CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This Question paper contains **38** questions. **All** questions are **compulsory**.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In Section **A** – Question Number **1 to 18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.

(iv) In Section **B** – Question Number **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.

(v) In Section **C** – Question Number **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In Section **D** – Question Number **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.

(vii) In Section **E** – Question Number **36 to 38** are case study based questions, carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.

(ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

*This section comprises of **20** Multiple Choice Questions (MCQs) of **1** mark each.*

1. If $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then A^{-1} is [1 mark]

(A) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

Concept:

- The inverse of a nonsingular matrix A is the matrix A^{-1} satisfying $AA^{-1} = A^{-1}A = I$.
- A diagonal matrix with nonzero diagonal entries has an inverse obtained by taking the reciprocal of each diagonal entry.

Step 1: Take the reciprocals of the diagonal entries The given matrix has diagonal entries $-1, 1, 1$, all nonzero. Their reciprocals are again $-1, 1, 1$. Thus the inverse has the same diagonal entries and all off-diagonal entries remain zero.

Step 2: Verify the product and compare the choices Multiplying the proposed inverse by A gives

$$A^2 = \begin{pmatrix} (-1)^2 & 0 & 0 \\ 0 & 1^2 & 0 \\ 0 & 0 & 1^2 \end{pmatrix} = I_3.$$

This verifies the inverse directly. Option (A) changes both positive diagonal entries, option (B) changes all three signs, and option (C) changes the second entry; their products with A are not the identity.

Step 3: Conclusion $A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$. Hence, the correct option is (D).

Quick Tip to Solve: For a diagonal matrix whose entries are all 1 or -1, square the matrix: it is its own inverse.

2. If vector $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and vector $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, then which of the following is correct? [1 mark]

(A) $\vec{a} \parallel \vec{b}$

(B) $\vec{a} \perp \vec{b}$

(C) $|\vec{b}| > |\vec{a}|$

(D) $|\vec{a}| = |\vec{b}|$

Correct Answer: (B) $\vec{a} \perp \vec{b}$

Solution:

Concept:

- Two nonzero vectors are perpendicular exactly when their scalar product is zero: $\vec{a} \cdot \vec{b} = 0$.
- The magnitude of $x\hat{i} + y\hat{j} + z\hat{k}$ is $\sqrt{x^2 + y^2 + z^2}$. Parallel vectors have proportional components.

Step 1: Evaluate the scalar product Multiply corresponding components and add:

$$\vec{a} \cdot \vec{b} = 3(1) + 2(-1) + (-1)(1) = 3 - 2 - 1 = 0.$$

Both vectors are nonzero, so their angle is 90°.

Step 2: Check the other comparisons The component ratios are $3, -2, -1$, which are unequal; therefore the vectors are not parallel. Also,

$$|\vec{a}| = \sqrt{9 + 4 + 1} = \sqrt{14}, \qquad |\vec{b}| = \sqrt{1 + 1 + 1} = \sqrt{3}.$$

Hence $|\vec{a}| > |\vec{b}|$; options (C) and (D) both contradict the calculated magnitudes, while option (A) contradicts the component ratios.

Step 3: Conclusion $\vec{a} \perp \vec{b}$. Hence, the correct option is (B).

Quick Tip to Solve: When perpendicularity is an option, compute the dot product first; here the three terms cancel immediately.

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3. $\displaystyle\int_{-1}^{1} \frac{|x|}{x}\, dx,\ x \neq 0$ is equal to: [1 mark]

(A) -1

(B) 0

(C) 1

(D) 2

Correct Answer: (B) 0

Solution:

Concept:

- The modulus satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x > 0$.
- Split an integral at a point where its formula changes. A bounded integrand's value at one point does not change its definite integral.

Quick Tip to Solve: Recognise $|x|/x$ as an odd function away from zero: its integrals over equal opposite intervals cancel.

Step 1: Remove the modulus on each interval For $-1 \le x < 0$, the quotient $|x|/x = -1$. For $0 < x \le 1$, it equals 1. Although the displayed quotient is undefined at zero, both one-sided integrals exist, and their sum determines the stated integral.

Step 2: Add the signed contributions Evaluate separately on the two sides of zero:

$$\int_{-1}^{0} (-1)\,dx + \int_{0}^{1} 1\,dx = [-x]_{-1}^{0} + [x]_{0}^{1} = -1 + 1 = 0.$$

Option (A) counts only the negative part, option (C) only the positive part, and option (D) adds their unsigned areas. A definite integral adds signed areas, so the equal contributions cancel.

Step 3: Conclusion The integral equals 0. Hence, the correct option is (B).

Quick Tip to Solve: Recognise $|x|/x$ as an odd function away from zero: its integrals over equal opposite intervals cancel.

4. Which of the following is not a homogeneous function of x and y? [1 mark]

(A) $y^2 - xy$

(B) $x - 3y$

(C) $\sin^2 \dfrac{y}{x} + \dfrac{y}{x}$

(D) $\tan x - \sec y$

Correct Answer: (D) $\tan x - \sec y$

Solution:

Concept:

- A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$, wherever both sides are defined.
- A polynomial with terms of equal total degree is homogeneous. A function depending only on y/x is homogeneous of degree zero.

Step 1: Test the first three functions under scaling Replacing x, y by tx, ty, we get

$$(ty)^2 - (tx)(ty) = t^2(y^2 - xy), \qquad tx - 3ty = t(x - 3y).$$

Therefore (A) and (B) have degrees two and one. In (C), $(ty)/(tx) = y/x$, so both terms are unchanged; this function has degree zero.

Step 2: Check the trigonometric difference For $f(x, y) = \tan x - \sec y$, scaling gives $\tan(tx) - \sec(ty)$. To rule out any fixed degree, put $x = y = 0$: since $f(0, 0) = -1$, homogeneity would force $t^n = 1$, hence $n = 0$. But with $y = 0$, changing x to tx changes $\tan x - 1$ in general. Thus it is not homogeneous, whereas the other three pass the scaling test.

Step 3: Conclusion $\tan x - \sec y$ is not homogeneous. Hence, the correct option is (D).

Quick Tip to Solve: Scale both variables together: ratios stay fixed, but separate trigonometric arguments usually do not.

5. If $f(x) = |x| + |x-1|$, then which of the following is correct? [1 mark]

(A) $f(x)$ is both continuous and differentiable, at $x = 0$ and $x = 1$.

(B) $f(x)$ is differentiable but not continuous, at $x = 0$ and $x = 1$.

(C) $f(x)$ is continuous but not differentiable, at $x = 0$ and $x = 1$.

(D) $f(x)$ is neither continuous nor differentiable, at $x = 0$ and $x = 1$.

Correct Answer: (C) $f(x)$ is continuous but not differentiable, at $x = 0$ and $x = 1$.

Solution:

Concept:

- The modulus function is continuous everywhere, and a sum of continuous functions is continuous.
- Differentiability at a point requires the left-hand and right-hand derivatives to exist and be equal.

Step 1: Express the function on its three intervals Removing the absolute values according to their signs gives

$$f(x) = \begin{cases} 1 - 2x, & x < 0, \\ 1, & 0 \le x \le 1, \\ 2x - 1, & x > 1. \end{cases}$$

At zero both adjacent formulas have limiting value $1 = f(0)$; at one both have limiting value $1 = f(1)$. Thus the function is continuous at both specified points.

Step 2: Compare the one-sided slopes The slopes on the three open intervals are $-2, 0, 2$, respectively. Therefore

$$f'_-(0) = -2 \neq 0 = f'_+(0), \qquad f'_-(1) = 0 \neq 2 = f'_+(1).$$

Neither point has equal one-sided derivatives. Option (A) incorrectly asserts differentiability; options (B) and (D) incorrectly deny continuity. Moreover, differentiability without continuity, as in (B), is impossible.

Step 3: Conclusion The function is continuous but not differentiable at both $x = 0$ and $x = 1$. Hence, the correct option is (C).

Quick Tip to Solve: For a sum of absolute values, check slopes on the intervals cut out by the zeros of their arguments.

6. If A is a square matrix of order 2 such that $\det(A) = 4$, then $\det(4\,\text{adj}\,A)$ is equal to: [1 mark]

(A) 16

(B) 64

(C) 256

(D) 512

Correct Answer: (B) 64

Solution:

Concept:

- For a square matrix A of order n, $\det(\text{adj}\,A) = (\det A)^{n-1}$.
- Multiplying every entry of a matrix M of order n by a scalar k multiplies its determinant by k^n: $\det(kM) = k^n \det M$.

Step 1: Find the determinant of the adjoint Here A has order 2 and $\det A = 4$. Substituting these values in the adjoint identity gives

$$\det(\text{adj}\,A) = 4^{2-1} = 4.$$

The adjoint also has order two.

Step 2: Apply the scalar factor to both rows The scalar 4 multiplies every entry of $\text{adj}\,A$, so a factor of 4 comes from each of its two rows:

$$\det(4\,\text{adj}\,A) = 4^2 \det(\text{adj}\,A) = 16 \times 4 = 64.$$

The value 16 omits one row's factor. The values 256 and 512 do not follow the order-two determinant identities and cannot equal this expression.

Step 3: Conclusion $\det(4\,\text{adj}\,A) = 64$. Hence, the correct option is (B).

Quick Tip to Solve: Separate the two operations: first use $\det(\text{adj}\,A) = \det A$ for order two, then multiply by 4^2.

7. If E and F are two independent events such that $P(E) = \frac{2}{3}$, $P(F) = \frac{3}{7}$, then $P(E/\overline{F})$ is equal to: [1 mark]

(A) $\frac{1}{6}$

(B) $\frac{1}{2}$

(C) $\frac{2}{3}$

(D) $\frac{7}{9}$

Correct Answer: (C) $\frac{2}{3}$

Solution:

Concept:

- For independent events E, F, $P(E \cap F) = P(E)P(F)$.
- Conditional probability is $P(E \mid G) = P(E \cap G)/P(G)$ when $P(G) > 0$. Here the conditioning event is $\overline{F}$, the complement of F.

Step 1: Find the probabilities involving the complement Since $P(F) = 3/7$, its complement has probability $P(\overline{F}) = 1 - 3/7 = 4/7$. Independence gives

$$P(E \cap F) = \frac{2}{3} \cdot \frac{3}{7} = \frac{2}{7}.$$

The events $E \cap F$ and $E \cap \overline{F}$ partition E, so

$$P(E \cap \overline{F}) = \frac{2}{3} - \frac{2}{7} = \frac{8}{21}.$$

Step 2: Divide by the conditioning probability Using the nonzero probability $P(\overline{F}) = 4/7$, we obtain

$$P(E \mid \overline{F}) = \frac{8/21}{4/7} = \frac{8}{21} \cdot \frac{7}{4} = \frac{2}{3}.$$

The result equals $P(E)$, as expected because independence is preserved when one event is complemented. None of $1/6, 1/2, 7/9$ satisfies this conditional-probability calculation.

Step 3: Conclusion $P(E/\overline{F}) = \frac{2}{3}$. Hence, the correct option is (C).

Quick Tip to Solve: Independence of E and F also makes E independent of $\overline{F}$; conditioning leaves $P(E)$ unchanged.

8. The absolute maximum value of function $f(x) = x^3 - 3x + 2$ in $[0, 2]$ is: [1 mark]

(A) 0

(B) 2

(C) 4

(D) 5

Correct Answer: (C) 4

Solution:

Concept:

- A continuous function on a closed interval attains an absolute maximum and minimum.
- For a differentiable function, compare its values at interior critical points and both endpoints to find an absolute extremum.

Step 1: Locate the critical point inside the interval Differentiate $f(x) = x^3 - 3x + 2$:

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

The derivative vanishes at $x = 1$ and $x = -1$. Only $x = 1$ lies in $[0, 2]$; the point -1 must be excluded from the comparison.

Step 2: Compare the endpoint and critical values As the polynomial is continuous throughout the interval, the candidate values are

$$f(0) = 2, \qquad f(1) = 1 - 3 + 2 = 0, \qquad f(2) = 8 - 6 + 2 = 4.$$

Among them, 4 is largest. The value 0 in option (A) is the minimum, and 2 in option (B) is only the left-endpoint value. Option (D), 5, is larger than every candidate and is not attained as the maximum.

Step 3: Conclusion The absolute maximum value is 4, attained at $x = 2$. Hence, the correct option is (C).

Quick Tip to Solve: Never stop at $f'(x) = 0$ for a closed interval: check both endpoints, since the maximum here occurs at $x = 2$.

9. Let $A = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 4 & -1 \\ -3 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ -5 \\ -7 \end{pmatrix}$, $C = [9\ 8\ 7]$, **which of the following is defined?** [1 mark]

(A) Only AB

(B) Only AC

(C) Only BA

(D) All $AB,\ AC$ and BA

Correct Answer: (A) Only AB

Solution:

Concept:

- If P has order $m \times n$ and Q has order $r \times s$, the product PQ is defined precisely when $n = r$; its order is then $m \times s$.
- Matrix multiplication is order-sensitive. The existence of AB does not imply that BA exists.

Step 1: Read the order of each matrix Counting rows and columns in the displayed matrices gives

$$A : 3 \times 3, \qquad B : 3 \times 1, \qquad C : 1 \times 3.$$

In particular, B is a column matrix and C is a row matrix. The numerical entries do not affect whether the products are defined.

Step 2: Match the inner dimensions for each product For AB, the orders are $(3 \times 3)(3 \times 1)$; the inner dimensions match, so the result has order 3×1. For AC, the orders are $(3 \times 3)(1 \times 3)$, whose inner dimensions $3, 1$ differ. For BA, they are $(3 \times 1)(3 \times 3)$, whose inner dimensions $1, 3$ also differ. Thus neither product claimed in (B) or (C) exists, and the assertion in (D) is false.

Step 3: Conclusion Only AB is defined. Hence, the correct option is (A).

Quick Tip to Solve: Write the dimensions beside each factor and check only the adjacent inner numbers before doing any multiplication.

10. If $\int \frac{2^{1/x}}{x^2}\,dx = k \cdot 2^{1/x} + C$, then k is equal to [1 mark]

(A) $-\frac{1}{\log 2}$

(B) $-\log 2$

(C) -1

(D) $\frac{1}{2}$

Correct Answer: (A) $-\frac{1}{\log 2}$

Solution:

Concept:

- For exponential integration, $\int a^t\,dt = a^t/\log a + C$, where $a > 0$, $a \neq 1$, and $\log$ denotes the natural logarithm.
- Use substitution when the integrand contains a function together with its derivative: for $t = 1/x$, $dt = -dx/x^2$.

Step 1: Substitute the exponent Put $t = 1/x$, with $x \neq 0$. The negative sign in the differential is essential:

$$\int \frac{2^{1/x}}{x^2}\,dx = -\int 2^t\,dt.$$

Step 2: Integrate and compare the coefficient Apply the exponential integration formula and return to x:

$$-\int 2^t\,dt = -\frac{2^t}{\log 2} + C = -\frac{1}{\log 2}\,2^{1/x} + C.$$

Option (B) multiplies by $\log 2$ instead of dividing; (C) omits the logarithmic factor; (D) also loses the negative sign.

Step 3: Conclusion Comparing with $k\,2^{1/x} + C$, $k = -1/\log 2$. Hence, the correct option is (A).

Quick Tip to Solve: For $2^{1/x}/x^2$, set $t = 1/x$ immediately and retain the minus sign from dt.

11. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=\sqrt{37}$, $|\vec{b}|=3$ and $|\vec{c}|=4$, then angle between $\vec{b}$ and $\vec{c}$ is [1 mark]

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

Correct Answer: (C) $\frac{\pi}{3}$

Solution:

Concept:

- The angle θ between two nonzero vectors satisfies $\vec{b}\cdot\vec{c}=|\vec{b}|\,|\vec{c}|\cos\theta$, with $0\le\theta\le\pi$.
- The squared magnitude of a sum is $|\vec{b}+\vec{c}|^2=|\vec{b}|^2+|\vec{c}|^2+2\vec{b}\cdot\vec{c}$.

Step 1: Use the zero vector sum From $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have $\vec{b}+\vec{c}=-\vec{a}$. Thus

$$37=9+16+2\vec{b}\cdot\vec{c}, \qquad \vec{b}\cdot\vec{c}=6.$$

Step 2: Calculate the angle Substitute the magnitudes into the dot-product formula:

$$\cos\theta=\frac{6}{3\times4}=\frac{1}{2}.$$

Among the listed angles, only $\pi/3$ has this cosine; $\pi/6$, $\pi/4$ and $\pi/2$ have cosines $\sqrt{3}/2$, $1/\sqrt{2}$ and 0, respectively.

Step 3: Conclusion The angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$. Hence, the correct option is (C).

Quick Tip to Solve: When three vectors sum to zero, square the sum of the two vectors whose angle is required.

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12. The integrating factor of differential equation $(x+2y^3)\frac{dy}{dx}=2y$ is [1 mark]

(A) $e^{y^2/2}$

(B) $\frac{1}{\sqrt{y}}$

(C) $\frac{1}{y^2}$

(D) e^{-1/y^2}

Correct Answer: (B) $\frac{1}{\sqrt{y}}$

Solution:

Concept:

- A linear differential equation in x as a function of y has the form $dx/dy+P(y)x=Q(y)$.

Quick Tip to Solve: If a differential equation is nonlinear in y, check whether reversing dy/dx makes it linear in x.

- Its integrating factor is I.F. $= e^{\int P(y)\,dy}$; identify the dependent variable before applying this formula.

Step 1: Rewrite the equation with y independent The original equation is not linear in y as a function of x. Treat x as the dependent variable and rearrange, for $y \neq 0$:

$$\frac{dx}{dy} = \frac{x + 2y^3}{2y}, \qquad \frac{dx}{dy} - \frac{x}{2y} = y^2.$$

Step 2: Integrate the coefficient of x Here $P(y) = -1/(2y)$, so on $y > 0$,

$$\text{I.F.} = e^{-\frac{1}{2}\log y} = y^{-1/2} = \frac{1}{\sqrt{y}}.$$

More generally, one may use $|y|^{-1/2}$ on any interval avoiding zero. The printed choice is its positive-domain form. Option (C) would require coefficient $-2/y$; (A) and (D) have logarithmic derivatives y and $2/y^3$, not $-1/(2y)$.

Step 3: Conclusion The integrating factor matching the given choices is $1/\sqrt{y}$. Hence, the correct option is (B).

Quick Tip to Solve: If a differential equation is nonlinear in y, check whether reversing dy/dx makes it linear in x.

13. If $A = \begin{pmatrix} 7 & 0 & x \\ 0 & 7 & 0 \\ 0 & 0 & y \end{pmatrix}$ is a scalar matrix, then y^x is equal to [1 mark]

(A) 0

(B) 1

(C) 7

(D) ± 7

Correct Answer: (B) 1

Solution:

Concept:

- A scalar matrix is of the form λI: all off-diagonal entries are zero and all diagonal entries equal the same scalar λ.
- For any nonzero number a, the zero-exponent rule gives $a^0 = 1$.

Step 1: Apply the scalar-matrix conditions The entry x is off the main diagonal, so it must be zero. The first two diagonal entries are 7, so the third diagonal entry must also be 7:

$$x = 0, \qquad y = 7.$$

Step 2: Evaluate the requested power The question asks for y^x, not for y or the product xy.

Quick Tip to Solve: For a scalar matrix, first set every off-diagonal unknown to zero, then equate all diagonal entries.

Substitution gives

$$y^x = 7^0 = 1.$$

Option (A) confuses the exponent with the value of the power; (C) reports the base alone; (D) introduces a sign ambiguity that the scalar-matrix condition does not allow.

Step 3: Conclusion The value of y^x is 1. Hence, the correct option is (B).

Quick Tip to Solve: For a scalar matrix, first set every off-diagonal unknown to zero, then equate all diagonal entries.

14. The corner points of the feasible region in graphical representation of an L.P.P. are $(2, 72), (15, 20)$ and $(40, 15)$. If $Z = 18x + 9y$ be the objective function, then [1 mark]

(A) Z is maximum at $(2, 72)$, minimum at $(15, 20)$

(B) Z is maximum at $(15, 20)$, minimum at $(40, 15)$

(C) Z is maximum at $(40, 15)$, minimum at $(15, 20)$

(D) Z is maximum at $(40, 15)$, minimum at $(2, 72)$

Correct Answer: (C) Z is maximum at $(40, 15)$, minimum at $(15, 20)$

Solution:

Concept:

- For a linear objective function, an attained maximum or minimum over a polygonal feasible region occurs at a corner point.
- To compare the corners here, substitute each ordered pair (x, y) into $Z = 18x + 9y$.

Step 1: Evaluate the objective at each corner Keep the coordinates in their printed order. The three values are

$$Z(2, 72) = 18(2) + 9(72) = 684,$$
$$Z(15, 20) = 18(15) + 9(20) = 450,$$
$$Z(40, 15) = 18(40) + 9(15) = 855.$$

Step 2: Compare the three values Since $450 < 684 < 855$, the largest value is at $(40, 15)$ and the smallest at $(15, 20)$. Thus (A) names the wrong maximum, (B) reverses the actual extreme points, and (D) names the wrong minimum. A large value of just one coordinate does not by itself determine the objective value.

Step 3: Conclusion $Z_{\max} = 855$ at $(40, 15)$, and $Z_{\min} = 450$ at $(15, 20)$. Hence, the correct option is (C).

Quick Tip to Solve: Compare $2x + y$ at the three vertices; multiplying by the positive factor 9 preserves their order.

15. If A and B are invertible matrices, then which of the following is not correct? [1 mark]

(A) $(A+B)^{-1} = B^{-1} + A^{-1}$

(B) $(AB)^{-1} = B^{-1}A^{-1}$

(C) $\text{adj}(A) = |A|\,A^{-1}$

(D) $|A^{-1}| = |A|^{-1}$

Correct Answer: (A) $(A+B)^{-1} = B^{-1} + A^{-1}$

Solution:

Concept:

- The inverse of a product reverses the order: $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices of compatible order.
- For an invertible square matrix, $\text{adj}\,A = |A|A^{-1}$ and $|A^{-1}| = 1/|A|$. Inversion is not distributive over addition.

Step 1: Test the proposed sum rule Take $A = B = I$, where I is an identity matrix. Both are invertible, and their sum is also invertible, but

$$(A+B)^{-1} = (2I)^{-1} = \frac{1}{2}I, \qquad B^{-1} + A^{-1} = 2I.$$

These differ, so the asserted equality in (A) fails even when its left side exists.

Step 2: Check the remaining identities For (B), $(AB)(B^{-1}A^{-1}) = I$, confirming the reverse order. For (C), multiply $A\,\text{adj}\,A = |A|I$ by A^{-1}. For (D), take determinants in $AA^{-1} = I$ to get $|A|\,|A^{-1}| = 1$. Therefore each of (B), (C) and (D) is valid.

Step 3: Conclusion The identity $(A+B)^{-1} = B^{-1} + A^{-1}$ is not correct in general. Hence, the correct option is (A).

Quick Tip to Solve: Test a doubtful matrix identity with $A = B = I$; this disproves the inverse-of-a-sum rule immediately.

16. If the feasible region of a linear programming problem with objective function $Z = ax + by$, is bounded, then which of the following is correct? [1 mark]

(A) It will only have a maximum value.

(B) It will only have a minimum value.

(C) It will have both maximum and minimum values.

(D) It will have neither maximum nor minimum value.

Correct Answer: (C) It will have both maximum and minimum values.

Solution:

Concept:

- A nonempty feasible region defined by the usual non-strict linear inequalities is closed; if bounded, it is a closed and bounded polygonal region.

Quick Tip to Solve: For a nonempty bounded feasible polygon, list all corner values: both the largest and smallest are attained.

- The linear objective $Z = ax + by$ is continuous, so it attains both a maximum and a minimum on such a region.

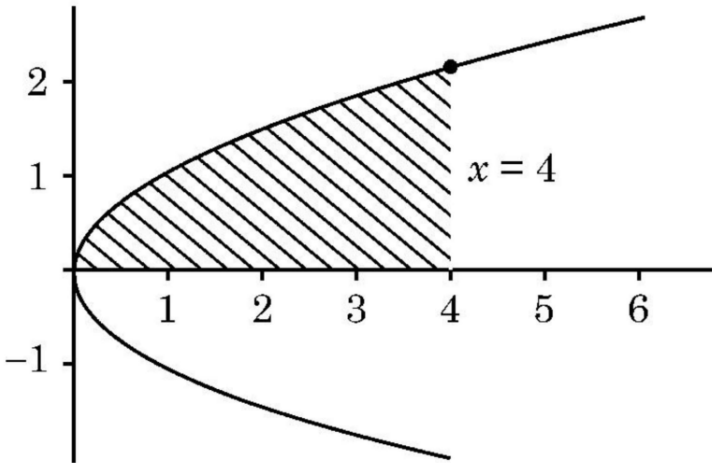
Step 1: Use boundedness of the feasible region Every feasible point lies within a finite region. Consequently the objective cannot increase or decrease without bound there. The boundary belongs to the feasible set under the usual linear-programming inequalities, so the limiting extreme values are attained.

Step 2: Apply the corner-point principle Evaluate Z at the corner points to obtain both its largest and smallest values. Either value may be shared by several points or a complete boundary segment, but that does not prevent its existence. Thus (A) and (B) wrongly exclude one extreme, while (D) excludes both. If the objective is constant, the maximum and minimum are equal and still both exist.

Step 3: Conclusion The bounded feasible region has both maximum and minimum values of Z. Hence, the correct option is (C).

Quick Tip to Solve: For a nonempty bounded feasible polygon, list all corner values: both the largest and smallest are attained.

17. The area of the shaded region bounded by the curves $y^2 = x$, $x = 4$ and the x-axis is given by [1 mark]



(A) $\int_0^4 x\,dx$

(B) $\int_0^2 y^2\,dy$

(C) $2\int_0^4 \sqrt{x}\,dx$

(D) $\int_0^4 \sqrt{x}\,dx$

Correct Answer: (D) $\int_0^4 \sqrt{x}\,dx$

Solution:

Concept:

- Area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$ is $\int_a^b [f(x) - g(x)]\,dx$, with limits set by the bounding intersections.
- The parabola $y^2 = x$ has branches $y = \pm\sqrt{x}$; the shaded region in the printed figure uses only the upper branch and the x-axis.

Step 1: Identify the strip and its limits The upper branch meets the x-axis at the origin, and the right boundary is $x = 4$. A vertical strip inside the shaded region therefore extends from $y = 0$ to $y = \sqrt{x}$, with height $\sqrt{x}$ and width dx. The limits are $x = 0$ and $x = 4$.

Step 2: Compare the proposed area expressions Adding these strips gives

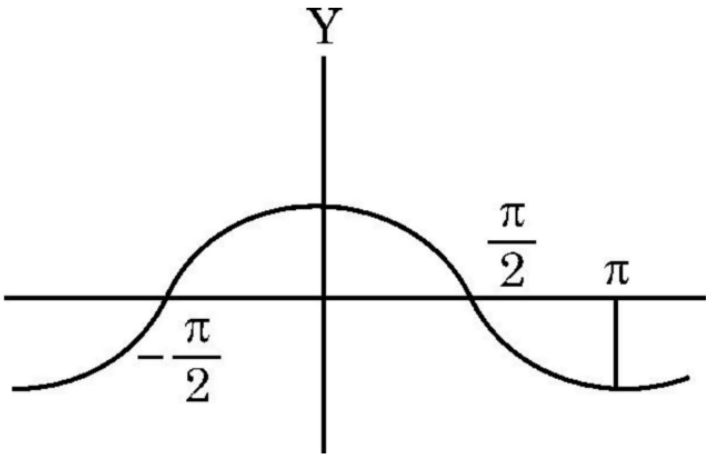
$$A = \int_0^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{16}{3}.$$

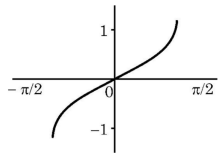
Option (A) integrates the wrong height. Option (B) measures the region to the left of the parabola between $y = 0$ and $y = 2$; horizontal strips for the shaded region instead have width $4 - y^2$. Option (C) doubles the area and would include the equal unshaded region below the x-axis.

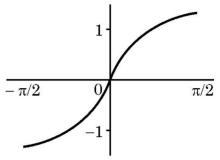
Step 3: Conclusion The shaded area is given by $\int_0^4 \sqrt{x}\,dx$. Hence, the correct option is (D).

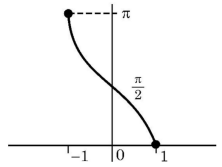
Quick Tip to Solve: Read the shading before using symmetry: shading above the axis requires only the positive square-root branch.

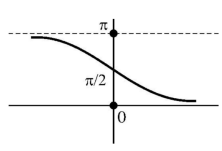
18. The graph of a trigonometric function is as shown . Which of the following will represent graph of its inverse? [1 mark]

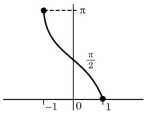


(A) 

(B) 

(C) 

(D) 

Correct Answer: (C) 

Solution:

Concept:

- The given curve is the cosine graph, $y = \cos x$, with its maximum at $x = 0$ and a zero at $x = \pi/2$.
- The principal inverse cosine uses the restriction $0 \le x \le \pi$; its domain is $[-1, 1]$, its range is $[0, \pi]$, and its graph is decreasing.

Step 1: Interchange the coordinates Reflection of the restricted cosine graph in $y = x$ gives the inverse graph. The cosine points $(0, 1)$, $(\pi/2, 0)$ and $(\pi, -1)$ become, respectively,

$$(1, 0), \qquad (0, \pi/2), \qquad (-1, \pi).$$

Thus the required graph has finite endpoints at $(-1, \pi)$ and $(1, 0)$.

Step 2: Compare all four options Options (A) and (B) are increasing curves through the origin, so neither represents inverse cosine. Option (D) is decreasing but extends beyond $[-1, 1]$ and approaches horizontal levels instead of ending at the required points. Option (C) alone is decreasing on $[-1, 1]$, passes through $(0, \pi/2)$, and has the two required endpoints.

Step 3: Conclusion The required graph is $y = \cos^{-1} x$, shown in (C). Hence, the correct option is (C).

Quick Tip to Solve: Identify inverse cosine by its three points: $(-1, \pi)$, $(0, \pi/2)$, and $(1, 0)$.

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19. Assertion (A): Let Z be the set of integers. A function $f : Z \to Z$ defined as $f(x) = 3x - 5,\ \forall x \in Z$ is a bijective.

Reason (R): A function is a bijective if it is both surjective and injective. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A function is injective if equal outputs imply equal inputs. It is surjective if every element of the stated codomain has a preimage in the domain.
- A bijection must be both injective and surjective; the domain and codomain here are the integers $\mathbb{Z}$.

Step 1: Test Assertion (A) If $f(m) = f(n)$, then $3m - 5 = 3n - 5$, so $m = n$; thus f is injective.

Quick Tip to Solve: For an integer-to-integer linear function, test whether 0 has an integer preimage before claiming it is onto.

However, the integer 0 has no integer preimage, since

$$3x - 5 = 0 \quad \Rightarrow \quad x = \frac{5}{3} \notin \mathbb{Z}.$$

Hence f is not surjective onto $\mathbb{Z}$, and the assertion that it is bijective is false.

Step 2: Test Reason (R) The reason states the definition of a bijection: a function is bijective precisely when it is both injective and surjective. This statement is true.

Step 3: Relate the two statements The true definition cannot establish the false assertion: this function fails its surjectivity requirement. Options (A) and (B) wrongly treat the assertion as true, while (C) also wrongly calls the reason false.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For an integer-to-integer linear function, test whether 0 has an integer preimage before claiming it is onto.

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20. Assertion (A): $f(x) = \begin{cases} 3x - 8, & x \leq 5 \\ 2k, & x > 5 \end{cases}$ **is continuous at** $x = 5$ **for** $k = \frac{5}{2}$.

Reason (R): For a function f **to be continuous at** $x = a$, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- At a joining point a, continuity requires the left-hand limit, right-hand limit and function value to be equal.
- For the given piecewise function, the branch containing the equality sign determines $f(5)$, while the branch for $x > 5$ determines the right-hand limit.

Step 1: Test Assertion (A) The left branch gives

$$f(5) = \lim_{x \to 5^-} f(x) = 3(5) - 8 = 7.$$

For the printed value $k = 5/2$, the right-hand limit is

$$\lim_{x \to 5^+} f(x) = 2k = 2\left(\frac{5}{2}\right) = 5.$$

Quick Tip to Solve: At $x = 5$, calculate $3(5) - 8$ first, then compare it directly with $2k$; continuity needs equality.

Since $7 \neq 5$, the function is not continuous at $x = 5$. Assertion (A) is false.

Step 2: Test Reason (R) The stated equality of the two one-sided limits with $f(a)$ is exactly the continuity criterion. Therefore Reason (R) is true.

Step 3: Apply the reason to this function The criterion would require $2k = 7$, that is, $k = 7/2$, rather than the value in the assertion. Thus the reason reveals why the assertion fails. Options (A), (B) and (C) all incorrectly regard the assertion as true.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: At $x = 5$, calculate $3(5) - 8$ first, then compare it directly with $2k$; continuity needs equality.

Section B (10 marks)

*This section comprises of **5** Very Short Answer (VSA) type questions of **2** marks each.*

21(a). Differentiate $2^{\cos^2 x}$ w.r.t. $\cos^2 x$. [2 marks]

Solution:

Concept:

- For a constant positive base b, differentiating b^u with respect to its exponent u gives $\dfrac{d}{du}b^u = b^u \ln b$.
- The requested independent quantity is $u = \cos^2 x$, so the task is to differentiate the exponential as a function of u.

Step 1: Express the function in the requested variable Put $u = \cos^2 x$ and let $y = 2^{\cos^2 x}$. Then $y = 2^u$. This substitution is useful because the derivative is requested with respect to $\cos^2 x$, not directly with respect to x. Equivalently, write the exponential using the natural base:

$$y = e^{u \ln 2}.$$

Step 2: Differentiate with respect to the exponent Apply the exponential derivative and the chain rule. Since $\ln 2$ is constant,

$$\begin{aligned}\frac{dy}{du} &= e^{u \ln 2}\frac{d}{du}(u \ln 2)\\ &= e^{u \ln 2}\ln 2\\ &= 2^u \ln 2.\end{aligned}$$

Finally replace u by $\cos^2 x$. No factor $-2\sin x \cos x$ is needed: that factor would arise only when differentiating with respect to x.

Step 3: Conclusion $\dfrac{d(2^{\cos^2 x})}{d(\cos^2 x)} = 2^{\cos^2 x}\ln 2.$

Quick Tip to Solve: Replace the entire quantity after "with respect to" by u; here this immediately turns the problem into differentiating 2^u.

OR

21(b). **If $\tan^{-1}(x^2+y^2)=a^2$, then find $\dfrac{dy}{dx}$.** [2 marks]

Solution:

Concept:

- For a differentiable expression $u(x)$, $\dfrac{d}{dx}\tan^{-1}u=\dfrac{u'}{1+u^2}$.
- In implicit differentiation, y depends on x, so $\dfrac{d}{dx}y^2=2y\dfrac{dy}{dx}$. The parameter a is constant.

Step 1: Differentiate both sides implicitly Starting from $\tan^{-1}(x^2+y^2)=a^2$, apply the chain rule to the left side. The derivative of the constant on the right is zero, giving

$$\frac{2x+2y\dfrac{dy}{dx}}{1+(x^2+y^2)^2}=0.$$

The term $2y\,dy/dx$ is essential because y, unlike a, varies with x.

Step 2: Isolate the derivative For real x, y, the denominator $1+(x^2+y^2)^2$ is strictly positive. Therefore the numerator must vanish:

$$2x+2y\frac{dy}{dx}=0,$$
$$y\frac{dy}{dx}=-x.$$

Dividing by y gives the derivative wherever $y\neq 0$. This agrees with the slope obtained from $x^2+y^2=\tan(a^2)$, which follows by taking tangent on both sides of the original relation. At $y=0$, division by y is not valid and the formula does not assert a finite derivative.

Step 3: Conclusion $\dfrac{dy}{dx}=-\dfrac{x}{y}$, wherever $y\neq 0$.

Quick Tip to Solve: When an inverse tangent equals a constant, its positive derivative denominator can be cleared immediately after implicit differentiation.

22. **Evaluate:** $\tan^{-1}\left[2\sin\left(2\cos^{-1}\dfrac{\sqrt{3}}{2}\right)\right]$. [2 marks]

Solution:

Concept:

- The principal value of $\cos^{-1}t$ lies in $[0,\pi]$. In particular, $\cos^{-1}(\sqrt{3}/2)=\pi/6$.
- The principal value of $\tan^{-1}t$ lies in $(-\pi/2,\pi/2)$; use this interval when identifying the final angle.

Step 1: Evaluate the inner inverse cosine Let $\theta=\cos^{-1}(\sqrt{3}/2)$. Since $\cos(\pi/6)=\sqrt{3}/2$ and $\pi/6$ is in

Quick Tip to Solve: Work from the innermost inverse function outward, preserving both factors of 2, and check the principal range at the last step.

the principal range of inverse cosine, $\theta = \pi/6$. Thus the doubled angle inside the sine is

$$2\cos^{-1}\frac{\sqrt{3}}{2} = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}.$$

Step 2: Simplify the sine and evaluate the inverse tangent Substitute the inner value before applying the outer inverse function:

$$2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right) = 2\sin\frac{\pi}{3}$$
$$= 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}.$$

Now $\tan(\pi/3) = \sqrt{3}$, and $\pi/3$ belongs to $(-\pi/2, \pi/2)$. Hence it is the unique principal angle selected by the outer inverse tangent; adding multiples of π is not appropriate for an inverse-function value.

Step 3: Conclusion $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \frac{\pi}{3}$.

Quick Tip to Solve: Work from the innermost inverse function outward, preserving both factors of 2, and check the principal range at the last step.

23. The diagonals of a parallelogram are given by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$. Find the area of the parallelogram. [2 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{u}, \vec{v}$ is $|\vec{u} \times \vec{v}|$.
- If its diagonal vectors are $\vec{a}, \vec{b}$, its area is $\frac{1}{2}|\vec{a} \times \vec{b}|$. Indeed, the diagonals can be written as $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$, whose cross product has magnitude $2|\vec{u} \times \vec{v}|$.

Step 1: Calculate the cross product of the diagonals Use the given components in a determinant, keeping the minus sign in the middle cofactor:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$
$$= [1-3]\hat{i} - [-2-1]\hat{j} + [6+1]\hat{k}$$
$$= -2\hat{i} + 3\hat{j} + 7\hat{k}.$$

Step 2: Take the magnitude and apply the diagonal formula The magnitude of this cross product is

$$|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 3^2 + 7^2} = \sqrt{62}.$$

These are diagonal vectors, so this magnitude is twice the required area; the factor $\frac{1}{2}$ must be retained.

Quick Tip to Solve: Identify whether the supplied vectors are sides or diagonals before calculating: diagonal cross products give twice the parallelogram's area.

Step 3: Conclusion The area of the parallelogram is $\boxed{\frac{\sqrt{62}}{2}}$ square units.

Quick Tip to Solve: Identify whether the supplied vectors are sides or diagonals before calculating: diagonal cross products give twice the parallelogram's area.

24. Find the intervals in which function $f(x) = 5x^{3/2} - 3x^{5/2}$ is (i) increasing (ii) decreasing. [2 marks]

Solution:

Concept:

- A function is increasing on an interval where $f'(x) > 0$, and decreasing where $f'(x) < 0$.
- Fractional powers with denominator 2 restrict this real-valued function to $x \geq 0$. Use the power rule $\frac{d}{dx}x^n = nx^{n-1}$ in the interior of its domain.

Step 1: Differentiate and factor the derivative For $f(x) = 5x^{3/2} - 3x^{5/2}$, differentiation gives

$$\begin{aligned} f'(x) &= \frac{15}{2}x^{1/2} - \frac{15}{2}x^{3/2} \\ &= \frac{15}{2}\sqrt{x}(1-x), \qquad x > 0. \end{aligned}$$

The interior stationary point is $x = 1$; $x = 0$ is the endpoint of the domain, not a boundary of any negative-domain interval.

Step 2: Determine the sign on each interval On $0 < x < 1$, both $\sqrt{x}$ and $1 - x$ are positive, so $f'(x) > 0$. On $x > 1$, the square root remains positive but $1 - x < 0$, so $f'(x) < 0$. Thus the sign changes from positive to negative at $x = 1$. There is no interval with $x < 0$, since the given real-valued function is undefined there.

Step 3: Conclusion (i) The function is increasing on $\boxed{(0, 1)}$. (ii) It is decreasing on $\boxed{(1, \infty)}$. Including boundary values, it is also strictly increasing on $[0, 1]$ and strictly decreasing on $[1, \infty)$.

Quick Tip to Solve: For half-integer powers, establish $x \geq 0$ first, then factor out $\sqrt{x}$; only $1 - x$ changes sign inside the domain.

25(a). Two friends while flying kites from different locations, find the strings of their kites crossing each other. The strings can be represented by vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$. Determine the angle formed between the kite strings. Assume there is no slack in the strings. [2 marks]

Solution:

Concept:

- For nonzero vectors $\vec{a}$ and $\vec{b}$, their angle θ satisfies $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where $0 \leq \theta \leq \pi$.
- The dot product is the sum of products of corresponding components. The magnitude is the square root of the sum of their squares. Taut kite strings have the directions represented by the given

Quick Tip to Solve: Keep the magnitudes in radical form until the final step; here $\sqrt{14}\sqrt{24} = 4\sqrt{21}$ simplifies the cosine immediately.

vectors.

Step 1: Calculate the dot product and magnitudes For $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$, we obtain

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 3(2) + 1(-2) + 2(4) = 12, \\ |\vec{a}| &= \sqrt{9 + 1 + 4} = \sqrt{14}, \\ |\vec{b}| &= \sqrt{4 + 4 + 16} = \sqrt{24} = 2\sqrt{6}. \end{aligned}$$

Step 2: Substitute in the angle formula Divide the scalar product by the product of the magnitudes:

$$\cos\theta = \frac{12}{\sqrt{14}\sqrt{24}} = \frac{12}{4\sqrt{21}} = \frac{3}{\sqrt{21}} = \sqrt{\frac{3}{7}}.$$

This value lies between 0 and 1, so the angle between the supplied directions is acute, consistent with the positive dot product.

Step 3: Conclusion The angle between the kite strings is $\boxed{\theta = \cos^{-1}\sqrt{\frac{3}{7}}}$, approximately 49.1°.

Quick Tip to Solve: Keep the magnitudes in radical form until the final step; here $\sqrt{14}\sqrt{24} = 4\sqrt{21}$ simplifies the cosine immediately.

OR

25(b). Find a vector of magnitude 21 units in the direction opposite to that of $\overrightarrow{AB}$ where A and B are the points $A(2, 1, 3)$ and $B(8, -1, 0)$ respectively. [2 marks]

Solution:

Concept:

- The displacement $\overrightarrow{AB}$ is found by subtracting the coordinates of A from those of B.
- The unit vector opposite to a nonzero vector $\vec{v}$ is $-\vec{v}/|\vec{v}|$. Multiplying this unit vector by the desired magnitude gives the required vector.

Step 1: Obtain the direction vector and its length With $A(2, 1, 3)$ and $B(8, -1, 0)$, subtract corresponding coordinates:

$$\begin{aligned} \overrightarrow{AB} &= (8 - 2)\hat{i} + (-1 - 1)\hat{j} + (0 - 3)\hat{k} \\ &= 6\hat{i} - 2\hat{j} - 3\hat{k}, \\ |\overrightarrow{AB}| &= \sqrt{6^2 + (-2)^2 + (-3)^2} = \sqrt{49} = 7. \end{aligned}$$

Step 2: Reverse the direction and scale to the required magnitude The unit vector in the opposite direction is

$$-\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{-6\hat{i} + 2\hat{j} + 3\hat{k}}{7}.$$

Multiplying by 21 gives components $(-18, 6, 9)$. Its magnitude is $\sqrt{324 + 36 + 81} = \sqrt{441} = 21$, and it

Quick Tip to Solve: Since $|\overrightarrow{AB}| = 7$, multiply $\overrightarrow{AB}$ directly by $-21/7 = -3$; the negative sign reverses its direction.

is a negative multiple of $\overrightarrow{AB}$, as required.

Step 3: Conclusion The required vector is $\boxed{-18\hat{i}+6\hat{j}+9\hat{k}}$, of magnitude 21 units opposite to $\overrightarrow{AB}$.

Quick Tip to Solve: Since $|\overrightarrow{AB}| = 7$, multiply $\overrightarrow{AB}$ directly by $-21/7 = -3$; the negative sign reverses its direction.

Section C (18 marks)

*This section comprises of **6** Short Answer (SA) type questions of **3** marks each.*

26. The side of an equilateral triangle is increasing at the rate of 3 cm/s. At what rate its area increasing when the side of the triangle is 15 cm? [3 marks]

Solution:

Concept:

- An equilateral triangle of side a has area $A = \dfrac{\sqrt{3}}{4}a^2$. Here both the side a and the area A vary with time t.
- For related rates, differentiate with respect to time using the chain rule: $\dfrac{dA}{dt} = \dfrac{dA}{da}\dfrac{da}{dt}$. A length rate has units cm/s, while an area rate has units cm^2/s.

Step 1: Express the area in terms of the changing side Let $a(t)$ be the side length in centimetres at time t seconds. Since the triangle remains equilateral, its altitude is $\sqrt{3}a/2$, giving

$$A = \frac{1}{2}a\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{4}a^2.$$

The given rate is $da/dt = 3$ cm/s. We need the instantaneous area rate when $a = 15$ cm, so keep a variable while differentiating.

Step 2: Differentiate with respect to time Apply the chain rule to $a(t)^2$:

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\,2a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}.$$

The factor da/dt is essential: differentiating area with respect to side alone would not produce a rate with respect to time.

Step 3: Evaluate at the specified side length Substitute the instantaneous length and its rate:

$$\left.\frac{dA}{dt}\right|_{a=15} = \frac{\sqrt{3}}{2}(15)(3) = \frac{45\sqrt{3}}{2}\ \text{cm}^2/\text{s}.$$

The positive value agrees with the increasing side length. The units also follow from multiplying a length in centimetres by a rate in centimetres per second.

Step 4: Conclusion The area is increasing at $\boxed{\dfrac{45\sqrt{3}}{2}\ \text{cm}^2/\text{s}}$ when the side is 15 cm.

Quick Tip to Solve: Differentiate $A = (\sqrt{3}/4)a^2$ before inserting $a = 15$; inserting the length too early hides the changing quantity and loses the chain-rule factor.

27. Solve the following linear programming problem graphically:

Maximise $Z = x + 2y$

Subject to the constraints:

$x - y \geq 0$

$x - 2y \geq -2$

$x \geq 0,\ y \geq 0$ [3 marks]

Solution:

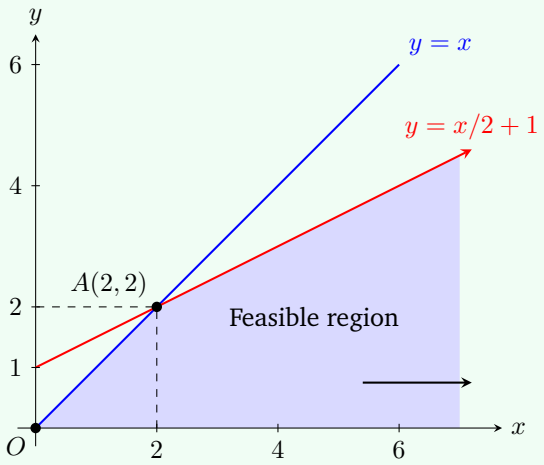
Concept:

- Each linear inequality represents a closed half-plane. The feasible region is their common part in the first quadrant.
- For an unbounded feasible region, corner values alone do not establish a maximum. Check whether the objective $Z = x + 2y$ can increase indefinitely along a feasible ray.

Step 1: Draw the boundary lines and shade the common region Rewrite the restrictions as $y \leq x$ and $y \leq x/2 + 1$, together with $x, y \geq 0$. Draw $y = x$ through $(0, 0)$ and $(2, 2)$, and draw $y = x/2 + 1$ through $(0, 1)$ and $(2, 2)$. The required side is below each line. Their intersection satisfies

$$x = \frac{x}{2} + 1 \quad \Longrightarrow \quad x = 2, \qquad y = 2.$$

The feasible region has corners $O(0, 0)$ and $A(2, 2)$, and extends indefinitely to the right between the upper boundary shown and the x-axis.



Shaded feasible region, continuing without bound to the right.

Step 2: Test whether the objective is bounded above At the corners, $Z(O) = 0$ and $Z(A) = 6$. Since the region is unbounded, this does not prove a maximum exists. Every point $(t, 0)$, with $t \geq 0$, is feasible because

$$t - 0 \geq 0, \qquad t - 2(0) \geq -2.$$

Along this feasible ray, $Z = t$, which can exceed any prescribed real number by choosing a larger t.

Step 3: Conclusion The objective $Z = x + 2y$ is unbounded above on the feasible region. Hence, this

Quick Tip to Solve: After finding the corners, test the positive x-axis: here $(t, 0)$ stays feasible while $Z = t$ grows without bound.

linear programming problem has no finite maximum value.

Quick Tip to Solve: After finding the corners, test the positive x-axis: here $(t, 0)$ stays feasible while $Z = t$ grows without bound.

28(a). Find: $\displaystyle\int \frac{x + \sin x}{1 + \cos x}\, dx$. [3 marks]

Solution:

Concept:

- The half-angle identities are $1 + \cos x = 2\cos^2(x/2)$ and $\sin x/(1 + \cos x) = \tan(x/2)$.
- Integration by parts gives $\int u\, dv = uv - \int v\, du$. Here choose the algebraic factor $u = x$, and use $d[\tan(x/2)]/dx = \frac{1}{2}\sec^2(x/2)$.

Step 1: Separate the integral and use half-angle identities Let the required integral be I. On any interval where $1 + \cos x \neq 0$, separate the numerator to obtain

$$\begin{aligned} I &= \int \frac{x}{1 + \cos x}\, dx + \int \frac{\sin x}{1 + \cos x}\, dx \\ &= \frac{1}{2}\int x \sec^2 \frac{x}{2}\, dx + \int \tan \frac{x}{2}\, dx. \end{aligned}$$

This form is useful because the derivative of the tangent in the second integral supplies the trigonometric factor in the first integral.

Step 2: Apply integration by parts to the first term Choose $u = x$ and $dv = \frac{1}{2}\sec^2(x/2)\, dx$. Then $du = dx$ and $v = \tan(x/2)$, so

$$\frac{1}{2}\int x \sec^2 \frac{x}{2}\, dx = x \tan \frac{x}{2} - \int \tan \frac{x}{2}\, dx.$$

Substitute this expression into I. The two remaining tangent integrals cancel exactly. In particular, no separate logarithmic term remains in the antiderivative. The cancellation can also be checked directly by differentiating the resulting product:

$$\frac{d}{dx}\left(x \tan \frac{x}{2}\right) = \tan \frac{x}{2} + \frac{x}{2}\sec^2 \frac{x}{2} = \frac{\sin x + x}{1 + \cos x}.$$

Step 3: Conclusion

$$\boxed{\int \frac{x + \sin x}{1 + \cos x}\, dx = x \tan \frac{x}{2} + C.}$$

Quick Tip to Solve: Convert $1 + \cos x$ using the half-angle identity, then look for the product derivative of $x \tan(x/2)$.

OR

28(b). Evaluate: $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^3 x\sqrt{2\sin 2x}}\,dx.$ [3 marks]

Solution:

Concept:

- For the substitution $t = \tan x$, $dx = dt/(1+t^2)$, $\cos^2 x = 1/(1+t^2)$ and $\sin 2x = 2t/(1+t^2)$.
- In a definite integral, transform both limits with the variable. The integrand is singular at $x = 0$; its integral is evaluated as a right-hand limit of proper integrals.

Step 1: Transform the integrand and the limits Put $t = \tan x$. The lower and upper limits become $t = 0$ and $t = 1$, respectively. On $0 < x \le \pi/4$, both t and $\cos x$ are positive, so the square roots have their positive values. Consequently,

$$\cos^3 x = (1+t^2)^{-3/2},$$
$$\sqrt{2\sin 2x} = \sqrt{\frac{4t}{1+t^2}} = \frac{2\sqrt{t}}{\sqrt{1+t^2}}.$$

Multiplying these denominator factors and then including $dx = dt/(1+t^2)$ gives

$$\frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \frac{1+t^2}{2\sqrt{t}}\,dt = \frac{1}{2}\left(t^{-1/2} + t^{3/2}\right)dt.$$

Step 2: Integrate the powers and evaluate the endpoint limit Write the transformed integral with a positive lower limit a, then let a decrease to zero:

$$\begin{aligned} I &= \frac{1}{2}\lim_{a\to 0^+}\int_a^1 \left(t^{-1/2} + t^{3/2}\right)dt \\ &= \lim_{a\to 0^+}\left[\sqrt{t} + \frac{t^{5/2}}{5}\right]_a^1 \\ &= \lim_{a\to 0^+}\left(1 + \frac{1}{5} - \sqrt{a} - \frac{a^{5/2}}{5}\right). \end{aligned}$$

Both lower-end terms tend to zero. This also verifies convergence: although the original integrand is undefined at the lower endpoint, the singular term after substitution is $t^{-1/2}$, whose integral has a finite limit there.

Step 3: Conclusion

$$\boxed{\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \frac{6}{5}.}$$

Quick Tip to Solve: With $t = \tan x$, combine all powers of $1 + t^2$ before integrating; retain the factor 2 from $\sqrt{2\sin 2x}$.

29(a). Verify that lines given by $\vec{r} = (1-\lambda)\hat{i} + (\lambda-2)\hat{j} + (3-2\lambda)\hat{k}$ and $\vec{r} = (\mu+1)\hat{i} + (2\mu-1)\hat{j} - (2\mu+1)\hat{k}$ are skew lines. Hence, find shortest distance between the lines. [3 marks]

Solution:

Concept:

- Two lines are skew if they are neither parallel nor intersecting. For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, nonproportional direction vectors rule out parallelism.
- The shortest distance between skew lines is $d = |(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|/|\vec{b}_1 \times \vec{b}_2|$, the projection of a joining vector onto their common normal.

Step 1: Identify the points and direction vectors Separate the constant and parameter terms in each equation:

$$\vec{a}_1 = (1, -2, 3), \qquad \vec{b}_1 = (-1, 1, -2),$$
$$\vec{a}_2 = (1, -1, -1), \quad \vec{b}_2 = (1, 2, -2).$$

The first components of the direction vectors would require a proportionality factor of -1, whereas the second components would require 2. Therefore the lines are not parallel.

Step 2: Check whether the lines intersect At an intersection, all three coordinates must agree. Equality of the first two coordinates gives

$$1 - \lambda = 1 + \mu, \qquad \lambda - 2 = 2\mu - 1.$$

Hence $\mu = -\lambda$, and $3\lambda = 1$, so $\lambda = 1/3$, $\mu = -1/3$. But their third coordinates would then be

$$3 - 2\lambda = \frac{7}{3}, \qquad -2\mu - 1 = -\frac{1}{3}.$$

These are unequal. Thus the lines do not intersect and, since they are not parallel, they are skew.

Step 3: Compute the common normal and its projection Calculate the cross product and joining vector:

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = 2\hat{i} - 4\hat{j} - 3\hat{k},$$
$$\vec{a}_2 - \vec{a}_1 = (0, 1, -4).$$

Their scalar product and the normal's magnitude are

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = -4 + 12 = 8, \qquad |\vec{b}_1 \times \vec{b}_2| = \sqrt{4 + 16 + 9} = \sqrt{29}.$$

The nonzero scalar product also confirms that the lines are not coplanar.

Step 4: Conclusion The given lines are skew, and their shortest distance is

$$\boxed{d = \frac{8}{\sqrt{29}} \text{ units.}}$$

Quick Tip to Solve: An intersection must satisfy all three coordinates. After solving the first two, check the third before declaring that the lines meet.

OR

29(b). During a cricket match, the position of the bowler, the wicket keeper and the leg slip fielder are in a line given by $\vec{B} = 2\hat{i} + 8\hat{j}$, $\vec{W} = 6\hat{i} + 12\hat{j}$ and $\vec{F} = 12\hat{i} + 18\hat{j}$ respectively. Calculate the ratio in which the wicketkeeper divides the line segment joining the bowler and the leg slip fielder. [3 marks]

Solution:

Concept:

- If a point W divides the segment BF internally in the ratio $BW : WF = m : n$, its position vector is $\vec{W} = (n\vec{B} + m\vec{F})/(m + n)$.
- Equal position vectors have equal corresponding components. For internal division, the point lies between the two endpoints and both ratio terms are positive.

Step 1: Identify the points and set the ratio The given position vectors give $B = (2, 8, 0)$, $W = (6, 12, 0)$ and $F = (12, 18, 0)$. Let the required ratio, measured from the bowler towards the fielder, be $BW : WF = m : n$. The wicketkeeper's coordinates lie between the corresponding coordinates of the two endpoints, so we use internal division.

Step 2: Apply the section formula Substituting the given vectors into the internal section formula gives

$$6\hat{i} + 12\hat{j} = \frac{n(2\hat{i} + 8\hat{j}) + m(12\hat{i} + 18\hat{j})}{m + n}.$$

Equating the coefficients of $\hat{i}$,

$$\begin{aligned} 6(m + n) &= 2n + 12m, \\ 4n &= 6m, \\ \frac{m}{n} &= \frac{2}{3}. \end{aligned}$$

The $\hat{j}$ components independently give $12(m + n) = 8n + 18m$, again yielding $4n = 6m$, so both coordinates give the same ratio.

Step 3: Check the order of the ratio The displacement vectors are

$$\overrightarrow{BW} = 4\hat{i} + 4\hat{j}, \qquad \overrightarrow{WF} = 6\hat{i} + 6\hat{j}.$$

They point in the same direction, confirming that W lies on the segment, and their lengths are $4\sqrt{2}$ and $6\sqrt{2}$. Thus the shorter part is the distance from the bowler to the wicketkeeper, consistent with the order $BW : WF$.

Step 4: Conclusion The wicketkeeper divides the line segment joining the bowler to the leg slip fielder internally in the ratio $\boxed{BW : WF = 2 : 3}$.

Quick Tip to Solve: Use coordinate differences in the same order: $(6 - 2) : (12 - 6) = 4 : 6$. Confirm that the second coordinate gives the same ratio before concluding internal division.

30(a). The probability distribution for the number of students being absent in a class on a Saturday is as follows:

X	0	2	4	5
$P(X)$	p	$2p$	$3p$	p

Where X is the number of students absent. [3 marks]

30(a)(i). Calculate p. [1 mark]

Solution:

Concept:

- A probability distribution lists mutually exclusive values of a random variable X; the probabilities of all its possible values add to 1.
- Each probability must lie between 0 and 1, so the value obtained from normalization must also satisfy this requirement.

Step 1: Normalize the given distribution The possible numbers of absent students are $0, 2, 4, 5$, with respective probabilities $p, 2p, 3p, p$. These exhaust the stated distribution. Therefore

$$p + 2p + 3p + p = 1.$$

Step 2: Solve and check the probability parameter Collecting the coefficients gives $7p = 1$, and hence $p = 1/7$. The resulting probabilities are $1/7, 2/7, 3/7, 1/7$. All are non-negative, no probability exceeds 1, and their sum is $7/7 = 1$, so this value defines a valid distribution.

Step 3: Conclusion The required value is $\boxed{p = \frac{1}{7}}$.

Quick Tip to Solve: To find the unknown probability parameter, add the probabilities rather than the values of the random variable.

30(a)(ii). Calculate the mean of the number of absent students on Saturday. [2 marks]

Solution:

Concept:

- The mean, or expectation, of a discrete random variable is $E(X) = \sum_x xP(X = x)$. Each possible value is weighted by its own probability.
- The distribution must first be normalized: $p + 2p + 3p + p = 1$, giving $p = 1/7$. Here X counts absent students, so its expectation describes the long-run average count on Saturdays.

Step 1: Obtain the probabilities for the weighted mean For completeness, the sum of the four probabilities is $7p = 1$, so $p = 1/7$, as found in part (i). Thus

$$P(X = 0) = \frac{1}{7}, \quad P(X = 2) = \frac{2}{7},$$
$$P(X = 4) = \frac{3}{7}, \quad P(X = 5) = \frac{1}{7}.$$

Quick Tip to Solve: Pair each count with its own probability; the arithmetic average of the four listed counts is not the expected value.

The total probability is 1; these are the weights to use, rather than giving equal weight to the four values.

Step 2: Calculate the expectation Multiply each number of absent students by its corresponding probability and add:

$$E(X) = 0\left(\frac{1}{7}\right) + 2\left(\frac{2}{7}\right) + 4\left(\frac{3}{7}\right) + 5\left(\frac{1}{7}\right)$$
$$= \frac{0+4+12+5}{7} = \frac{21}{7} = 3.$$

The value 0 contributes nothing to the numerator, but its probability was still needed to normalize the distribution. The result lies between the smallest and largest possible counts, 0 and 5, as a weighted average must.

Step 3: Conclusion The mean number of students absent on Saturday is $\boxed{3}$.

Quick Tip to Solve: Pair each count with its own probability; the arithmetic average of the four listed counts is not the expected value.

OR

30(b). For the vacancy advertised in the newspaper, 3000 candidates submitted their applications. From the data it was revealed that two third of the total applicants were females and other were males. The selection for the job was done through a written test. The performance of the applicants indicates that the probability of a male getting a distinction in written test is 0.4 and that a female getting a distinction is 0.35. Find the probability that the candidate chosen at random will have a distinction in the written test. [3 marks]

Solution:

Concept:

- If events M and F form a partition of the sample space, then $P(M) + P(F) = 1$, and no outcome belongs to both events.
- The law of total probability gives $P(D) = P(M)P(D \mid M) + P(F)P(D \mid F)$, where D is the event that the selected candidate has a distinction.

Step 1: Find the probabilities of selecting each group Let M, F and D denote the events that the randomly chosen applicant is male, is female, and has a distinction, respectively. Of the 3000 applicants, the number of females is $(2/3)(3000) = 2000$; the remaining 1000 are males. Therefore

$$P(F) = \frac{2000}{3000} = \frac{2}{3}, \qquad P(M) = \frac{1000}{3000} = \frac{1}{3}.$$

The given performance probabilities are conditional probabilities: $P(D \mid M) = 0.4$ and $P(D \mid F) = 0.35$.

Step 2: Combine the mutually exclusive routes to a distinction A candidate with a distinction belongs either to the male group or to the female group. These two possibilities are disjoint and include all

Quick Tip to Solve: Weight the male distinction rate by $1/3$ and the female distinction rate by $2/3$; the applicant total cancels when these group probabilities are formed.

applicants with a distinction. Hence the total probability is

$$\begin{aligned} P(D) &= P(M)P(D \mid M) + P(F)P(D \mid F) \\ &= \frac{1}{3}\left(\frac{2}{5}\right) + \frac{2}{3}\left(\frac{7}{20}\right) \\ &= \frac{2}{15} + \frac{7}{30} \\ &= \frac{11}{30}. \end{aligned}$$

The group probabilities supply the necessary weights: an unweighted average of 0.4 and 0.35 would incorrectly treat the two groups as equally numerous. The computed probability lies between these two conditional probabilities and is closer to 0.35, as expected because females form the larger group.

Step 3: Conclusion The probability that the randomly chosen candidate has a distinction in the written test is $\boxed{\frac{11}{30}}$, approximately 0.3667.

Quick Tip to Solve: Weight the male distinction rate by $1/3$ and the female distinction rate by $2/3$; the applicant total cancels when these group probabilities are formed.

31. Sketch the graph of $y = |x + 3|$ and find the area of the region enclosed by the curve, x-axis, between $x = -6$ and $x = 0$, using integration. [3 marks]

Solution:

Concept:

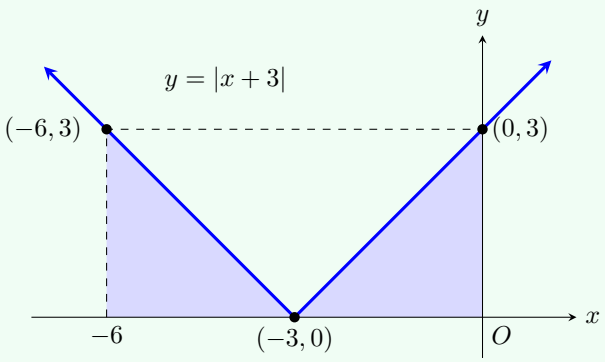
- The modulus satisfies $|u| = -u$ for $u < 0$ and $|u| = u$ for $u \geq 0$. Thus the graph changes its linear rule where the expression inside the modulus vanishes.
- If $y = f(x) \geq 0$ on $[a, b]$, the area between its graph and the x-axis is $\int_a^b f(x)\,dx$, in square units.

Step 1: Write the branches and sketch the graph The expression $x + 3$ is zero at $x = -3$. Hence

$$y = |x + 3| = \begin{cases} -x - 3, & x < -3, \\ x + 3, & x \geq -3. \end{cases}$$

The two branches form a V with vertex $(-3, 0)$, passing through $(-6, 3)$ and $(0, 3)$. Shade the region above the x-axis and below the curve between $x = -6$ and $x = 0$.

Quick Tip to Solve: Locate the zero of the expression inside the modulus first: split at $x = -3$. The two equal triangular regions provide a quick check on the integral.



The graph $y = |x + 3|$ and the two shaded regions between $x = -6$ and $x = 0$.

Step 2: Split and evaluate the area integral The curve is non-negative throughout the interval, but its algebraic rule changes at -3. Split the integral there and use the appropriate rule on each side:

$$\begin{aligned} A &= \int_{-6}^{0} |x+3|\,dx \\ &= \int_{-6}^{-3} (-x-3)\,dx + \int_{-3}^{0} (x+3)\,dx \\ &= \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} \\ &= \left(-\frac{9}{2} + 9 - (-18+18)\right) + \left(0 - \left(\frac{9}{2} - 9\right)\right) \\ &= \frac{9}{2} + \frac{9}{2} = 9. \end{aligned}$$

Both contributions are positive. Each shaded triangle has base 3 and height 3, independently confirming the area $9/2$ of each portion.

Step 3: Conclusion The required area is $\boxed{9 \text{ square units}}$; the sketch has vertex $(-3, 0)$.

Quick Tip to Solve: Locate the zero of the expression inside the modulus first: split at $x = -3$. The two equal triangular regions provide a quick check on the integral.

Section D (20 marks)

*This section comprises of **4** Long Answer (LA) type questions of **5** marks each.*

32(a). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$. [5 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x. Thus $\dfrac{d}{dx}\sqrt{1-y^2} = -\dfrac{y}{\sqrt{1-y^2}}\dfrac{dy}{dx}$ by the chain rule.

Quick Tip to Solve: After implicit differentiation, replace a using the original relation: both coefficients reduce to the common numerator $1 + \sqrt{1-x^2}\sqrt{1-y^2} - xy$.

- The given relation can eliminate the constant a after differentiation. Use $(\sqrt{1-x^2})^2 + x^2 = 1$ and the analogous identity for y to simplify the resulting ratio.

Step 1: Introduce notation and identify the domain Let $A = \sqrt{1-x^2}$, $B = \sqrt{1-y^2}$, and $p = \dfrac{dy}{dx}$. On a differentiable interior portion of the curve, $|x| < 1$ and $|y| < 1$, so $A, B > 0$. The given equation becomes

$$A + B = a(x - y).$$

Since $A+B > 0$, we have $x-y \neq 0$. We may therefore express the constant in terms of the coordinates as

$$a = \frac{A+B}{x-y}.$$

These domain conditions also ensure that differentiating the square roots does not introduce a zero denominator.

Step 2: Differentiate the relation implicitly Differentiate both sides with respect to x, keeping a constant and applying the chain rule to the term containing y:

$$-\frac{x}{A} - \frac{y}{B}p = a(1-p).$$

Move all the terms containing p to one side and the remaining terms to the other:

$$\left(a - \frac{y}{B}\right)p = a + \frac{x}{A}.$$

We next simplify both coefficients using the original equation, rather than treating a, x, y as independent quantities.

Step 3: Simplify the two coefficients Substituting $a = (A+B)/(x-y)$ and using $A^2 = 1 - x^2$ gives

$$\begin{aligned} a + \frac{x}{A} &= \frac{A(A+B) + x(x-y)}{A(x-y)} \\ &= \frac{1 + AB - xy}{A(x-y)}. \end{aligned}$$

Similarly, using $B^2 = 1 - y^2$,

$$\begin{aligned} a - \frac{y}{B} &= \frac{B(A+B) - y(x-y)}{B(x-y)} \\ &= \frac{1 + AB - xy}{B(x-y)}. \end{aligned}$$

The same numerator appears in both expressions. It is positive in the interior because $AB > 0$ and $xy < 1$, so its cancellation is valid.

Step 4: Divide and restore the square roots Dividing the rearranged derivative equation by its nonzero coefficient yields

$$p = \frac{a + x/A}{a - y/B} = \frac{1 + AB - xy}{A(x-y)} \, \frac{B(x-y)}{1 + AB - xy} = \frac{B}{A}.$$

Since both square roots are positive on the differentiable interior portion, their quotient is the nonnegative square root of the quotient of their squares.

Quick Tip to Solve: After implicit differentiation, replace a using the original relation: both coefficients reduce to the common numerator $1 + \sqrt{1-x^2}\sqrt{1-y^2} - xy$.

Step 5: Conclusion Therefore, wherever the above derivatives are defined,

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}}.$$

Hence the required identity is proved.

Quick Tip to Solve: After implicit differentiation, replace a using the original relation: both coefficients reduce to the common numerator $1 + \sqrt{1-x^2}\sqrt{1-y^2} - xy$.

OR

32(b). **If** $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ **and** $y = \sin\theta$**, then find** $\frac{d^2y}{dx^2}$ **at** $\theta = \frac{\pi}{4}$**.** [5 marks]

Solution:

Concept:

- For parametric equations $x = x(\theta)$ and $y = y(\theta)$, the first derivative is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, provided $dx/d\theta \neq 0$.
- The second derivative must also be taken with respect to x: $\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}(dy/dx)}{dx/d\theta}$. Differentiating the slope with respect to θ alone does not give the required second derivative.

Step 1: Differentiate the logarithmic term carefully The given parametric equations are

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right), \qquad y = \sin\theta.$$

Here a is a constant, and $\log$ denotes the natural logarithm. Apply the chain rule to the logarithmic term, including the factor $1/2$ from the inner angle:

$$\begin{aligned}\frac{d}{d\theta}\log\tan\frac{\theta}{2} &= \frac{1}{\tan(\theta/2)}\cdot\frac{1}{2}\sec^2\frac{\theta}{2}\\ &= \frac{1}{2\sin(\theta/2)\cos(\theta/2)}\\ &= \frac{1}{\sin\theta}.\end{aligned}$$

The double-angle identity in the last line is what makes the subsequent simplification straightforward.

Step 2: Obtain the first derivative Differentiate both coordinates with respect to the parameter:

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = \frac{a(1-\sin^2\theta)}{\sin\theta} = \frac{a\cos^2\theta}{\sin\theta},$$
$$\frac{dy}{d\theta} = \cos\theta.$$

Quick Tip to Solve: Simplify the first derivative to $\tan\theta/a$ before differentiating again, and remember to divide by $dx/d\theta$ a second time before setting $\theta = \pi/4$.

Divide the second coordinate derivative by the first:

$$\frac{dy}{dx} = \frac{\cos\theta}{a\cos^2\theta/\sin\theta} = \frac{\tan\theta}{a}.$$

At $\theta = \pi/4$, these divisions are valid for $a \neq 0$, since $dx/d\theta = a/\sqrt{2} \neq 0$. If $a = 0$, x is constant and the requested derivative with respect to x is not defined.

Step 3: Differentiate the slope with respect to the parameter Keep θ variable until the second derivative has been obtained. Differentiating the slope gives

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{1}{a}\sec^2\theta.$$

Now divide once more by $dx/d\theta$ to convert differentiation with respect to θ into differentiation with respect to x:

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta/a}{a\cos^2\theta/\sin\theta} = \frac{\sin\theta}{a^2\cos^4\theta}.$$

Step 4: Substitute the specified parameter value At $\theta = \pi/4$, use $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$ and hence $\cos^4(\pi/4) = 1/4$. Thus

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = \frac{1/\sqrt{2}}{a^2/4} = \frac{4}{a^2\sqrt{2}}.$$

Rationalising the numerical factor gives the required exact value. The factor a^2 arises because division by the coordinate derivative occurs at each differentiation with respect to x.

Step 5: Conclusion The required second derivative is

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = \frac{2\sqrt{2}}{a^2}}, \qquad a \neq 0.$$

Quick Tip to Solve: Simplify the first derivative to $\tan\theta/a$ before differentiating again, and remember to divide by $dx/d\theta$ a second time before setting $\theta = \pi/4$.

33. Find the absolute maximum and absolute minimum of function $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $[1, 5]$. [5 marks]

Solution:

Concept:

- The extreme value theorem guarantees an absolute maximum and an absolute minimum for a continuous function on a closed, bounded interval.
- For a differentiable function on $[a, b]$, compare the values at both endpoints and at every interior critical point satisfying $f'(x) = 0$. A local extremum need not be an absolute extremum on the interval.

Quick Tip to Solve: After finding $x = 2, 3$, make the four-entry list $f(1), f(2), f(3), f(5)$. Keeping both endpoints prevents confusing local extrema with absolute extrema.

Step 1: Check continuity on the given interval The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is a polynomial, so it is continuous throughout $[1, 5]$ and differentiable at every point of $(1, 5)$. Thus its absolute extreme values exist. Since its derivative is defined everywhere, the only interior candidates are the zeros of the derivative; the endpoints 1 and 5 must also be included.

Step 2: Find all interior critical points Differentiate the polynomial term by term and factor the resulting quadratic:

$$\begin{aligned} f'(x) &= 6x^2 - 30x + 36 \\ &= 6(x^2 - 5x + 6) \\ &= 6(x - 2)(x - 3). \end{aligned}$$

Hence $f'(x) = 0$ gives $x = 2$ and $x = 3$. Both lie inside $(1, 5)$, so the complete candidate list is $1, 2, 3, 5$.

Step 3: Evaluate the function at every candidate Substitute into the original function, including its constant term:

$$\begin{aligned} f(1) &= 2 - 15 + 36 + 1 = 24, \\ f(2) &= 16 - 60 + 72 + 1 = 29, \\ f(3) &= 54 - 135 + 108 + 1 = 28, \\ f(5) &= 250 - 375 + 180 + 1 = 56. \end{aligned}$$

These four values exhaust the possibilities for an absolute extremum. Compare function values, not just the locations of the critical points.

Step 4: Check the comparison using the derivative sign On $(1, 2)$, both factors in $f'(x) = 6(x-2)(x-3)$ are negative, so f increases. On $(2, 3)$, the factors have opposite signs, so f decreases. On $(3, 5)$, both are positive, so f increases again. Thus $x = 2$ is a local maximum and $x = 3$ is a local minimum, but the endpoint values must still be compared with them. The ordering is $24 < 28 < 29 < 56$.

Step 5: Conclusion The absolute maximum value is $\boxed{56}$, attained at $\boxed{x = 5}$, and the absolute minimum value is $\boxed{24}$, attained at $\boxed{x = 1}$, on $[1, 5]$.

Quick Tip to Solve: After finding $x = 2, 3$, make the four-entry list $f(1), f(2), f(3), f(5)$. Keeping both endpoints prevents confusing local extrema with absolute extrema.

34(a). Find the image A' of the point $A(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, find the equation of the line joining A and A'. [5 marks]

Solution:

Concept:

- If H is the foot of the perpendicular from a point A to a line, then its image A' in that line satisfies $\overrightarrow{OA'} = 2\overrightarrow{OH} - \overrightarrow{OA}$. Thus the line bisects AA' at right angles.
- A vector perpendicular to a line with direction vector $\mathbf{d}$ has zero dot product with $\mathbf{d}$. A line through position vector $\mathbf{a}$ with direction $\mathbf{v}$ is $\mathbf{r} = \mathbf{a} + s\mathbf{v}$.

Step 1: Express the foot of the perpendicular parametrically Set the common ratio $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ equal to t. Every point on it has coordinates $(t, 1 + 2t, 2 + 3t)$, and its direction vector is $\mathbf{d} = (1, 2, 3)$.

Quick Tip to Solve: First find the perpendicular foot using a dot product, then use $A' = 2H - A$. A zero direction ratio means that coordinate is constant in the joining line.

In particular, write the foot of the perpendicular as $H = (t, 1+2t, 2+3t)$. Since $A = (1, 6, 3)$,

$$\overrightarrow{AH} = (t-1,\ 2t-5,\ 3t-1).$$

Step 2: Use perpendicularity to locate the foot The condition AH perpendicular to the given line is $\overrightarrow{AH} \cdot \mathbf{d} = 0$. Therefore,

$$(t-1) + 2(2t-5) + 3(3t-1) = 0,$$
$$14t - 14 = 0.$$

Thus $t = 1$, giving $H = (1, 3, 5)$. The vector $\overrightarrow{AH} = (0, -3, 2)$ indeed has dot product $0 - 6 + 6 = 0$ with $(1, 2, 3)$.

Step 3: Apply the midpoint relation to obtain the image Let $A' = (x', y', z')$. Reflection makes H the midpoint of AA', so

$$\left(\frac{1+x'}{2}, \frac{6+y'}{2}, \frac{3+z'}{2}\right) = (1, 3, 5).$$

Equating coordinates gives $x' = 1$, $y' = 0$, and $z' = 7$. The midpoint construction places the image the same distance beyond the foot.

Step 4: Form the equation of the joining line A direction vector of the line joining A and its image is

$$\overrightarrow{AA'} = (0, -6, 4) = 2(0, -3, 2).$$

Hence a parametric equation through $A = (1, 6, 3)$ is

$$x = 1, \qquad y = 6 - 3s, \qquad z = 3 + 2s, \qquad s \in \mathbb{R}.$$

It contains A, H, and A' at $s = 0, 1, 2$, respectively. Since its first direction ratio is zero, retain $x = 1$ separately when eliminating the parameter; do not divide by zero in a symmetric equation.

Step 5: Conclusion The image is $\boxed{A' = (1, 0, 7)}$. Its joining line with A is $\boxed{x = 1, \quad \frac{y-6}{-3} = \frac{z-3}{2}}$.

Quick Tip to Solve: First find the perpendicular foot using a dot product, then use $A' = 2H - A$. A zero direction ratio means that coordinate is constant in the joining line.

OR

34(b). Find a point P on the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ such that its distance from point $Q(2, 4, -1)$ is 7 units. Also, find the equation of line joining P and Q. [5 marks]

Solution:

Concept:

- A point on a line in symmetric form is obtained by setting all three ratios equal to one real parameter. For direction ratios (a, b, c) and a point (x_0, y_0, z_0), its coordinates are $(x_0 + at, y_0 + bt, z_0 + ct)$.

Quick Tip to Solve: Set the three ratios equal to one parameter before applying the distance formula. Here the resulting perfect square gives a unique point; use coordinate differences, not the original line's direction ratios, for the joining line.

- The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.
- A line joining two distinct points has direction ratios given by their coordinate differences. Any common nonzero factor in these ratios may be cancelled.

Step 1: Express the point in terms of a parameter Let the common value of the ratios in the given line be t. Then

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = t,$$

so every point on this line has the form

$$P(t-5,\ 4t-3,\ 6-9t).$$

The required distance condition will determine which value of t is allowed.

Step 2: Apply the distance condition Since $Q = (2, 4, -1)$ and $PQ = 7$, squaring the distance formula gives

$$(t-7)^2 + (4t-7)^2 + (7-9t)^2 = 49.$$

Expand the three squares separately and collect like powers of t:

$$\begin{aligned} t^2 - 14t + 49 + 16t^2 - 56t + 49 + 81t^2 - 126t + 49 &= 49, \\ 98t^2 - 196t + 147 &= 49, \\ 98(t^2 - 2t + 1) &= 0, \\ (t-1)^2 &= 0. \end{aligned}$$

Therefore $t = 1$. The repeated root shows that only one point on the given line is at this distance from Q.

Step 3: Calculate and check the coordinates Substituting $t = 1$ gives

$$P = (1-5,\ 4-3,\ 6-9) = (-4, 1, -3).$$

Each ratio in the original line equals 1. Also,

$$PQ = \sqrt{(2+4)^2 + (4-1)^2 + (-1+3)^2} = \sqrt{36+9+4} = 7.$$

Thus both the line condition and the required distance are satisfied.

Step 4: Form the joining line The vector from P to Q is

$$\overrightarrow{PQ} = (2+4,\ 4-1,\ -1+3) = (6, 3, 2).$$

Using $P = (-4, 1, -3)$ as a point and $(6, 3, 2)$ as direction ratios, the required line is

$$\frac{x+4}{6} = \frac{y-1}{3} = \frac{z+3}{2}.$$

At common ratio 0 it passes through P, and at common ratio 1 it passes through Q.

Quick Tip to Solve: Set the three ratios equal to one parameter before applying the distance formula. Here the resulting perfect square gives a unique point; use coordinate differences, not the original line's direction ratios, for the joining line.

Step 5: Conclusion The required point is $\boxed{P = (-4, 1, -3)}$, and the line joining P and Q is

$$\boxed{\frac{x+4}{6} = \frac{y-1}{3} = \frac{z+3}{2}}.$$

Quick Tip to Solve: Set the three ratios equal to one parameter before applying the distance formula. Here the resulting perfect square gives a unique point; use coordinate differences, not the original line's direction ratios, for the joining line.

35. A school wants to allocate students into three clubs: Sports, Music and Drama, under following conditions:

- **The number of students in Sports club should be equal to the sum of the number of students in Music and Drama club.**
- **The number of students in Music club should be 20 more than half the number of students in Sports club.**
- **The total number of students to be allocated in all three clubs are 180.**

Find the number of students allocated to different clubs, using matrix method. [5 marks]

Solution:

Concept:

- Translate each club condition into a linear equation, using one variable for the number of students in each club. A consistent ordering of these variables fixes the columns of the coefficient matrix.
- A system of linear equations can be written as $AX = B$, where A is the coefficient matrix, X the column of unknowns and B the constants column.
- If $|A| \neq 0$, the unique solution is $X = A^{-1}B$, with $A^{-1} = \dfrac{1}{|A|}\operatorname{adj} A$. The adjugate is the transpose of the cofactor matrix.

Step 1: Translate the three conditions Let x, y, z be the numbers of students in Sports, Music and Drama, respectively. The first condition gives $x = y + z$. The second gives $y = \frac{x}{2} + 20$. The total is $x + y + z = 180$. Thus

$$\begin{aligned} x - y - z &= 0, \\ -x + 2y &= 40, \\ x + y + z &= 180. \end{aligned}$$

Multiplication of the Music condition by 2 clears the fraction and produces the constant 40, not 20.

Step 2: Write the matrix equation and test invertibility With the variables kept in Sports, Music, Drama order, write

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 40 \\ 180 \end{pmatrix}.$$

Quick Tip to Solve: Clear the fraction in the Music equation first, then keep the same club order in every matrix row. Check the final column against all three original conditions to catch sign or cofactor errors.

Then $AX = B$. Expanding the determinant along the first row,

$$\begin{aligned} |A| &= 1(2\cdot 1 - 0\cdot 1) - (-1)((-1)\cdot 1 - 0\cdot 1) \\ &\quad + (-1)((-1)\cdot 1 - 2\cdot 1) \\ &= 2 - 1 + 3 = 4 \neq 0. \end{aligned}$$

Hence the inverse exists and the allocation is uniquely determined.

Step 3: Obtain the inverse from the cofactors Taking signed minors gives the cofactor matrix

$$C = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix}.$$

For example, $C_{12} = -((-1)\cdot 1 - 0\cdot 1) = 1$ and $C_{23} = -\big(1\cdot 1 - (-1)\cdot 1\big) = -2$. Transpose this matrix and divide by the determinant:

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & 1 \end{pmatrix}.$$

Step 4: Multiply by the constants column Applying the matrix method,

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 40 \\ 180 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 360 \\ 260 \\ 100 \end{pmatrix} = \begin{pmatrix} 90 \\ 65 \\ 25 \end{pmatrix}. \end{aligned}$$

As a check, $65 + 25 = 90$, $90/2 + 20 = 65$, and $90 + 65 + 25 = 180$. All three original conditions hold, and each allocation is a nonnegative integer.

Step 5: Conclusion The school should allocate $\boxed{90}$ students to Sports, $\boxed{65}$ to Music and $\boxed{25}$ to Drama.

Quick Tip to Solve: Clear the fraction in the Music equation first, then keep the same club order in every matrix row. Check the final column against all three original conditions to catch sign or cofactor errors.

Section E (12 marks)

*This section comprises of **3** case study based questions of **4** marks each.*

36. A technical company is designing a rectangular solar panel installation on a roof using 300 metres of boundary material. The design includes a partition running parallel to one of the sides dividing the area (roof) into two sections.

Let the length of the side perpendicular to the partition be x metres and with parallel to the partition be y metres.

Based on this information, answer the following questions: [4 marks]

36(i). Write the equation for the total boundary material used in the boundary and parallel to the partition in terms of x and y. [1 mark]

Solution:

Concept:

- The perimeter of a rectangle of side lengths x and y is $2x + 2y$.
- The internal partition is parallel to the side of length y, so its length is y. Count it once in the material budget.

Step 1: Count the outer boundary The two sides perpendicular to the partition each have length x; the other two each have length y. Thus the external boundary consumes $2x + 2y$ metres of material.

Step 2: Include the internal partition The partition spans the rectangle in the direction parallel to the sides of length y, so it uses a further y metres. The total is therefore $2x + 2y + y$. Equating this total to the available 300 metres gives the constraint on the dimensions.

Step 3: Conclusion $\boxed{2x + 3y = 300}$, with $x, y > 0$.

Quick Tip to Solve: Count the two outer sides parallel to the partition plus the partition itself: there are three lengths y.

36(ii). Write the area of the solar panel as a function of x. [1 mark]

Solution:

Concept:

- The rectangular area is $A = xy$; the partition divides the roof without adding to or subtracting from its enclosed area.
- The material constraint is $2x + 3y = 300$, so eliminate y before expressing area as a function of x.

Step 1: Express the other dimension From $2x + 3y = 300$, we obtain

$$y = \frac{300 - 2x}{3}.$$

Since both side lengths must be positive, the admissible values satisfy $0 < x < 150$.

Quick Tip to Solve: First solve the material equation for y, then substitute into xy; the partition affects the constraint, not the area formula.

Step 2: Substitute into the area Multiply the perpendicular side x by the parallel side y:

$$A(x) = x\left(\frac{300 - 2x}{3}\right) = 100x - \frac{2}{3}x^2.$$

The same rectangle encloses both sections, so no extra factor for the partition is needed.

Step 3: Conclusion $\boxed{A(x) = 100x - \frac{2}{3}x^2 \text{ square metres}}$, $0 < x < 150$.

Quick Tip to Solve: First solve the material equation for y, then substitute into xy; the partition affects the constraint, not the area formula.

36(iii)(a). Find the critical points of the area function. Use second derivative test to determine critical points at the maximum area. Also, find the maximum area. [2 marks]

Solution:

Concept:

- The boundary constraint $2x + 3y = 300$ gives $A(x) = xy = 100x - \frac{2}{3}x^2$, with $0 < x < 150$.
- For an interior extremum of a differentiable area function, first solve $A'(x) = 0$ and then establish that the stationary point is a maximum.

Step 1: Find the stationary point Differentiate the area function with respect to x:

$$A'(x) = 100 - \frac{4}{3}x.$$

Setting this equal to zero gives $x = 75$, the only critical value in the feasible interval. The corresponding other dimension is $y = (300 - 150)/3 = 50$ metres.

Step 2: Apply the second derivative test The second derivative is

$$A''(x) = -\frac{4}{3} < 0.$$

In particular it is negative at $x = 75$, so the stationary point is a strict maximum. The area function is a downward-opening quadratic on the entire feasible interval, so this maximum is also the greatest feasible area.

Step 3: Evaluate the maximum area Substituting the dimensions gives $A(75) = 75 \times 50 = 3750$ square metres. As a check, the boundary and partition together use $2(75) + 3(50) = 300$ metres, as required.

Step 4: Conclusion The unique critical point is $\boxed{x = 75 \text{ m}}$, with $y = 50$ m. The maximum area is $\boxed{3750 \text{ m}^2}$.

Quick Tip to Solve: The negative constant second derivative proves the stationary point is the maximum throughout the physical interval.

OR

36(iii)(b). Using first derivative test, calculate the maximum area the company can enclose with the 300 metres of boundary material, considering the parallel partition. [2 marks]

Solution:

Concept:

- The boundary constraint $2x + 3y = 300$ gives $A(x) = xy = 100x - \frac{2}{3}x^2$, with $0 < x < 150$.
- For an interior extremum of a differentiable area function, first solve $A'(x) = 0$ and then establish that the stationary point is a maximum.

Step 1: Locate the critical value The derivative is

$$A'(x) = 100 - \frac{4}{3}x = \frac{4}{3}(75 - x).$$

It vanishes only at $x = 75$ metres, which lies in the feasible interval $0 < x < 150$.

Step 2: Use the first derivative sign test For $0 < x < 75$, the factor $75 - x$ is positive, so $A'(x) > 0$ and the area increases. For $75 < x < 150$, that factor is negative, so $A'(x) < 0$ and the area decreases. Therefore the derivative changes from positive to negative at 75, proving that this value gives the maximum area.

Step 3: Calculate the enclosed area At $x = 75$, the constraint yields $y = (300 - 2 \times 75)/3 = 50$ metres. Hence $A = xy = 75 \times 50 = 3750$ square metres. These dimensions satisfy the full material budget, including the internal partition.

Step 4: Conclusion The company can enclose a maximum area of $\boxed{3750\ \text{m}^2}$, using dimensions 75 m and 50 m.

Quick Tip to Solve: Factor the first derivative as (4/3)(75-x); its sign on each side of 75 gives the required test directly.

37. A class-room teacher is keen to assess the learning of her students the concept of "relations" taught to them. She writes the following five relations each defined on the set $A = \{1, 2, 3\}$:

$R_1 = \{(2, 3), (3, 2)\}$

$R_2 = \{(1, 2), (1, 3), (3, 2)\}$

$R_3 = \{(1, 2), (2, 1), (1, 1)\}$

$R_4 = \{(1, 1), (1, 2), (3, 3), (2, 2)\}$

$R_5 = \{(1, 1), (1, 2), (3, 3), (2, 2), (2, 1), (2, 3), (3, 2)\}$

The students are asked to answer the following questions about the above relations: [4 marks]

37(i). Identify the relation which is reflexive, transitive but not symmetric. [1 mark]

Solution:

Concept:

- A relation on A is reflexive if (a, a) belongs to it for every $a \in A$.

Quick Tip to Solve: First find all three diagonal pairs, then test the lone pair $(1, 2)$ for its reverse and for possible chains.

- Symmetry requires $(a, b) \Rightarrow (b, a)$; transitivity requires $(a, b), (b, c) \Rightarrow (a, c)$.

Step 1: Check reflexivity and symmetry In $R_4 = \{(1,1), (1,2), (3,3), (2,2)\}$, all three diagonal pairs are present, so it is reflexive on $A = \{1, 2, 3\}$. However, $(1,2) \in R_4$ while $(2,1) \notin R_4$, so symmetry fails.

Step 2: Verify transitivity The only non-diagonal pair is $(1,2)$. Composing it with $(1,1)$ or $(2,2)$ only requires $(1,2)$, which is already present. Compositions of diagonal pairs require those same diagonal pairs. Thus every possible chain has its required closing pair and R_4 is transitive.

Step 3: Conclusion The required relation is $\boxed{R_4}$.

Quick Tip to Solve: First find all three diagonal pairs, then test the lone pair $(1,2)$ for its reverse and for possible chains.

37(ii). Identify the relation which is reflexive and symmetric but not transitive. [1 mark]

Solution:

Concept:

- Reflexivity on $A = \{1, 2, 3\}$ requires $(1,1), (2,2), (3,3)$.
- Symmetry requires every ordered pair to have its reverse. One chain $(a, b), (b, c)$ with (a, c) missing disproves transitivity.

Step 1: Check the diagonal and reverse pairs Relation R_5 contains all three diagonal pairs, so it is reflexive. Its non-diagonal pairs are $(1,2), (2,1), (2,3), (3,2)$, and each appears together with its reverse. The diagonal pairs reverse to themselves, so R_5 is also symmetric.

Step 2: Exhibit a chain that violates transitivity We have $(1,2) \in R_5$ and $(2,3) \in R_5$. Transitivity would therefore require $(1,3) \in R_5$, but this ordered pair is absent. This single counterexample establishes that the relation is not transitive.

Step 3: Conclusion The required relation is $\boxed{R_5}$.

Quick Tip to Solve: After checking diagonal and reverse pairs, use the chain $1 \to 2 \to 3$ to test transitivity immediately.

37(iii)(a). Identify the relations which are symmetric but neither reflexive nor transitive. [2 marks]

Solution:

Concept:

- A relation is symmetric when every (a, b) is accompanied by (b, a). A diagonal pair is its own reverse.
- Reflexivity requires all diagonal pairs of the given set. Transitivity requires (a, c) whenever both (a, b) and (b, c) are present; one missing required pair disproves it.

Step 1: Test the relation R1 In $R_1 = \{(2,3), (3,2)\}$, the two ordered pairs are reverses of one another,

Quick Tip to Solve: A pair and its reverse force a diagonal pair under transitivity; use $2 \to 3 \to 2$ for R_1 and $2 \to 1 \to 2$ for R_3.

so the relation is symmetric. It is not reflexive on A, since the diagonal pairs, including $(1,1)$, are absent. Also, $(2,3)$ and $(3,2)$ would require $(2,2)$ for transitivity, but $(2,2) \notin R_1$. Hence transitivity fails.

Step 2: Test the relation R3 In $R_3 = \{(1,2),(2,1),(1,1)\}$, the first two pairs reverse each other and $(1,1)$ reverses itself. Thus it is symmetric. It is not reflexive because $(2,2)$ and $(3,3)$ are missing. The chain $(2,1),(1,2)$ requires $(2,2)$, which is absent, so it is not transitive.

Step 3: Exclude the remaining relations Relations R_2 and R_4 contain $(1,2)$ but not $(2,1)$, so neither is symmetric. Relation R_5 is reflexive and therefore cannot meet the requirement of being non-reflexive.

Step 4: Conclusion The required relations are $\boxed{R_1 \text{ and } R_3}$.

Quick Tip to Solve: A pair and its reverse force a diagonal pair under transitivity; use $2 \to 3 \to 2$ for R_1 and $2 \to 1 \to 2$ for R_3.

OR

37(iii)(b). What pairs should be added to the relation R_2 to make it an equivalence relation? [2 marks]

Solution:

Concept:

- An equivalence relation is reflexive, symmetric and transitive on its underlying set.
- For an equivalence relation on $A = \{1,2,3\}$, reflexivity forces all diagonal pairs and symmetry forces the reverse of every existing pair.

Step 1: Add the missing diagonal pairs The given relation is $R_2 = \{(1,2),(1,3),(3,2)\}$. None of its diagonal pairs is present. Reflexivity therefore requires adding

$$(1,1), \quad (2,2), \quad (3,3).$$

Step 2: Add the missing reverse pairs The reverse of $(1,2)$ is $(2,1)$, the reverse of $(1,3)$ is $(3,1)$, and the reverse of $(3,2)$ is $(2,3)$. All three are missing, so symmetry requires adding each of them.

Step 3: Check that transitivity also holds After these additions the relation contains all nine ordered pairs in $A \times A$. Thus for any $a,b,c \in A$, the pair (a,c) is present whenever (a,b) and (b,c) are present. The resulting universal relation is transitive as well as reflexive and symmetric. Every added pair was forced by one of the first two properties, so none of the six additions can be omitted.

Step 4: Conclusion Add the six ordered pairs

$$\boxed{\{(1,1),(2,2),(3,3),(2,1),(3,1),(2,3)\}}.$$

The resulting equivalence relation is $A \times A$.

Quick Tip to Solve: For this R_2, add the three diagonal pairs and the three missing reverses; the completed relation already equals $A \times A$.

38. A bank offers loan to its customers on different types of interest namely, fixed rate, floating rate and variable rate. From the past data with the bank, it is known that a customer avails loan on fixed rate, floating rate or variable rate with probabilities 10%, 20% and 70% respectively. A customer after availing loan can pay the loan or default on loan repayment. The bank data suggests that the probability that a person defaults on loan after availing it at fixed rate, floating rate and variable rate is 5%, 3% and 1% respectively.

Based on the above information, answer the following: [4 marks]

38(i). What is the probability that a customer after availing the loan will default on the loan repayment? [2 marks]

Solution:

Concept:

- Let F, L, V denote fixed, floating and variable rate loans, respectively, and let D denote default. These loan categories are mutually exclusive and exhaustive.
- The law of total probability is $P(D) = P(F)P(D \mid F) + P(L)P(D \mid L) + P(V)P(D \mid V)$.

Step 1: Convert the data to probabilities The loan-type probabilities are $P(F) = 0.10$, $P(L) = 0.20$, and $P(V) = 0.70$. The respective conditional default probabilities are 0.05, 0.03, and 0.01. These are probabilities of default within each category, so they must be weighted by the proportions taking that category.

Step 2: Add the disjoint default cases The events $D \cap F$, $D \cap L$, and $D \cap V$ are disjoint and together cover every default. Thus

$$\begin{aligned} P(D) &= (0.10)(0.05) + (0.20)(0.03) + (0.70)(0.01) \\ &= 0.005 + 0.006 + 0.007 = 0.018. \end{aligned}$$

Simply adding the three conditional rates would ignore how frequently each kind of loan is taken.

Step 3: Conclusion The probability of default is $\boxed{0.018 = \frac{9}{500}}$, or $\boxed{1.8\%}$.

Quick Tip to Solve: Multiply each loan category share by its own default rate, then add the three products.

38(ii). A customer after availing the loan, defaults on loan repayment. What is the probability that he availed the loan at a variable rate of interest? [2 marks]

Solution:

Concept:

- Let F, L, V denote fixed, floating and variable rate loans, respectively, and let D denote default. These loan categories are mutually exclusive and exhaustive.
- The law of total probability is $P(D) = P(F)P(D \mid F) + P(L)P(D \mid L) + P(V)P(D \mid V)$.
- Bayes' theorem gives $P(V \mid D) = \dfrac{P(V)P(D \mid V)}{P(D)}$.

Step 1: Calculate the variable-rate default probability The event of taking a variable-rate loan and

Quick Tip to Solve: Use the variable-rate contribution 0.007 as the numerator and the total default probability 0.018 as the denominator.

defaulting has probability

$$P(V \cap D) = P(V)P(D \mid V) = (0.70)(0.01) = 0.007.$$

This is a joint probability before conditioning on the fact that default has occurred.

Step 2: Find the probability of any default Considering all three mutually exclusive loan types,

$$P(D) = (0.10)(0.05) + (0.20)(0.03) + (0.70)(0.01) = 0.018.$$

This denominator includes defaults from fixed, floating and variable rate borrowers.

Step 3: Condition on the observed default Among customers who defaulted, the required share is the variable-rate default contribution divided by all defaults:

$$P(V \mid D) = \frac{0.007}{0.018} = \frac{7}{18}.$$

The given one percent rate is $P(D \mid V)$, which has the opposite conditioning and is not the answer to this question.

Step 4: Conclusion The probability that the defaulting customer took a variable-rate loan is $\boxed{\frac{7}{18}}$.

Quick Tip to Solve: Use the variable-rate contribution 0.007 as the numerator and the total default probability 0.018 as the denominator.