

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-2-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In **Section A** – Question Number **1 to 18** are Multiple Choice Questions (MCQs) type and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B** – Question Number **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C** – Question Number **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D** – Question Number **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In **Section E** – Question Number **36 to 38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

(This section comprises of 20 multiple choice questions (MCQs) of 1 mark each.)

1. The projection vector of vector $\vec{a}$ on vector $\vec{b}$ is

[1 mark]

(A) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$

(B) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

(C) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

(D) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{b}$

Correct Answer: (A) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $(\vec{a} \cdot \vec{b})/|\vec{b}|$.
- The unit vector along $\vec{b}$ is $\hat{b} = \vec{b}/|\vec{b}|$; multiplying the scalar projection by this unit vector gives the vector projection.

Step 1: Find the projection vector Let θ be the angle between the vectors. The component of $\vec{a}$ along $\vec{b}$ has signed magnitude $|\vec{a}| \cos \theta = (\vec{a} \cdot \vec{b})/|\vec{b}|$. Thus its vector form is

$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}.$$

Step 2: Check the alternatives Option (A) has the correct vector direction and coefficient. Option (B) is only the scalar projection, so it cannot represent the requested vector. Option (C) is also a scalar and instead gives the scalar projection of $\vec{b}$ on $\vec{a}$. Option (D) uses $|\vec{a}|^2$ in the denominator, which does not give the component along $\vec{b}$ in general.

Step 3: Conclusion The projection vector is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$. Hence, the correct option is (A).

Quick Tip to Solve: For projection on $\vec{b}$, use $\vec{b}$ as the direction and $|\vec{b}|^2$ in the denominator.

2. The function $f(x) = x^2 - 4x + 6$ is increasing in the interval

[1 mark]

- (A) $(0, 2)$
- (B) $(-\infty, 2]$
- (C) $[1, 2]$
- (D) $[2, \infty)$

Correct Answer: (D) $[2, \infty)$

Solution:

Concept:

- A differentiable function increases on an interval when its derivative is positive throughout the interior.
- A stationary endpoint, where the derivative is zero, can belong to an interval on which the function is strictly increasing.

Step 1: Determine the sign of the derivative For $f(x) = x^2 - 4x + 6$, differentiate term by term:

$$f'(x) = 2x - 4 = 2(x - 2).$$

The derivative is negative for $x < 2$, zero at $x = 2$, and positive for $x > 2$. Therefore the function decreases to its minimum at $x = 2$ and increases thereafter.

Step 2: Check the endpoint and options Write $f(x) = (x - 2)^2 + 2$. For any $2 \leq x_1 < x_2$, we have $0 \leq x_1 - 2 < x_2 - 2$, so $(x_1 - 2)^2 < (x_2 - 2)^2$. Hence $f(x_1) < f(x_2)$, including when $x_1 = 2$. This proves strict increase on $[2, \infty)$. Options (A), (B), and (C) lie to the left of the minimum, where the function decreases.

Step 3: Conclusion The function is increasing on $[2, \infty)$. Hence, the correct option is (D).

Quick Tip to Solve: Factor the derivative as $2(x - 2)$; its sign changes at the vertex, and the right-hand interval includes the vertex.

3. If $f(2a - x) = f(x)$, then $\int_0^{2a} f(x) dx$ is

[1 mark]

(A) $\int_0^{2a} f\left(\frac{x}{2}\right) dx$

(B) $\int_0^a f(x) dx$

(C) $2 \int_a^0 f(x) dx$

(D) $2 \int_0^a f(x) dx$

Correct Answer: Both printed options (A) and (D) equal $2 \int_0^a f(x) dx$. Option (D) follows directly from symmetry; option (A) is equivalent by the substitution $u = x/2$. Hence the printed question has two correct options.

Solution:

Concept:

- The condition $f(2a - x) = f(x)$ expresses symmetry about $x = a$.
- For an integrable function, substitution $u = 2a - x$ makes the integrals over the two halves of $[0, 2a]$ equal.

Step 1: Split the integral and use symmetry Let $J = \int_0^a f(x) dx$. Split the required integral at a . In its second part, set $u = 2a - x$, so $dx = -du$; the limits $x = a, 2a$ become $u = a, 0$. Hence

$$\int_a^{2a} f(x) dx = \int_a^a f(2a - u) du = \int_0^a f(u) du = J.$$

Thus the required integral is $J + J = 2J$.

Step 2: Check every printed expression Option (D) is $2J$, the standard symmetry identity. Option (B) is only J , and option (C) reverses the limits, giving $-2J$; neither equals the required integral in general. The printed option (A) also evaluates to $2J$: putting $u = x/2$, with $dx = 2du$, gives

$$\int_0^{2a} f(x/2) dx = 2 \int_0^a f(u) du = 2J.$$

Thus (A) and (D) are mathematically equivalent as printed.

Step 3: Conclusion $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$. The standard symmetry answer is option (D); printed option (A) is also equivalent by substitution.

Quick Tip to Solve: For $f(2a - x) = f(x)$, split at a and double one half; also check substitution before rejecting another integral option.

4. If $A = \begin{pmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{pmatrix}$ is a symmetric matrix, then $(2x + y)$ is [1 mark]

- (A) -8
(B) 0
(C) 6
(D) 8

Correct Answer: (D) 8

Solution:

Concept:

- A square matrix is symmetric when $A^T = A$, so corresponding entries across its main diagonal are equal.
- The diagonal entries impose no extra condition; compare $a_{12} = a_{21}$, $a_{13} = a_{31}$, and $a_{23} = a_{32}$.

Step 1: Equate the first two off-diagonal pairs From $a_{12} = a_{21}$, we obtain $12 = 6x$, so $x = 2$. Next use $a_{13} = a_{31}$:

$$4y = 8x \implies y = 2x = 4.$$

These equations determine the values required for the expression.

Step 2: Check consistency and evaluate The remaining off-diagonal pair gives $a_{23} = a_{32}$, or $2x = 4$. This agrees with $x = 2$, so all three symmetry conditions hold simultaneously. Substituting the values,

$$2x + y = 2(2) + 4 = 8.$$

Therefore option (D) has the required value; options (A), (B), and (C), namely -8 , 0 , and 6 , do not equal the evaluated expression.

Step 3: Conclusion $2x + y = 8$. Hence, the correct option is (D).

Quick Tip to Solve: Compare entries reflected across the main diagonal; first obtain $x = 12/6$, then use $4y = 8x$.

5. If $y = \sin^{-1} x$, $-1 \leq x \leq 0$, then the range of y is

[1 mark]

- (A) $\left(-\frac{\pi}{2}, 0\right)$
- (B) $\left[-\frac{\pi}{2}, 0\right]$
- (C) $\left[-\frac{\pi}{2}, 0\right)$
- (D) $\left(-\frac{\pi}{2}, 0\right]$

Correct Answer: (B) $\left[-\frac{\pi}{2}, 0\right]$

Solution:

Concept:

- The principal inverse sine has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
- The function $\sin^{-1} x$ is strictly increasing; hence included domain endpoints give included range endpoints.

Step 1: Evaluate the endpoints At $x = -1$, $y = \sin^{-1}(-1) = -\pi/2$. At $x = 0$, $y = \sin^{-1} 0 = 0$. Both values of x belong to the given interval, so both endpoint values of y occur.

Step 2: Determine the interval and check the options As x varies continuously from -1 to 0 , inverse sine increases continuously from $-\pi/2$ to 0 . Every intermediate value is attained. Option (A) wrongly omits both endpoints; (C) omits 0 , and (D) omits $-\pi/2$. Only (B) includes exactly the attained endpoints.

Step 3: Conclusion The range is $\boxed{[-\pi/2, 0]}$. Hence, the correct option is (B).

Quick Tip to Solve: First check whether each endpoint of the given domain is included; preserve that inclusion under increasing inverse sine.

6. If a line makes angles of $\frac{3\pi}{4}$, $\frac{\pi}{3}$ and θ with the positive directions of x , y and z -axis respectively, then θ is

[1 mark]

- (A) $-\frac{\pi}{3}$ only
- (B) $\frac{\pi}{3}$ only
- (C) $\frac{\pi}{6}$
- (D) $\pm \frac{\pi}{3}$

Correct Answer: (B) $\frac{\pi}{3}$ only

Solution:

Concept:

- For direction angles α, β, γ , the direction cosines satisfy $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
- A direction angle with a positive coordinate axis lies in $[0, \pi]$.

Quick Tip to Solve: After solving for a squared direction cosine, retain both signs and restrict direction angles to the interval from zero to π .

Step 1: Substitute the two known direction angles Here $\alpha = 3\pi/4$, $\beta = \pi/3$, and $\gamma = \theta$. Thus

$$\cos^2 \theta = 1 - \left(-\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Step 2: Apply the direction-angle range Therefore $\cos \theta = \pm 1/2$, giving $\theta = \pi/3$ or $2\pi/3$ in $[0, \pi]$. Of the printed options, only (B) is a valid direction angle satisfying the equation. The negative angle in (A) and (D) is outside the direction-angle range; (C) has $\cos^2(\pi/6) = 3/4$, so the sum would exceed one. The word "only" in (B) overlooks the other valid angle $2\pi/3$, which is absent from the choices.

Step 3: Conclusion The listed valid value is $\theta = \pi/3$, option (B). The complete mathematical set is $\{\pi/3, 2\pi/3\}$.

Quick Tip to Solve: After solving for a squared direction cosine, retain both signs and restrict direction angles to the interval from zero to pi.

7. If E and F are two events such that $P(E) > 0$ and $P(F) \neq 1$, then $P(\bar{E}/\bar{F})$ is

[1 mark]

- (A) $\frac{P(\bar{E})}{P(\bar{F})}$
- (B) $1 - P(\bar{E}/F)$
- (C) $1 - P(E/F)$
- (D) $\frac{1 - P(E \cup F)}{P(\bar{F})}$

Correct Answer: (D) $\frac{1 - P(E \cup F)}{P(\bar{F})}$

Solution:

Concept:

- Conditional probability is $P(A/B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- De Morgan's law gives $\bar{E} \cap \bar{F} = \overline{E \cup F}$.

Step 1: Write the conditional probability Since $P(F) \neq 1$, we have $P(\bar{F}) = 1 - P(F) > 0$. Therefore the required conditional probability exists and is

$$P(\bar{E}/\bar{F}) = \frac{P(\bar{E} \cap \bar{F})}{P(\bar{F})}.$$

Step 2: Simplify the numerator and test the choices By De Morgan's law, $P(\bar{E} \cap \bar{F}) = 1 - P(E \cup F)$. Option (A) uses $P(\bar{E})$ without intersecting it with the conditioning event, so it does not give the conditional probability in general. Options (B) and (C) condition on F , whereas the required event is $\bar{F}$; changing that event changes the denominator and the intersection.

Step 3: Conclusion $P(\bar{E}/\bar{F}) = \frac{1 - P(E \cup F)}{P(\bar{F})}$. Hence, the correct option is (D).

Quick Tip to Solve: Keep the conditioning event unchanged, then turn the intersection of two complements into the complement of a union.

8. Which of the following can be both a symmetric and skew-symmetric matrix?

[1 mark]

- (A) Unit Matrix
- (B) Diagonal Matrix
- (C) Null Matrix
- (D) Row Matrix

Correct Answer: (C) Null Matrix

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, and a skew-symmetric matrix satisfies $A^T = -A$.
- A real matrix satisfying both conditions must have every entry zero.

Step 1: Combine the two matrix identities If the same matrix is symmetric and skew-symmetric, then $A = A^T = -A$. Adding A to both sides gives $2A = O$, where O is the zero matrix. Hence $A = O$; this conclusion applies to every entry, including all diagonal and off-diagonal entries.

Step 2: Check the listed matrix types The null matrix indeed satisfies $O^T = O = -O$. A nonzero unit matrix is symmetric but fails $I^T = -I$. A general diagonal matrix is symmetric, but skew symmetry forces each of its diagonal entries to be zero. A general row matrix is not even square, so the transpose identities cannot both hold. Thus the null matrix is the specific listed answer.

Step 3: Conclusion The matrix can be the Null Matrix. Hence, the correct option is (C).

Quick Tip to Solve: Equate the two expressions for the transpose immediately; A equals negative A forces every real entry to vanish.

9. The equation of a line parallel to the vector $3\hat{i} + \hat{j} + 2\hat{k}$ and passing through the point $(4, -3, 7)$ is

[1 mark]

- (A) $x = 4t + 3, y = -3t + 1, z = 7t + 2$
- (B) $x = 3t + 4, y = t + 3, z = 2t + 7$
- (C) $x = 3t + 4, y = t - 3, z = 2t + 7$
- (D) $x = 3t + 4, y = -t + 3, z = 2t + 7$

Correct Answer: (C) $x = 3t + 4, y = t - 3, z = 2t + 7$

Solution:

Concept:

- A line through (x_0, y_0, z_0) , parallel to a vector with components (a, b, c) , has equations $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$.
- The constant terms locate the point at $t = 0$; the coefficients of t give the direction ratios.

Step 1: Insert the point and direction The specified point is $(4, -3, 7)$, and the given vector has components $(3, 1, 2)$. Hence the required parametrisation is

$$x = 4 + 3t, \quad y = -3 + t, \quad z = 7 + 2t.$$

Quick Tip to Solve: For a parametric line, compare the coefficients with the direction vector and set the parameter to zero to check the base point.

The same parameter must be used in all three coordinates.

Step 2: Check the alternatives Option (C) has exactly these equations. In (A), the direction is $(4, -3, 7)$, which is not proportional to $(3, 1, 2)$. In (B), the point at $t = 0$ is $(4, 3, 7)$; to have $x = 4$ requires $t = 0$, so it cannot give $y = -3$. In (D), the direction $(3, -1, 2)$ is not parallel to the given vector.

Step 3: Conclusion The required line is $x = 3t + 4$, $y = t - 3$, $z = 2t + 7$. Hence, the correct option is (C).

Quick Tip to Solve: For a parametric line, compare the coefficients with the direction vector and set the parameter to zero to check the base point.

10. Four friends Abhay, Bina, Chhaya and Devesh were asked to simplify $4AB + 3(AB + BA) - 4BA$, where A and B are both matrices of order 2×2 . It is known that $A \neq B \neq I$ and $A^{-1} \neq B$.

Their answers are:

Abhay: $6AB$

Bina: $7AB - BA$

Chhaya: $8AB$

Devesh: $7BA - AB$

Who answered it correctly?

[1 mark]

(A) Abhay

(B) Bina

(C) Chhaya

(D) Devesh

Correct Answer: (B) Bina

Solution:

Concept:

- Matrix addition and scalar multiplication are distributive, so coefficients of the same ordered product can be collected.
- For arbitrary matrices, AB and BA need not be equal: the order of multiplication must be preserved.

Step 1: Expand the bracket All products have order 2×2 , so the indicated additions are defined. Distribute the scalar 3 without changing multiplication order:

$$4AB + 3(AB + BA) - 4BA = 4AB + 3AB + 3BA - 4BA.$$

Step 2: Collect like ordered products The coefficients of AB add to $4 + 3 = 7$, while those of BA add to $3 - 4 = -1$. Therefore the expression simplifies to $7AB - BA$, Bina's answer. Abhay's and Chhaya's answers improperly reduce the two distinct products to one product; Devesh's answer reverses their coefficients. The given conditions supply no identity allowing this change.

Quick Tip to Solve: Keep AB and BA as separate like-term groups; combine only scalar coefficients multiplying the same ordered product.

Step 3: Conclusion The simplified expression is $7AB - BA$. Bina answered correctly; hence, the correct option is (B).

Quick Tip to Solve: Keep AB and BA as separate like-term groups; combine only scalar coefficients multiplying the same ordered product.

11. A cylindrical tank of radius 10 cm is being filled with sugar at the rate of $100\pi \text{ cm}^3/\text{s}$. The rate, at which the height of the sugar inside the tank is increasing, is [1 mark]

- (A) 0.1 cm/s
- (B) 0.5 cm/s
- (C) 1 cm/s
- (D) 1.1 cm/s

Correct Answer: (C) 1 cm/s

Solution:

Concept:

- The volume of a cylinder is $V = \pi r^2 h$, where r is its radius and h is the height of its contents.
- When the radius is constant, $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$; volume inflow equals cross-sectional area times height increase.

Step 1: Relate volume to height The tank has fixed radius $r = 10 \text{ cm}$, so its cross-sectional area is $\pi(10)^2 = 100\pi \text{ cm}^2$. Thus $V = 100\pi h$. Differentiation with respect to time gives

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}.$$

Step 2: Use the filling rate and check units Substitute the stated inflow $\frac{dV}{dt} = 100\pi \text{ cm}^3/\text{s}$:

$$\frac{dh}{dt} = \frac{100\pi}{100\pi} = 1 \text{ cm/s}.$$

Options (A), (B) and (D) would respectively imply volume rates 10π , 50π and $110\pi \text{ cm}^3/\text{s}$, contradicting the given inflow.

Step 3: Conclusion The height increases at 1 cm/s. Hence, the correct option is (C).

Quick Tip to Solve: Divide the volume inflow by the fixed cross-sectional area; remember to square the radius before cancelling pi.

12. Let $\vec{p}$ and $\vec{q}$ be two unit vectors and α be the angle between them. Then $(\vec{p} + \vec{q})$ will be a unit vector for what value of α ? [1 mark]

- (A) $\frac{\pi}{4}$
- (B) $\frac{\pi}{3}$
- (C) $\frac{\pi}{2}$
- (D) $\frac{2\pi}{3}$

Correct Answer: (D) $\frac{2\pi}{3}$

Solution:

Concept:

- For unit vectors, $|\vec{p}| = |\vec{q}| = 1$, and $\vec{p} \cdot \vec{q} = \cos \alpha$, where α is their angle.
- The squared magnitude of the sum is $|\vec{p} + \vec{q}|^2 = |\vec{p}|^2 + |\vec{q}|^2 + 2\vec{p} \cdot \vec{q}$.

Step 1: Impose the unit magnitude The sum is a unit vector precisely when its squared magnitude is 1. Expanding the dot product gives

$$1 = (\vec{p} + \vec{q}) \cdot (\vec{p} + \vec{q}) = 2 + 2 \cos \alpha.$$

Hence $\cos \alpha = -\frac{1}{2}$.

Step 2: Select the angle and exclude the others The angle between vectors lies in $[0, \pi]$, so $\cos \alpha = -\frac{1}{2}$ gives $\alpha = \frac{2\pi}{3}$. At $\pi/4$, $\pi/3$ and $\pi/2$, the squared magnitudes are respectively $2 + \sqrt{2}$, 3 and 2, none equal to 1. Thus all three other printed choices fail the unit-vector condition.

Step 3: Conclusion $\alpha = \frac{2\pi}{3}$. Hence, the correct option is (D).

Quick Tip to Solve: Square the required magnitude first: a unit sum of two unit vectors requires their dot product to be minus one-half.

13. The line $x = 1 + 5\mu$, $y = -5 + \mu$, $z = -6 - 3\mu$ passes through which of the following point? [1 mark]

- (A) $(1, -5, 6)$
- (B) $(1, 5, 6)$
- (C) $(1, -5, -6)$
- (D) $(-1, -5, 6)$

Correct Answer: (C) $(1, -5, -6)$

Solution:

Concept:

- A point lies on a parametric line only if one common parameter value satisfies all three coordinate equations.
- Setting the parameter to zero gives the point formed by the constant terms, which is always on the

Quick Tip to Solve: First put the common parameter equal to zero; if that triple appears among the options, check it before solving any equations.

line.

Step 1: Find an immediate point The printed equations are $x = 1 + 5\mu$, $y = -5 + \mu$, $z = -6 - 3\mu$. Setting $\mu = 0$ gives

$$x = 1, \quad y = -5, \quad z = -6.$$

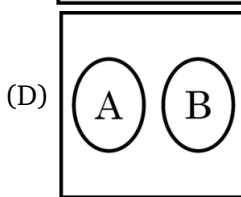
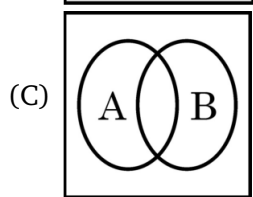
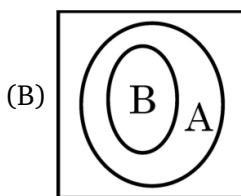
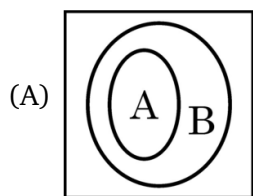
This triple occurs in option (C).

Step 2: Check the remaining triples For options (A) and (D), $y = -5$ forces $\mu = 0$, and therefore $x = 1$ and $z = -6$; their printed coordinates do not match. For (B), $y = 5$ forces $\mu = 10$, giving $x = 51$ and $z = -36$, rather than $(1, 5, 6)$. The parameter cannot be chosen separately for different coordinates.

Step 3: Conclusion The line passes through $(1, -5, -6)$. Hence, the correct option is (C).

Quick Tip to Solve: First put the common parameter equal to zero; if that triple appears among the options, check it before solving any equations.

14. If A denotes the set of continuous functions and B denotes the set of differentiable functions, then which of the following depicts the correct relation between set A and B ? [1 mark]



Correct Answer: (B)



Solution:

Concept:

- Every function differentiable at a point is continuous at that point; thus differentiability throughout a common domain implies continuity there.
- The converse is false: $f(x) = |x|$ is continuous on $\mathbb{R}$ but is not differentiable at $x = 0$.

Step 1: Establish the containment Let f belong to B , the set of differentiable functions. At each point of its domain, differentiability implies continuity. Consequently f also belongs to A , and $B \subseteq A$. This is depicted by the circle labelled B lying inside the circle labelled A .

Step 2: Check the printed diagrams The printed option (B) shows precisely this nesting. Option (A) reverses the inclusion, claiming that every continuous function is differentiable, contradicted by $|x|$.

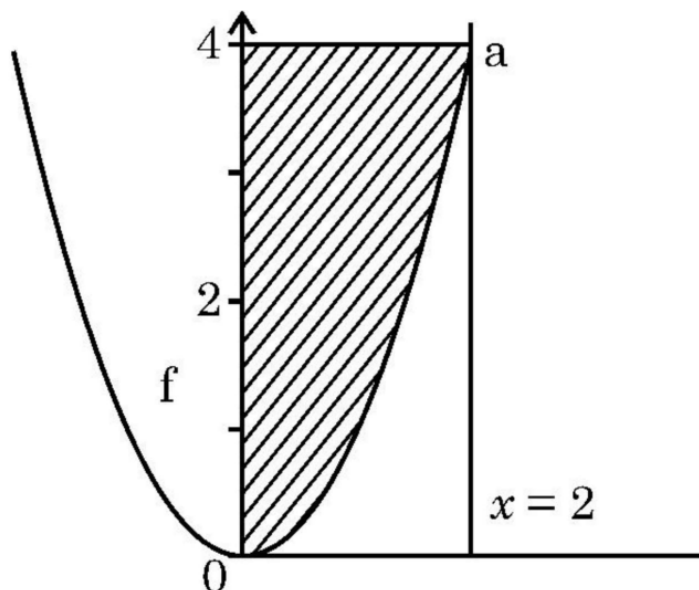
Quick Tip to Solve: Remember the direction of implication: differentiable implies continuous, so put the differentiable set inside the continuous set.

Option (C) places some differentiable functions outside the continuous set, violating the theorem. Option (D) makes the sets disjoint, whereas differentiable functions such as $f(x) = x$ belong to both.

Step 3: Conclusion $B \subset A$: the differentiable functions lie inside the continuous functions. Hence, the correct option is (B).

Quick Tip to Solve: Remember the direction of implication: differentiable implies continuous, so put the differentiable set inside the continuous set.

15. The area of the shaded region (figure) represented by the curves $y = x^2$, $0 \leq x \leq 2$ and the y -axis is given by [1 mark]



(A) $\int_0^2 x^2 dx$

(C) $\int_0^4 x^2 dx$

(B) $\int_0^2 \sqrt{y} dy$

(D) $\int_0^4 \sqrt{y} dy$

Correct Answer: (D) $\int_0^4 \sqrt{y} dy$

Solution:

Concept:

- For horizontal strips, area is $\int (x_{\text{right}} - x_{\text{left}}) dy$; the limits are the vertical extent of the shaded region.

Quick Tip to Solve: Use horizontal strips for a region between the y -axis and a curve; convert $x=2$ to the upper y -limit $y=4$.

- On the nonnegative branch of $y = x^2$, the inverse relation is $x = \sqrt{y}$.

Step 1: Read the shaded boundaries The shaded region is between the y -axis $x = 0$ and the right branch $x = \sqrt{y}$, from $y = 0$ to $y = 4$. The upper height is $2^2 = 4$, obtained at the labelled point on $x = 2$. A horizontal strip therefore has width $\sqrt{y} - 0$.

Step 2: Form the integral and check the choices The required area is

$$A = \int_0^4 \sqrt{y} \, dy = \left[\frac{2}{3} y^{3/2} \right]_0^4 = \frac{16}{3}.$$

Option (D) uses both the correct width and the correct limits. Option (A) instead gives the area below the parabola from $x = 0$ to 2 , which is unshaded. Option (B) stops at $y = 2$, omitting the upper shaded portion. Option (C) integrates x^2 to $x = 4$, beyond the given horizontal extent.

Step 3: Conclusion The area is represented by $\int_0^4 \sqrt{y} \, dy$. Hence, the correct option is (D).

Quick Tip to Solve: Use horizontal strips for a region between the y -axis and a curve; convert $x=2$ to the upper y -limit $y=4$.

16. A factory produces two products X and Y . The profit earned by selling X and Y is represented by the objective function $Z = 5x + 7y$, where x and y are the number of units of X and Y respectively sold. Which of the following statement is correct? [1 mark]

- (A) The objective function maximizes the difference of the profit earned from products X and Y .
- (B) The objective function measures the total production of products X and Y .
- (C) The objective function maximizes the combined profit earned from selling X and Y .
- (D) The objective function ensures the company produces more of product X than product Y .

Correct Answer: (C) The objective function maximizes the combined profit earned from selling X and Y .

Solution:

Concept:

- An objective function in linear programming gives the quantity to be maximised or minimised, such as total profit.
- In $Z = 5x + 7y$, the coefficients 5 and 7 represent profit per unit of products X and Y , respectively.

Step 1: Interpret both terms Selling x units of X yields profit $5x$, and selling y units of Y yields profit $7y$. Their sum $Z = 5x + 7y$ therefore measures the combined profit. In the stated profit objective, one chooses feasible production or sales quantities to maximise this sum.

Step 2: Check the alternatives Option (C) matches the combined-profit interpretation. A difference of profits would involve subtraction, so (A) is incorrect. Total units sold or produced would be represented by $x + y$, rather than the profit-weighted sum, so (B) is incorrect. The objective itself places no condition $x > y$; such a condition would need a separate constraint, so (D) is also incorrect.

Step 3: Conclusion The objective function maximises the combined profit earned from selling X and Y . Hence, the correct option is (C).

Quick Tip to Solve: Interpret the coefficients as profit per unit: add the two profit contributions, and do not infer constraints from the objective alone.

17. If A and B are square matrices of order m such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following is always correct? [1 mark]

- (A) $A = B$
- (B) $AB = BA$
- (C) $A = O$ or $B = O$
- (D) $A = I$ or $B = I$

Correct Answer: (B) $AB = BA$

Solution:

Concept:

- Matrix multiplication is distributive, but generally $AB \neq BA$.
- Commuting matrices satisfy $AB = BA$; the usual difference-of-squares factorisation requires this condition.

Step 1: Expand the ordered product Keep the order of the matrix factors while distributing:

$$(A - B)(A + B) = A^2 + AB - BA - B^2.$$

The middle terms cannot be cancelled until commutativity is established.

Step 2: Compare with the given identity Substituting the stated equality and subtracting $A^2 - B^2$ from both sides gives

$$O = AB - BA.$$

Thus the hypothesis precisely forces the matrices to commute. It does not force equality, a zero matrix or an identity matrix: for example, $A = 2I$, $B = 3I$ satisfy the identity while failing all three of those additional conditions.

Step 3: Conclusion $AB = BA$ is always correct. Hence, the correct option is (B).

Quick Tip to Solve: Expand the matrix product without reversing any factors; the leftover term $AB - BA$ detects commutativity.

18. If p and q are respectively the order and degree of the differential equation $\frac{d}{dx} \left(\frac{dy}{dx} \right)^3 = 0$, then $(p - q)$ is [1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (B) 1

Solution:

Concept:

- The order is the order of the highest derivative present after carrying out any indicated differentiation.
- The degree is the power of the highest-order derivative when the equation is polynomial in derivatives.

Step 1: Carry out the outer differentiation Apply the chain rule to the cube of dy/dx :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right)^3 = 3 \left(\frac{dy}{dx} \right)^2 \frac{d^2y}{dx^2} = 0.$$

This is a polynomial equation in the first and second derivatives. Do not divide by $(dy/dx)^2$, since it may vanish.

Step 2: Identify order and degree The highest derivative is d^2y/dx^2 , so $p = 2$. Its exponent is one, so $q = 1$. The exponent three in the original expression is not the degree of the expanded differential equation, and the exponent two belongs to the lower-order derivative.

Step 3: Conclusion $p - q = 2 - 1 = 1$. Hence, the correct option is (B).

Quick Tip to Solve: Perform the outer derivative first; identify degree from the exponent of the highest derivative, not the original cube.

19. Assertion (A): $A = \text{diag}[3 \ 5 \ 2]$ is a scalar matrix of order 3×3 .

Reason (R): If a diagonal matrix has all non-zero elements equal, it is known as a scalar matrix.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true but Reason (R) is false.
- (D) Assertion (A) is false but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false but Reason (R) is true.

Solution:

Concept:

- A diagonal matrix has zero entries outside its principal diagonal.
- A scalar matrix has the form kI : all principal diagonal entries are the same scalar k .

Step 1: Test the assertion The notation $\text{diag}[3 \ 5 \ 2]$ means a matrix whose principal diagonal entries are 3, 5, 2 and whose other entries vanish. It is diagonal, but the three diagonal entries are unequal. Therefore it is not a scalar matrix and Assertion (A) is false.

Step 2: Test the reason Reason (R) states the scalar-matrix criterion for the diagonal entries: they must all be equal. In the stated CBSE definition this is true; the diagonal form alone is insufficient to make a matrix scalar.

Step 3: Choose the truth combination The reason cannot explain the assertion as a true statement, because the assertion fails its equality test. Options (A) and (B) require a true assertion, and option (C) also requires it, so none of those combinations applies.

Step 4: Conclusion Assertion is false and Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For a diagonal matrix, compare the principal diagonal entries before calling it scalar; 3, 5, 2 immediately fail equality.

20. Assertion (A): Every point of the feasible region of a Linear Programming Problem is an optimal solution.

Reason (R): The optimal solution for a Linear Programming Problem exists only at one or more corner point(s) of the feasible region. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true but Reason (R) is false.
(D) Assertion (A) is false but Reason (R) is true.

Correct Answer: No listed option is mathematically correct as printed: both statements are false. The intended corner-point interpretation would give (D).

Solution:

Concept:

- A feasible solution satisfies all constraints; an optimal solution also attains the required greatest or least objective value.
- The corner-point theorem says an existing optimum can be attained at a vertex in the usual CBSE setting; it does not exclude optimal points along an entire edge.

Step 1: Evaluate the assertion Consider maximising $Z = x + y$ subject to $x \geq 0, y \geq 0, x + y \leq 1$. The feasible origin gives $Z = 0$, while $(1, 0)$ gives $Z = 1$. Therefore every feasible point is not necessarily optimal, and Assertion (A) is false.

Step 2: Evaluate the printed reason literally In the same example, every point on $x + y = 1$ with nonnegative coordinates is optimal. In particular, $(1/2, 1/2)$ gives the greatest value $Z = 1$, but is not a corner point. Hence the printed claim that an optimal solution exists **only** at corner points is false. The true theorem guarantees an optimal corner point when an optimum exists under its usual conditions; it does not say that all optimal points must be corners.

Step 3: Compare with the available combinations Both printed statements are false, and none of the four combinations lists that case. If Reason (R) were interpreted as saying an optimum *can be attained* at a corner point, it would express the intended theorem and the intended combination would be (D). That interpretation changes the restrictive force of the printed word “only”.

Step 4: Conclusion No listed option correctly describes both statements as printed. Under the intended corner-point interpretation, the expected option would be (D).

Quick Tip to Solve: When an objective takes equal optimum values at adjacent vertices, their entire joining edge is optimal; test claims containing “only” against this case.

Section B (10 marks)

(This section comprises of 5 Very Short Answer (VSA) type questions of 2 marks each.)

21(a). A vector $\vec{a}$ makes equal angles with all the three axes. If the magnitude of the vector is $5\sqrt{3}$ units, then find $\vec{a}$. [2 marks]

Solution:

Concept:

- If α, β, γ are angles with the positive coordinate axes, the direction cosines are $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$, and $\vec{a} = |\vec{a}|(l\hat{i} + m\hat{j} + n\hat{k})$.

Step 1: Use equality of the direction angles Since the three angles are equal, their cosines are equal. Write $l = m = n = t$. The direction-cosine identity then gives

$$3t^2 = 1, \quad t = \pm \frac{1}{\sqrt{3}}.$$

All three signs must be the same; independently choosing signs would produce unequal angles with the positive axes.

Step 2: Find the Cartesian components Multiply each direction cosine by the given magnitude $5\sqrt{3}$:

$$\vec{a} = 5\sqrt{3}t(\hat{i} + \hat{j} + \hat{k}).$$

For the common acute angle, $t = 1/\sqrt{3}$, so every component is 5. If the common angle is obtuse, $t = -1/\sqrt{3}$, so every component is -5 . Both choices have magnitude $\sqrt{25 + 25 + 25} = 5\sqrt{3}$, as required.

Step 3: Conclusion

$$\vec{a} = \pm 5(\hat{i} + \hat{j} + \hat{k}).$$

The usual positive-direction vector is $5\hat{i} + 5\hat{j} + 5\hat{k}$; its negative also makes equal angles with all three positive axes.

Quick Tip to Solve: Equal angles mean equal direction cosines; use their squared sum to get the common component and retain the common sign.

OR

21(b). If $\vec{a}$ and $\vec{b}$ are position vectors of two points P and Q respectively, then find the position vector of a point R in QP produced such that $QR = \frac{3}{2}QP$. [2 marks]

Solution:

Concept:

- For position vectors $\vec{OP} = \vec{a}, \vec{OQ} = \vec{b}$, the displacement from Q to P is $\vec{QP} = \vec{a} - \vec{b}$.

Quick Tip to Solve: Start at Q , then add $3/2$ times the displacement $\vec{a} - \vec{b}$; this automatically gives the external point.

- A point on the ray QP has position vector $\vec{b} + t(\vec{a} - \vec{b})$; values $t > 1$ put it beyond P .

Step 1: Use the given direction and length ratio The phrase QP produced places R beyond P on the ray beginning at Q . Thus $\vec{QR}$ and $\vec{QP}$ have the same direction. The stated length ratio gives the corresponding vector relation

$$\vec{QR} = \frac{3}{2}\vec{QP} = \frac{3}{2}(\vec{a} - \vec{b}).$$

Since the multiplier exceeds one, this correctly extends past P , rather than placing R inside the segment.

Step 2: Add the displacement to the starting position Follow the path from the origin to Q , then from Q to R :

$$\begin{aligned}\vec{OR} &= \vec{OQ} + \vec{QR} \\ &= \vec{b} + \frac{3}{2}(\vec{a} - \vec{b}) \\ &= \frac{3}{2}\vec{a} - \frac{1}{2}\vec{b}.\end{aligned}$$

As a check, $PR = QR - QP = QP/2$; hence R divides PQ externally in the ratio $PR : RQ = 1 : 3$, consistent with the result.

Step 3: Conclusion The required position vector is $\vec{OR} = \frac{3\vec{a} - \vec{b}}{2}$.

Quick Tip to Solve: Start at Q , then add $3/2$ times the displacement $\vec{a} - \vec{b}$; this automatically gives the external point.

22. Evaluate: $\int_0^{\pi/4} \sqrt{1 + \sin 2x} \, dx$

[2 marks]

Solution:

Concept:

- The identity $1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2$ turns the radicand into a perfect square.
- For a real quantity u , $\sqrt{u^2} = |u|$. The sign must be checked on the entire interval before removing the absolute value.

Step 1: Simplify the square root on the given interval For $0 \leq x \leq \pi/4$, $\sin x \geq 0$ and $\cos x > 0$. Therefore their sum is positive throughout the interval, so

$$\sqrt{1 + \sin 2x} = |\sin x + \cos x| = \sin x + \cos x.$$

No interval split is needed, because the sign does not change between either endpoint.

Step 2: Integrate and substitute both limits Integrate the two trigonometric terms separately and

Quick Tip to Solve: Replace $1 + \sin 2x$ by $(\sin x + \cos x)^2$, check the sign on the limits, then integrate the sum directly.

evaluate the antiderivative:

$$\begin{aligned} I &= \int_0^{\pi/4} (\sin x + \cos x) dx \\ &= [-\cos x + \sin x]_0^{\pi/4} \\ &= \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (-1 + 0). \end{aligned}$$

The upper-limit terms cancel, while the lower-limit antiderivative is -1 . Subtracting that lower value is essential to obtain the positive integral of a positive integrand.

Step 3: Conclusion $\int_0^{\pi/4} \sqrt{1 + \sin 2x} dx = 1$.

Quick Tip to Solve: Replace $1 + \sin 2x$ by $(\sin x + \cos x)^2$, check the sign on the limits, then integrate the sum directly.

23. Find the values of 'a' for which $f(x) = \sin x - ax + b$ is increasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval if its derivative is non-negative throughout that interval. For a differentiable increasing function, the derivative cannot be negative at any point.
- For every real x , $-1 \leq \cos x \leq 1$. A constant added to a function does not change its derivative or its monotonicity.

Step 1: Differentiate and use the minimum value of cosine Differentiating $f(x) = \sin x - ax + b$, where a and b are constants, gives

$$f'(x) = \cos x - a.$$

The least possible value of this derivative is $-1 - a$, since $\cos x = -1$ at $x = (2k + 1)\pi$, with k an integer. Thus the required condition is

$$-1 - a \geq 0, \quad \text{that is, } a \leq -1.$$

Step 2: Check the boundary and exclude the other values If $a < -1$, then $f'(x) > 0$ everywhere. At $a = -1$, the derivative is $1 + \cos x \geq 0$; it vanishes only at isolated odd multiples of π , so the function is still strictly increasing. Its derivative has a positive integral over every interval of positive length. If $a > -1$, then $f'(\pi) = -1 - a < 0$, and continuity of the derivative makes the function decrease near π . Such values cannot satisfy the requirement on all of $\mathbb{R}$.

Step 3: Conclusion The required values are $a \leq -1$, with no restriction on the constant b .

Quick Tip to Solve: For monotonicity on the entire real line, test the smallest possible derivative; here use the minimum cosine value, -1 .

24. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then find x , such that $\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear. [2 marks]

Solution:

Concept:

- Two non-collinear vectors $\vec{a}$ and $\vec{b}$ are linearly independent: if $u\vec{a} + v\vec{b} = \vec{0}$, then $u = v = 0$.
- Two nonzero vectors are collinear precisely when one is a scalar multiple of the other. Compare coefficients only after expressing both in the same independent pair.

Step 1: Set up the scalar multiple relation Let $\vec{\beta} = \lambda\vec{\alpha}$. Substituting the given expressions yields

$$(3+2x)\vec{a} - 2\vec{b} = \lambda((x-2)\vec{a} + \vec{b}).$$

Neither vector is zero, since their respective coefficients of $\vec{b}$ are 1 and -2 . By independence of $\vec{a}$ and $\vec{b}$, the coefficients must satisfy

$$3+2x = \lambda(x-2), \quad -2 = \lambda.$$

Step 2: Determine x and verify collinearity Using $\lambda = -2$ in the first equation,

$$3+2x = -2(x-2) = -2x+4 \implies 4x = 1.$$

For this value,

$$\vec{\alpha} = -\frac{7}{4}\vec{a} + \vec{b}, \quad \vec{\beta} = \frac{7}{2}\vec{a} - 2\vec{b} = -2\vec{\alpha}.$$

Thus the vectors are collinear and point in opposite directions. The coefficient comparison gives a unique value because the underlying two vectors are non-collinear.

Step 3: Conclusion The required value is $x = \frac{1}{4}$.

Quick Tip to Solve: Use the coefficients of the second independent vector first: they immediately give the scalar multiplier -2.

25(a). If $x = e^{\frac{x}{y}}$, then prove that $\frac{dy}{dx} = \frac{x-y}{x \log x}$. [2 marks]

Solution:

Concept:

- Taking natural logarithms of $x = e^{x/y}$ gives $\log x = x/y$. For a real solution, $x > 0$, $y \neq 0$, and $x \neq 1$.
- When y depends on x , use the product rule: $\frac{d}{dx}(y \log x) = \log x \frac{dy}{dx} + \frac{y}{x}$.

Step 1: Rewrite the exponential relation using logarithms The given exponential is positive, so its left

Quick Tip to Solve: Take logarithms, then multiply by y before differentiating; the product rule avoids a longer quotient calculation.

side x must be positive and its natural logarithm is defined. Taking logarithms on both sides gives

$$\log x = \frac{x}{y} \implies y \log x = x.$$

The original expression contains x/y , so $y \neq 0$. Also $x = 1$ would require $1 = e^{1/y}$, impossible for finite nonzero real y ; thus division by $\log x$ is permitted.

Step 2: Differentiate the product and isolate the derivative Differentiating $y \log x = x$ with respect to x , keeping in mind that y is a function of x , gives

$$\log x \frac{dy}{dx} + \frac{y}{x} = 1.$$

Therefore

$$\log x \frac{dy}{dx} = 1 - \frac{y}{x} = \frac{x - y}{x}.$$

Dividing both sides by the nonzero quantity $\log x$ establishes the required expression on the domain of the given relation.

Step 3: Conclusion Hence $\boxed{\frac{dy}{dx} = \frac{x - y}{x \log x}}$, as required.

Quick Tip to Solve: Take logarithms, then multiply by y before differentiating; the product rule avoids a longer quotient calculation.

OR

25(b). If $f(x) = \begin{cases} 2x - 3, & -3 \leq x \leq -2 \\ x + 1, & -2 < x \leq 0 \end{cases}$, check the differentiability of $f(x)$ at $x = -2$. [2 marks]

Solution:

Concept:

- Differentiability at an interior point implies continuity there. A jump in the one-sided limits is sufficient to rule out differentiability.
- The difference quotient must use the actual value $f(-2)$ in both one-sided derivatives; the slopes of the two branches alone do not give the derivatives at a discontinuity.

Step 1: Check continuity at the joining point The first branch includes $x = -2$, so

$$f(-2) = 2(-2) - 3 = -7.$$

The one-sided limits are

$$\lim_{x \rightarrow -2^-} f(x) = -7, \quad \lim_{x \rightarrow -2^+} f(x) = -2 + 1 = -1.$$

Since these are unequal, the function has a jump at -2 and is not continuous there. Consequently the

Quick Tip to Solve: Check the one-sided function values before comparing slopes; a jump makes differentiability impossible.

necessary condition for differentiability already fails.

Step 2: Confirm the failure with difference quotients For a sufficiently small negative h , $f(-2 + h) = -7 + 2h$, whereas for positive h , $f(-2 + h) = -1 + h$. Hence

$$\lim_{h \rightarrow 0^-} \frac{f(-2 + h) - f(-2)}{h} = 2,$$
$$\frac{f(-2 + h) - f(-2)}{h} = \frac{6 + h}{h} \quad (h > 0).$$

The second quotient tends to $+\infty$ as $h \rightarrow 0^+$, so no finite right derivative exists. This confirms the conclusion from continuity.

Step 3: Conclusion The function is not differentiable at $x = -2$, because it is discontinuous there.

Quick Tip to Solve: Check the one-sided function values before comparing slopes; a jump makes differentiability impossible.

Section C (38 marks)

(This section comprises of 6 Short Answer (SA) type questions of 3 marks each.)

26(a). Solve the differential equation $2(y + 3) - xy \frac{dy}{dx} = 0$; given $y(1) = -2$.

[3 marks]

Solution:

Concept:

- A separable differential equation can be written with all y -dependent factors multiplying dy and all x -dependent factors multiplying dx , then integrated.
- The identity $\frac{y}{y+3} = 1 - \frac{3}{y+3}$ reduces the integral to a linear term and a logarithm. An initial condition determines the integration constant.
- Division by a factor requires checking the excluded case; here $y = -3$ is a constant solution of the equation but does not satisfy $y(1) = -2$.

Step 1: Separate the variables near the initial point The given equation is

$$xy \frac{dy}{dx} = 2(y + 3).$$

At the initial point, $x = 1$ and $y = -2$, so the factors used below are nonzero. On the solution branch through this point, separate the variables as

$$\frac{y}{y+3} dy = \frac{2}{x} dx.$$

The exceptional constant solution $y = -3$, lost when dividing by $y + 3$, cannot be the required solution since its initial value would be -3 .

Quick Tip to Solve: Rewrite $y/(y+3)$ as $1-3/(y+3)$ before integrating, and use the initial point only after obtaining the constant.

Step 2: Integrate both sides Using $y/(y+3) = 1 - 3/(y+3)$, integrate to obtain

$$\int \left(1 - \frac{3}{y+3}\right) dy = \int \frac{2}{x} dx,$$
$$y - 3 \log |y+3| = 2 \log |x| + C.$$

Absolute values make the logarithmic antiderivatives valid on intervals where their arguments have fixed nonzero signs. The initial point lies on the branch with $x > 0$ and $y + 3 > 0$.

Step 3: Apply the initial condition and verify Substitution of $x = 1$ and $y = -2$ gives

$$-2 - 3 \log 1 = 2 \log 1 + C, \quad C = -2.$$

Differentiating the resulting implicit relation gives

$$\left(1 - \frac{3}{y+3}\right) \frac{dy}{dx} = \frac{2}{x} \implies xy \frac{dy}{dx} = 2(y+3),$$

which reproduces the original equation. The relation also contains the required point $(1, -2)$, confirming both the differential equation and the initial condition.

Step 4: Conclusion The required particular solution is $y - 3 \log |y+3| = 2 \log |x| - 2$, on the branch through $(1, -2)$.

Quick Tip to Solve: Rewrite $y/(y+3)$ as $1 - 3/(y+3)$ before integrating, and use the initial point only after obtaining the constant.

OR

26(b). Solve the following differential equation: $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$.

[3 marks]

Solution:

Concept:

- A first-order linear equation has the form $dy/dx + P(x)y = Q(x)$, where x is the independent variable and y is the dependent variable.
- The integrating factor is $I = e^{\int P(x) dx}$. It converts the equation into $d(Iy)/dx = IQ$, which can be integrated directly.

Step 1: Identify the coefficients of the linear equation The printed equation is $(1+x^2)dy/dx + 2xy = 4x^2$. Divide by $1+x^2$, which is positive for every real x , to obtain

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}.$$

Comparing with the standard form gives $P(x) = 2x/(1+x^2)$ and $Q(x) = 4x^2/(1+x^2)$. This division does not remove any real value of x from the domain.

Step 2: Multiply by the integrating factor and integrate Substitute $u = 1+x^2$, so $du = 2x dx$, in the

Quick Tip to Solve: Recognise the product derivative: the derivative of $1+x^2$ is $2x$, the coefficient multiplying y .

integral defining the integrating factor:

$$I = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2.$$

Multiplying the equation by I turns its left side into a product derivative:

$$\frac{d}{dx}[(1+x^2)y] = (1+x^2)\frac{dy}{dx} + 2xy = 4x^2.$$

On integrating with respect to x ,

$$(1+x^2)y = \int 4x^2 dx = \frac{4x^3}{3} + C.$$

Here C is the arbitrary constant of integration. As no initial condition is supplied, it remains unrestricted. Differentiating this integrated identity gives the original equation, which verifies the solution.

Step 3: Conclusion The general solution is $y = \frac{4x^3/3 + C}{1+x^2}$, for all real x .

Quick Tip to Solve: Recognise the product derivative: the derivative of $1+x^2$ is $2x$, the coefficient multiplying y .

27. Let R be a relation defined over $\mathbb{N}$, where $\mathbb{N}$ is set of natural numbers, defined as “ mRn if and only if m is a multiple of n , $m, n \in \mathbb{N}$ ”. Find whether R is reflexive, symmetric and transitive or not.

[3 marks]

Solution:

Concept:

- For a relation on $\mathbb{N}$, reflexivity means mRm for every natural number m ; symmetry means $mRn \Rightarrow nRm$.
- Transitivity means that mRn and nRp together imply mRp , for all $m, n, p \in \mathbb{N}$.
- The statement that m is a multiple of n means $m = kn$ for some natural-number multiplier k .

Step 1: Check reflexivity Choose any $m \in \mathbb{N}$. Since $m = 1 \cdot m$, every natural number is a multiple of itself. Hence the defining condition for mRm holds for every member of the set, proving reflexivity. It is necessary to use an arbitrary element here, because checking only a few specific numbers would not establish the property on all of $\mathbb{N}$.

Step 2: Check symmetry with a counterexample Take $m = 2$ and $n = 1$. We have $2 = 2 \cdot 1$, so $2R1$. But 1 is not a multiple of 2: the equation $1 = 2k$ has no natural-number solution k . Therefore $1R2$ is false. This pair disproves the implication $mRn \Rightarrow nRm$, so the relation is not symmetric. Although equal numbers satisfy the condition in both directions, symmetry would have to hold for every related pair.

Step 3: Check transitivity for arbitrary elements Suppose mRn and nRp , with $m, n, p \in \mathbb{N}$. By the

Quick Tip to Solve: Keep the direction straight: mRn means n divides m . The pair $(2, 1)$ immediately tests symmetry.

definition, there exist $a, b \in \mathbb{N}$ such that

$$m = an, \quad n = bp.$$

Substituting the second equality into the first gives $m = a(bp) = (ab)p$. The product ab is a natural number, so m is a multiple of p . Thus mRp follows whenever the two hypotheses hold, proving transitivity.

Step 4: Conclusion The relation R is reflexive and transitive, but not symmetric.

Quick Tip to Solve: Keep the direction straight: mRn means n divides m . The pair $(2, 1)$ immediately tests symmetry.

28. Solve the following linear programming problem graphically:

Minimise $Z = x - 5y$ **subject to the constraints:**

$$x - y \geq 0$$

$$-x + 2y \geq 2$$

$$x \geq 3, \quad y \leq 4, \quad y \geq 0$$

[3 marks]

Solution:

Concept:

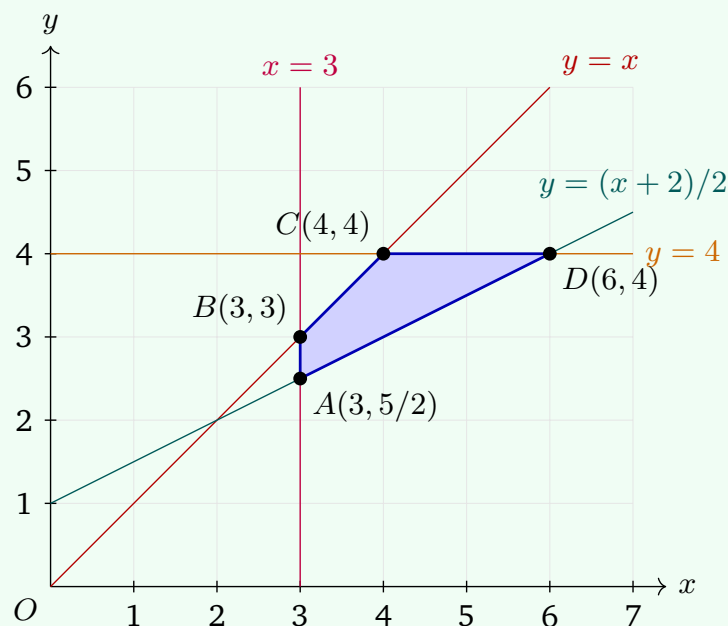
- The feasible region is the common part of the half-planes specified by every constraint. A point must satisfy all the inequalities to be feasible.
- A linear objective $Z = x - 5y$ attains its minimum at a vertex of a nonempty bounded polygonal feasible region. Evaluate Z at each vertex to compare them.

Step 1: Plot the boundary lines and identify the feasible polygon Rewrite the inequalities as

$$y \leq x, \quad y \geq \frac{x+2}{2}, \quad x \geq 3, \quad 0 \leq y \leq 4.$$

Plot the boundaries $y = x$, $y = (x+2)/2$, $x = 3$ and $y = 4$. All boundaries are included because the inequalities are non-strict. Shade the common region to the right of $x = 3$, below $y = x$ and $y = 4$, and above $y = (x+2)/2$. This region automatically lies above $y = 0$. Its vertices, in boundary order, are $A(3, 5/2)$, $B(3, 3)$, $C(4, 4)$ and $D(6, 4)$. For example, D follows by solving $4 = (x+2)/2$, giving $x = 6$. The intersection $(2, 2)$ of the two sloping lines is excluded by $x \geq 3$.

Quick Tip to Solve: Check every vertex; the negative coefficient of y favours larger y , and along $y = 4$ the smallest feasible x gives the minimum.



The shaded quadrilateral is the feasible region; its vertices are labelled.

Step 2: Compare the objective values at all vertices The shaded quadrilateral is nonempty and bounded, so the vertex method applies. Substitute each vertex into $Z = x - 5y$:

Vertex	$Z = x - 5y$
$A(3, 5/2)$	$3 - 5(5/2) = -19/2$
$B(3, 3)$	$3 - 15 = -12$
$C(4, 4)$	$4 - 20 = -16$
$D(6, 4)$	$6 - 20 = -14$

The smallest value in the table occurs at C . This point satisfies the constraints directly: $4 - 4 = 0$, $-4 + 2(4) = 4 \geq 2$, $4 \geq 3$, and $0 \leq 4 \leq 4$. Thus the candidate lies in the feasible region, as required.

Step 3: Conclusion The minimum value is $Z_{\min} = -16$, attained at $(x, y) = (4, 4)$.

Quick Tip to Solve: Check every vertex; the negative coefficient of y favours larger y , and along $y = 4$ the smallest feasible x gives the minimum.

29(a). If $y = \log \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2$, then show that $x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$.

[3 marks]

Solution:

Concept:

- For positive arguments, $\log(a/b) = \log a - \log b$. Here $x > 0$ is required by the square roots and the reciprocal square root.
- The first and second derivatives are denoted by $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$. In differentiation,

Quick Tip to Solve: Rewrite the squared radical argument as $(x+1)^2/x$ before taking either derivative; the logarithm then separates into two simple terms.

$$d(\log x)/dx = 1/x.$$

Step 1: Simplify the argument before differentiating Since $x > 0$, combine the terms inside the parentheses:

$$\sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}.$$

Therefore the squared argument is $(x+1)^2/x$, and logarithm laws give

$$y = \log \frac{(x+1)^2}{x} = 2 \log(x+1) - \log x.$$

Both x and $x+1$ are positive in this domain, so these logarithmic identities apply directly.

Step 2: Obtain the two derivatives Differentiate the simplified expression term by term:

$$y_1 = \frac{2}{x+1} - \frac{1}{x}.$$

Differentiating once again gives

$$y_2 = -\frac{2}{(x+1)^2} + \frac{1}{x^2}.$$

The negative sign in the derivative of $2/(x+1)$ and the positive sign in the derivative of $-1/x$ are essential for the required cancellation.

Step 3: Substitute into the required expression Using these derivatives, the left-hand side becomes

$$\begin{aligned} x(x+1)^2 y_2 + (x+1)^2 y_1 &= -2x + \frac{(x+1)^2}{x} + 2(x+1) - \frac{(x+1)^2}{x} \\ &= -2x + 2(x+1) = 2. \end{aligned}$$

Thus the reciprocal terms cancel exactly, leaving a constant independent of x .

Step 4: Conclusion Hence $x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$, as required.

Quick Tip to Solve: Rewrite the squared radical argument as $(x+1)^2/x$ before taking either derivative; the logarithm then separates into two simple terms.

OR

29(b). If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, $-1 < x < 1$, $x \neq y$, then prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$. [3 marks]

Solution:

Concept:

- Introducing $u = \sqrt{1+x}$ and $v = \sqrt{1+y}$ converts the radical relation into a factorable polynomial relation.
- Once y is expressed explicitly as a function of x , the quotient rule gives $(f/g)' = (f'g - fg')/g^2$,

Quick Tip to Solve: Replace the two square roots by u, v ; the original relation factors as $(u+v)(uv-1) = 0$, immediately removing both radicals.

for $g \neq 0$.

Step 1: Substitute the square-root variables Put $u = \sqrt{1+x}$, $v = \sqrt{1+y}$. Then $x = u^2 - 1$ and $y = v^2 - 1$. The given equation is

$$(u^2 - 1)v + (v^2 - 1)u = 0.$$

Collecting terms and factoring yields

$$u^2v + uv^2 - u - v = (u + v)(uv - 1) = 0.$$

Because $-1 < x < 1$, we have $u > 0$. The real square root defining v is nonnegative, so $u + v > 0$. Therefore it is legitimate to divide by this factor, giving $uv = 1$.

Step 2: Express y as a rational function The relation $uv = 1$ means

$$\sqrt{(1+x)(1+y)} = 1.$$

Squaring this nonnegative equality gives $(1+x)(1+y) = 1$. Since $1+x > 0$, division gives

$$1 + y = \frac{1}{1+x}, \quad y = \frac{1}{1+x} - 1 = -\frac{x}{1+x}.$$

This expression also satisfies the original unsquared equation: $\sqrt{1+y} = 1/\sqrt{1+x}$, so its two terms cancel. Thus no extraneous branch has been introduced by squaring. The condition $x \neq y$ excludes their common point in the stated interval and does not alter this expression on the remaining domain.

Step 3: Differentiate the explicit expression Apply the quotient rule to $y = -x/(1+x)$:

$$\frac{dy}{dx} = \frac{-(1+x) - (-x)(1)}{(1+x)^2} = \frac{-1-x+x}{(1+x)^2}.$$

The denominator is nonzero throughout the given interval, so the derivative exists at every permitted point of the curve.

Step 4: Conclusion Hence $\boxed{\frac{dy}{dx} = -\frac{1}{(1+x)^2}}$, as required.

Quick Tip to Solve: Replace the two square roots by u, v ; the original relation factors as $(u+v)(uv-1) = 0$, immediately removing both radicals.

30(a). A die with number 1 to 6 is biased such that $P(2) = \frac{3}{10}$ and probability of other numbers is equal. Find the mean of the number of times number 2 appears on the dice, if the dice is thrown twice. [3 marks]

Solution:

Concept:

- Let X count the occurrences of number 2 in two independent throws. Then X can take only the values 0, 1, 2.
- For a discrete random variable, the mean is $E(X) = \sum_x xP(X = x)$. Independent throws allow probabilities of specified sequences to be multiplied.

Step 1: Identify success and failure probabilities A success means obtaining number 2, so the printed probability is

$$p = P(2) = \frac{3}{10}.$$

The probability of any other number, regarded collectively as a failure, is

$$q = 1 - p = \frac{7}{10}.$$

Since the five other faces are equally likely, each has probability $(7/10)/5 = 7/50$. For counting occurrences of 2, only their combined probability $7/10$ is needed.

Step 2: Construct the probability distribution In two independent throws, no success means two failures; exactly one success can occur in either order; and two successes means obtaining 2 twice. Thus

$$P(X = 0) = q^2 = \frac{49}{100},$$

$$P(X = 1) = pq + qp = 2pq = \frac{42}{100},$$

$$P(X = 2) = p^2 = \frac{9}{100}.$$

These probabilities add to $(49 + 42 + 9)/100 = 1$, confirming a complete distribution. The factor of two for $X = 1$ counts the distinct orders success-failure and failure-success.

Step 3: Evaluate the expected count Weight each possible count by its probability:

$$\begin{aligned} E(X) &= 0 \left(\frac{49}{100} \right) + 1 \left(\frac{42}{100} \right) + 2 \left(\frac{9}{100} \right) \\ &= \frac{42 + 18}{100} = \frac{60}{100} = \frac{3}{5}. \end{aligned}$$

This also agrees with the binomial mean np for $n = 2$ throws and success probability $p = 3/10$. The mean is an average over repetitions, so it need not itself be one of the integer counts 0, 1, 2.

Step 4: Conclusion The mean number of appearances of 2 in two throws is $\boxed{\frac{3}{5}}$, or 0.6.

Quick Tip to Solve: Count a throw of 2 as a success: with two independent throws, the expected success count is immediately $np = 2(3/10) = 3/5$.

30(b). Two dice are thrown. Defined are the following two events A and B :

$A = \{(x, y) : x + y = 9\}$, $B = \{(x, y) : x \neq 3\}$, where (x, y) denote a point in the sample space.

Check if events A and B are independent or mutually exclusive.

[3 marks]

Solution:

Concept:

- Two fair dice have 36 equally likely ordered outcomes (x, y) , where each coordinate ranges from 1 to 6.
- Events are independent if $P(A \cap B) = P(A)P(B)$. They are mutually exclusive if $A \cap B = \emptyset$; these are different tests.

Step 1: Count the outcomes in each event The outcomes whose sum is 9 are

$$A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}.$$

Consequently, $P(A) = 4/36 = 1/9$. For B , the first die must not show 3. It therefore has five permitted values 1, 2, 4, 5, 6, while the second die has all six values. Hence

$$|B| = 5 \times 6 = 30, \quad P(B) = \frac{30}{36} = \frac{5}{6}.$$

The exclusion concerns only the first coordinate, so outcomes with $y = 3$ remain in B when $x \neq 3$.

Step 2: Find the intersection and test mutual exclusivity Of the four outcomes in A , only $(3, 6)$ violates the condition $x \neq 3$. Thus

$$A \cap B = \{(4, 5), (5, 4), (6, 3)\}, \quad P(A \cap B) = \frac{3}{36} = \frac{1}{12}.$$

The intersection contains three outcomes and has positive probability. Therefore the two events can occur together and do not satisfy the condition for mutual exclusivity.

Step 3: Test the product condition for independence Compute the product of their individual probabilities:

$$P(A)P(B) = \frac{1}{9} \times \frac{5}{6} = \frac{5}{54}.$$

Compare on a common denominator:

$$P(A \cap B) = \frac{9}{108}, \quad P(A)P(B) = \frac{10}{108}.$$

These values differ, so the independence condition fails. Equivalently, after knowing that the sum is 9, the probability of B becomes $3/4$, which differs from its original probability $5/6$.

Step 4: Conclusion Events A and B are neither independent nor mutually exclusive.

Quick Tip to Solve: List the four ordered outcomes with sum 9, then remove only $(3, 6)$ for B ; use the remaining three for both event tests.

31. Find: $\int \frac{1}{x} \sqrt{\frac{x+a}{x-a}} dx.$

[3 marks]

Solution:

Concept:

- For a square root of a ratio of linear expressions, use that entire square root as the new variable. This turns the radical and differential into rational expressions.
- Use $\int dt/(1+t^2) = \tan^{-1} t + C$ and $\int dt/(t^2-1) = \frac{1}{2} \log |(t-1)/(t+1)| + C$, where C is arbitrary.

Step 1: Substitute the square root For $a \neq 0$, put $t = \sqrt{(x+a)/(x-a)}$ on a real interval where the integrand is defined. Squaring and rearranging gives

$$t^2(x-a) = x+a, \quad x = \frac{a(t^2+1)}{t^2-1}.$$

Differentiating this expression,

$$dx = \frac{-4at}{(t^2-1)^2} dt.$$

Substitute x , the radical t , and dx together so that the factor t from the original integrand is retained.

Step 2: Resolve and integrate the rational expression The integral becomes

$$\begin{aligned} I &= \int \frac{t}{x} dx = -4 \int \frac{t^2}{(t^2+1)(t^2-1)} dt \\ &= -2 \int \left(\frac{1}{t^2-1} + \frac{1}{t^2+1} \right) dt. \end{aligned}$$

The partial-fraction identity follows because the two numerators add to $2t^2$. Therefore,

$$I = \log \left| \frac{t+1}{t-1} \right| - 2 \tan^{-1} t + C.$$

Retain the absolute value in the logarithm to obtain a real antiderivative on either admissible interval.

If $a = 0$, the radical is 1 for $x \neq 0$, and the integral directly reduces to $\log |x| + C$.

Step 3: Conclusion For $a \neq 0$, the required integral is

$$\log \left| \frac{\sqrt{\frac{x+a}{x-a}} + 1}{\sqrt{\frac{x+a}{x-a}} - 1} \right| - 2 \tan^{-1} \sqrt{\frac{x+a}{x-a}} + C.$$

For $a = 0$, it is $\log |x| + C$.

Quick Tip to Solve: Substitute the entire radical first, then express x and dx in that variable before simplifying the integrand.

32. Using integration, find the area of the region bounded by the line $y = 5x + 2$, the x -axis and the ordinates $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- The geometrical area between a curve $y = f(x)$ and the x -axis over $[a, b]$ is $\int_a^b |f(x)| dx$. The absolute value makes each part of the area nonnegative.
- If the curve crosses the axis inside the interval, split the integral at its zero. On each resulting interval integrate upper boundary minus lower boundary.

Step 1: Locate the intersection with the axis The bounding line is $y = 5x + 2$. Its intersection with the x -axis satisfies

$$5x + 2 = 0, \quad x = -\frac{2}{5}.$$

This zero lies between the prescribed ordinates $x = -2$ and $x = 2$. Since the slope is positive, $5x + 2$ is negative on $[-2, -2/5)$ and positive on $(-2/5, 2]$. Add the positive areas of these two parts.

Step 2: Calculate the area below the axis On the left interval the upper boundary is $y = 0$ and the lower boundary is $y = 5x + 2$. Hence

$$\begin{aligned} A_1 &= \int_{-2}^{-2/5} \{0 - (5x + 2)\} dx \\ &= - \left[\frac{5}{2}x^2 + 2x \right]_{-2}^{-2/5} \\ &= - \left\{ \left(\frac{2}{5} - \frac{4}{5} \right) - (10 - 4) \right\} = \frac{32}{5}. \end{aligned}$$

The strip height $-(5x + 2)$ is positive on this interval.

Step 3: Calculate the area above the axis and combine On the right interval the upper boundary is the line and the lower boundary is the axis. Therefore

$$\begin{aligned} A_2 &= \int_{-2/5}^2 (5x + 2) dx \\ &= \left[\frac{5}{2}x^2 + 2x \right]_{-2/5}^2 \\ &= (10 + 4) - \left(\frac{2}{5} - \frac{4}{5} \right) = \frac{72}{5}. \end{aligned}$$

Adding the two positive contributions gives

$$A = A_1 + A_2 = \frac{32}{5} + \frac{72}{5} = \frac{104}{5}.$$

A single integral over the whole interval gives signed area; splitting at $-2/5$ prevents cancellation.

Step 4: Conclusion The required area is $\boxed{\frac{104}{5}}$ square units, or 20.8 square units.

Quick Tip to Solve: Solve $5x+2=0$ before integrating; split there and reverse the sign of the below-axis integral.

33. Find: $\int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx.$

[5 marks]

Solution:

Concept:

- Resolve a proper rational function with denominator $(x + 2)(x^2 + 1)$ into $\frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1}$, with constant A, B, C .
- Use $\int dx/(x + a) = \ln|x + a| + C$, $\int dx/(1 + x^2) = \tan^{-1} x + C$, and the substitution $u = x^2 + 1$ for the numerator $2x$.

Step 1: Set up the partial fractions The numerator degree is two and the denominator degree is three, so polynomial division is unnecessary. The real linear factor requires a constant numerator, while the irreducible quadratic requires a general linear numerator. Write

$$\frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1}.$$

Multiplication by the denominator gives

$$x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2).$$

This is a polynomial identity, so corresponding coefficients on both sides must be equal.

Step 2: Determine the constants Expanding the right side and grouping powers of x gives

$$x^2 + x + 1 = (A + B)x^2 + (2B + C)x + (A + 2C).$$

Thus

$$A + B = 1, \quad 2B + C = 1, \quad A + 2C = 1.$$

The first two equations imply $B = 1 - A$ and $C = 2A - 1$. Inserting the latter into the third gives $A + 2(2A - 1) = 1$, or $5A = 3$. Consequently

$$A = \frac{3}{5}, \quad B = \frac{2}{5}, \quad C = \frac{1}{5}.$$

These values reproduce the constant, linear and quadratic coefficients of the given numerator.

Step 3: Integrate the three terms Substitute the constants into the decomposition:

$$I = \frac{3}{5} \int \frac{dx}{x + 2} + \frac{1}{5} \int \frac{2x}{x^2 + 1} dx + \frac{1}{5} \int \frac{dx}{x^2 + 1}.$$

For the middle term, let $u = x^2 + 1$, so $du = 2x dx$. Its integral is $\ln(x^2 + 1)$. Since $x^2 + 1 > 0$ for real x , no absolute value is needed there. The first term integrates to $\ln|x + 2|$, and the third to $\tan^{-1} x$. Differentiating these three expressions recovers respectively the three partial fractions, providing a check on both logarithmic coefficients.

Quick Tip to Solve: Substitute $x = -2$ in the polynomial identity first to find $A = 3/5$; then compare the quadratic and linear coefficients to obtain B and C quickly.

Step 4: Conclusion Hence, for $x \neq -2$,

$$I = \frac{3}{5} \ln |x + 2| + \frac{1}{5} \ln(x^2 + 1) + \frac{1}{5} \tan^{-1} x + C.$$

The integration constant is arbitrary on each interval of the domain.

Quick Tip to Solve: Substitute $x = -2$ in the polynomial identity first to find $A = 3/5$; then compare the quadratic and linear coefficients to obtain B and C quickly.

34(a). Find the shortest distance between the lines:

$$\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3} \text{ and } \frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}.$$

[5 marks]

Solution:

Concept:

- For lines through points with position vectors $\mathbf{a}_1, \mathbf{a}_2$ and nonparallel direction vectors $\mathbf{b}_1, \mathbf{b}_2$, their shortest distance is $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$.
- The cross product is perpendicular to both line directions. Projecting the vector between any two base points onto this common perpendicular gives the separation of the lines.

Step 1: Identify base points and direction vectors Comparing each equation with the symmetric form $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the first line passes through $A(-1, 1, 9)$ and has direction vector $\mathbf{b}_1 = (2, 1, -3)$. The second passes through $B(3, -15, 9)$ and has direction vector $\mathbf{b}_2 = (2, -7, 5)$. In particular, $y + 15 = y - (-15)$, so the second base point has negative y-coordinate. Therefore

$$\mathbf{a}_2 - \mathbf{a}_1 = (3 + 1, -15 - 1, 9 - 9) = (4, -16, 0).$$

Step 2: Find the common perpendicular direction Calculate the cross product using a determinant:

$$\begin{aligned} \mathbf{b}_1 \times \mathbf{b}_2 &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -3 \\ 2 & -7 & 5 \end{vmatrix} \\ &= (5 - 21)\hat{\mathbf{i}} - (10 + 6)\hat{\mathbf{j}} + (-14 - 2)\hat{\mathbf{k}} \\ &= -16\hat{\mathbf{i}} - 16\hat{\mathbf{j}} - 16\hat{\mathbf{k}}. \end{aligned}$$

Its magnitude is

$$|\mathbf{b}_1 \times \mathbf{b}_2| = \sqrt{256 + 256 + 256} = 16\sqrt{3}.$$

The cross product is nonzero, so the lines are nonparallel and the stated formula applies.

Step 3: Compute the scalar triple product The numerator of the distance formula is the absolute value of

$$(4, -16, 0) \cdot (-16, -16, -16) = -64 + 256 + 0 = 192.$$

Thus $d = 192/(16\sqrt{3}) = 12/\sqrt{3}$. As a geometric check, the point $P(3, 3, 3)$ on the first line corresponds

Quick Tip to Solve: Read base-point signs carefully from the symmetric equations, and take the absolute value of the scalar triple product so the final distance is positive.

to parameter 2, and $Q(-1, -1, -1)$ on the second to parameter -2 . The connecting vector $\overrightarrow{PQ} = (-4, -4, -4)$ has zero dot product with each direction vector: $-8 - 4 + 12 = 0$ and $-8 + 28 - 20 = 0$. Its length is $\sqrt{48}$, agreeing with the projection calculation.

Step 4: Conclusion The shortest distance between the two given lines is

$$d = \frac{12}{\sqrt{3}} = 4\sqrt{3} \text{ units.}$$

Quick Tip to Solve: Read base-point signs carefully from the symmetric equations, and take the absolute value of the scalar triple product so the final distance is positive.

OR

34(b). Find the image A' of the point $A(2, 1, 2)$ in the line $l : \vec{r} = 4\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$. Also, find the equation of line joining AA' . Find the foot of perpendicular from point A on the line l . [5 marks]

Solution:

Concept:

- The perpendicular foot lies on the given line. The vector joining the point to its foot is perpendicular to the direction of the line, so their dot product is zero.
- Reflection in a line makes the foot the midpoint of the point and its image: $\vec{a}' = 2\vec{h} - \vec{a}$, where $\vec{h}$ is the position vector of the foot.
- A line through $\vec{a}$ with direction $\vec{d}$ is $\vec{r} = \vec{a} + t\vec{d}$, for real t .

Step 1: Locate a general point on the given line Let H denote the foot of the perpendicular from $A(2, 1, 2)$ to l . Since H is on l , its coordinates have the form

$$H = (4 + \lambda, 2 - \lambda, 2 - \lambda).$$

The direction vector of the given line is $\vec{d} = (1, -1, -1)$, and the displacement from A to this point is

$$\overrightarrow{AH} = (2 + \lambda, 1 - \lambda, -\lambda).$$

We determine λ by requiring this displacement to be perpendicular to $\vec{d}$.

Step 2: Determine the perpendicular foot Using $\overrightarrow{AH} \cdot \vec{d} = 0$, we obtain

$$\begin{aligned} (2 + \lambda) - (1 - \lambda) - \lambda &= 0, \\ 1 + 3\lambda &= 0, \\ \lambda &= -\frac{1}{3}. \end{aligned}$$

Therefore the foot is

$$H = \left(\frac{11}{3}, \frac{7}{3}, \frac{7}{3} \right).$$

Quick Tip to Solve: Find the foot with a dot product, then use $A' = 2H - A$; the vector $H - A$ immediately supplies the direction of the joining line.

This gives $\overrightarrow{AH} = \frac{1}{3}(5, 4, 1)$. Its dot product with the line's direction is $\frac{1}{3}(5 - 4 - 1) = 0$, checking the perpendicularity condition.

Step 3: Find the image using the midpoint relation For reflection in l , the points A, H, A' are collinear, A and A' are on opposite sides of H , and $HA = HA'$. Thus H is the midpoint of AA' . Consequently,

$$\begin{aligned} A' &= 2H - A \\ &= \left(\frac{22}{3} - 2, \frac{14}{3} - 1, \frac{14}{3} - 2 \right) \\ &= \left(\frac{16}{3}, \frac{11}{3}, \frac{8}{3} \right). \end{aligned}$$

Taking the midpoint of A and these coordinates reproduces H , confirming that the image is at the correct distance across the line.

Step 4: Write the equation of the line joining the point and its image The vector from A to A' is

$$\overrightarrow{AA'} = \left(\frac{10}{3}, \frac{8}{3}, \frac{2}{3} \right) = \frac{2}{3}(5, 4, 1).$$

Hence a direction vector of AA' is $(5, 4, 1)$. Its equation through A is

$$\vec{r} = 2\hat{i} + \hat{j} + 2\hat{k} + t(5\hat{i} + 4\hat{j} + \hat{k}), \quad t \in \mathbb{R}.$$

In symmetric form, this is $\frac{x-2}{5} = \frac{y-1}{4} = \frac{z-2}{1}$. The foot occurs at $t = \frac{1}{3}$ and the image at $t = \frac{2}{3}$, so both lie on the joining line.

Step 5: Conclusion The image is $A' \left(\frac{16}{3}, \frac{11}{3}, \frac{8}{3} \right)$. The line AA' is $\frac{x-2}{5} = \frac{y-1}{4} = z - 2$, and the foot is $H \left(\frac{11}{3}, \frac{7}{3}, \frac{7}{3} \right)$.

Quick Tip to Solve: Find the foot with a dot product, then use $A' = 2H - A$; the vector $H - A$ immediately supplies the direction of the joining line.

35(a). Given $A = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$, find AB . Hence, solve the system of linear equations:

$$x - y + z = 4$$

$$x - 2y - 2z = 9$$

$$2x + y + 3z = 1$$

[5 marks]

Solution:

Concept:

- In a matrix product, entry (i, j) is the dot product of row i of the first matrix with column j of the second matrix. Thus $(AB)_{ij} = \sum_{k=1}^3 a_{ik}b_{kj}$.
- A linear system is written as $BX = C$, where B contains the coefficients, X is the column of

Quick Tip to Solve: Recognise the coefficient matrix as B ; once AB is a nonzero scalar multiple of the identity, multiply the constant column by A and divide by that scalar.

unknowns and C is the column of constants. If $AB = kI$ for nonzero k , multiplying by A gives $kX = AC$.

Step 1: Compute all entries of the product Multiply each row of A by each column of B , preserving the order AB . The first row gives

$$\begin{aligned}(AB)_{11} &= (-4)(1) + 4(1) + 4(2) = 8, \\(AB)_{12} &= (-4)(-1) + 4(-2) + 4(1) = 0, \\(AB)_{13} &= (-4)(1) + 4(-2) + 4(3) = 0.\end{aligned}$$

Similarly, the second row gives

$$\begin{aligned}(AB)_{21} &= (-7)(1) + 1(1) + 3(2) = 0, \\(AB)_{22} &= (-7)(-1) + 1(-2) + 3(1) = 8, \\(AB)_{23} &= (-7)(1) + 1(-2) + 3(3) = 0.\end{aligned}$$

For the third row,

$$\begin{aligned}(AB)_{31} &= 5(1) + (-3)(1) + (-1)(2) = 0, \\(AB)_{32} &= 5(-1) + (-3)(-2) + (-1)(1) = 0, \\(AB)_{33} &= 5(1) + (-3)(-2) + (-1)(3) = 8.\end{aligned}$$

Hence the product is $AB = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8I$.

Step 2: Express the equations in matrix form Reading the coefficients in the order x, y, z , the three equations have coefficient matrix B . Put

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad C = \begin{pmatrix} 4 \\ 9 \\ 1 \end{pmatrix}.$$

The system is therefore $BX = C$. Multiplying both sides on the left by A , and using associativity, gives

$$A(BX) = AC \implies (AB)X = AC \implies 8X = AC.$$

Since $AB = 8I$, the square matrix B is invertible and $B^{-1} = A/8$. Thus this procedure gives the unique solution.

Step 3: Multiply the constant column and verify the values Calculate

$$AC = \begin{pmatrix} -4(4) + 4(9) + 4(1) \\ -7(4) + 1(9) + 3(1) \\ 5(4) - 3(9) - 1(1) \end{pmatrix} = \begin{pmatrix} 24 \\ -16 \\ -8 \end{pmatrix}.$$

Quick Tip to Solve: Recognise the coefficient matrix as B ; once AB is a nonzero scalar multiple of the identity, multiply the constant column by A and divide by that scalar.

Dividing the column by 8 gives $X = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$. Substitute these values into each original equation to check the signs:

$$3 - (-2) + (-1) = 4,$$

$$3 - 2(-2) - 2(-1) = 9,$$

$$2(3) + (-2) + 3(-1) = 1.$$

All three left-hand sides agree with the corresponding constants.

Step 4: Conclusion $AB = 8I$, and the unique solution of the given system is $x = 3, y = -2, z = -1$.

Quick Tip to Solve: Recognise the coefficient matrix as B; once AB is a nonzero scalar multiple of the identity, multiply the constant column by A and divide by that scalar.

OR

35(b). If $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$, then find A^{-1} . Hence, solve the system of linear equations:

$$x - 2y = 10$$

$$2x - y - z = 8$$

$$-2y + z = 7$$

[5 marks]

Solution:

Concept:

- For a square matrix A , the inverse exists when $|A| \neq 0$, and $A^{-1} = \frac{\text{adj } A}{|A|}$. The adjugate is the transpose of the matrix of cofactors.
- A system of linear equations is written as $BX = C$, where B contains the coefficients of the unknowns in the chosen order. Its unique solution is $X = B^{-1}C$, provided $|B| \neq 0$.
- Transposition preserves products in reverse order, giving $(A^T)^{-1} = (A^{-1})^T$. This is useful because the coefficient matrix here is A^T .

Step 1: Calculate the determinant Expanding along the first row, with the alternating cofactor signs, gives

$$\begin{aligned} |A| &= 1 \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} \\ &= 1(-1 - 2) - 2(-2) = 1. \end{aligned}$$

Thus the determinant is nonzero and the inverse exists. The zero in the third position of the first row makes its contribution vanish.

Step 2: Form the cofactors and inverse Deleting the row and column of each entry and multiplying

Quick Tip to Solve: Compare the coefficient matrix with the given matrix before multiplying: here it is A^T , so use $(A^{-1})^T$, not A^{-1} .

its minor by $(-1)^{i+j}$, the cofactor matrix is

$$K = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

For example, $K_{13} = (-2)(-1) - (-1)(0) = 2$, and $K_{31} = 2(-2) - 0(-1) = -4$. Transposing the cofactor matrix and dividing by the determinant gives

$$A^{-1} = \frac{K^T}{1} = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Direct multiplication also confirms $AA^{-1} = I_3$.

Step 3: Express the given system using the correct coefficient matrix In the order x, y, z , the three equations have coefficient matrix

$$B = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix} = A^T.$$

Hence, writing $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}$, the system is $A^T X = C$. We must use the inverse of the transpose, because the rows of A itself are not the coefficient rows of the printed equations.

Step 4: Solve by matrix multiplication and check Multiplying by $(A^{-1})^T$,

$$\begin{aligned} X &= (A^{-1})^T C \\ &= \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}. \end{aligned}$$

Checking the original equations gives $0 - 2(-5) = 10$, $2(0) - (-5) - (-3) = 8$, and $-2(-5) + (-3) = 7$. All three right-hand sides agree, confirming the result.

Step 5: Conclusion

$$A^{-1} = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad x = 0, y = -5, z = -3.$$

Quick Tip to Solve: Compare the coefficient matrix with the given matrix before multiplying: here it is A^T , so use $(A^{-1})^T$, not A^{-1} .

Section E (12 marks)

(This section comprises of 3 case study based questions of 4 marks each.)

36. A school is organizing a debate competition with participants as speakers $S = \{S_1, S_2, S_3, S_4\}$ and these are judged by judges $J = \{J_1, J_2, J_3\}$. Each speaker can be assigned one judge. Let R be a relation from set S to J defined as $R = \{(x, y) : \text{speaker } x \text{ is judged by judge } y, x \in S, y \in J\}$.

Based on the above, answer the following:

[4 marks]

36(i). How many relations can be there from S to J ?

[1 mark]

Solution:

Concept:

- A relation from a set S to a set J is any subset of the Cartesian product $S \times J$.
- If a finite set contains n elements, it has 2^n subsets: each element can independently be included or excluded.

Step 1: Count the available ordered pairs There are four speakers in S and three judges in J . Each speaker can be paired with each of the three judges, so the Cartesian product contains

$$|S \times J| = |S| |J| = 4 \times 3 = 12$$

distinct ordered pairs.

Step 2: Count all subsets of the Cartesian product For each of the twelve ordered pairs, there are two independent choices: include it in the relation or omit it. Therefore the number of relations is $2^{12} = 4096$. This count includes the empty relation and the full Cartesian product. Although the context mentions assigning a judge, this part asks for relations; requiring exactly one judge for every speaker would instead count functions, giving 3^4 .

Step 3: Conclusion The number of relations from S to J is 4096.

Quick Tip to Solve: For relations count subsets, using $2^{|S||J|}$; use $|J|^{|S|}$ only when counting functions from S to J .

36(ii). A student identifies a function from S to J as $f = \{(S_1, J_1), (S_2, J_2), (S_3, J_2), (S_4, J_3)\}$. Check if it is bijective.

[1 mark]

Solution:

Concept:

- A function is one-one if distinct elements of its domain have distinct images.
- A function is onto if every element of its codomain occurs as an image. It is bijective only when it is both one-one and onto.

Step 1: Test whether the function is one-one The printed mapping gives $f(S_2) = J_2 = f(S_3)$, although $S_2 \neq S_3$. Thus two different speakers are assigned the same judge, so f is not one-one.

Quick Tip to Solve: To disprove bijectivity here, point directly to the two distinct speakers S_2 and S_3 who share J_2 .

Step 2: Check the images against the codomain The image set is $f(S) = \{J_1, J_2, J_3\} = J$: J_1 is the image of S_1 , J_2 is the image of S_2 and S_3 , and J_3 is the image of S_4 . Therefore f is onto. Each speaker still has exactly one image, so the repeated judge does not prevent this relation from being a function.

Step 3: Conclusion f is onto but not one-one; hence it is not bijective.

Quick Tip to Solve: To disprove bijectivity here, point directly to the two distinct speakers S_2 and S_3 who share J_2 .

36(iii)(a). How many one-one functions can be there from set S to set J ?

[2 marks]

Solution:

Concept:

- In a one-one function, no two distinct domain elements can have the same image.
- For finite sets, an injection from a set with m elements to one with n elements can exist only if $m \leq n$. This is the pigeonhole principle.

Step 1: Count the domain and codomain elements The speaker set is $S = \{S_1, S_2, S_3, S_4\}$, so $|S| = 4$. The judge set is $J = \{J_1, J_2, J_3\}$, so $|J| = 3$. A function from S to J must assign one of these three judges to each of the four speakers.

Step 2: Apply the condition for an injection To make the function one-one, all four speakers would need different judges. Assigning distinct judges to the first three speakers uses every element of J , leaving no unused judge for the fourth speaker. Thus any assignment necessarily repeats at least one image. Equivalently, the required inequality $4 \leq 3$ is false. Although functions from S to J exist, none of them can satisfy the extra one-one condition.

Step 3: Conclusion The number of one-one functions from S to J is $\boxed{0}$.

Quick Tip to Solve: Compare set sizes before counting injections: a larger domain cannot map injectively into a smaller codomain.

OR

36(iii)(b). Another student considers a relation $R_1 = \{(S_1, S_2), (S_2, S_4)\}$ in set S . Write minimum ordered pairs to be included in R_1 so that R_1 is reflexive but not symmetric.

[2 marks]

Solution:

Concept:

- A relation on S is reflexive if (x, x) belongs to it for every $x \in S$.
- A relation is symmetric if (x, y) in the relation always implies (y, x) is also in it. One pair with a missing reverse disproves symmetry.

Step 1: Identify all required self-pairs Here $S = \{S_1, S_2, S_3, S_4\}$ and the printed relation is $R_1 = \{(S_1, S_2), (S_2, S_4)\}$. It contains no diagonal pair. Reflexivity therefore requires adding (S_1, S_1) ,

Quick Tip to Solve: Add missing diagonal pairs for reflexivity, then use an existing pair whose reverse is absent to disprove symmetry.

(S_2, S_2) , (S_3, S_3) and (S_4, S_4) . All four are necessary: omitting any one leaves that particular element unrelated to itself.

Step 2: Check that adding these pairs preserves the lack of symmetry Add exactly the four required self-pairs and no others. The resulting relation contains (S_1, S_2) , but does not contain its reverse (S_2, S_1) . It also contains (S_2, S_4) without (S_4, S_2) . Adding self-pairs does not supply either reverse pair, because each self-pair is its own reverse. Consequently the enlarged relation is reflexive and remains non-symmetric. No extra off-diagonal pair is needed to achieve the requested properties.

Step 3: Conclusion The minimum addition is $\{(S_1, S_1), (S_2, S_2), (S_3, S_3), (S_4, S_4)\}$, comprising four ordered pairs.

Quick Tip to Solve: Add missing diagonal pairs for reflexivity, then use an existing pair whose reverse is absent to disprove symmetry.

37. Three persons viz. Amber, Bonzi and Comet are manufacturing cars which run on petrol and on battery as well. Their production share in the market is 60%, 30% and 10% respectively. Of their respective production capacities, 20%, 10% and 5% cars respectively are electric (or battery operated). Based on the above, answer the following: [4 marks]

37(i)(a). What is the probability that a randomly selected car is an electric car? [2 marks]

Solution:

Concept:

- Let A, B, C be the events that the car is manufactured by Amber, Bonzi and Comet respectively, and let E denote an electric car.
- The manufacturers form a mutually exclusive and exhaustive partition. Thus $P(E) = P(A)P(E | A) + P(B)P(E | B) + P(C)P(E | C)$.

Step 1: Express the given percentages as probabilities The market shares give $P(A) = 0.60$, $P(B) = 0.30$, and $P(C) = 0.10$. The electric fractions within the respective manufacturers' production are $P(E | A) = 0.20$, $P(E | B) = 0.10$, and $P(E | C) = 0.05$. These are conditional probabilities; the electric fractions alone cannot be added or averaged without weighting by the production shares.

Step 2: Add the contributions from the three manufacturers Every selected electric car belongs to exactly one of the three manufacturers. Applying the law of total probability gives

$$\begin{aligned} P(E) &= (0.60)(0.20) + (0.30)(0.10) + (0.10)(0.05) \\ &= 0.120 + 0.030 + 0.005 = 0.155. \end{aligned}$$

Each product is the probability of selecting an electric car from the specified manufacturer, so their sum covers all electric cars.

Step 3: Conclusion The required probability is $\frac{31}{200} = 0.155$, or 15.5%.

Quick Tip to Solve: Multiply each manufacturer's market share by its electric fraction before adding the three products.

37(i)(b). What is the probability that a randomly selected car is a petrol car?

[2 marks]

Solution:

Concept:

- Let A, B, C denote manufacture by Amber, Bonzi and Comet respectively, and let F denote a petrol car. Their market shares are the probabilities of these manufacturer events.
- The petrol fraction of each manufacturer's cars is the complement of its electric fraction. For the exhaustive manufacturer partition, $P(F) = \sum_{M=A,B,C} P(M)P(F | M)$.

Step 1: Determine the petrol fractions The given electric percentages are 20%, 10%, and 5% for Amber, Bonzi and Comet. Consequently the corresponding petrol percentages are 80%, 90%, and 95%. Thus $P(F | A) = 0.80$, $P(F | B) = 0.90$, and $P(F | C) = 0.95$. The market shares remain $P(A) = 0.60$, $P(B) = 0.30$, and $P(C) = 0.10$.

Step 2: Weight each fraction by its market share Since a car comes from exactly one of these manufacturers, the three ways of selecting a petrol car are disjoint. Therefore

$$\begin{aligned} P(F) &= (0.60)(0.80) + (0.30)(0.90) + (0.10)(0.95) \\ &= 0.480 + 0.270 + 0.095 = 0.845. \end{aligned}$$

As a check, the overall electric probability is 0.155, and $0.845 + 0.155 = 1$, consistent with the two car types covering the production.

Step 3: Conclusion The required probability is $\frac{169}{200} = 0.845$, or 84.5%.

Quick Tip to Solve: Take the complement of each electric fraction before weighting, or subtract the total electric probability from one.

37(ii). A car is selected at random and is found to be electric. What is the probability that it was manufactured by Comet?

[1 mark]

Solution:

Concept:

- Let C mean manufacture by Comet and E mean that the car is electric. Bayes' theorem gives $P(C | E) = P(C)P(E | C)/P(E)$.
- Conditioning on an electric car restricts the sample space to all electric cars, so the denominator must be the total electric probability.

Step 1: Find the joint and total electric probabilities Comet's market share is 0.10 and its electric fraction is 0.05. Thus $P(C \cap E) = 0.10 \times 0.05 = 0.005$. Including the other manufacturers, $P(E) = 0.60 \times 0.20 + 0.30 \times 0.10 + 0.10 \times 0.05 = 0.155$.

Step 2: Apply the conditional-probability formula Among the electric cars, Comet's contribution is

Quick Tip to Solve: For manufacturer given electric, divide that manufacturer's electric contribution by the total electric contribution.

divided by the total electric contribution:

$$P(C | E) = \frac{0.005}{0.155} = \frac{5}{155} = \frac{1}{31}.$$

Using 0.05 alone would instead answer the reverse question: whether a Comet car is electric.

Step 3: Conclusion The required probability is $\boxed{\frac{1}{31}}$.

Quick Tip to Solve: For manufacturer given electric, divide that manufacturer's electric contribution by the total electric contribution.

37(iii). A car is selected at random and is found to be electric. What is the probability that it was manufactured by Amber or Bonzi? [1 mark]

Solution:

Concept:

- Let A, B, C denote manufacture by Amber, Bonzi and Comet, and let E denote an electric car. Since a car has only one manufacturer, A and B are disjoint.
- Conditional probability gives $P(A \cup B | E) = P((A \cup B) \cap E) / P(E)$; the numerator is the sum of Amber's and Bonzi's electric contributions.

Step 1: Find the electric contributions Amber contributes $0.60 \times 0.20 = 0.120$, Bonzi contributes $0.30 \times 0.10 = 0.030$, and Comet contributes $0.10 \times 0.05 = 0.005$. The total electric probability is therefore $P(E) = 0.120 + 0.030 + 0.005 = 0.155$.

Step 2: Condition on the car being electric The favourable electric contribution for Amber or Bonzi is $0.120 + 0.030 = 0.150$. Hence

$$P(A \cup B | E) = \frac{0.150}{0.155} = \frac{150}{155} = \frac{30}{31}.$$

This also equals $1 - P(C | E) = 1 - 1/31$, since the manufacturer events are exhaustive even after conditioning on an electric car.

Step 3: Conclusion The required probability is $\boxed{\frac{30}{31}}$.

Quick Tip to Solve: Add the joint electric contributions of the two manufacturers first, then divide by the overall electric probability.

38. A small town is analyzing the pattern of a new street light installation. The lights are set up in such a way that the intensity of light at any point x metres from the start of the street can be modelled by $f(x) = e^x \sin x$, where x is in metres.

Based on the above, answer the following:

[4 marks]

38(i). Find the intervals on which the $f(x)$ is increasing or decreasing, $x \in [0, \pi]$.

[2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where $f'(x) > 0$ and decreasing where $f'(x) < 0$. A stationary point divides the interval into regions in which the derivative can be tested.
- The product rule gives $(uv)' = u'v + uv'$. Here $u = e^x$, $v = \sin x$, and the positive factor e^x never changes the sign of the derivative.

Step 1: Differentiate the intensity function For $f(x) = e^x \sin x$, apply the product rule:

$$f'(x) = e^x \sin x + e^x \cos x = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right).$$

Since $e^x > 0$, the sign of $f'(x)$ is the sign of $\sin(x + \pi/4)$. On the given domain, the angle $x + \pi/4$ ranges from $\pi/4$ to $5\pi/4$.

Step 2: Locate the stationary point and test signs The only zero of the derivative in $[0, \pi]$ occurs when $x + \pi/4 = \pi$, giving $x = 3\pi/4$. For $0 < x < 3\pi/4$, the angle lies in $(\pi/4, \pi)$, where sine is positive, so $f'(x) > 0$. For $3\pi/4 < x < \pi$, the angle lies in $(\pi, 5\pi/4)$, where sine is negative, so $f'(x) < 0$. Thus the derivative changes from positive to negative at the stationary point.

Step 3: Conclusion The function is increasing on $(0, \frac{3\pi}{4})$ and decreasing on $(\frac{3\pi}{4}, \pi)$. Including the boundary points, it is strictly increasing on $[0, 3\pi/4]$ and strictly decreasing on $[3\pi/4, \pi]$.

Quick Tip to Solve: Rewrite $\sin x + \cos x$ as $\sqrt{2} \sin(x + \pi/4)$; this makes the zero and the sign on each interval immediate.

38(ii). Verify, whether each critical point when $x \in [0, \pi]$ is a point of local maximum or local minimum or a point of inflexion.

[2 marks]

Solution:

Concept:

- An interior critical point of a differentiable function satisfies $f'(x) = 0$. If $f''(c) < 0$ at a stationary point c , the second derivative test identifies a strict local maximum; if $f''(c) > 0$, it identifies a strict local minimum.
- A point of inflexion requires a change in concavity, hence a change in the sign of f'' . A zero of f' does not by itself imply inflexion.

Step 1: Find the critical point in the stated interval Differentiate the given function to obtain

$$f'(x) = e^x(\sin x + \cos x) = \sqrt{2}e^x \sin(x + \pi/4).$$

Quick Tip to Solve: After finding $f' = 0$, use $f'' = 2e^x \cos x$; its sign at $3\pi/4$ gives the classification immediately.

This derivative exists everywhere. As e^x is positive, its only zero for $0 \leq x \leq \pi$ is $c = 3\pi/4$. Thus there is precisely one interior critical point to classify.

Step 2: Apply the second derivative test Differentiating once more gives

$$\begin{aligned}f''(x) &= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) \\&= 2e^x \cos x, \\f''(3\pi/4) &= 2e^{3\pi/4} \left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}e^{3\pi/4} < 0.\end{aligned}$$

Hence the critical point is a strict local maximum. This agrees with the positive-to-negative change of f' . Moreover, f'' stays negative near this point, so it is not a point of inflexion. For completeness, concavity changes at $x = \pi/2$, but $f'(\pi/2) = e^{\pi/2} \neq 0$; that inflexion point is not a critical point.

Step 3: Conclusion The sole critical point is $x = 3\pi/4$, a strict local maximum with value $f(3\pi/4) = e^{3\pi/4}/\sqrt{2}$. There is no interior critical point that is a local minimum or a point of inflexion.

Quick Tip to Solve: After finding $f' = 0$, use $f'' = 2e^x \cos x$; its sign at $3\pi/4$ gives the classification immediately.