

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1** to **18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

This section comprises of **20 Multiple Choice Questions (MCQs)** of **1 mark** each.

1. If the feasible region of a linear programming problem with objective function $Z = ax + by$, is bounded, then which of the following is correct? [1 mark]

- (A) It will only have a maximum value.
- (B) It will only have a minimum value.
- (C) It will have both maximum and minimum values.
- (D) It will have neither maximum nor minimum value.

Correct Answer: (C) It will have both maximum and minimum values.

Solution:

Concept:

- A feasible region consists of points satisfying all the constraints; for a feasible linear programming problem this region is nonempty.
- A linear objective function $Z = ax + by$ attains both its maximum and minimum on a bounded feasible region.

Step 1: Apply the bounded-region theorem The feasible region of a linear programming problem is the intersection of its closed constraint half-planes. When this nonempty region is bounded, it is closed and bounded. Since $Z = ax + by$ is continuous, both its greatest and least values are attained on this region.

Step 2: Compare the statements At least one corner point gives a maximum and at least one gives a minimum. If the objective is constant along an edge, more than one point may give an extremum, but existence is unchanged. Thus (A) and (B) incorrectly exclude one extremum, while (D) excludes both.

Step 3: Conclusion The objective function has both maximum and minimum values. Hence, the correct option is (C).

Quick Tip to Solve: For a bounded nonempty feasible region, evaluate the objective at every corner point to locate both extrema.

2. The unit vector perpendicular to the vectors $\hat{i} - \hat{j}$ and $\hat{i} + \hat{j}$ is

[1 mark]

- (A) $\hat{k}$
- (B) $-\hat{k} + \hat{j}$
- (C) $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$
- (D) $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$

Correct Answer: (A) $\hat{k}$

Solution:

Concept:

- A vector perpendicular to two nonparallel vectors is parallel to their cross product.
- A unit vector has magnitude 1; perpendicularity is checked by a zero dot product.

Step 1: Compute the perpendicular direction Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$. Then

$$\vec{a} \times \vec{b} = \hat{i} \times \hat{j} - \hat{j} \times \hat{i} = 2\hat{k}.$$

Its magnitude is 2, so the two unit normals are $\hat{k}$ and $-\hat{k}$. Only the former is listed.

Step 2: Check the other options Option (B) has magnitude $\sqrt{2}$, and its dot product with $\vec{a}$ is -1 , so it fails both tests. Option (C) is parallel to $\vec{a}$, with dot product $\sqrt{2}$ with $\vec{a}$. Option (D) is parallel to $\vec{b}$, with dot product $\sqrt{2}$ with $\vec{b}$. Neither is perpendicular to both vectors.

Step 3: Conclusion The listed unit vector perpendicular to both vectors is $\hat{k}$. Hence, the correct option is (A).

Quick Tip to Solve: Both given vectors lie in the xy -plane, so first test the unit vectors along the z -axis.

3. If $\int_0^1 \frac{e^x}{1+x} dx = \alpha$, then $\int_0^1 \frac{e^x}{(1+x)^2} dx$ is equal to

[1 mark]

- (A) $\alpha - 1 + \frac{e}{2}$
- (B) $\alpha + 1 - \frac{e}{2}$
- (C) $\alpha - 1 - \frac{e}{2}$
- (D) $\alpha + 1 + \frac{e}{2}$

Correct Answer: (B) $\alpha + 1 - \frac{e}{2}$

Solution:

Concept:

- The quotient rule gives $\frac{d}{dx}(u/v) = (vu' - uv')/v^2$.
- The fundamental theorem gives $\int_a^b F'(x) dx = F(b) - F(a)$.

Quick Tip to Solve: When two integrands differ by an extra denominator factor, differentiate the simpler quotient before trying integration by parts.

Step 1: Relate the two integrands Differentiate the quotient appearing in the given integral:

$$\frac{d}{dx} \left(\frac{e^x}{1+x} \right) = \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2}.$$

Thus the desired integrand is the given integrand minus this derivative.

Step 2: Integrate and use the limits Writing the required integral as I , integration from 0 to 1 gives

$$\alpha - I = \left[\frac{e^x}{1+x} \right]_0^1 = \frac{e}{2} - 1.$$

The boundary term is subtracted from α , so the lower-end contribution becomes positive and the upper-end contribution negative. Options (A), (C) and (D) each reverse at least one of these signs.

Step 3: Conclusion $I = \alpha + 1 - \frac{e}{2}$. Hence, the correct option is (B).

Quick Tip to Solve: When two integrands differ by an extra denominator factor, differentiate the simpler quotient before trying integration by parts.

4. If $\int \frac{2^{1/x}}{x^2} dx = k \cdot 2^{1/x} + C$, then k is equal to

[1 mark]

- (A) $-\frac{1}{\log 2}$
- (B) $-\log 2$
- (C) -1
- (D) $\frac{1}{2}$

Correct Answer: (A) $-\frac{1}{\log 2}$

Solution:

Concept:

- For $a > 0$, $a \neq 1$, we have $\int a^u du = a^u / \log a + C$, where $\log$ denotes natural logarithm.
- The substitution $u = 1/x$ gives $du = -dx/x^2$, on any interval avoiding $x = 0$.

Step 1: Substitute the reciprocal variable Take $u = 1/x$. The negative sign in $du = -x^{-2} dx$ is essential. Therefore

$$\int \frac{2^{1/x}}{x^2} dx = - \int 2^u du = -\frac{2^u}{\log 2} + C.$$

Replace u by $1/x$ to compare with the prescribed form.

Step 2: Check the constant by differentiation Differentiating $k 2^{1/x}$ gives

$$\frac{d}{dx} (k 2^{1/x}) = -k \log 2 \frac{2^{1/x}}{x^2}.$$

Matching the integrand requires $-k \log 2 = 1$. Option (B) gives the coefficient $(\log 2)^2$, option (C) gives $\log 2$, and option (D) gives $-\log 2/2$; none equals 1.

Quick Tip to Solve: The pair $1/x$ in an exponent and dx/x^2 outside it signals $u = 1/x$; retain the minus sign.

Step 3: Conclusion $k = -\frac{1}{\log 2}$. Hence, the correct option is (A).

Quick Tip to Solve: The pair $1/x$ in an exponent and dx/x^2 outside it signals $u = 1/x$; retain the minus sign.

5. If $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then A^{-1} is

[1 mark]

(A) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(C) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

Concept:

- The inverse of a nonsingular matrix A is the unique matrix B satisfying $AB = BA = I$.
- The inverse of a diagonal matrix with nonzero diagonal entries is obtained by reciprocating each diagonal entry.

Step 1: Invert the diagonal entries The given matrix is $A = \text{diag}(-1, 1, 1)$. Its determinant is $(-1)(1)(1) = -1 \neq 0$, so an inverse exists. The reciprocals of its entries are $-1, 1, 1$, respectively; zero off-diagonal entries remain zero.

Step 2: Verify the product and compare Squaring the matrix gives

$$A^2 = \text{diag}((-1)^2, 1^2, 1^2) = I.$$

Thus the inverse has the same diagonal signs as A . The products of A with options (A), (B) and (C) have diagonals $(1, -1, -1)$, $(-1, -1, -1)$ and $(1, -1, 1)$, respectively, so none is the identity.

Step 3: Conclusion $A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$. Hence, the correct option is (D).

Quick Tip to Solve: For a diagonal matrix whose entries are only 1 and -1 , check $A^2 = I$: it is its own inverse.

6. If $\begin{pmatrix} x+y & 3y \\ 3x & x+3 \end{pmatrix} = \begin{pmatrix} 9 & 4x+y \\ x+6 & y \end{pmatrix}$, then $(x-y)$ is equal to

[1 mark]

(A) -7

(B) -3

(C) 3

(D) 7

Correct Answer: (B) -3

Solution:

Concept:

- Two matrices are equal exactly when they have the same order and all corresponding entries are equal.
- Use simple corresponding-entry equations to find unknowns, then check them in the remaining entries.

Step 1: Compare the second-row entries The lower-left entries give $3x = x + 6$, hence $2x = 6$ and $x = 3$. The lower-right entries then give $y = x + 3 = 6$. These equations determine the unknowns directly without simultaneous elimination.

Step 2: Verify all entries and subtract For $x = 3$ and $y = 6$, the upper-left entries satisfy $x + y = 9$, and the upper-right entries satisfy

$$3y = 18 = 4x + y.$$

All four corresponding entries agree. Subtracting in the order asked gives $x - y = 3 - 6$. Option (C) reverses the subtraction, while options (A) and (D) are incompatible with these verified values.

Step 3: Conclusion $x - y = -3$. Hence, the correct option is (B).

Quick Tip to Solve: Start with $3x = x + 6$, which contains only one unknown; then use $y = x + 3$ and verify the first row.

7. Let M and N be two events such that $P(M) = 0.6$, $P(N) = 0.2$ and $P(M \cap N) = 0.5$, then $P(M'/N')$ is [1 mark]

- (A) $\frac{7}{8}$
- (B) $\frac{2}{5}$
- (C) $\frac{1}{2}$
- (D) $\frac{2}{3}$

Correct Answer: No valid option: the printed intersection probability 0.5 exceeds $P(N) = 0.2$. Formal substitution gives 7/8, option (A).

Solution:

Concept:

- Conditional probability is $P(M' | N') = P(M' \cap N')/P(N')$ when $P(N') > 0$.
- The complement-of-union identity gives $P(M' \cap N') = 1 - P(M \cup N)$, while $P(M \cap N) \leq P(N)$ must hold for any events.

Step 1: Check whether the data define events The printed values give $P(M \cap N) = 0.5$ but $P(N) = 0.2$. Since $M \cap N \subseteq N$, an intersection cannot have a larger probability than N . The inequality $0.5 > 0.2$ therefore makes the supplied data inconsistent.

Step 2: Identify the formal numerical result If one substitutes the numbers algebraically, the complement formula gives

$$\begin{aligned} P(M \cup N) &= 0.6 + 0.2 - 0.5 = 0.3, \\ 1 - P(M \cup N) &= 0.7, \quad 1 - P(N) = 0.8. \end{aligned}$$

Their ratio is $0.7/0.8 = 7/8$, which matches (A), rather than (B), (C) or (D). However, the intermediate union probability 0.3 is also smaller than $P(M) = 0.6$, confirming that this calculation cannot represent actual events.

Step 3: Conclusion As printed, the data admit no valid probability model and hence no mathematically valid option. The intended formal calculation yields 7/8, option (A).

Quick Tip to Solve: Before using a conditional-probability formula, check that the intersection does not exceed either individual event probability.

8. Which of the following is not a homogeneous function of x and y ?

[1 mark]

- (A) $y^2 - xy$
- (B) $x - 3y$
- (C) $\sin^2 \frac{y}{x} + \frac{y}{x}$
- (D) $\tan x - \sec y$

Correct Answer: (D) $\tan x - \sec y$

Solution:

Concept:

- A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$, whenever the scaled arguments are in its domain.
- A polynomial whose terms have the same total degree is homogeneous; a function of y/x alone has degree zero.

Step 1: Test the first three functions by scaling In (A), replacing x, y by tx, ty gives $t^2(y^2 - xy)$, so its degree is two. In (B), the result is $t(x - 3y)$, so its degree is one. In (C), the ratio $ty/(tx) = y/x$ is unchanged for nonzero t , so the whole expression is homogeneous of degree zero on $x \neq 0$.

Step 2: Examine the trigonometric arguments in option D Scaling produces $\tan(tx) - \sec(ty)$, which does not factor as a fixed power of t times $\tan x - \sec y$. For example, at $(0, 0)$ its value is -1 , which would force degree zero. But $f(\pi/6, 0) = 1/\sqrt{3} - 1$, whereas $f(\pi/3, 0) = \sqrt{3} - 1$, so doubling both arguments does not leave the value unchanged. Thus (A), (B) and (C) all satisfy homogeneity, while (D) fails it.

Step 3: Conclusion $\tan x - \sec y$ is not homogeneous. Hence, the correct option is (D).

Quick Tip to Solve: Scale both variables together: powers reveal the degree, while the ratio y/x stays unchanged and gives degree zero.

9. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = \sqrt{37}$, $|\vec{b}| = 3$ and $|\vec{c}| = 4$, then angle between $\vec{b}$ and $\vec{c}$ is

[1 mark]

- (A) $\frac{\pi}{6}$
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{3}$
- (D) $\frac{\pi}{2}$

Correct Answer: (C) $\frac{\pi}{3}$

Solution:

Concept:

- For vectors $\vec{b}$ and $\vec{c}$ with angle θ , their dot product is $\vec{b} \cdot \vec{c} = |\vec{b}||\vec{c}| \cos \theta$.
- Squaring a vector sum means taking its dot product with itself: $|\vec{b} + \vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c}$.

Step 1: Relate the magnitudes using the zero sum From $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we have $\vec{b} + \vec{c} = -\vec{a}$. Reversing

Quick Tip to Solve: Isolate the vector with known resultant magnitude, then square its norm; do not add the three magnitudes as scalar quantities.

a vector does not change its magnitude, so

$$|\vec{b} + \vec{c}|^2 = |\vec{a}|^2 = 37.$$

Step 2: Compute the cosine of the required angle Substitute the given magnitudes into the squared-sum identity:

$$\begin{aligned} 37 &= 3^2 + 4^2 + 2(3)(4) \cos \theta, \\ 12 &= 24 \cos \theta, \\ \cos \theta &= \frac{1}{2}. \end{aligned}$$

The angle between vectors lies in $[0, \pi]$. The cosines of options (A), (B) and (D) are $\sqrt{3}/2$, $1/\sqrt{2}$ and zero, respectively, so none satisfies the required value.

Step 3: Conclusion The angle between $\vec{b}$ and $\vec{c}$ is $\theta = \pi/3$. Hence, the correct option is (C).

Quick Tip to Solve: Isolate the vector with known resultant magnitude, then square its norm; do not add the three magnitudes as scalar quantities.

10. If $f(x) = |x| + |x - 1|$, then which of the following is correct?

[1 mark]

- (A) $f(x)$ is both continuous and differentiable, at $x = 0$ and $x = 1$.
- (B) $f(x)$ is differentiable but not continuous, at $x = 0$ and $x = 1$.
- (C) $f(x)$ is continuous but not differentiable, at $x = 0$ and $x = 1$.
- (D) $f(x)$ is neither continuous nor differentiable, at $x = 0$ and $x = 1$.

Correct Answer: (C) $f(x)$ is continuous but not differentiable, at $x = 0$ and $x = 1$.

Solution:

Concept:

- The modulus function is continuous everywhere, and a sum of continuous functions is continuous.
- Differentiability at a point requires equal left-hand and right-hand derivatives there; unequal one-sided slopes give a corner.

Step 1: Remove the modulus signs interval by interval The signs change at $x = 0$ and $x = 1$. Therefore,

$$f(x) = \begin{cases} 1 - 2x, & x < 0, \\ 1, & 0 \leq x \leq 1, \\ 2x - 1, & x > 1. \end{cases}$$

At each joining point, both neighbouring formulae approach the value one, which equals $f(0) = f(1) = 1$. Thus continuity holds at both points.

Step 2: Compare the one-sided derivatives The three open intervals have slopes -2 , zero and 2 ,

Quick Tip to Solve: For a sum of modulus terms, split at their zeros and compare adjacent slopes; continuity is already guaranteed by the modulus functions.

respectively. Hence,

$$\begin{aligned}f'_-(0) &= -2, & f'_+(0) &= 0, \\f'_-(1) &= 0, & f'_+(1) &= 2.\end{aligned}$$

The derivatives disagree at each joining point. Option (A) wrongly asserts differentiability; (B) and (D) wrongly deny continuity. Also, differentiability could never hold without continuity, as suggested by (B).

Step 3: Conclusion f is continuous but not differentiable at both $x = 0$ and $x = 1$. Hence, the correct option is (C).

Quick Tip to Solve: For a sum of modulus terms, split at their zeros and compare adjacent slopes; continuity is already guaranteed by the modulus functions.

11. A system of linear equations is represented as $AX = B$, where A is coefficient matrix, X is variable matrix and B is the constant matrix. Then the system of equations is [1 mark]

- (A) Consistent, if $|A| \neq 0$, solution is given by $X = BA^{-1}$.
- (B) Inconsistent if $|A| = 0$ and $(\text{adj } A)B = 0$.
- (C) Inconsistent if $|A| \neq 0$.
- (D) May or may not be consistent if $|A| = 0$ and $(\text{adj } A)B = 0$.

Correct Answer: (D) May or may not be consistent if $|A| = 0$ and $(\text{adj } A)B = 0$.

Solution:

Concept:

- A linear system is consistent if it has at least one solution. If $|A| \neq 0$, the inverse exists and $AX = B$ has the unique solution $X = A^{-1}B$.
- The identity $(\text{adj } A)A = |A|I$ gives $|A|X = (\text{adj } A)B$. For a singular matrix, a zero right-hand side in this identity alone does not guarantee consistency.

Step 1: Check the nonsingular alternatives When $|A| \neq 0$, multiply $AX = B$ on the left by A^{-1} :

$$A^{-1}AX = A^{-1}B \implies X = A^{-1}B.$$

The order of matrix multiplication matters. Thus (A) is wrong because it writes BA^{-1} , while (C) is wrong because a nonsingular coefficient matrix always gives a consistent system.

Step 2: Examine the singular condition If $|A| = 0$ and $(\text{adj } A)B = 0$, the derived identity reduces to $0 = 0$ and cannot decide whether the original equations agree. For example, take A to be the 2×2 zero matrix, whose adjugate is also zero. If B is the zero column, every X solves the system. If $B = (1, 0)^T$, the first equation reads $0 = 1$, so no solution exists. Both examples satisfy the stated determinant and adjugate conditions. Hence (B) is too strong.

Step 3: Conclusion Under $|A| = 0$ and $(\text{adj } A)B = 0$, the system may or may not be consistent. Hence, the correct option is (D).

Quick Tip to Solve: For $AX = B$, keep the inverse on the left of B ; when both determinant and adjugate product vanish, check the original equations.

12. The absolute maximum value of function $f(x) = x^3 - 3x + 2$ in $[0, 2]$ is:

[1 mark]

- (A) 0
- (B) 2
- (C) 4
- (D) 5

Correct Answer: (C) 4

Solution:

Concept:

- A continuous function on a closed interval attains an absolute maximum and an absolute minimum.
- For a differentiable function, compare its values at every interior stationary point and at both endpoints to find the absolute extrema.

Step 1: Find the stationary points in the interval Differentiate the polynomial:

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1).$$

Setting the derivative to zero gives $x = 1$ and $x = -1$. Only $x = 1$ belongs to the interval $[0, 2]$; the other root must be discarded. The endpoints 0 and 2 must also be tested even though their derivatives are nonzero.

Step 2: Compare the candidate function values Substitute the two endpoints and the interior stationary point:

$$f(0) = 0 - 0 + 2 = 2,$$

$$f(1) = 1 - 3 + 2 = 0,$$

$$f(2) = 8 - 6 + 2 = 4.$$

Option (A) is the minimum value, and (B) is only the left-endpoint value. Option (D) exceeds every candidate and is not attained. Checking only the stationary point would miss the absolute maximum at the right endpoint.

Step 3: Conclusion The absolute maximum value is 4, attained at $x = 2$. Hence, the correct option is (C).

Quick Tip to Solve: On a closed interval, always list the two endpoints alongside the roots of $f'(x) = 0$ before comparing function values.

13. The order and degree of differential function $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^5 = \frac{d^2y}{dx^2}$ are

[1 mark]

- (A) order 1, degree 1
- (B) order 1, degree 2
- (C) order 2, degree 1
- (D) order 2, degree 2

Correct Answer: (C) order 2, degree 1

Solution:

Concept:

- The order of a differential equation is the order of the highest derivative occurring in it.
- When the equation is polynomial in its derivatives, its degree is the exponent of the highest-order derivative, not the largest exponent of any derivative.

Step 1: Identify the highest derivative The printed equation contains dy/dx inside the brackets and d^2y/dx^2 on the right. The highest derivative is therefore the second derivative. Raising an expression involving the first derivative to a power does not increase its derivative order, so options (A) and (B) are excluded.

Step 2: Read the power of the highest-order derivative Write the equation with all terms on one side:

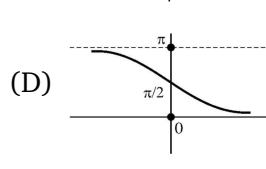
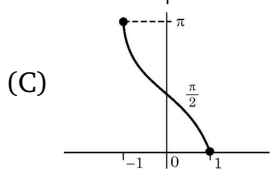
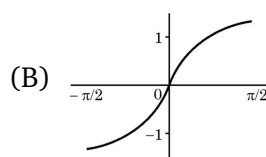
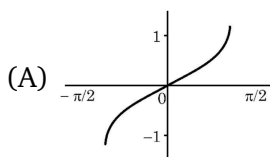
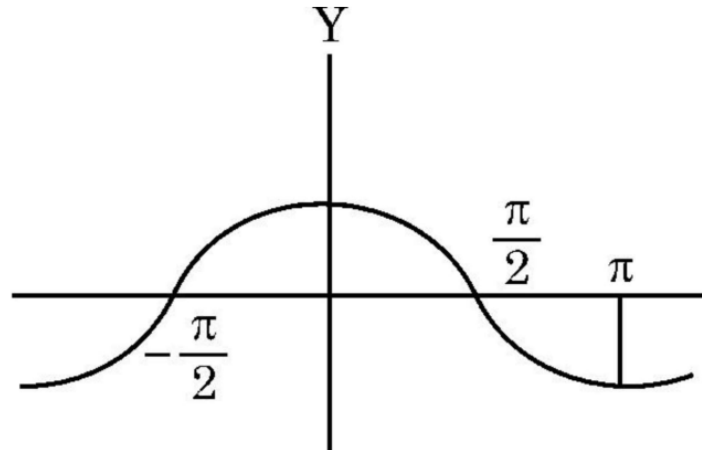
$$\frac{d^2y}{dx^2} - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^5 = 0.$$

The bracket has the positive integer exponent five, so its expansion is already a polynomial in dy/dx . The second derivative occurs to the first power only. Although expansion produces a tenth power of the first derivative, that lower-order derivative does not determine the degree. Option (D) incorrectly assigns degree two.

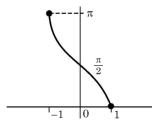
Step 3: Conclusion The differential equation has order 2 and degree 1. Hence, the correct option is (C).

Quick Tip to Solve: First identify the highest-order derivative, then read only its exponent after checking that the equation is polynomial in derivatives.

14. The graph of a trigonometric function is as shown. Which of the following will represent the graph of its inverse? [1 mark]



Correct Answer: (C)



Solution:

Concept:

- The inverse of a one-one function interchanges input and output; its graph is reflected in the line $y = x$.
- Cosine restricted to $[0, \pi]$ has range $[-1, 1]$. Hence its principal inverse $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$.

Step 1: Identify the given trigonometric graph The curve has a maximum at $x = 0$, zeros at $x = \pm\pi/2$, and a minimum at $x = \pi$. These are the features of $y = \cos x$. To obtain its principal inverse, use the decreasing branch from $x = 0$ to $x = \pi$.

Step 2: Compare the inverse graph with the options Interchanging coordinates gives the points $(1, 0)$, $(0, \pi/2)$, and $(-1, \pi)$. The inverse is decreasing and has finite endpoints at $x = \pm 1$. Graphs (A) and (B) are increasing, while (D) approaches horizontal levels and does not have these endpoints. Graph (C) has all the required features.

Step 3: Conclusion The required graph is that of $y = \cos^{-1} x$, shown in (C). Hence, the correct option is (C).

Quick Tip to Solve: For the inverse cosine graph, check the three points $(-1, \pi)$, $(0, \pi/2)$ and $(1, 0)$ before comparing curve shapes.

15. The corner points of the feasible region in graphical representation of an L.P.P. are $(2, 72)$, $(15, 20)$ and $(40, 15)$. If $Z = 18x + 9y$ is the objective function, then [1 mark]

- (A) Z is maximum at $(2, 72)$, minimum at $(15, 20)$
- (B) Z is maximum at $(15, 20)$, minimum at $(40, 15)$
- (C) Z is maximum at $(40, 15)$, minimum at $(15, 20)$
- (D) Z is maximum at $(40, 15)$, minimum at $(2, 72)$

Correct Answer: (C) Z is maximum at $(40, 15)$, minimum at $(15, 20)$

Solution:

Concept:

- For a linear objective function, an attained maximum or minimum over a bounded polygonal feasible region occurs at a corner point.
- Evaluate the same objective $Z = 18x + 9y$ at every listed corner and compare the numerical values.

Step 1: Evaluate the objective at each corner For the three vertices, substitution gives

$$\begin{aligned}Z(2, 72) &= 18(2) + 9(72) = 684, \\Z(15, 20) &= 18(15) + 9(20) = 450, \\Z(40, 15) &= 18(40) + 9(15) = 855.\end{aligned}$$

Step 2: Compare the values with the alternatives The values satisfy $450 < 684 < 855$. Thus $(15, 20)$ gives the smallest value and $(40, 15)$ gives the largest. Option (A) incorrectly selects the first vertex for the maximum; (B) reverses the extremal vertices, and (D) incorrectly selects the first vertex for the minimum.

Step 3: Conclusion The maximum is 855 at $(40, 15)$, and the minimum is 450 at $(15, 20)$. Hence, the correct option is (C).

Quick Tip to Solve: Factor $Z = 9(2x + y)$ and compare $2x + y$ at all three vertices to reduce the arithmetic.

16. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 4 & -1 \\ -3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ -5 \\ -7 \end{bmatrix}$ and $C = [9 \ 8 \ 7]$, then which of the following is defined?

[1 mark]

- (A) Only AB
- (B) Only AC
- (C) Only BA
- (D) All AB , AC and BA

Correct Answer: (A) Only AB

Solution:

Concept:

- If P has order $m \times n$ and Q has order $r \times s$, then PQ exists precisely when $n = r$.
- When the inner dimensions match, the product has order $m \times s$; reversing the order may destroy compatibility.

Step 1: Write the orders of the matrices The matrix A has three rows and three columns, B is a column with three entries, and C is a row with three entries. Thus their orders are 3×3 , 3×1 and 1×3 , respectively.

Step 2: Check all three proposed products For AB , the inner dimensions are both 3, so the product exists and has order 3×1 . For AC , the inner dimensions are 3 and 1, so it is undefined. For BA , the inner dimensions are 1 and 3, so it is also undefined. Therefore (B), (C) and (D) each include an undefined product.

Step 3: Conclusion Only AB is defined. Hence, the correct option is (A).

Quick Tip to Solve: Write the orders side by side and compare only the adjacent inner numbers; actual matrix entries need not be multiplied.

17. If A and B are invertible matrices, then which of the following is not correct?

[1 mark]

- (A) $(A + B)^{-1} = B^{-1} + A^{-1}$
- (B) $(AB)^{-1} = B^{-1}A^{-1}$
- (C) $\text{adj}(A) = |A|A^{-1}$
- (D) $|A|^{-1} = |A^{-1}|$

Correct Answer: (A) $(A + B)^{-1} = B^{-1} + A^{-1}$

Solution:

Concept:

- For invertible square matrices, $(AB)^{-1} = B^{-1}A^{-1}$: taking the inverse reverses the order of multiplication.
- The standard identities are $\text{adj}(A) = |A|A^{-1}$ and $|A^{-1}| = 1/|A|$; there is no corresponding additive inverse rule.

Quick Tip to Solve: Test a suspected inverse identity using $A = B = I$; this quickly disproves the additive rule without lengthy matrix calculations.

Step 1: Test the proposed rule for a sum Take $A = B = I$, where I is an identity matrix. Both matrices and their sum are invertible, but

$$(A + B)^{-1} = (2I)^{-1} = \frac{1}{2}I, \quad B^{-1} + A^{-1} = 2I.$$

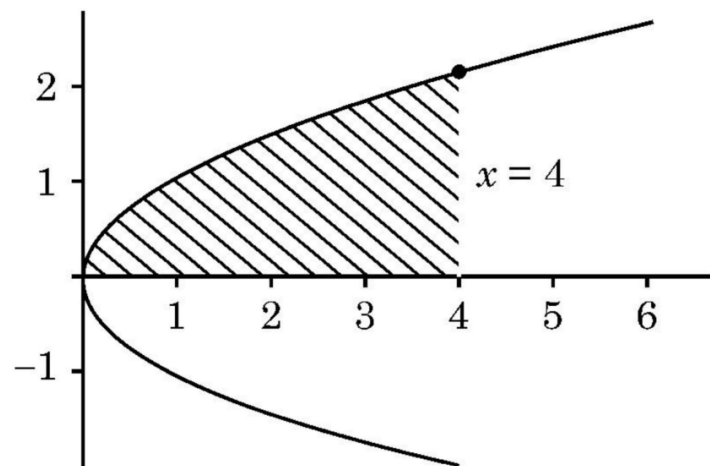
These are unequal, so the identity in (A) is false. In general, even the sum of two invertible matrices need not be invertible.

Step 2: Verify the other identities Option (B) follows because $(AB)(B^{-1}A^{-1}) = I$. From $A \operatorname{adj}(A) = |A|I$, multiplying by A^{-1} gives (C). Taking determinants in $AA^{-1} = I$ yields $|A||A^{-1}| = 1$, which proves (D). Thus all three remaining statements are valid.

Step 3: Conclusion The statement $(A + B)^{-1} = B^{-1} + A^{-1}$ is not correct. Hence, the correct option is (A).

Quick Tip to Solve: Test a suspected inverse identity using $A = B = I$; this quickly disproves the additive rule without lengthy matrix calculations.

18. The area of the shaded region bounded by the curves $y^2 = x$, $x = 4$ and the x -axis is given by: [1 mark]



(A) $\int_0^4 x \, dx$

(B) $\int_0^2 y^2 \, dy$

(C) $2 \int_0^4 \sqrt{x} \, dx$

(D) $\int_0^4 \sqrt{x} \, dx$

Correct Answer: (D) $\int_0^4 \sqrt{x} \, dx$

Solution:

Concept:

- For vertical strips, area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$ is $\int_a^b [u(x) - l(x)] dx$.
- The upper branch of $y^2 = x$ is $y = \sqrt{x}$. The shaded part lies above $y = 0$, from $x = 0$ to $x = 4$.

Step 1: Choose the height and limits of a vertical strip The parabola meets the x -axis at the origin, and the right boundary is $x = 4$. At each x in this interval, the shaded strip has height $\sqrt{x} - 0$. Its elemental area is therefore $\sqrt{x} dx$.

Step 2: Identify the correct integral Adding the strips gives $\int_0^4 \sqrt{x} dx$. Option (A) uses $y = x$ as the height. Option (B) integrates y^2 from the y -axis to the parabola, whereas a horizontal strip here has width $4 - y^2$. Option (C) doubles the desired area and includes the unshaded lower half.

Step 3: Conclusion The shaded area is represented by $\int_0^4 \sqrt{x} dx$. Hence, the correct option is (D).

Quick Tip to Solve: Check which side of the x -axis is shaded before multiplying by 2; only the upper half is required here.

19. Assertion (A): $f(x) = \begin{cases} 3x - 8, & x \leq 5, \\ 2k, & x > 5 \end{cases}$ is continuous at $x = 5$ for $k = \frac{5}{2}$.

Reason (R): For a function f to be continuous at $x = a$, $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$. [1 mark]

- (A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true, but R is not the correct explanation of A.
(C) A is true, but R is false.
(D) A is false, but R is true.

Correct Answer: (D) A is false, but R is true.

Solution:

Concept:

- Continuity at $x = a$ requires the left-hand limit, right-hand limit and function value to exist and be equal.
- For a piecewise function, compute each one-sided limit using the branch defined on that side; the equality sign determines the value at the joining point.

Step 1: Evaluate the assertion At $x = 5$, the first branch gives $f(5) = 3(5) - 8 = 7$, and $\lim_{x \rightarrow 5^-} f(x) = 7$. The right-hand limit is $2k$. For the asserted value $k = 5/2$, it equals 5, so it differs from 7. The function is not continuous, and Assertion (A) is false.

Step 2: Evaluate the reason The equation $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$ is exactly the condition for continuity at the interior point a . Thus Reason (R) is true.

Step 3: Determine the relationship and option Applying the reason would require $2k = 7$, giving $k = 7/2$, rather than the asserted $5/2$. Hence the true reason exposes the false assertion. Options (A),

Quick Tip to Solve: At a piecewise join, first evaluate the branch containing the equality sign; here $f(5) = 7$, so the other branch must also tend to 7.

(B) and (C) all require a true assertion and therefore cannot apply.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: At a piecewise join, first evaluate the branch containing the equality sign; here $f(5) = 7$, so the other branch must also tend to 7.

20. Assertion (A): Let $\mathbb{Z}$ be the set of integers. A function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined as $f(x) = 3x - 5, \forall x \in \mathbb{Z}$, is a bijective.

Reason (R): A function is a bijective if it is both surjective and injective.

[1 mark]

- (A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true, but R is not the correct explanation of A.
(C) A is true, but R is false.
(D) A is false, but R is true.

Correct Answer: (D) A is false, but R is true.

Solution:

Concept:

- A function is injective when equal outputs imply equal inputs.
- A function is surjective when every element of its codomain has a preimage; a bijection satisfies both properties.

Step 1: Evaluate the assertion If $f(m) = f(n)$, then $3m - 5 = 3n - 5$, giving $m = n$; hence the map is injective. However, $0 \in \mathbb{Z}$ has no integer preimage, since $3x - 5 = 0$ requires $x = 5/3 \notin \mathbb{Z}$. Thus the map is not surjective onto $\mathbb{Z}$ and is not bijective. Assertion (A) is false.

Step 2: Evaluate the reason Reason (R) states the defining condition for a bijective function: it must be both injective and surjective. This statement is true.

Step 3: Relate the two statements The true definition does not establish the assertion because this function fails the surjectivity requirement. Options (A), (B) and (C) each require a true assertion, so none applies.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For an integer-domain linear map, test whether an integer such as zero has an integer preimage before claiming surjectivity.

Section B (10 marks)

This section comprises of 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Find the area of a parallelogram whose diagonals are given by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$.

[2 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.
- If its diagonal vectors are $\vec{a}$ and $\vec{b}$, the area equals $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Step 1: Relate the diagonals to the area The diagonals can be represented by $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$. Their cross product is $-2(\vec{u} \times \vec{v})$, so its magnitude is twice the area. Reversing a diagonal changes only the sign of the cross product, not the required area.

Step 2: Compute the cross product Substitute the components of the given diagonals into the determinant:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ &= \hat{i}(1 - 3) - \hat{j}(-2 - 1) + \hat{k}(6 + 1) \\ &= -2\hat{i} + 3\hat{j} + 7\hat{k}.\end{aligned}$$

Therefore $|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 3^2 + 7^2} = \sqrt{62}$.

Step 3: Conclusion The area of the parallelogram is $\frac{\sqrt{62}}{2}$ square units.

Quick Tip to Solve: When the given vectors are diagonals, halve the cross-product magnitude; for adjacent side vectors, do not halve it.

22. Find the values of a for which $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- For a differentiable function to be decreasing throughout $\mathbb{R}$, its derivative must be nonpositive at every real input.
- The expression $A \cos x + B \sin x$ has maximum $\sqrt{A^2 + B^2}$; a constant term in a function disappears on differentiation.

Step 1: Differentiate the function Differentiating each term, with a and b constant, gives

$$f'(x) = \sqrt{3} \cos x + \sin x - 2a = 2 \cos \left(x - \frac{\pi}{6} \right) - 2a.$$

Thus the value of b has no effect on whether the function decreases.

Quick Tip to Solve: Combine the sine and cosine in the derivative into one sinusoid; compare its maximum with the constant being subtracted.

Step 2: Impose the condition for every real input Since the maximum value of the cosine is 1, the largest derivative is $2 - 2a$, attained when $x = \pi/6 + 2n\pi$, $n \in \mathbb{Z}$. Consequently,

$$f'(x) \leq 0 \text{ for every } x \iff 2 - 2a \leq 0.$$

At the boundary $a = 1$, the derivative is negative except at isolated points, so the function is still strictly decreasing. If $a < 1$, the derivative is positive at its maximum, which rules out decrease on all of $\mathbb{R}$.

Step 3: Conclusion The required values are $\boxed{a \geq 1}$, with no restriction on the constant b .

Quick Tip to Solve: Combine the sine and cosine in the derivative into one sinusoid; compare its maximum with the constant being subtracted.

23(a). Two friends while flying kites from different locations, find the strings of their kites crossing each other. The strings can be represented by vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$. Determine the angle formed between the kite strings, assuming there is no slack in the strings. [2 marks]

Solution:

Concept:

- For two nonzero vectors $\vec{a}$ and $\vec{b}$, their angle θ satisfies $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.
- In Cartesian form, the dot product is the sum of products of corresponding components, and the magnitude is the square root of the sum of squared components.

Step 1: Calculate the dot product and magnitudes Since the strings have no slack, their directions are represented by the given vectors. Using $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$, we obtain

$$\begin{aligned}\vec{a} \cdot \vec{b} &= 3(2) + 1(-2) + 2(4) = 12, \\ |\vec{a}| &= \sqrt{9 + 1 + 4} = \sqrt{14}, \\ |\vec{b}| &= \sqrt{4 + 4 + 16} = \sqrt{24} = 2\sqrt{6}.\end{aligned}$$

Step 2: Apply the angle formula Dividing the dot product by the product of the magnitudes gives

$$\cos \theta = \frac{12}{\sqrt{14}\sqrt{24}} = \frac{12}{4\sqrt{21}} = \frac{3}{\sqrt{21}}.$$

The dot product is positive, so the angle between these directed vectors is acute; it is also the smaller angle between the crossing strings.

Step 3: Conclusion The required angle is $\boxed{\theta = \cos^{-1} \left(\frac{3}{\sqrt{21}} \right)}$, approximately 49.1° .

Quick Tip to Solve: Compute the dot product first: its positive sign immediately tells you to choose the acute angle between the strings.

23(b). Find a vector of magnitude 21 units in the direction opposite to that of $\vec{AB}$ where $A(2, 1, 3)$ and $B(8, -1, 0)$. [2 marks]

Solution:

Concept:

- The displacement vector $\vec{AB}$ is obtained by subtracting the coordinates of A from those of B .
- For a nonzero vector $\vec{v}$, the unit vector opposite to its direction is $-\vec{v}/|\vec{v}|$. Multiply this by the required magnitude to obtain the desired vector.

Step 1: Find the displacement and its magnitude With $A = (2, 1, 3)$ and $B = (8, -1, 0)$, coordinate subtraction gives

$$\begin{aligned}\vec{AB} &= (8 - 2)\hat{i} + (-1 - 1)\hat{j} + (0 - 3)\hat{k} \\ &= 6\hat{i} - 2\hat{j} - 3\hat{k}, \\ |\vec{AB}| &= \sqrt{6^2 + (-2)^2 + (-3)^2} = 7.\end{aligned}$$

Step 2: Reverse the direction and scale the vector The required direction is from B towards A , so use a negative multiple of $\vec{AB}$. Normalising and then multiplying by 21 gives

$$\vec{r} = -\frac{21}{7}(6\hat{i} - 2\hat{j} - 3\hat{k}) = -18\hat{i} + 6\hat{j} + 9\hat{k}.$$

As a check, $|\vec{r}| = \sqrt{324 + 36 + 81} = 21$. Also, $\vec{r} = -3\vec{AB}$, verifying that its direction is opposite to the given displacement.

Step 3: Conclusion The required vector is $\boxed{-18\hat{i} + 6\hat{j} + 9\hat{k}}$, of magnitude 21 units.

Quick Tip to Solve: First calculate $B - A$; then multiply by $-21/|B - A|$ so that both the requested magnitude and opposite direction are enforced.

24. Solve for x : $2 \tan^{-1} x + \sin^{-1} \left(\frac{2x}{1+x^2} \right) = 4\sqrt{3}$. [2 marks]

Solution:

Concept:

- Principal values satisfy $-\pi/2 < \tan^{-1} x < \pi/2$ and $-\pi/2 \leq \sin^{-1} u \leq \pi/2$, for real arguments in their domains.
- Writing $t = \tan^{-1} x$ gives $2x/(1+x^2) = \sin 2t$; the principal value of $\sin^{-1}(\sin 2t)$ must be chosen in its specified range.

Step 1: Check the domain and the possible range For every real x , the inverse sine is defined because

$$1 + x^2 - 2|x| = (|x| - 1)^2 \geq 0, \quad \left| \frac{2x}{1+x^2} \right| \leq 1.$$

Quick Tip to Solve: Before manipulating inverse trigonometric equations, compare the requested constant with the principal-value ranges; here that check rules out every real value immediately.

The two inverse trigonometric terms are principal values. Therefore their sum obeys the upper bound

$$2 \tan^{-1} x + \sin^{-1} \left(\frac{2x}{1+x^2} \right) < \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

Step 2: Compare the printed right-hand side with this bound The printed constant is $4\sqrt{3}$. Numerically,

$$\frac{3\pi}{2} \approx 4.7124, \quad 4\sqrt{3} \approx 6.9282.$$

Thus the required value exceeds even the upper bound allowed by the principal-value ranges. In particular, replacing $\sin^{-1}(\sin 2t)$ by $2t$ without checking its branch cannot produce a valid solution.

Step 3: Conclusion There is no real value of x satisfying the printed equation; the real solution set is $\emptyset$.

Quick Tip to Solve: Before manipulating inverse trigonometric equations, compare the requested constant with the principal-value ranges; here that check rules out every real value immediately.

25(a). Differentiate $2^{\cos^2 x}$ w.r.t. $\cos^2 x$.

[2 marks]

Solution:

Concept:

- For a positive constant base a , the derivative of an exponential is $\frac{d}{du} a^u = a^u \ln a$, where $\ln$ denotes the natural logarithm.
- When differentiating with respect to an expression, regard that expression as the independent variable. Here the differentiation variable is $u = \cos^2 x$.

Step 1: Rewrite the function using the requested variable Let $u = \cos^2 x$ and $y = 2^{\cos^2 x}$. The function to be differentiated is then simply

$$y = 2^u = e^{u \ln 2}.$$

The question asks for dy/du , rather than dy/dx . Thus no factor arising from differentiating $\cos^2 x$ is needed in the requested derivative.

Step 2: Differentiate the exponential Since $\ln 2$ is constant, the chain rule applied to $e^{u \ln 2}$ gives

$$\frac{dy}{du} = e^{u \ln 2} \frac{d}{du} (u \ln 2) = e^{u \ln 2} \ln 2 = 2^u \ln 2.$$

This computation directly treats the given expression as a function of u , so it avoids division by du/dx , which may vanish at some values of x .

Step 3: Conclusion Substituting $u = \cos^2 x$, the required derivative is

$$\boxed{\frac{d(2^{\cos^2 x})}{d(\cos^2 x)} = 2^{\cos^2 x} \ln 2}.$$

Quick Tip to Solve: Set the expression after “with respect to” equal to a new variable first; for this question the function becomes just 2^u .

25(b). If $\tan^{-1}(x^2 + y^2) = a^2$, then find $\frac{dy}{dx}$.

[2 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x ; in particular, $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.
- The chain rule for inverse tangent is $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1 + u^2}$. The parameter a is constant.

Step 1: Differentiate both sides with respect to x Starting with $\tan^{-1}(x^2 + y^2) = a^2$, differentiate the left-hand side as a composite function. Since the derivative of the constant a^2 is zero, we obtain

$$\frac{1}{1 + (x^2 + y^2)^2} \left(2x + 2y \frac{dy}{dx} \right) = 0.$$

The term $2y \frac{dy}{dx}$ is essential because y depends on x .

Step 2: Isolate the required derivative The denominator $1 + (x^2 + y^2)^2$ is strictly positive for real x, y . Multiplying by it and simplifying gives

$$2x + 2y \frac{dy}{dx} = 0, \quad y \frac{dy}{dx} = -x.$$

We may now divide by y wherever $y \neq 0$. The expression for the slope is valid at such points of the implicitly defined curve. At $y = 0$, this division is not valid, so the formula does not assert a finite derivative there.

Step 3: Conclusion The required derivative is $\boxed{\frac{dy}{dx} = -\frac{x}{y}}$, for $y \neq 0$.

Quick Tip to Solve: When the inverse tangent of an expression is constant, its positive derivative denominator can be cleared; focus on differentiating the inner expression correctly.

Section C (18 marks)

This section comprises of 6 Short Answer (SA) type questions of 3 marks each.

26. Solve the following linear programming problem graphically:

Maximize $Z = 8x + 9y$

Subject to the constraints:

$$2x + 3y \leq 6$$

$$3x - 2y \leq 6$$

$$y \leq 1$$

$$x \geq 0, y \geq 0$$

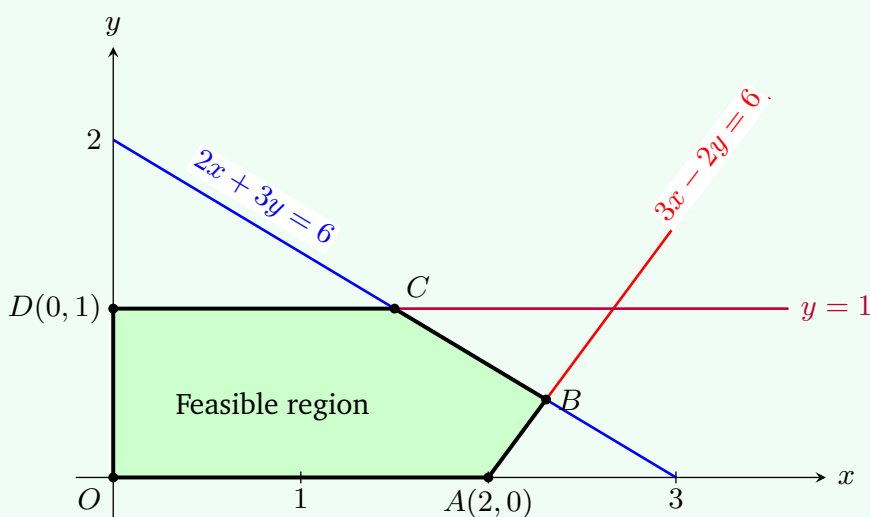
[3 marks]

Solution:

Concept:

- Each linear inequality describes a closed half-plane. Their intersection with $x \geq 0, y \geq 0$ is the feasible region.
- A linear objective function $Z = 8x + 9y$ attains its maximum at a corner point of a nonempty bounded polygonal feasible region.

Step 1: Draw the boundary lines and shade the common region Plot $2x + 3y = 6$ through $(3, 0)$ and $(0, 2)$. The origin satisfies $2x + 3y \leq 6$, so retain the side containing it. Plot $3x - 2y = 6$ through $(2, 0)$ and $(3, 3/2)$; again retain the side containing the origin. Finally draw $y = 1$ and retain the region below it in the first quadrant. The shaded common region in the graph is the bounded polygon $OABCD$, including its boundary. Points outside any one of these half-planes are excluded even if they satisfy the other inequalities.



Shaded feasible polygon $OABCD$ in the first quadrant.

Step 2: Find the corner points On the axes the feasible vertices are $O = (0, 0)$, $A = (2, 0)$ and

Quick Tip to Solve: Check every intersection against all the inequalities before including it in the corner-point table; the intersection $(8/3, 1)$ is infeasible here.

$D = (0, 1)$. At the intersection B of the two oblique lines, solve

$$\begin{aligned} 2x + 3y &= 6, & 3x - 2y &= 6, \\ 4x + 6y &= 12, & 9x - 6y &= 18. \end{aligned}$$

Adding gives $13x = 30$; substitution then gives $y = 6/13$. Thus $B = (30/13, 6/13)$, which obeys $y \leq 1$. The intersection of $y = 1$ and $2x + 3y = 6$ gives $C = (3/2, 1)$. The point $(8/3, 1)$, where the other oblique line meets $y = 1$, violates $2x + 3y \leq 6$ and is not a feasible vertex.

Step 3: Evaluate the objective at each feasible vertex Substitute the five vertices in $Z = 8x + 9y$:

Vertex (x, y)	$Z = 8x + 9y$
$O (0, 0)$	0
$A (2, 0)$	16
$B (30/13, 6/13)$	$240/13 + 54/13 = 294/13$
$C (3/2, 1)$	$12 + 9 = 21$
$D (0, 1)$	9

The largest entry is $294/13$, approximately 22.615, and it occurs at B . Since the adjacent vertices have strictly smaller objective values, the maximum is attained at this single vertex.

Step 4: Conclusion The maximum value is $Z_{\max} = \frac{294}{13}$, attained at $x = \frac{30}{13}, y = \frac{6}{13}$.

Quick Tip to Solve: Check every intersection against all the inequalities before including it in the corner-point table; the intersection $(8/3, 1)$ is infeasible here.

27(a). Find: $\int \frac{2x - 1}{(x - 1)(x + 2)(x - 3)} dx$

[3 marks]

Solution:

Concept:

- A proper rational function with distinct linear factors in its denominator can be expressed as a sum of partial fractions, one for each factor.
- The logarithmic integral is $\int \frac{1}{x - a} dx = \ln |x - a| + C$. Absolute values keep the expression valid wherever the denominator is nonzero.

Step 1: Set up the partial fractions The numerator has lower degree than the denominator, so no polynomial division is needed. Write

$$\frac{2x - 1}{(x - 1)(x + 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C_1}{x - 3}.$$

Multiplying by the product of the three factors gives the polynomial identity

$$2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C_1(x - 1)(x + 2).$$

Quick Tip to Solve: After clearing the denominator, substitute $x = 1, -2, 3$ to obtain the coefficients directly instead of solving simultaneous equations.

Step 2: Determine the three coefficients Substitute each zero of the denominator into this polynomial identity. Each substitution removes two terms and isolates one coefficient:

$$\begin{aligned}x = 1 : \quad 1 &= -6A, \quad A = -\frac{1}{6}, \\x = -2 : \quad -5 &= 15B, \quad B = -\frac{1}{3}, \\x = 3 : \quad 5 &= 10C_1, \quad C_1 = \frac{1}{2}.\end{aligned}$$

These substitutions are used in the polynomial identity, not in the original rational function, which is undefined at those three points.

Step 3: Integrate the separated terms By linearity of integration, the required integral is

$$-\frac{1}{6} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{dx}{x+2} + \frac{1}{2} \int \frac{dx}{x-3}.$$

Apply the logarithmic formula to each term. A single arbitrary constant accounts for all three integration constants. The antiderivative is understood on any interval avoiding $-2, 1, 3$.

Step 4: Conclusion

$$-\frac{1}{6} \ln |x-1| - \frac{1}{3} \ln |x+2| + \frac{1}{2} \ln |x-3| + C.$$

Quick Tip to Solve: After clearing the denominator, substitute $x = 1, -2, 3$ to obtain the coefficients directly instead of solving simultaneous equations.

OR

27(b). Evaluate: $\int_0^5 (|x-1| + |x-2| + |x-5|) dx$

[3 marks]

Solution:

Concept:

- For a real number u , $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$. Thus each modulus changes its algebraic form at its zero.
- Split a definite integral at every interior point where an expression inside a modulus changes sign, then integrate the resulting polynomials on their respective intervals.

Step 1: Identify the sign changes The expressions inside the moduli vanish at $x = 1, 2, 5$. The points 1 and 2 are inside the integration interval, while 5 is its right endpoint. Therefore use the intervals $[0, 1]$, $[1, 2]$ and $[2, 5]$. In particular, $|x-5| = 5-x$ throughout this whole interval.

Step 2: Remove the moduli interval by interval On $[0, 1]$, all three expressions are nonpositive; on $[1, 2]$, only $x-1$ is nonnegative; and on $[2, 5]$, both $x-1$ and $x-2$ are nonnegative. Hence the

Quick Tip to Solve: Split only at 1 and 2: the zero 5 is already an endpoint, and $|x-5|$ remains $5-x$ throughout.

integrand becomes

$$|x-1| + |x-2| + |x-5| = \begin{cases} 8-3x, & 0 \leq x \leq 1, \\ 6-x, & 1 \leq x \leq 2, \\ x+2, & 2 \leq x \leq 5. \end{cases}$$

The formulas agree at the endpoints shared by adjacent intervals, as the original modulus function is continuous there.

Step 3: Evaluate and add the three integrals Denoting the required integral by I , we obtain

$$\begin{aligned} I &= \int_0^1 (8-3x) dx + \int_1^2 (6-x) dx + \int_2^5 (x+2) dx \\ &= \left[8x - \frac{3}{2}x^2 \right]_0^1 + \left[6x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} + 2x \right]_2^5 \\ &= \frac{13}{2} + \left(10 - \frac{11}{2} \right) + \left(\frac{45}{2} - 6 \right) \\ &= \frac{13}{2} + \frac{9}{2} + \frac{33}{2}. \end{aligned}$$

Each contribution is positive, consistent with the nonnegative integrand.

Step 4: Conclusion The required definite integral is $\boxed{\frac{55}{2}}$.

Quick Tip to Solve: Split only at 1 and 2: the zero 5 is already an endpoint, and $|x-5|$ remains $5-x$ throughout.

28. A spherical medicine ball when dropped in water dissolves in such a way that the rate of decrease of volume at any instant is proportional to its surface area. Calculate the rate of decrease of its radius.

[3 marks]

Solution:

Concept:

- For a sphere of radius r , its volume is $V = \frac{4}{3}\pi r^3$ and its surface area is $S = 4\pi r^2$.
- When r depends on time t , the chain rule gives $\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$. A decreasing quantity has a negative time derivative.

Step 1: Express the given proportionality Let $r(t) > 0$ be the radius of the medicine ball while it is dissolving, and let $k > 0$ be the constant of proportionality. The rate of decrease of volume is the positive quantity $-dV/dt$. The stated condition therefore gives

$$-\frac{dV}{dt} = kS = k(4\pi r^2), \quad \text{or} \quad \frac{dV}{dt} = -4\pi k r^2.$$

The negative sign is necessary because the volume becomes smaller as time increases.

Step 2: Relate volume change to radius change Differentiate the volume formula with respect to

Quick Tip to Solve: For a sphere, $dV/dr = 4\pi r^2 = S$; substitute $dV/dt = -kS$ and cancel the surface area to obtain the radius rate immediately.

time, remembering that the radius is a function of time:

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) = 4\pi r^2 \frac{dr}{dt}.$$

Equating this expression to the rate supplied by the proportionality condition gives

$$4\pi r^2 \frac{dr}{dt} = -4\pi k r^2.$$

For every instant before the ball dissolves completely, $r > 0$, so the common factor $4\pi r^2$ can be cancelled.

Step 3: Interpret the resulting rate After cancellation, the signed derivative is $dr/dt = -k$. It contains neither the instantaneous radius nor time, so the radius decreases uniformly even though the volume does not. If r_0 is the radius at $t = 0$, integration gives $r(t) = r_0 - kt$ during dissolution, which confirms the constant rate. The constant k has dimensions of length per unit time; no numerical value for it is supplied in the question.

Step 4: Conclusion The radius decreases at the constant rate $-\frac{dr}{dt} = k$, where $k > 0$ is the given proportionality constant. Equivalently, its signed rate of change is $\frac{dr}{dt} = -k$.

Quick Tip to Solve: For a sphere, $dV/dr = 4\pi r^2 = S$; substitute $dV/dt = -kS$ and cancel the surface area to obtain the radius rate immediately.

29. Sketch the graph of $y = |x + 3|$ and find the area of the region enclosed by the curve and the x -axis, between $x = -6$ and $x = 0$, using integration. [3 marks]

Solution:

Concept:

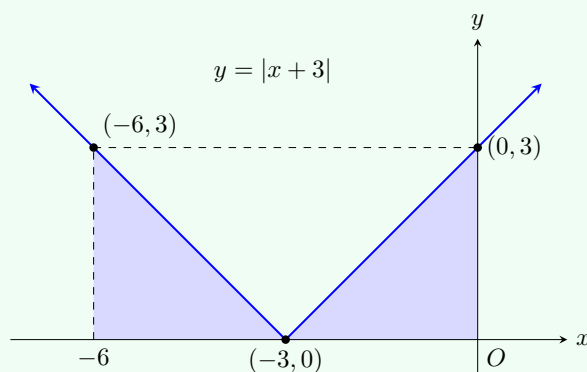
- The modulus function satisfies $|u| = -u$ for $u < 0$ and $|u| = u$ for $u \geq 0$; therefore its formula changes where its argument vanishes.
- The area between a nonnegative curve $y = f(x)$ and the x -axis on $[a, b]$ is $A = \int_a^b f(x) dx$. Split the integral wherever the defining formula changes.

Step 1: Express the modulus piecewise and sketch The expression $x + 3$ is zero at $x = -3$, negative to its left and positive to its right. Thus

$$y = |x + 3| = \begin{cases} -x - 3, & x < -3, \\ x + 3, & x \geq -3. \end{cases}$$

The graph is V-shaped with vertex $(-3, 0)$. Its left and right arms have slopes -1 and 1 , respectively. At the endpoints of the required interval, the curve passes through $(-6, 3)$ and $(0, 3)$. The shaded portions lie above the x -axis and between the vertical boundaries $x = -6$ and $x = 0$.

Quick Tip to Solve: Split at $x = -3$, where the modulus changes sign; integrating $x + 3$ unchanged from -6 to 0 would incorrectly cancel the two areas.



The shaded region under $y = |x + 3|$ between $x = -6$ and $x = 0$.

Step 2: Integrate separately on the two intervals Both parts contribute positive area, because the modulus curve never goes below the x -axis. Splitting at its vertex gives

$$\begin{aligned} A &= \int_{-6}^{-3} (-x - 3) dx + \int_{-3}^0 (x + 3) dx \\ &= \left[-\frac{x^2}{2} - 3x \right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x \right]_{-3}^0 \\ &= \left(\frac{9}{2} - 0 \right) + \left(0 + \frac{9}{2} \right). \end{aligned}$$

As a check, each shaded region is a right triangle with base 3 and height 3, consistent with the two equal integral contributions.

Step 3: Conclusion The graph has vertex $(-3, 0)$, and the required enclosed area is 9 square units.

Quick Tip to Solve: Split at $x = -3$, where the modulus changes sign; integrating $x + 3$ unchanged from -6 to 0 would incorrectly cancel the two areas.

30(a). Verify that lines given by $\vec{r} = (1 - \lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$ and $\vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$ are skew lines. Hence, find the shortest distance between the lines. [3 marks]

Solution:

Concept:

- Two lines are skew if they are nonparallel and nonintersecting. For nonparallel direction vectors $\vec{b}_1, \vec{b}_2$, a nonzero scalar triple product $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)$ proves noncoplanarity and hence skewness.
- For skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$.

Step 1: Identify points and direction vectors Separate each constant position vector from the coefficient of its parameter:

$$\begin{aligned} \vec{a}_1 &= (1, -2, 3), & \vec{b}_1 &= (-1, 1, -2), \\ \vec{a}_2 &= (1, -1, -1), & \vec{b}_2 &= (1, 2, -2). \end{aligned}$$

Quick Tip to Solve: Reuse the nonzero scalar triple product: it proves the lines are skew and supplies the numerator of the shortest-distance formula.

Thus $\vec{a}_2 - \vec{a}_1 = (0, 1, -4)$. The direction vectors are not proportional: their first components would require a multiplier -1 , whereas their second components would require 2 . Hence the lines are not parallel.

Step 2: Verify skewness using the scalar triple product A vector perpendicular to both line directions is

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= 2\hat{i} - 4\hat{j} - 3\hat{k}.\end{aligned}$$

Its scalar product with the displacement between the selected points is

$$(0, 1, -4) \cdot (2, -4, -3) = 0 - 4 + 12 = 8 \neq 0.$$

Therefore the displacement does not lie in the plane spanned by the two directions. The lines are noncoplanar, so they cannot intersect and are skew.

Step 3: Project the displacement onto the common normal The magnitude of the common normal vector is

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}.$$

The shortest separation is the absolute component of $\vec{a}_2 - \vec{a}_1$ along the unit common normal. Substituting the scalar triple product and this magnitude in the distance formula gives $d = |8|/\sqrt{29}$.

Step 4: Conclusion The two lines are skew, and their shortest distance is $\boxed{\frac{8}{\sqrt{29}} \text{ units}}.$

Quick Tip to Solve: Reuse the nonzero scalar triple product: it proves the lines are skew and supplies the numerator of the shortest-distance formula.

OR

30(b). During a cricket match, the position of the bowler, the wicket keeper and the leg slip fielder are in a line given by $\vec{B} = 2\hat{i} + 8\hat{j}$, $\vec{W} = 6\hat{i} + 12\hat{j}$ and $\vec{F} = 12\hat{i} + 18\hat{j}$ respectively. Calculate the ratio in which the wicket keeper divides the line segment joining the bowler and the leg slip fielder. [3 marks]

Solution:

Concept:

- If a point W divides BF internally in the ratio $BW : WF = m : n$, then its position vector is $\vec{W} = \frac{n\vec{B} + m\vec{F}}{m + n}.$
- For three collinear points in the order B, W, F , the positive ratio of the lengths $|\vec{BW}| : |\vec{WF}|$ gives the required internal division ratio.

Step 1: Identify the coordinates and the order of points The given position vectors represent $B = (2, 8)$, $W = (6, 12)$ and $F = (12, 18)$. All three satisfy $y - x = 6$, confirming that they lie on one

Quick Tip to Solve: Since the points are collinear, compare the consecutive x -coordinate differences: $(6 - 2) : (12 - 6) = 4 : 6$. Keep the order bowler-to-keeper, then keeper-to-fielder.

straight line. Both coordinates of W lie between the corresponding coordinates of B and F , so the wicket keeper divides the segment internally.

Step 2: Apply the section formula Let the desired ratio be $BW : WF = m : n$, with $m, n > 0$. Using the x -coordinates in the internal section formula,

$$6 = \frac{2n + 12m}{m + n},$$

$$6m + 6n = 2n + 12m,$$

$$4n = 6m, \quad \frac{m}{n} = \frac{2}{3}.$$

The y -coordinate independently gives $12(m + n) = 8n + 18m$, which reduces to the same relation $4n = 6m$. Thus both coordinates are consistent with one internal ratio.

Step 3: Check the ratio directly from the displacements The consecutive displacement vectors are

$$\overrightarrow{BW} = 4\hat{i} + 4\hat{j}, \quad \overrightarrow{WF} = 6\hat{i} + 6\hat{j}.$$

Their positive lengths are $4\sqrt{2}$ and $6\sqrt{2}$, respectively. Hence their length ratio is $4\sqrt{2} : 6\sqrt{2}$, agreeing with the section formula. Notice that the smaller segment is from the bowler to the wicket keeper; reversing the point order would reverse the requested ratio.

Step 4: Conclusion The wicket keeper divides the segment joining the bowler and the leg slip fielder internally in the ratio $\boxed{BW : WF = 2 : 3}$.

Quick Tip to Solve: Since the points are collinear, compare the consecutive x -coordinate differences: $(6 - 2) : (12 - 6) = 4 : 6$. Keep the order bowler-to-keeper, then keeper-to-fielder.

31(a). The probability distribution for the number of students being absent in a class on a Saturday is as follows:

X	0	2	4	5
$P(X)$	p	$2p$	$3p$	p

Where X is the number of students absent.

[3 marks]

31(a)(i). Calculate p .

[1 mark]

Solution:

Concept:

- A probability distribution lists all possible values of a random variable; their probabilities must add to 1.
- Every probability must lie between 0 and 1, which provides a check on the value of the unknown constant.

Step 1: Apply the total-probability condition The possible numbers of absent students are 0, 2, 4, 5. Their corresponding probabilities are $p, 2p, 3p, p$. Since these are all the possible outcomes in the

Quick Tip to Solve: Add the coefficients of p in the probability row, then take the reciprocal; do not add the values in the X row.

distribution,

$$p + 2p + 3p + p = 1.$$

Step 2: Simplify and check the probabilities Combining the coefficients gives $7p = 1$. Dividing both sides by 7 makes the probabilities $1/7, 2/7, 3/7, 1/7$; all are non-negative and their sum is $7/7 = 1$, as required.

Step 3: Conclusion The required constant is $p = \frac{1}{7}$.

Quick Tip to Solve: Add the coefficients of p in the probability row, then take the reciprocal; do not add the values in the X row.

31(a)(ii). Calculate the mean of the number of absent students on Saturday.

[2 marks]

Solution:

Concept:

- The mean or expectation of a discrete random variable is $E(X) = \sum xP(X = x)$: each possible value is weighted by its probability.
- A probability distribution satisfies $\sum P(X = x) = 1$; here this determines $p = 1/7$ before the mean is evaluated.

Step 1: Identify the probability weights The probability row gives $p + 2p + 3p + p = 1$, so $7p = 1$ and $p = 1/7$, consistent with part (i). Therefore the probabilities of 0, 2, 4, 5 absent students are respectively $1/7, 2/7, 3/7, 1/7$. The mean must use these weights because the four possible counts are not equally likely.

Step 2: Evaluate the weighted sum Multiply each count by its corresponding probability and add:

$$\begin{aligned} E(X) &= 0 \cdot p + 2 \cdot (2p) + 4 \cdot (3p) + 5 \cdot p \\ &= (0 + 4 + 12 + 5)p \\ &= 21 \left(\frac{1}{7} \right). \end{aligned}$$

The zero-absence outcome contributes zero to the mean, although its probability was included when normalising the distribution. The weighted mean lies between the smallest count 0 and the largest count 5, providing a check on the calculation.

Step 3: Conclusion The mean number of students absent on Saturday is 3 students.

Quick Tip to Solve: Pair each X value with the probability directly below it; averaging the four counts without these weights gives the wrong mean.

31(b). For the vacancy advertised in the newspaper, 3000 candidates submitted their applications. From the data it was revealed that two third of the total applicants were females and other were males. The selection for the job was done through a written test. The performance of the applicants indicates that the probability of a male getting distinction in written test is 0.4 and that of a female getting distinction is 0.35. Find the probability that the candidate chosen at random will have a distinction in the written test. [3 marks]

Solution:

Concept:

- The events F (female applicant) and M (male applicant) partition the applicant population: they are mutually exclusive and exhaustive.
- If D is the event of getting a distinction, the theorem of total probability gives $P(D) = P(F)P(D | F) + P(M)P(D | M)$.

Step 1: Identify the population proportions and conditional probabilities Two thirds of the 3000 applicants are females, so there are 2000 female applicants and 1000 male applicants. A candidate is selected at random from all 3000 applicants; hence

$$P(F) = \frac{2000}{3000} = \frac{2}{3}, \quad P(M) = \frac{1000}{3000} = \frac{1}{3}.$$

The given distinction probabilities apply within each group:

$$P(D | F) = 0.35 = \frac{7}{20}, \quad P(D | M) = 0.4 = \frac{2}{5}.$$

These are conditional probabilities, not probabilities of the joint events of belonging to a group and earning a distinction.

Step 2: Add the two mutually exclusive ways to obtain a distinction A candidate with a distinction is either a female with a distinction or a male with a distinction. Multiplying each group proportion by its conditional distinction probability and then adding gives

$$\begin{aligned} P(D) &= \frac{2}{3} \cdot \frac{7}{20} + \frac{1}{3} \cdot \frac{2}{5} \\ &= \frac{7}{30} + \frac{2}{15} = \frac{7}{30} + \frac{4}{30} = \frac{11}{30}. \end{aligned}$$

This weighted probability lies between 0.35 and 0.4, as expected. Since females form the larger group, the result is closer to their distinction probability; the unweighted average of the two given probabilities is inappropriate.

Step 3: Conclusion The probability that a randomly chosen candidate has a distinction in the written test is $\frac{11}{30}$, approximately 0.3667.

Quick Tip to Solve: Weight 0.35 by $\frac{2}{3}$ and 0.4 by $\frac{1}{3}$; the applicant total cancels when the population proportions are formed.

Section D (20 marks)

This section comprises of 4 Long Answer (LA) type questions of 5 marks each.

32. A school wants to allocate students into three clubs: Sports, Music and Drama, under the following conditions:

- The number of students in Sports club should be equal to the sum of the number of students in Music and Drama club.
- The number of students in Music club should be 20 more than half the number of students in Sports club.
- The total number of students to be allocated in all three clubs are 180.

Find the number of students allocated to different clubs, using matrix method.

[5 marks]

Solution:

Concept:

- Translate the conditions into simultaneous linear equations. If x, y, z denote the numbers of students in Sports, Music and Drama respectively, all three unknowns count students.
- A system can be written as $AX = B$, where A is the coefficient matrix, X the column of unknowns and B the constant column. When $|A| \neq 0$, its unique solution is $X = A^{-1}B$.
- The inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.

Step 1: Form the three equations The Sports condition gives $x = y + z$. The Music condition gives $y = \frac{x}{2} + 20$; multiplying by 2 clears the fraction. The total gives $x + y + z = 180$. Thus,

$$\begin{aligned}x - y - z &= 0, \\ -x + 2y &= 40, \\ x + y + z &= 180.\end{aligned}$$

The constant in the second equation is 40 because the entire equation, including the extra 20 students, was multiplied by 2.

Step 2: Express the system in matrix form Keeping the unknowns in the order Sports, Music, Drama, write

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 40 \\ 180 \end{pmatrix}, \quad AX = B.$$

Expanding the determinant along the first row,

$$|A| = 1(2) - (-1)(-1) + (-1)(-3) = 2 - 1 + 3 = 4.$$

Since the determinant is nonzero, the matrix inverse exists and the allocation is uniquely determined.

Step 3: Calculate the inverse matrix Using the signed minors, the cofactor matrix and its transpose

Quick Tip to Solve: Keep the column order Sports, Music, Drama throughout the matrix calculation, and double both terms in the Music condition before writing its row.

are

$$C = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \text{adj } A = C^T = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & 1 \end{pmatrix}.$$

For example, the cofactor of the entry in row 1, column 3 is $(-1)(1) - 2(1) = -3$. Hence,

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & 1 \end{pmatrix}.$$

Step 4: Multiply by the constant column and verify Premultiplying $AX = B$ by A^{-1} gives

$$X = A^{-1}B = \frac{1}{4} \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 40 \\ 180 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 360 \\ 260 \\ 100 \end{pmatrix} = \begin{pmatrix} 90 \\ 65 \\ 25 \end{pmatrix}.$$

All entries are nonnegative integers. They satisfy $90 = 65 + 25$, $65 = 90/2 + 20$, and $90 + 65 + 25 = 180$, so every stated condition has been checked.

Step 5: Conclusion The school should allocate 90 students to Sports, 65 students to Music and 25 students to Drama.

Quick Tip to Solve: Keep the column order Sports, Music, Drama throughout the matrix calculation, and double both terms in the Music condition before writing its row.

33. Find: $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx.$

[5 marks]

Solution:

Concept:

- For an integral containing an inverse trigonometric function, integration by parts is useful: $\int u dv = uv - \int v du$. Choose the inverse trigonometric expression as u .
- The substitution $t = \sqrt{x/(a+x)}$ gives $x = at^2/(1-t^2)$. It changes the inverse function into $\sin^{-1} t$, whose derivative is $1/\sqrt{1-t^2}$.
- The identities $t^2 = 1 - (1-t^2)$ and $\frac{d}{dt} \left(\frac{t}{\sqrt{1-t^2}} \right) = (1-t^2)^{-3/2}$ reduce the remaining integral to standard forms.

Step 1: Set up the substitution Denote the required integral by I . For $a \neq 0$, put

$$t = \sqrt{\frac{x}{a+x}}, \quad t^2(a+x) = x, \quad x = \frac{at^2}{1-t^2}.$$

On the real domain of the original inverse sine, $0 \leq t < 1$. We work on the interior of this domain so that the derivatives exist. This choice also fixes the nonnegative square-root branch.

Quick Tip to Solve: After setting $t = \sqrt{x/(a+x)}$, use parts with $dv = dx$; split t^2 as $1 - (1-t^2)$ to integrate the remainder immediately.

Step 2: Apply integration by parts Take $u = \sin^{-1} t$ and $dv = dx$. Then $v = x$, while $du = dt/\sqrt{1-t^2}$. Consequently,

$$\begin{aligned} I &= x \sin^{-1} t - \int \frac{x}{\sqrt{1-t^2}} dt \\ &= x \sin^{-1} t - a \int \frac{t^2}{(1-t^2)^{3/2}} dt. \end{aligned}$$

The substitution for x is made only in the remaining integral; the first term can be retained for convenient back-substitution.

Step 3: Integrate the remaining expression Replacing t^2 by $1 - (1 - t^2)$ separates the fraction:

$$\begin{aligned} \int \frac{t^2}{(1-t^2)^{3/2}} dt &= \int \left[\frac{1}{(1-t^2)^{3/2}} - \frac{1}{\sqrt{1-t^2}} \right] dt \\ &= \frac{t}{\sqrt{1-t^2}} - \sin^{-1} t. \end{aligned}$$

Therefore the integration-by-parts expression becomes

$$I = (x + a) \sin^{-1} t - \frac{at}{\sqrt{1-t^2}} + C.$$

Step 4: Restore the original variable and check the branch From the substitution,

$$\frac{t^2}{1-t^2} = \frac{x}{a}, \quad \frac{t}{\sqrt{1-t^2}} = \sqrt{\frac{x}{a}}.$$

For a positive parameter a , the second term is $a\sqrt{x/a} = \sqrt{ax}$, giving the familiar form. If $a < 0$, retaining $a\sqrt{x/a}$ preserves the correct sign on its real domain $x < 0$. The printed question does not restrict the sign of a . For $a = 0$, the integrand is simply $\sin^{-1} 1 = \pi/2$ wherever $x \neq 0$, and its integral is $\pi x/2 + C$.

Step 5: Conclusion For $a > 0$, the required integral is

$$(x + a) \sin^{-1} \sqrt{\frac{x}{a+x}} - \sqrt{ax} + C.$$

More generally, for $a \neq 0$ on the real domain, it is

$$(x + a) \sin^{-1} \sqrt{\frac{x}{a+x}} - a\sqrt{\frac{x}{a}} + C.$$

For $a = 0$, it is $\pi x/2 + C$ on each interval excluding zero.

Quick Tip to Solve: After setting $t = \sqrt{x/(a+x)}$, use parts with $dv = dx$; split t^2 as $1 - (1 - t^2)$ to integrate the remainder immediately.

34(a). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[5 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x , while a is a constant. Thus $\frac{d}{dx}\sqrt{1-y^2} = -\frac{y}{\sqrt{1-y^2}}\frac{dy}{dx}$.
- The difference of squares identity $(u-v)(u+v) = u^2 - v^2$ can be used with the original relation to simplify the derivative.
- Work where $|x| < 1$ and $|y| < 1$, so both radicals are positive.

Step 1: Introduce notation and differentiate the relation Put $u = \sqrt{1-x^2}$, $v = \sqrt{1-y^2}$, and $p = dy/dx$. The given equation is $u + v = a(x-y)$. Differentiating both sides with respect to x , using the chain rule on the second square root, gives

$$-\frac{x}{u} - \frac{y}{v}p = a(1-p).$$

Collect the terms containing p on one side and divide to isolate the derivative. Clearing the small fractions separates the radical ratio from the factor still to be simplified:

$$\left(a - \frac{y}{v}\right)p = a + \frac{x}{u} \implies p = \frac{v}{u} \frac{au + x}{av - y}.$$

Step 2: Obtain an identity from the original equation The two radicals have squares whose difference factors conveniently:

$$u^2 - v^2 = (1-x^2) - (1-y^2) = -(x-y)(x+y).$$

On the other hand, replacing $u + v$ by its given value yields

$$(u-v)(u+v) = a(x-y)(u-v).$$

Since $u + v > 0$, the original equation ensures that $x - y \neq 0$. Equating the last two expressions and cancelling $x - y$, we obtain

$$a(u-v) = -(x+y), \quad au + x = av - y.$$

Step 3: Simplify the derivative Since $au + x = av - y$, the extra factor is unity. The common value is nonzero: otherwise $x = -au$, $y = av$ would give $u + v = -a^2(u+v)$, impossible since $u + v > 0$. Thus cancellation is valid.

Step 4: Conclusion Therefore, wherever the stated derivative is defined,

$$\boxed{\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}}.$$

Quick Tip to Solve: After implicit differentiation, use the difference of the squares of the two radicals to show that $a\sqrt{1-x^2} + x = a\sqrt{1-y^2} - y$; this cancels the extra factor immediately.

34(b). If $x = a \left(\cos \theta + \log \tan \frac{\theta}{2} \right)$ and $y = \sin \theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.

[5 marks]

Solution:**Concept:**

- For parametric equations, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, provided $dx/d\theta \neq 0$.
- The second derivative is obtained by differentiating the first derivative with respect to the parameter and dividing by $dx/d\theta$ again: $\frac{d^2y}{dx^2} = \frac{d}{d\theta} (dy/dx) / \frac{dx}{d\theta}$.
- Here a is a nonzero constant, $\log$ denotes the natural logarithm, and $2 \sin(\theta/2) \cos(\theta/2) = \sin \theta$.

Step 1: Differentiate the logarithmic term Apply the chain rule carefully, including the factor $1/2$ from the argument $\theta/2$:

$$\begin{aligned} \frac{d}{d\theta} \log \tan \frac{\theta}{2} &= \frac{1}{\tan(\theta/2)} \cdot \frac{1}{2} \sec^2 \frac{\theta}{2} \\ &= \frac{1}{2 \sin(\theta/2) \cos(\theta/2)} = \frac{1}{\sin \theta}. \end{aligned}$$

Step 2: Find both parameter derivatives and their ratio Differentiate the given equations with respect to θ , keeping a constant:

$$\begin{aligned} \frac{dx}{d\theta} &= a \left(-\sin \theta + \frac{1}{\sin \theta} \right) \\ &= \frac{a \cos^2 \theta}{\sin \theta}, \\ \frac{dy}{d\theta} &= \cos \theta. \end{aligned}$$

Thus, in a neighbourhood of $\theta = \pi/4$, the first derivative is

$$\frac{dy}{dx} = \frac{\cos \theta}{a \cos^2 \theta / \sin \theta} = \frac{\tan \theta}{a}.$$

Since $dx/d\theta = a/\sqrt{2}$ at the specified parameter value, division is valid for $a \neq 0$.

Step 3: Differentiate the slope with respect to the parameter Differentiating $\tan \theta/a$ with respect to θ gives $\sec^2 \theta/a$. To convert this rate of change into differentiation with respect to x , divide once more by $dx/d\theta$:

$$\frac{d^2y}{dx^2} = \frac{\sec^2 \theta/a}{a \cos^2 \theta / \sin \theta} = \frac{\sin \theta}{a^2 \cos^4 \theta}.$$

At $\theta = \pi/4$, use $\sin \theta = \cos \theta = 1/\sqrt{2}$ and hence $\cos^4 \theta = 1/4$.

Step 4: Conclusion The required value is

$$\left. \frac{d^2y}{dx^2} \right|_{\theta=\pi/4} = \frac{4}{a^2 \sqrt{2}} = \frac{2\sqrt{2}}{a^2}.$$

Quick Tip to Solve: Simplify dy/dx to $\tan \theta/a$ before differentiating again. Then divide $\sec^2 \theta/a$ by $dx/d\theta$, and only afterwards substitute $\theta = \pi/4$.

35(a). Find the image A' of the point $A(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, find the equation of the line joining A and A' . [5 marks]

Solution:

Concept:

- The image of a point A in a line is the point A' for which the given line passes through the midpoint H of AA' , and AH is perpendicular to that line.
- For perpendicular vectors, their scalar product is zero: $\vec{AH} \cdot \vec{d} = 0$, where $\vec{d}$ is a direction vector of the given line.
- The midpoint relation gives $A' = 2H - A$. A line through position vector $\vec{a}$ with direction vector $\vec{v}$ is $\vec{r} = \vec{a} + s\vec{v}$, where $s \in \mathbb{R}$.

Step 1: Express the foot of the perpendicular using a parameter Let H be the foot of the perpendicular from $A(1, 6, 3)$ to the given line. Equating its symmetric ratios to t , we have

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = t.$$

Thus $H = (t, 1 + 2t, 2 + 3t)$, and the given line has direction vector $\vec{d} = (1, 2, 3)$. The displacement from A to H is

$$\vec{AH} = (t - 1, 2t - 5, 3t - 1).$$

Step 2: Use the perpendicularity condition Since AH is perpendicular to the given line, the dot product of their direction vectors vanishes. Therefore

$$\begin{aligned}(t - 1) + 2(2t - 5) + 3(3t - 1) &= 0, \\ t - 1 + 4t - 10 + 9t - 3 &= 0, \\ 14t - 14 &= 0.\end{aligned}$$

Hence $t = 1$, giving $H = (1, 3, 5)$. The vector $\vec{AH} = (0, -3, 2)$ has dot product $0 - 6 + 6 = 0$ with $(1, 2, 3)$, confirming perpendicularity.

Step 3: Apply the midpoint relation for the image Write $A' = (x', y', z')$. As H is the midpoint of AA' ,

$$\left(\frac{1 + x'}{2}, \frac{6 + y'}{2}, \frac{3 + z'}{2} \right) = (1, 3, 5).$$

Solving coordinate by coordinate gives $x' = 1$, $y' = 0$, and $z' = 7$. The displacements $\vec{AH} = (0, -3, 2)$ and $\vec{HA'} = (0, -3, 2)$ are equal, so the midpoint and equal-distance requirements both hold.

Step 4: Form the line through the point and its image The joining line has direction vector $\vec{AA'} = (0, -6, 4)$, which can be reduced to $(0, -3, 2)$. Using A as the point on this line, its parametric equations are

$$x = 1, \quad y = 6 - 3s, \quad z = 3 + 2s, \quad s \in \mathbb{R}.$$

The original point occurs at $s = 0$, the foot at $s = 1$, and the image at $s = 2$. Since the first direction ratio is zero, write $x = 1$ separately rather than dividing by zero in a symmetric equation.

Quick Tip to Solve: First find the perpendicular foot with a dot product, then double its coordinates and subtract the original point. Keep a constant coordinate separate when its direction ratio is zero.

Step 5: Conclusion The image is $A' = (1, 0, 7)$. The line joining A and A' is

$$x = 1, \quad \frac{y - 6}{-3} = \frac{z - 3}{2}.$$

Quick Tip to Solve: First find the perpendicular foot with a dot product, then double its coordinates and subtract the original point. Keep a constant coordinate separate when its direction ratio is zero.

OR

35(b). Find a point P on the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ such that its distance from point $Q(2, 4, -1)$ is 7 units. Also, find the equation of the line joining P and Q . [5 marks]

Solution:

Concept:

- A point on a line in symmetric form is obtained by equating all three ratios to a real parameter t , then expressing each coordinate in terms of t .
- The distance between $P(x, y, z)$ and $Q(a, b, c)$ satisfies $PQ^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$. A prescribed distance determines the allowed parameter values.
- The vector joining two distinct points provides the direction ratios of their joining line. Any nonzero common factor in these ratios may be cancelled.

Step 1: Parameterise the given line Equate the ratios to t :

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = t.$$

Every point on this line consequently has coordinates

$$P = (t - 5, 4t - 3, 6 - 9t).$$

In particular, the negative denominator in the third ratio gives $z = 6 - 9t$. We must choose t so that this point is exactly 7 units from $Q(2, 4, -1)$.

Step 2: Impose the distance condition and expand The coordinate differences $P - Q$ are $t - 7$, $4t - 7$, and $7 - 9t$. Substituting them into the squared distance formula gives

$$(t - 7)^2 + (4t - 7)^2 + (7 - 9t)^2 = 49.$$

Expanding each square,

$$\begin{aligned}(t - 7)^2 &= t^2 - 14t + 49, \\(4t - 7)^2 &= 16t^2 - 56t + 49, \\(7 - 9t)^2 &= 81t^2 - 126t + 49.\end{aligned}$$

Quick Tip to Solve: Use one parameter for all three coordinates before applying the distance formula. Here the resulting perfect square gives a repeated root, so there is only one admissible point.

Adding these expressions and subtracting the right-hand side yields

$$98t^2 - 196t + 98 = 0.$$

Step 3: Determine and verify the point Dividing the quadratic by 98 and factorising,

$$t^2 - 2t + 1 = (t - 1)^2 = 0.$$

The only parameter value is $t = 1$. Substitution in the parametric coordinates gives $P = (-4, 1, -3)$. To check membership of the original line, its three ratios at this point are

$$\frac{-4 + 5}{1} = \frac{1 + 3}{4} = \frac{-3 - 6}{-9} = 1.$$

Also, the required distance is verified directly:

$$PQ = \sqrt{(-4 - 2)^2 + (1 - 4)^2 + (-3 + 1)^2} = \sqrt{36 + 9 + 4} = 7.$$

The repeated root represents one point, rather than two distinct possible points.

Step 4: Form the line joining the two points From $P = (-4, 1, -3)$ to $Q = (2, 4, -1)$, the direction vector is

$$\overrightarrow{PQ} = (2 + 4, 4 - 1, -1 + 3) = (6, 3, 2).$$

A line through Q in this direction has parametric equations

$$x = 2 + 6s, \quad y = 4 + 3s, \quad z = -1 + 2s.$$

It contains Q at $s = 0$ and P at $s = -1$. Eliminating s gives its symmetric form. As an additional check, $(6, 3, 2) \cdot (1, 4, -9) = 6 + 12 - 18 = 0$, consistent with the unique nearest point on the given line.

Step 5: Conclusion The required point is $P = (-4, 1, -3)$, and the line joining P and Q is

$$\frac{x - 2}{6} = \frac{y - 4}{3} = \frac{z + 1}{2}.$$

Quick Tip to Solve: Use one parameter for all three coordinates before applying the distance formula. Here the resulting perfect square gives a repeated root, so there is only one admissible point.

Section E (12 marks)

This section comprises of 3 case study/passage based questions of 4 marks each.

36. A class-room teacher is keen to assess the learning of her students on the concept of “relations” taught to them. She writes the following five relations each defined on the set $A = \{1, 2, 3\}$:

$$R_1 = \{(2, 3), (3, 2)\}$$

$$R_2 = \{(1, 2), (1, 3), (3, 2)\}$$

$$R_3 = \{(1, 2), (2, 1), (1, 1)\}$$

$$R_4 = \{(1, 1), (1, 2), (3, 3), (2, 2)\}$$

$$R_5 = \{(1, 1), (1, 2), (3, 3), (2, 2), (2, 1), (2, 3), (3, 2)\}$$

The students are asked to answer the following questions about the above relations:

[4 marks]

36(i). Identify the relation which is reflexive, transitive but not symmetric.

[1 mark]

Solution:

Concept:

- A relation on A is reflexive if every pair (a, a) , with $a \in A$, belongs to it.
- Symmetry requires (b, a) whenever (a, b) occurs; transitivity requires (a, c) whenever (a, b) and (b, c) occur.

Step 1: Check reflexivity and symmetry In R_4 , all three diagonal pairs $(1, 1)$, $(2, 2)$, $(3, 3)$ occur, so it is reflexive on $A = \{1, 2, 3\}$. However, $(1, 2) \in R_4$ whereas $(2, 1) \notin R_4$, so it is not symmetric.

Step 2: Check all possible compositions The only non-diagonal pair of R_4 is $(1, 2)$. Composing it with $(1, 1)$ on the left or $(2, 2)$ on the right gives $(1, 2)$, which is already present. Composing diagonal pairs gives those same diagonal pairs. These exhaust the possible chains, so R_4 is transitive.

Step 3: Conclusion The required relation is R_4 .

Quick Tip to Solve: First look for all three diagonal pairs; then use the missing reverse of $(1, 2)$ to reject symmetry.

36(ii). Identify the relation which is reflexive and symmetric but not transitive.

[1 mark]

Solution:

Concept:

- Reflexivity on $A = \{1, 2, 3\}$ requires $(1, 1)$, $(2, 2)$, $(3, 3)$; symmetry requires every pair to have its reverse.
- To disprove transitivity, find (a, b) and (b, c) in the relation but (a, c) absent.

Step 1: Verify reflexivity and symmetry In R_5 , all three diagonal pairs occur. The remaining pairs are $(1, 2)$, $(2, 1)$, $(2, 3)$, $(3, 2)$, arranged in reverse pairs. Thus R_5 is both reflexive and symmetric.

Step 2: Find a chain that fails transitivity Although $(1, 2) \in R_5$ and $(2, 3) \in R_5$, the required pair $(1, 3)$ does not belong to R_5 . This single counterexample disproves transitivity; checking further chains is unnecessary.

Quick Tip to Solve: In R_5 , test the chain $1 \rightarrow 2 \rightarrow 3$: its missing direct pair immediately disproves transitivity.

Step 3: Conclusion The required relation is R_5 .

Quick Tip to Solve: In R_5 , test the chain $1 \rightarrow 2 \rightarrow 3$: its missing direct pair immediately disproves transitivity.

36(iii)(a). Identify the relations which are symmetric but neither reflexive nor transitive. [2 marks]

Solution:

Concept:

- A symmetric relation contains (b, a) for every (a, b) in it. A diagonal pair is its own reverse.
- Reflexivity requires every diagonal pair on A . Transitivity requires (a, c) for every two-pair chain $(a, b), (b, c)$.

Step 1: Test the first relation In $R_1 = \{(2, 3), (3, 2)\}$, both pairs have their reverses, so it is symmetric. It is not reflexive because, for example, $(1, 1)$ is missing. Also $(2, 3), (3, 2) \in R_1$ would require $(2, 2) \in R_1$ for transitivity, but that pair is absent. Thus R_1 satisfies all the requested conditions.

Step 2: Test the third relation In $R_3 = \{(1, 2), (2, 1), (1, 1)\}$, the first two pairs reverse each other and the last reverses itself. Thus it is symmetric. Since $(2, 2)$ and $(3, 3)$ are absent, it is not reflexive. The chain $(2, 1), (1, 2)$ requires the missing pair $(2, 2)$, so it is not transitive either.

Step 3: Exclude the remaining relations Neither R_2 nor R_4 is symmetric: each contains $(1, 2)$ without $(2, 1)$. Although R_5 is symmetric, it is reflexive, so it does not meet the requirement of being neither reflexive nor transitive.

Step 4: Conclusion The required relations are R_1 and R_3 .

Quick Tip to Solve: For a relation containing a reverse pair $(a, b), (b, a)$, check whether the forced diagonal pairs $(a, a), (b, b)$ are present.

OR

36(iii)(b). What pairs should be added to the relation R_2 to make it an equivalence relation? [2 marks]

Solution:

Concept:

- An equivalence relation is reflexive, symmetric and transitive. Missing pairs must be added without removing any original pair.
- In an equivalence relation, elements joined by the given pairs belong to the same equivalence class, and every ordered pair within that class must occur.

Step 1: Add the diagonal and reverse pairs The given relation is $R_2 = \{(1, 2), (1, 3), (3, 2)\}$. Reflexivity on $A = \{1, 2, 3\}$ requires the missing diagonal pairs $(1, 1), (2, 2), (3, 3)$. Symmetry applied to the three original pairs requires $(2, 1), (3, 1), (2, 3)$, respectively. None of these six required pairs is already in R_2 .

Quick Tip to Solve: The pairs $(1, 2)$ and $(1, 3)$ connect all three elements, so the equivalence class is all of A ; complete all nine ordered pairs.

Step 2: Verify transitivity after the additions After these additions, the resulting relation contains every ordered pair on A ; hence it equals $A \times A$, which has $3^2 = 9$ pairs. For any $a, b, c \in A$, the pair (a, c) is present whenever (a, b) and (b, c) are present. Thus transitivity holds as well. Every proposed pair was forced by reflexivity or symmetry, so the list is also minimal.

Step 3: Conclusion Add the following six pairs:

$$\{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1), (2, 3)\}.$$

The resulting equivalence relation is $A \times A$.

Quick Tip to Solve: The pairs $(1, 2)$ and $(1, 3)$ connect all three elements, so the equivalence class is all of A ; complete all nine ordered pairs.

37. A bank offers loan to its customers on different types of interest namely, fixed rate, floating rate and variable rate. From the past data with the bank, it is known that a customer avails a loan on fixed rate, floating rate or variable rate with probabilities 10%, 20% and 70% respectively. A customer after availing a loan can pay the loan or default on loan repayment. The bank data suggests that the probability that a person defaults on loan after availing it at fixed rate, floating rate and variable rate is 5%, 3% and 1% respectively.

Based on the above information, answer the following:

[4 marks]

37(i). What is the probability that a customer after availing the loan will default on the loan repayment?

[2 marks]

Solution:

Concept:

- The fixed, floating and variable loan types form mutually exclusive and exhaustive events F, L, V , respectively.
- For the default event D , the law of total probability gives $P(D) = P(F)P(D | F) + P(L)P(D | L) + P(V)P(D | V)$.

Step 1: Express the given percentages as probabilities The loan-type probabilities are $P(F) = 0.10$, $P(L) = 0.20$ and $P(V) = 0.70$, whose sum is 1. The corresponding conditional probabilities of default are $P(D | F) = 0.05$, $P(D | L) = 0.03$ and $P(D | V) = 0.01$. These default rates apply within their respective loan categories.

Step 2: Add the three disjoint default probabilities A defaulting customer must belong to exactly one loan category. Multiplying each category's probability by its conditional default rate gives its contribution to the total:

$$\begin{aligned} P(D) &= (0.10)(0.05) + (0.20)(0.03) + (0.70)(0.01) \\ &= 0.005 + 0.006 + 0.007 \\ &= 0.018. \end{aligned}$$

The category rates must be weighted because the three types of loan are not equally likely.

Quick Tip to Solve: Multiply each default rate by its loan-type share, then add; do not take the ordinary average of 5%, 3%, 1%.

Step 3: Conclusion The probability of default is $0.018 = \frac{9}{500}$, or 1.8% .

Quick Tip to Solve: Multiply each default rate by its loan-type share, then add; do not take the ordinary average of 5%, 3%, 1%.

37(ii). A customer after availing the loan, defaults on loan repayment. What is the probability that he availed the loan at a variable rate of interest? [2 marks]

Solution:

Concept:

- Let F, L, V denote fixed, floating and variable rate loans, respectively, and let D denote default on repayment.
- Bayes' theorem gives $P(V | D) = \frac{P(V)P(D | V)}{P(D)}$; the denominator includes defaults from every loan category.

Step 1: Calculate the overall default probability The three loan categories are mutually exclusive and exhaustive. Therefore the total probability of default is

$$\begin{aligned}P(D) &= P(F)P(D | F) + P(L)P(D | L) + P(V)P(D | V) \\&= (0.10)(0.05) + (0.20)(0.03) + (0.70)(0.01) \\&= 0.005 + 0.006 + 0.007 = 0.018.\end{aligned}$$

Step 2: Restrict the sample space to defaulters The probability that a randomly chosen borrower has both a variable rate loan and a default is $P(V \cap D) = (0.70)(0.01) = 0.007$. Because the customer is known to have defaulted, divide this joint probability by the total default probability:

$$P(V | D) = \frac{P(V \cap D)}{P(D)} = \frac{0.007}{0.018} = \frac{7}{18}.$$

This reverses the given conditioning: the requested event is the loan type given default, whereas 0.01 is the default probability given a variable rate loan.

Step 3: Conclusion The required probability is $\frac{7}{18}$.

Quick Tip to Solve: Among all default contributions 0.005, 0.006, 0.007, the variable rate contribution is 0.007; divide it by their sum.

38. A technical company is designing a rectangular solar panel installation on a roof using 300 metres of boundary material. The design includes a partition running parallel to one of the sides dividing the area (roof) into two sections.

Let the length of the side perpendicular to the partition be x metres and with parallel to the partition be y metres.

Based on this information, answer the following questions:

[4 marks]

38(i). Write the equation for the total boundary material used in the boundary and parallel to the partition in terms of x and y .

[1 mark]

Solution:

Concept:

- The perimeter of a rectangle with perpendicular side lengths x and y is $2x + 2y$.
- An internal partition parallel to the side of length y also has length y . Its material must be added to the outer perimeter.

Step 1: Count the outer sides There are two opposite sides each of length x and two opposite sides each of length y . Thus the outer boundary uses $2x + 2y$ metres of material.

Step 2: Include the internal partition The partition spans the rectangle parallel to its y -sides, so it requires another y metres. The available material is used for both the boundary and this single partition:

$$(2x + 2y) + y = 300.$$

Step 3: Conclusion The required boundary constraint is $2x + 3y = 300$, with x and y measured in metres.

Quick Tip to Solve: Count three lengths parallel to the partition, but only two lengths perpendicular to it.

38(ii). Write the area of the solar panel as a function of x .

[1 mark]

Solution:

Concept:

- The area of a rectangle is the product of its side lengths: $A = xy$, in square metres.
- The boundary and one partition together impose $2x + 3y = 300$. This relation expresses one side in terms of the other.

Step 1: Express the parallel side in terms of x Two outer sides have length x , while the two other outer sides and the partition each have length y . Rearrange their material constraint:

$$3y = 300 - 2x, \quad y = \frac{300 - 2x}{3}.$$

Step 2: Substitute into the rectangle's area The partition divides the installation without changing its

Quick Tip to Solve: Eliminate y using $2x + 3y = 300$; using the perimeter alone would omit the partition.

total rectangular area. Therefore,

$$A(x) = x \left(\frac{300 - 2x}{3} \right) = 100x - \frac{2}{3}x^2.$$

Both side lengths must be positive; hence the physical domain is $0 < x < 150$.

Step 3: Conclusion The area function is $A(x) = 100x - \frac{2}{3}x^2$ square metres, for $0 < x < 150$.

Quick Tip to Solve: Eliminate y using $2x + 3y = 300$; using the perimeter alone would omit the partition.

38(iii)(a). Find the critical points of the area function. Use second derivative test to determine critical points at the maximum area. Also, find the maximum area. [2 marks]

Solution:

Concept:

- An interior critical point of a differentiable area function occurs where $A'(x) = 0$, with x in its physical domain.
- The second derivative test gives a strict local maximum at a critical point if $A''(x) < 0$. A negative second derivative throughout an interval makes the function strictly concave there.

Step 1: Form and differentiate the area function The boundary constraint, including the parallel partition, is $2x + 3y = 300$. Thus $y = (300 - 2x)/3$, and

$$A(x) = xy = 100x - \frac{2}{3}x^2, \quad 0 < x < 150.$$

Differentiating with respect to x gives

$$A'(x) = 100 - \frac{4x}{3}.$$

Step 2: Locate and classify the critical point Solving $A'(x) = 0$ gives $x = 75$ metres, which lies inside the physical domain. The derivative exists everywhere, so there are no other critical points. Also,

$$A''(x) = -\frac{4}{3} < 0.$$

Hence this point is a strict maximum. Since the area function is strictly concave on the entire domain, this is the unique absolute maximum as well.

Step 3: Evaluate the dimensions and area At the critical point,

$$y = \frac{300 - 2(75)}{3} = 50 \text{ m}, \quad A(75) = 75 \times 50 = 3750 \text{ m}^2.$$

These dimensions use $2(75) + 3(50) = 300$ metres of material, as required.

Step 4: Conclusion The unique critical point is $x = 75$ m, where $y = 50$ m. The maximum area is

Quick Tip to Solve: For this quadratic, $A'' = -4/3$ is negative everywhere, so its only stationary point gives the maximum area.

$$3750 \text{ m}^2.$$

Quick Tip to Solve: For this quadratic, $A'' = -4/3$ is negative everywhere, so its only stationary point gives the maximum area.

OR

38(iii)(b). Using first derivative test, calculate the maximum area the company can enclose with the 300 metres of boundary material, considering the parallel partition. [2 marks]

Solution:

Concept:

- The area of the rectangular installation is $A = xy$. Counting the partition as well as the outer boundary gives $2x + 3y = 300$.
- By the first derivative test, a stationary point is a maximum when the derivative changes sign from positive on its left to negative on its right.

Step 1: Express the area in terms of x The side parallel to the partition is $y = (300 - 2x)/3$, so

$$A(x) = x \left(\frac{300 - 2x}{3} \right) = 100x - \frac{2}{3}x^2.$$

The physical domain is $0 < x < 150$, because both dimensions must be positive.

Step 2: Find the stationary point Differentiate and set the derivative equal to zero:

$$A'(x) = 100 - \frac{4x}{3} = \frac{4}{3}(75 - x), \quad A'(x) = 0 \implies x = 75.$$

This is the only stationary point in the allowed interval.

Step 3: Apply the first derivative test For $0 < x < 75$, we have $75 - x > 0$, so $A'(x) > 0$ and the area increases. For $75 < x < 150$, we have $75 - x < 0$, so $A'(x) < 0$ and the area decreases. Thus the area increases up to $x = 75$ and decreases afterwards, proving that it has its absolute maximum there.

Step 4: Calculate the maximum area Substitute the maximizing value into the boundary constraint:

$$y = \frac{300 - 2(75)}{3} = 50 \text{ m}.$$

The corresponding enclosed area is

$$A(75) = 75 \times 50 = 3750 \text{ m}^2.$$

Step 5: Conclusion By the first derivative test, the maximum enclosed area is 3750 m^2 , attained for $x = 75 \text{ m}$ and $y = 50 \text{ m}$.

Quick Tip to Solve: Factor the derivative as $A'(x) = \frac{4}{3}(75 - x)$; its sign on either side of 75 is then immediate.