -8)$, the vector from the given point to the prospective foot is

$$\overrightarrow{PH} = (2 - 2t, 6t - 3, 9 - 3t).$$

Its scalar product with the direction vector must vanish:

$$\begin{aligned} (-2)(2 - 2t) + 6(6t - 3) - 3(9 - 3t) &= 0 \\ -4 + 4t + 36t - 18 - 27 + 9t &= 0 \\ 49t - 49 &= 0. \end{aligned}$$

Hence $t = 1$.

Step 3: Calculate and verify the foot and distance Substituting $t = 1$ into the line gives $H = (2, 6, -2)$. Check that this point lies on the given line: Each of its three ratios equals 1: $\frac{4-2}{2} = \frac{6}{6} = \frac{1-(-2)}{3} = 1$.

Quick Tip to Solve: Equate the three ratios to one parameter first; this preserves the direction-vector signs before imposing the zero dot product.

Also $\overrightarrow{PH} = (0, 3, 6)$, and $(0, 3, 6) \cdot (-2, 6, -3) = 18 - 18 = 0$. This independently confirms that PH is perpendicular to the line. Its length is

$$PH = \sqrt{(2-2)^2 + (6-3)^2 + (-2+8)^2} = \sqrt{9+36}.$$

Step 4: Conclusion The foot of the perpendicular is $(2, 6, -2)$, and the perpendicular distance from the point to the line is $3\sqrt{5}$ units.

Quick Tip to Solve: Equate the three ratios to one parameter first; this preserves the direction-vector signs before imposing the zero dot product.

OR

33(b). Find the shortest distance between the lines L_1 and L_2 given below:

L_1 : the line passing through $(2, -1, 1)$ and parallel to $\frac{x}{1} = \frac{y}{1} = \frac{z}{3}$, and

L_2 : $\vec{r} = \hat{i} + (2\mu + 1)\hat{j} - (\mu + 2)\hat{k}$.

[5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has vector equation $\vec{r} = \vec{a} + t\vec{b}$.
- For two nonparallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, their shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. The cross product gives a direction perpendicular to both lines.

Step 1: Express both lines in vector form The line $x/1 = y/1 = z/3$ has direction ratios 1, 1, 3. Since L_1 passes through $(2, -1, 1)$, write

$$L_1 : \vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} + 3\hat{k}).$$

Expanding the given equation of L_2 separates its fixed point from its direction:

$$L_2 : \vec{r} = (\hat{i} + \hat{j} - 2\hat{k}) + \mu(2\hat{j} - \hat{k}).$$

Thus choose $\vec{a}_1 = (2, -1, 1)$, $\vec{a}_2 = (1, 1, -2)$, $\vec{b}_1 = (1, 1, 3)$ and $\vec{b}_2 = (0, 2, -1)$.

Step 2: Find the common perpendicular direction Compute the cross product using the determinant:

$$\begin{aligned} \vec{n} = \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 0 & 2 & -1 \end{vmatrix} \\ &= (-1-6)\hat{i} - (-1-0)\hat{j} + (2-0)\hat{k} \\ &= -7\hat{i} + \hat{j} + 2\hat{k}. \end{aligned}$$

Quick Tip to Solve: In the second line, separate the constant vector from the coefficient of the parameter before taking the cross product; its direction is $(0, 2, -1)$.

It is nonzero, so the lines are not parallel. Its magnitude is

$$|\vec{n}| = \sqrt{49 + 1 + 4} = \sqrt{54} = 3\sqrt{6}.$$

As a check, $(1, 1, 3) \cdot (-7, 1, 2) = 0$ and $(0, 2, -1) \cdot (-7, 1, 2) = 0$.

Step 3: Project the joining vector onto this direction A vector joining the chosen points of the two lines is

$$\vec{a}_2 - \vec{a}_1 = -\hat{i} + 2\hat{j} - 3\hat{k}.$$

Its scalar product with $\vec{n}$ is

$$(\vec{a}_2 - \vec{a}_1) \cdot \vec{n} = (-1)(-7) + (2)(1) + (-3)(2) = 3.$$

Moving either endpoint along its line adds a multiple of $\vec{b}_1$ or $\vec{b}_2$, whose scalar product with $\vec{n}$ is zero. Hence this perpendicular component is fixed and gives the shortest separation. Substitution in the distance formula yields

$$d = \frac{|3|}{3\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

The nonzero scalar triple product also confirms that these nonparallel lines are skew, rather than intersecting.

Step 4: Conclusion The shortest distance between L_1 and L_2 is $\boxed{\frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \text{ units}}.$

Quick Tip to Solve: In the second line, separate the constant vector from the coefficient of the parameter before taking the cross product; its direction is $(0, 2, -1)$.

34(a). If $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{pmatrix}$, then find A^{-1} and hence solve the following system of equations:

$$x + 2y - 3z = 1$$

$$2x - 3z = 2$$

$$x + 2y = 3$$

[5 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$, and $A^{-1} = \frac{1}{|A|} \text{adj } A$. The adjugate is the transpose of the matrix of cofactors $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the corresponding minor.
- A system of linear equations with coefficient matrix A , unknown column X , and constant column B is $AX = B$. If A is invertible, its unique solution is $X = A^{-1}B$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then multiply the inverse by the constant column $(1, 2, 3)$ in the given order.

Step 1: Calculate the determinant Expanding the given matrix along its first row gives

$$\begin{aligned}|A| &= 1 \begin{vmatrix} 0 & -3 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} + (-3) \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \\ &= 1(6) - 2(3) - 3(4) = -12.\end{aligned}$$

Since the determinant is nonzero, the inverse exists and the given system will have exactly one solution.

Step 2: Form the cofactor matrix and adjugate Evaluating the signed minors gives the nine cofactors:

$$\begin{aligned}C_{11} &= 0(0) - (-3)(2) = 6, & C_{12} &= -[2(0) - (-3)(1)] = -3, \\C_{13} &= 2(2) - 0(1) = 4, & C_{21} &= -[2(0) - (-3)(2)] = -6, \\C_{22} &= 1(0) - (-3)(1) = 3, & C_{23} &= -[1(2) - 2(1)] = 0, \\C_{31} &= 2(-3) - (-3)(0) = -6, & C_{32} &= -[1(-3) - (-3)(2)] = -3, \\C_{33} &= 1(0) - 2(2) = -4.\end{aligned}$$

Therefore the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 6 & -3 & 4 \\ -6 & 3 & 0 \\ -6 & -3 & -4 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} 6 & -6 & -6 \\ -3 & 3 & -3 \\ 4 & 0 & -4 \end{pmatrix}.$$

Step 3: Obtain the inverse and solve by matrix multiplication Divide each entry of the adjugate by the determinant:

$$A^{-1} = \frac{-1}{12} \begin{pmatrix} 6 & -6 & -6 \\ -3 & 3 & -3 \\ 4 & 0 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

The equations have precisely the coefficient matrix A . Put $X = (x, y, z)^T$ and $B = (1, 2, 3)^T$, so $AX = B$. Multiplying on the left by A^{-1} gives

$$X = A^{-1}B = \begin{pmatrix} -\frac{1}{2}(1) + \frac{1}{2}(2) + \frac{1}{2}(3) \\ \frac{1}{4}(1) - \frac{1}{4}(2) + \frac{1}{4}(3) \\ -\frac{1}{3}(1) + 0(2) + \frac{1}{3}(3) \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}.$$

Step 4: Check all three original equations Substituting the computed values verifies each equation:

$$\begin{aligned}x + 2y - 3z &= 2 + 2\left(\frac{1}{2}\right) - 3\left(\frac{2}{3}\right) = 1, \\2x - 3z &= 2(2) - 3\left(\frac{2}{3}\right) = 2, \\x + 2y &= 2 + 2\left(\frac{1}{2}\right) = 3.\end{aligned}$$

All three constants agree with the question, confirming the multiplication and signs in the inverse.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then multiply the inverse by the constant column $(1, 2, 3)$ in the given order.

Step 5: Conclusion

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}, \text{ and the unique solution is } \boxed{x = 2, y = \frac{1}{2}, z = \frac{2}{3}}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then multiply the inverse by the constant column (1,2,3) in the given order.

OR

34(b). Find the product of the matrices $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$ **and hence solve the following system of linear equations:**

$$x + 2y - 3z = -4$$

$$2x + 3y + 2z = 2$$

$$3x - 3y - 4z = 11$$

[5 marks]

Solution:

Concept:

- In a matrix product AB , entry (i, j) is the scalar product of row i of A with column j of B . The order of multiplication must be preserved.
- If $AB = kI$, where I is the identity matrix and $k \neq 0$, then $A^{-1} = \frac{1}{k}B$. Thus the system $AX = C$, with unknown column X and constant column C , has solution $X = A^{-1}C$.

Step 1: Define the matrices and calculate the first row of the product Let

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}.$$

Multiplying the first row of A by each column of B , we obtain

$$(AB)_{11} = 1(-6) + 2(14) + (-3)(-15) = 67,$$

$$(AB)_{12} = 1(17) + 2(5) + (-3)(9) = 0,$$

$$(AB)_{13} = 1(13) + 2(-8) + (-3)(-1) = 0.$$

Step 2: Calculate the remaining rows and obtain the inverse For the second row,

$$(AB)_{21} = 2(-6) + 3(14) + 2(-15) = 0,$$

$$(AB)_{22} = 2(17) + 3(5) + 2(9) = 67,$$

$$(AB)_{23} = 2(13) + 3(-8) + 2(-1) = 0.$$

Quick Tip to Solve: Once the product is $67I$, use the supplied second matrix divided by 67 as the inverse; multiply it by the constants in their original order, $(-4, 2, 11)^T$.

For the third row,

$$(AB)_{31} = 3(-6) + (-3)(14) + (-4)(-15) = 0,$$

$$(AB)_{32} = 3(17) + (-3)(5) + (-4)(9) = 0,$$

$$(AB)_{33} = 3(13) + (-3)(-8) + (-4)(-1) = 67.$$

Therefore $AB = 67I$. As $67 \neq 0$, A is invertible and

$$A^{-1} = \frac{1}{67} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}.$$

Step 3: Express the equations as a matrix equation and solve The coefficient matrix of the given system is exactly A . Setting

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad C = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix},$$

the system is $AX = C$. Premultiplying by A^{-1} gives

$$\begin{aligned} X &= \frac{1}{67} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix} \\ &= \frac{1}{67} \begin{pmatrix} 24 + 34 + 143 \\ -56 + 10 - 88 \\ 60 + 18 - 11 \end{pmatrix} = \frac{1}{67} \begin{pmatrix} 201 \\ -134 \\ 67 \end{pmatrix}. \end{aligned}$$

Check the resulting entries in all three original equations:

$$3 + 2(-2) - 3(1) = -4,$$

$$2(3) + 3(-2) + 2(1) = 2,$$

$$3(3) - 3(-2) - 4(1) = 11.$$

All three right-hand sides agree, confirming the solution.

Step 4: Conclusion The required product is $AB = \begin{pmatrix} 67 & 0 & 0 \\ 0 & 67 & 0 \\ 0 & 0 & 67 \end{pmatrix} = 67I$. Hence the unique solution is $\boxed{x = 3, y = -2, z = 1}$.

Quick Tip to Solve: Once the product is $67I$, use the supplied second matrix divided by 67 as the inverse; multiply it by the constants in their original order, $(-4, 2, 11)^T$.

35. Solve the following L.P.P. graphically:

Minimise $Z = 6x + 3y$

Subject to constraints $4x + y \geq 80$; $x + 5y \geq 115$; $3x + 2y \leq 150$; $x, y \geq 0$.

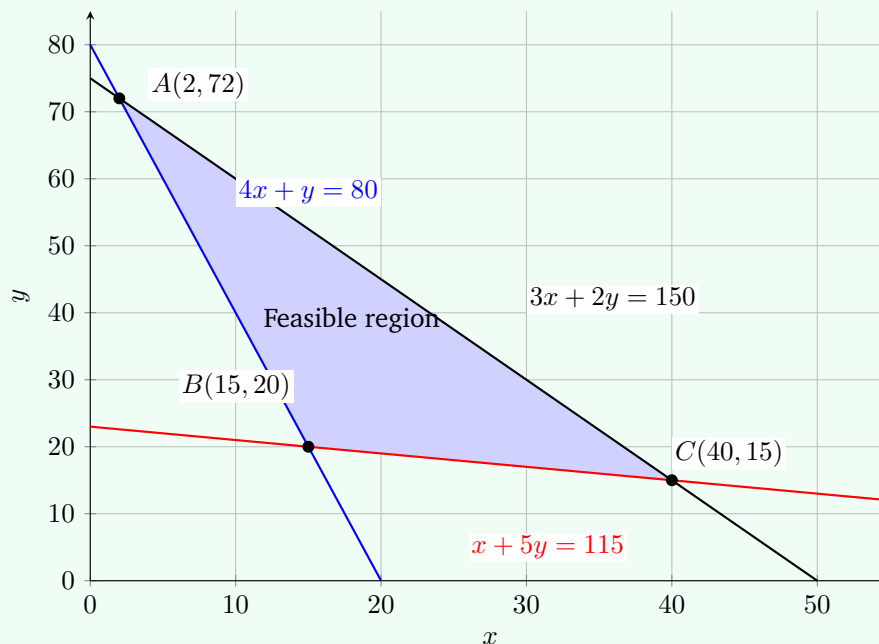
[5 marks]

Solution:

Concept:

- A linear inequality represents a half-plane bounded by its corresponding straight line. The feasible region is the common part satisfying every constraint, including $x \geq 0$ and $y \geq 0$.
- For a nonempty bounded feasible polygon, a linear objective $Z = 6x + 3y$ attains its minimum at a corner point. Thus locate all feasible vertices graphically and compare their objective values.

Step 1: Plot the boundary lines and choose the required sides Draw $4x + y = 80$ through $(20, 0)$ and $(0, 80)$, $x + 5y = 115$ through $(115, 0)$ and $(0, 23)$, and $3x + 2y = 150$ through $(50, 0)$ and $(0, 75)$. The origin gives values smaller than 80 and 115 in the first two constraints, so their required sides exclude the origin. The third constraint includes the origin. Restrict the graph to the first quadrant. Shade the common region above the first two lines and below the third line; this is the triangle ABC shown in the graph. The x -intercept 115 of the second line lies beyond the displayed window, so its displayed portion is drawn using $(0, 23)$ and another point $(40, 15)$.



The shaded feasible triangle has vertices $A(2, 72)$, $B(15, 20)$, and $C(40, 15)$.

Step 2: Calculate the three corner points For A, solve $4x + y = 80$ and $3x + 2y = 150$. Substituting $y = 80 - 4x$ gives $3x + 160 - 8x = 150$, hence $A = (2, 72)$. For B, solve $4x + y = 80$ and $x + 5y = 115$. Then $x + 5(80 - 4x) = 115$, so $-19x = -285$, giving $B = (15, 20)$. For C, solve $x + 5y = 115$ and $3x + 2y = 150$. Substituting $x = 115 - 5y$ gives $345 - 13y = 150$, hence $C = (40, 15)$.

Step 3: Verify feasibility and evaluate the objective The third, unused constraint at each intersection is also satisfied: at A, $x + 5y = 362 \geq 115$; at B, $3x + 2y = 85 \leq 150$; at C, $4x + y = 175 \geq 80$. All coordinates are positive. No point on the y -axis is feasible because it would require both $y \geq 80$ and $y \leq 75$; likewise the x -axis would require both $x \geq 115$ and $x \leq 50$. Thus there are no further axis

Quick Tip to Solve: For the two greater-than constraints shade away from the origin; for the less-than constraint shade towards it, then test every intersection against the remaining inequality.

vertices. The objective values are

$$Z(A) = 6(2) + 3(72) = 228,$$

$$Z(B) = 6(15) + 3(20) = 150,$$

$$Z(C) = 6(40) + 3(15) = 285.$$

Compare these values at the three vertices of the bounded triangle. The least value occurs at B , and neither adjacent edge has equal objective values at its endpoints, so this minimizing point is unique.

Step 4: Conclusion The minimum value is $Z_{\min} = 150$, attained at $x = 15, y = 20$.

Quick Tip to Solve: For the two greater-than constraints shade away from the origin; for the less-than constraint shade towards it, then test every intersection against the remaining inequality.

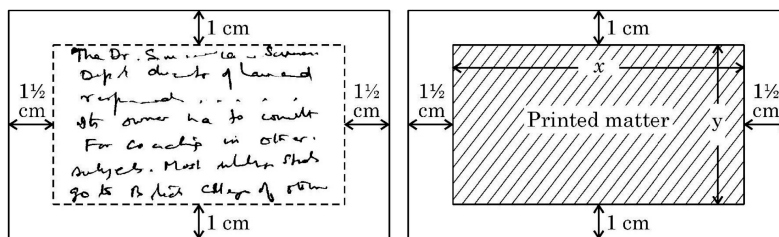
Section E (12 marks)

In this section there are 3 case study questions of 4 marks each.

36. A rectangular visiting card is to contain 24 sq.cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be $1\frac{1}{2}$ cm as shown below:

On the basis of the above information, answer the following questions:

[4 marks]



36(i). Write the expression for the area of the visiting card in terms of x .

[2 marks]

Solution:

Concept:

- The printed rectangle has horizontal dimension x cm and vertical dimension y cm, as labelled in the figure. Its fixed area gives $xy = 24$, hence $y = 24/x$, where $x > 0$.
- The area of the entire visiting card is the product of its outer dimensions. Opposite margins must both be added to the corresponding printed dimension.

Step 1: Express the outside dimensions using the margins Each of the left and right margins is $1\frac{1}{2} = 3/2$ cm, so the outer horizontal dimension is $x + 3/2 + 3/2 = x + 3$ cm. Each of the top and bottom margins is 1 cm, so the outer vertical dimension is $y + 1 + 1 = y + 2$ cm. These additions apply to the printed dimensions, rather than to the fixed area 24.

Step 2: Eliminate the other printed dimension Using $xy = 24$, substitute $y = 24/x$ into the product

Quick Tip to Solve: Add both side margins to x and both top and bottom margins to y before substituting $y = 24/x$; this includes the corner rectangles automatically.

of the outside dimensions:

$$\begin{aligned}A(x) &= (x + 3) \left(\frac{24}{x} + 2 \right) \\&= 24 + 2x + \frac{72}{x} + 6 \\&= 2x + \frac{72}{x} + 30.\end{aligned}$$

The terms include the printed area, the strips along the margins, and the corner rectangles. The restriction $x > 0$ ensures both printed dimensions are physically meaningful.

Step 3: Conclusion The area of the visiting card is $A(x) = 2x + \frac{72}{x} + 30 \text{ cm}^2$, for $x > 0$.

Quick Tip to Solve: Add both side margins to x and both top and bottom margins to y before substituting $y = 24/x$; this includes the corner rectangles automatically.

36(ii). Obtain the dimensions of the card of minimum area.

[2 marks]

Solution:

Concept:

- With printed dimensions x and y cm, $xy = 24$. Including the margins gives the card area $A(x) = (x + 3)(24/x + 2) = 2x + 72/x + 30$, where $x > 0$.
- A differentiable function has a stationary point where its first derivative vanishes. A positive second derivative throughout the domain establishes that this point is the unique minimum.

Step 1: Find the printed dimensions at the stationary point Differentiate the card area with respect to the positive printed width x :

$$A'(x) = 2 - \frac{72}{x^2}.$$

Setting $A'(x) = 0$ gives $2x^2 = 72$, hence $x^2 = 36$. Since a length is positive, take $x = 6$ cm; the negative root is inadmissible. From the fixed printed area, $y = 24/6 = 4$ cm.

Step 2: Confirm the minimum and add the margins The second derivative is $A''(x) = 144/x^3 > 0$ for every $x > 0$, and in particular $A''(6) = 2/3 > 0$. Moreover $A'(x) < 0$ for $0 < x < 6$ and $A'(x) > 0$ for $x > 6$, so the area decreases to this point and increases afterwards. Thus it gives the global minimum. The card's outer horizontal dimension is $6 + 3 = 9$ cm, while its outer vertical dimension is $4 + 2 = 6$ cm. As a check, their product is 54 cm^2 , also equal to $A(6) = 12 + 12 + 30$.

Step 3: Conclusion The card of minimum area has dimensions $9 \text{ cm} \times 6 \text{ cm}$. Its printed rectangle is $6 \text{ cm} \times 4 \text{ cm}$.

Quick Tip to Solve: The derivative determines the printed width $x = 6$ cm; add 3 cm horizontally and 2 cm vertically to report the dimensions of the whole card.

37. A departmental store sends bills to charge its customers once a month. Past experience shows that 70% of its customers pay their first month bill in time. The store also found that the customer who pays the bill in time has the probability of 0.8 of paying in time next month and the customer who doesn't pay in time has the probability of 0.4 of paying in time the next month.

Based on the above information, answer the following questions:

[4 marks]

37(i). Let E_1 and E_2 respectively denote the event of customer paying or not paying the first month bill in time. Find $P(E_1)$, $P(E_2)$.

[1 mark]

Solution:

Concept:

- The event E_1 means paying the first month bill in time, while E_2 means not paying it in time.
- Complementary events satisfy $P(E_1) + P(E_2) = 1$, because every customer belongs to exactly one of the two groups.

Step 1: Use the given percentage The store reports that 70% of customers pay the first month bill in time. Converting this percentage into a probability gives

$$P(E_1) = \frac{70}{100} = 0.7.$$

This is the initial first-month probability; the next-month conditional probabilities are not needed here.

Step 2: Use the complementary event A customer either pays the first bill in time or does not. Thus $E_2 = E_1'$, so

$$P(E_2) = 1 - P(E_1) = 1 - 0.7 = 0.3.$$

The two values add to 1, as required for exhaustive complementary events.

Step 3: Conclusion $P(E_1) = 0.7$, $P(E_2) = 0.3$.

Quick Tip to Solve: Convert the given 70 percent first-month rate to 0.7 and subtract it from 1 for the other group.

37(ii). Let A denotes the event of customer paying second month's bill in time, then find $P(A|E_1)$ and $P(A|E_2)$.

[1 mark]

Solution:

Concept:

- A denotes timely payment in the second month; E_1 and E_2 distinguish timely and late first-month payment.
- A conditional probability $P(A | E)$ describes the chance of A within customers already known to satisfy E .

Step 1: Read the timely first-month group The statement says that a customer who pays the bill in time has probability 0.8 of paying in time next month. Here the known first-month event is E_1 and

Quick Tip to Solve: The phrase who pays in time selects E_1 ; who does not pay in time selects E_2 before reading the next-month rate.

the next-month event is A . Hence

$$P(A | E_1) = 0.8.$$

Step 2: Read the late first-month group The statement also says that a customer who does not pay in time has probability 0.4 of paying in time next month. Therefore

$$P(A | E_2) = 0.4.$$

These are conditional rates, not joint probabilities; multiplying by the first-month group probabilities would answer a different question.

Step 3: Conclusion $P(A | E_1) = 0.8, \quad P(A | E_2) = 0.4.$

Quick Tip to Solve: The phrase who pays in time selects E_1 ; who does not pay in time selects E_2 before reading the next-month rate.

37(iii)(a). Find the probability of customer paying second month's bill in time.

[2 marks]

Solution:

Concept:

- Let A be timely second-month payment, E_1 timely first-month payment and E_2 its complement.
- The law of total probability for the mutually exclusive, exhaustive events E_1, E_2 is $P(A) = P(E_1)P(A | E_1) + P(E_2)P(A | E_2)$.

Step 1: Identify the two customer groups From the initial payment rate, $P(E_1) = 0.7$ and $P(E_2) = 1 - 0.7 = 0.3$. The transition rates are $P(A | E_1) = 0.8$ and $P(A | E_2) = 0.4$. Every timely second-month payer comes from exactly one of these first-month groups, so their joint probabilities can be added.

Step 2: Apply total probability Weight each conditional payment rate by the probability of its first-month group:

$$\begin{aligned} P(A) &= P(E_1)P(A | E_1) + P(E_2)P(A | E_2) \\ &= (0.7)(0.8) + (0.3)(0.4) \\ &= 0.56 + 0.12 = 0.68. \end{aligned}$$

The value lies between the two conditional rates 0.4 and 0.8, consistent with it being their weighted average. Simply adding the conditional rates would ignore the sizes of the groups.

Step 3: Conclusion The probability of paying the second month bill in time is $0.68 = \frac{17}{25}$.

Quick Tip to Solve: Weight 0.8 by 0.7 and 0.4 by 0.3; add the two paths leading to timely second-month payment.

37(iii)(b). Find the probability of customer paying first month's bill in time if it is found that customer has paid the second month's bill in time. [2 marks]

Solution:

Concept:

- Let A denote timely second-month payment and E_1 denote timely first-month payment; $E_2 = E_1'$.
- Bayes' theorem reverses the conditioning: $P(E_1 | A) = \frac{P(E_1)P(A | E_1)}{P(A)}$, provided $P(A) > 0$.

Step 1: Find the probability of the observed event The first-month probabilities are $P(E_1) = 0.7$, $P(E_2) = 0.3$, and the given conditional rates are $P(A | E_1) = 0.8$, $P(A | E_2) = 0.4$. By total probability,

$$P(A) = (0.7)(0.8) + (0.3)(0.4) = 0.56 + 0.12 = 0.68.$$

This denominator counts timely second-month customers from both first-month groups.

Step 2: Condition on timely second-month payment The probability that both months are paid in time is $P(E_1 \cap A) = P(E_1)P(A | E_1) = (0.7)(0.8) = 0.56$. Hence Bayes' theorem gives

$$P(E_1 | A) = \frac{P(E_1 \cap A)}{P(A)} = \frac{0.56}{0.68} = \frac{56}{68} = \frac{14}{17}.$$

The question asks for the first-month event given the second-month event, so the original conditional rate 0.8 cannot be used directly.

Step 3: Conclusion The required probability is $P(E_1 | A) = \frac{14}{17}$.

Quick Tip to Solve: For the reversed condition, divide the both-months timely path 0.56 by all second-month timely paths 0.68.

38. (a) Students of a school are taken to a railway museum to learn about railways heritage and its history.

An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by

$$R = \{(l_1, l_2) : l_1 \text{ is parallel to } l_2\}.$$

On the basis of the above information, answer the following questions:

[4 marks]

38(a)(i). Find whether the relation R is symmetric or not.

[1 mark]

Solution:

Concept:

- A relation R on L is symmetric if $(l_1, l_2) \in R$ implies $(l_2, l_1) \in R$ for every $l_1, l_2 \in L$.
- Parallel lines have the same direction: the statement $l_1 \parallel l_2$ is unchanged on interchanging the two

Quick Tip to Solve: To test symmetry, reverse an arbitrary related pair and check that the defining condition still holds.

lines.

Step 1: Interpret a related pair Choose any pair $(l_1, l_2) \in R$. From the definition printed in the question, this means that the rail line l_1 is parallel to the rail line l_2 . The choice of this pair is arbitrary, so the following argument applies to every pair belonging to R .

Step 2: Reverse the pair Since $l_1 \parallel l_2$, we also have $l_2 \parallel l_1$. Applying the same definition of R to this reversed statement gives $(l_2, l_1) \in R$. Thus reversing any related pair produces another related pair, which is precisely the condition for symmetry.

Step 3: Conclusion The relation R is symmetric.

Quick Tip to Solve: To test symmetry, reverse an arbitrary related pair and check that the defining condition still holds.

38(a)(ii). Find whether the relation R is transitive or not.

[1 mark]

Solution:

Concept:

- A relation R on L is transitive if $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$ imply $(l_1, l_3) \in R$.
- Parallelism means equality of direction. Lines parallel to the same line have the same direction as each other.

Step 1: Translate the two premises Let $l_1, l_2, l_3 \in L$ satisfy $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$. The definition gives $l_1 \parallel l_2$ and $l_2 \parallel l_3$. Thus the first and third rail lines both have the direction of the second rail line.

Step 2: Establish the required related pair Equality of these directions gives $l_1 \parallel l_3$. Therefore, by the definition of the relation, $(l_1, l_3) \in R$. This argument uses arbitrary lines satisfying the two premises, and hence verifies the transitive implication for all such triples of rail lines.

Step 3: Conclusion The relation R is transitive.

Quick Tip to Solve: For a transitivity test, link two relations through their shared middle line and compare the outside lines.

38(a)(iii). If one of the rail lines on the railway track is represented by the equation $y = 3x + 2$, then find the set of rail lines in R related to it.

[2 marks]

Solution:

Concept:

- In the slope-intercept equation $y = mx + c$, m is the slope and c is the intercept on the y -axis.
- Lines having the same slope have the same direction. Thus a line related by parallelism to a given nonvertical line must have that line's slope.
- The set sought is restricted to L , the rail lines on the railway track: an equation must represent a member of L to be included.

Quick Tip to Solve: Keep the coefficient of x equal to 3 and vary only the constant term to describe the parallel family.

Step 1: Determine the given line's direction Let l_0 denote the rail line $y = 3x + 2$. Comparing this with $y = mx + c$ gives slope $m = 3$ and intercept $c = 2$. The slope determines the direction of the line, whereas the intercept determines its position. Changing the intercept while keeping the slope fixed therefore produces lines with the same direction.

Step 2: Describe every related rail line If a rail line $l \in L$ is related to l_0 under R , it must be parallel to l_0 , so its slope is also 3. Its equation consequently has the form $y = 3x + c$, where c is a real constant. Conversely, every rail line in L of this form has the required direction and is related to l_0 . Thus the family is characterised by a fixed slope and a varying intercept, with membership in L retained.

Step 3: Conclusion The required set is $\{l \in L : l \text{ has equation } y = 3x + c, c \in \mathbb{R}\}$.

Quick Tip to Solve: Keep the coefficient of x equal to 3 and vary only the constant term to describe the parallel family.

OR

38(b). Let S be the relation defined by $S = \{(l_1, l_2) : l_1 \text{ is perpendicular to } l_2\}$, check whether the relation S is symmetric and transitive. [4 marks]

Solution:

Concept:

- A relation S is symmetric if $(l_1, l_2) \in S$ implies $(l_2, l_1) \in S$ for every pair of lines under consideration.
- A relation is transitive if $(l_1, l_2) \in S$ and $(l_2, l_3) \in S$ together imply $(l_1, l_3) \in S$. A single counterexample disproves transitivity.
- Perpendicular lines meet at a right angle. Two lines perpendicular to the same line are parallel, whereas a line is never perpendicular to itself.

Step 1: Check symmetry Let $(l_1, l_2) \in S$. By the definition of the relation, $l_1 \perp l_2$. The right angle between these lines is unchanged when their order is reversed, so $l_2 \perp l_1$. Consequently, $(l_2, l_1) \in S$. This verifies the implication required for symmetry for any perpendicular pair.

Step 2: Test the transitive implication Suppose $l_1 \perp l_2$ and $l_2 \perp l_3$. Then l_1 and l_3 have the same direction, so they are parallel (or coincide), not perpendicular. For a direct counterexample, take a perpendicular pair a, b . We have $(a, b) \in S$ and, by symmetry, $(b, a) \in S$. Transitivity would require $(a, a) \in S$, which is impossible because a line is not perpendicular to itself. Thus the required implication fails. Equivalently, the explicit lines $l_1 : y = 0$, $l_2 : x = 0$, and $l_3 : y = 1$ illustrate the same obstruction: both horizontal lines are perpendicular to the vertical line, but they are parallel to each other. The transitivity condition requires perpendicularity of the outside lines, which these premises cannot supply.

Step 3: Conclusion The perpendicularity relation S is $\boxed{\text{symmetric but not transitive}}$.

Quick Tip to Solve: For perpendicularity, reverse a pair to test symmetry; use that pair followed by its reverse to test transitivity.