CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-4-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All questions are compulsory.**

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E.**

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section B, **3** questions in Section C, **2** questions in Section D and **2** questions in Section E.

(ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The principal branch of $\cos^{-1} x$ is: [1 mark]

(A) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

(B) $[\pi, 2\pi]$

(C) $[0, \pi]$

(D) $[2\pi, 3\pi]$

Correct Answer: (C) $[0, \pi]$

Solution:

Concept:

- The principal branch is the chosen range of inverse cosine that makes its value unique.
- Cosine decreases from 1 to -1 on $[0, \pi]$, so its restriction there is one-one and onto $[-1, 1]$.

Step 1: Identify the inverse range For every $x \in [-1, 1]$, there is exactly one angle $\theta \in [0, \pi]$ satisfying $\cos\theta = x$. Thus $\cos^{-1} x$ takes its principal values in this interval, including both endpoints.

Step 2: Check the other intervals Option (A), $[\pi/2, 3\pi/2]$, is not one-one for cosine and does not cover positive cosine values. Options (B) and (D) are alternative restrictions but are not the standard principal branch. The conventional branch includes $\cos^{-1} 1 = 0$ and $\cos^{-1}(-1) = \pi$.

Step 3: Conclusion The principal branch is $\boxed{[0, \pi]}$. Hence, the correct option is (C).

Quick Tip to Solve: Remember the endpoint checks: inverse cosine of 1 is 0, and inverse cosine of -1 is pi.

---

2. The values of λ so that $f(x) = \sin x - \cos x - \lambda x + C$ decreases for all real values of x are: [1 mark]

(A) $1 < \lambda < \sqrt{2}$

(B) $\lambda \geq 1$

(C) $\lambda \geq \sqrt{2}$

(D) $\lambda < 1$

Correct Answer: (C) $\lambda \geq \sqrt{2}$

Solution:

Concept:

- A differentiable decreasing function must have $f'(x) \leq 0$. A derivative that is negative except at isolated points still gives a strictly decreasing function.
- The expression $\sin x + \cos x = \sqrt{2}\sin(x + \pi/4)$ has maximum $\sqrt{2}$ over the real numbers.

Step 1: Bound the derivative for every real input Differentiating, with C constant, gives

$$f'(x) = \cos x + \sin x - \lambda = \sqrt{2}\sin(x + \pi/4) - \lambda.$$

Quick Tip to Solve: For a derivative of the form $a\sin x + b\cos x - \lambda$, compare λ with the amplitude $\sqrt{a^2 + b^2}$.

Its greatest value is $\sqrt{2} - \lambda$. Therefore the derivative is nonpositive for every real x precisely when $\lambda \geq \sqrt{2}$. If $\lambda < \sqrt{2}$, then at $x = \pi/4$ the derivative is positive, so the function cannot decrease everywhere.

Step 2: Check the endpoint and the other options For $\lambda > \sqrt{2}$, the derivative is strictly negative everywhere. At $\lambda = \sqrt{2}$, it vanishes only at $x = \pi/4 + 2k\pi$, where k is an integer, and is negative elsewhere. Its integral over any interval of positive length is negative, so the function remains strictly decreasing. Option (A) lies below the necessary bound. Option (B) includes values such as $\lambda = 1$ for which the derivative becomes positive, and option (D) also lies below the bound.

Step 3: Conclusion The required values are $\lambda \geq \sqrt{2}$. Hence, the correct option is (C).

Quick Tip to Solve: For a derivative of the form $a \sin x + b \cos x - \lambda$, compare λ with the amplitude $\sqrt{a^2 + b^2}$.

3. If A and B are square matrices of same order such that $AB = A$ and $BA = B$, then $A^2 + B^2$ is equal to: [1 mark]

(A) $A + B$

(B) BA

(C) $2(A + B)$

(D) $2BA$

Correct Answer: (A) $A + B$

Solution:

Concept:

- Matrix multiplication is associative: $(AB)C = A(BC)$. The order of the factors must be retained.
- A square matrix M is idempotent when $M^2 = M$. Use the two given products to establish this property for both matrices.

Step 1: Obtain the square of each matrix From $AB = A$ and $BA = B$, associativity gives

$$A^2 = (AB)A = A(BA) = AB = A, \qquad B^2 = (BA)B = B(AB) = BA = B.$$

These equalities do not require assuming that A and B commute or that either matrix has an inverse.

Step 2: Compare the proposed expressions Adding the two identities yields $A^2 + B^2 = A + B$, exactly the expression in option (A). Since $BA = B$, option (B) would give only B, while option (D) would give $2B$; neither equals the required sum in general. Option (C) doubles the required expression. For instance, $A = B = I$ rules out (B) and (C). For (D), take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. These satisfy both given products, but $A + B \neq 2B$.

Step 3: Conclusion $A^2 + B^2 = A + B$. Hence, the correct option is (A).

Quick Tip to Solve: Replace one factor using the given product, then regroup by associativity; never interchange matrix factors.

4. If $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{for } x \neq \dfrac{\pi}{2} \\ k, & \text{for } x = \dfrac{\pi}{2} \end{cases}$ **is continuous at** $x = \dfrac{\pi}{2}$**, then the value of** k **is:** [1 mark]

(A) $\dfrac{3}{2}$

(B) $\dfrac{1}{6}$

(C) $\dfrac{1}{2}$

(D) 1

Correct Answer: (C) $\dfrac{1}{2}$

Solution:

Concept:

- Continuity at a requires $\lim_{x\to a} f(x) = f(a)$. Here $f(\pi/2) = k$.
- Use $1 - s^3 = (1-s)(1+s+s^2)$ and $\cos^2 x = (1-\sin x)(1+\sin x)$.

Step 1: Remove the indeterminate factor For x near but different from $\pi/2$, factorisation gives

$$\frac{1-\sin^3 x}{3\cos^2 x} = \frac{(1-\sin x)(1+\sin x+\sin^2 x)}{3(1-\sin x)(1+\sin x)} = \frac{1+\sin x+\sin^2 x}{3(1+\sin x)}.$$

This cancellation is valid in a sufficiently small punctured neighbourhood.

Step 2: Evaluate the limit As $x \to \pi/2$, $\sin x \to 1$, so the simplified expression tends to

$$\frac{1+1+1}{3(1+1)} = \frac{1}{2}.$$

Therefore continuity forces the assigned value k to equal this limit. The other printed values $3/2$, $1/6$, and 1 do not equal the limit.

Step 3: Conclusion $\boxed{k = \dfrac{1}{2}}$. Hence, the correct option is (C).

Quick Tip to Solve: Factor the numerator as a difference of cubes before substituting the limiting sine value.

5. For real x, let $f(x) = x^3 + 5x + 1$. Then: [1 mark]

(A) f is one-one but not onto on $\mathbb{R}$

(B) f is onto on $\mathbb{R}$ but not one-one

(C) f is one-one and onto on $\mathbb{R}$

(D) f is neither one-one nor onto on $\mathbb{R}$

Correct Answer: (C) f is one-one and onto on $\mathbb{R}$

Solution:

Concept:

- A function is one-one if distinct inputs have distinct outputs; a strictly increasing function on $\mathbb{R}$ has this property.
- A continuous function on $\mathbb{R}$ with limits $-\infty$ and $+\infty$ at the two ends attains every real value, so it is onto $\mathbb{R}$.

Step 1: Test whether the function is one-one Differentiating the polynomial gives $f'(x) = 3x^2 + 5$. Since $x^2 \geq 0$, we have $f'(x) \geq 5 > 0$ for every real x. Thus f is strictly increasing throughout its domain. In particular, if $x_1 < x_2$, then $f(x_1) < f(x_2)$, proving that the function is one-one: equal outputs therefore force equal inputs.

Step 2: Test whether every real output is attained The cubic polynomial is continuous, and its leading term gives

$$\lim_{x \to -\infty} f(x) = -\infty, \qquad \lim_{x \to +\infty} f(x) = +\infty.$$

Given any real number y, choose real $a < b$ such that $f(a) < y < f(b)$. The intermediate value theorem supplies an $x \in (a, b)$ with $f(x) = y$. Hence its range is all of $\mathbb{R}$. Options (A) and (B) each deny one established property, and option (D) denies both.

Step 3: Conclusion The function is both one-one and onto on $\mathbb{R}$. Hence, the correct option is (C).

Quick Tip to Solve: For an odd-degree polynomial, check the derivative for injectivity and the two infinite limits for surjectivity.

6. If the direction cosines of a line are $\lambda, \lambda, \lambda$, then λ is equal to: [1 mark]

(A) $-\frac{1}{\sqrt{3}}$

(B) 1

(C) $\frac{1}{\sqrt{3}}$

(D) $\pm\frac{1}{\sqrt{3}}$

Correct Answer: (D) $\pm\frac{1}{\sqrt{3}}$

Solution:

Concept:

- Direction cosines l, m, n satisfy $l^2 + m^2 + n^2 = 1$.
- Reversing the orientation of a line changes all three direction cosines in sign but preserves this identity.

Step 1: Apply the normalisation identity Here $l = m = n = \lambda$. Substitution into the direction-cosine identity gives

$$\lambda^2 + \lambda^2 + \lambda^2 = 1, \qquad 3\lambda^2 = 1.$$

The relation is a sum of squares, so it constrains the magnitude rather than the sign.

Step 2: Retain both signs Thus $\lambda^2 = 1/3$, whose real solutions are $\lambda = 1/\sqrt{3}$ and $\lambda = -1/\sqrt{3}$. Both triples have squared sum equal to one and describe opposite orientations along the same line. Options (A) and (C) each omit one allowed sign; option (B) gives squared sum three and is impossible.

Step 3: Conclusion $\boxed{\lambda = \pm\frac{1}{\sqrt{3}}}$. Hence, the correct option is (D).

Quick Tip to Solve: When equal direction cosines are given, use three times their square equals one and retain both signs.

7. If $\begin{vmatrix} -1 & 2 & 4 \\ 1 & x & 1 \\ 0 & 3 & 3x \end{vmatrix} = -57$, then the product of the possible values of x is: [1 mark]

(A) -24

(B) -16

(C) 16

(D) 24

Correct Answer: (A) -24

Solution:

Concept:

- Expand a third-order determinant along a convenient row, using cofactor signs $+, -, +$.

Quick Tip to Solve: After expanding, use the constant-to-leading coefficient ratio to find the product without solving.

- For $ax^2 + bx + c = 0$, the product of its two roots is c/a.

Step 1: Expand the determinant Expanding along the first row gives

$$\begin{aligned} D &= (-1)(3x^2 - 3) - 2(3x) + 4(3) \\ &= -3x^2 - 6x + 15. \end{aligned}$$

The second minor is $3x$ and the third minor is 3; the alternating cofactor sign makes the second term negative.

Step 2: Use the given value Set $-3x^2 - 6x + 15 = -57$. Rearranging and dividing by -3 yields $x^2 + 2x - 24 = 0$, or $(x + 6)(x - 4) = 0$. The possible values are -6 and 4, with product -24. This also follows directly from the constant term of the monic quadratic. Therefore the alternatives -16, 16, and 24 do not match.

Step 3: Conclusion The required product is $\boxed{-24}$. Hence, the correct option is (A).

Quick Tip to Solve: After expanding, use the constant-to-leading coefficient ratio to find the product without solving.

8. The matrix $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -7 \\ 2 & 7 & 0 \end{pmatrix}$ **is a:** [1 mark]

(A) diagonal matrix

(B) symmetric matrix

(C) skew symmetric matrix

(D) scalar matrix

Correct Answer: (C) skew symmetric matrix

Solution:

Concept:

- A real square matrix is skew symmetric if $A^T = -A$; all its diagonal entries are zero.
- A symmetric matrix satisfies $A^T = A$. A diagonal matrix has every off-diagonal entry zero, and a scalar matrix is diagonal with equal diagonal entries.

Step 1: Compare entries across the diagonal The given diagonal entries are all zero. Across the diagonal, the entry pairs are $(1, -1)$, $(-2, 2)$, and $(-7, 7)$. Each pair consists of opposite numbers, which is precisely the entrywise condition $a_{ji} = -a_{ij}$.

Step 2: Verify the matrix identity and alternatives Transposing gives

$$A^T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix} = -A.$$

It is not symmetric because, for example, $a_{12} = 1 \neq -1 = a_{21}$. Its nonzero off-diagonal entries also

Quick Tip to Solve: Check opposite entries across the main diagonal; a sign reversal in every pair identifies skew symmetry.

rule out both diagonal and scalar matrices.

Step 3: Conclusion The matrix is $\boxed{\text{skew symmetric}}$. Hence, the correct option is (C).

Quick Tip to Solve: Check opposite entries across the main diagonal; a sign reversal in every pair identifies skew symmetry.

9. If $f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 + ax, & 1 < x < 2 \end{cases}$ **is continuous for** $x \in (0, 2)$**, then** a **is equal to:** [1 mark]

(A) -4

(B) $-\frac{7}{2}$

(C) -2

(D) -1

Correct Answer: (D) -1

Solution:

Concept:

- Continuity at a joining point requires the left limit, the right limit and the function value to agree.
- Each polynomial branch is continuous inside its own interval, so only $x = 1$ needs checking.

Step 1: Evaluate the first branch The first branch includes $x = 1$, hence $f(1) = 3(1) - 2 = 1$. Its left-hand limit is also $\lim_{x \to 1^-}(3x - 2) = 1$. The endpoints 0 and 2 are excluded from the specified interval and impose no further conditions.

Step 2: Match the right-hand limit The second branch gives $\lim_{x \to 1^+}(2x^2 + ax) = 2 + a$. Continuity therefore requires $2 + a = 1$, giving $a = -1$. The values in (A), (B) and (C) give right limits $-2, -3/2, 0$, respectively, so none equals the required value 1.

Step 3: Conclusion $\boxed{a = -1}$. Hence, the correct option is (D).

Quick Tip to Solve: At the joining point, substitute into both branches and equate the resulting values.

10. If $f : \mathbb{N} \to W$ is defined as $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$, then f is: [1 mark]

(A) injective only

(B) surjective only

(C) a bijection

(D) neither surjective nor injective

Correct Answer: (B) surjective only

Solution:

Concept:

- Injectivity means distinct inputs have distinct images.
- Surjectivity means every element of the codomain $W = \{0, 1, 2, \ldots\}$ has a preimage in $\mathbb{N}$.

Step 1: Test injectivity The distinct natural numbers 1 and 3 are both odd, so $f(1) = f(3) = 0$. This is a counterexample to injectivity. Consequently, option (A) is false and the function cannot be a bijection as claimed in (C).

Step 2: Test every codomain value The value 0 is attained at $n = 1$. For any positive whole number k, choose the even natural number $n = 2k$; then $f(2k) = 2k/2 = k$. Thus every whole number occurs as an image. This proves surjectivity and rules out (D), which denies it.

Step 3: Conclusion The function is surjective only. Hence, the correct option is (B).

Quick Tip to Solve: Test repeated odd inputs for injectivity, then use n=2k to reach each positive whole number.

11. If $f(x) = 2x + \cos x$, then $f(x)$: [1 mark]

(A) has a maxima at $x = \pi$

(B) has a minima at $x = \pi$

(C) is an increasing function

(D) is a decreasing function

Correct Answer: (C) is an increasing function

Solution:

Concept:

- A positive derivative throughout an interval implies that a differentiable function is strictly increasing there.
- For every real x, $-1 \leq \sin x \leq 1$.

Step 1: Differentiate and bound the derivative For $f(x) = 2x + \cos x$, differentiation gives $f\prime(x) = 2 - \sin x$. Since the largest possible value of $\sin x$ is 1, the smallest possible derivative is $2 - 1 = 1$. Thus $f\prime(x) \geq 1 > 0$ for every real input.

Step 2: Check monotonicity and the distractors The derivative never changes sign and is always positive, so the function is strictly increasing on $\mathbb{R}$. At $x = \pi$, its derivative is $2 - \sin \pi = 2$, not zero;

Quick Tip to Solve: Bound sin x before solving any derivative equation: 2-sin x is always at least 1.

neither a maximum nor a minimum occurs there. Hence (A) and (B) fail, and the positive derivative rules out the decreasing behavior in (D).

Step 3: Conclusion $f(x)$ is an increasing function. Hence, the correct option is (C).

Quick Tip to Solve: Bound sin x before solving any derivative equation: 2-sin x is always at least 1.

---

12. If the sides AB and AC of $\triangle ABC$ are represented by vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ respectively, then the length of the median through A on BC is: [1 mark]

(A) $2\sqrt{2}$ units

(B) $\sqrt{18}$ units

(C) $\dfrac{\sqrt{34}}{2}$ units

(D) $\dfrac{\sqrt{48}}{2}$ units

Correct Answer: (C) $\dfrac{\sqrt{34}}{2}$ units

Solution:

Concept:

- If M is the midpoint of BC, then $\overrightarrow{AM}=(\overrightarrow{AB}+\overrightarrow{AC})/2$.
- The length of $a\hat{i}+b\hat{j}+c\hat{k}$ is $\sqrt{a^2+b^2+c^2}$.

Step 1: Obtain the median vector Let M be the midpoint of BC. Using the given vectors relative to the common vertex A, we obtain

$$\overrightarrow{AM}=\frac{(\hat{j}+\hat{k})+(3\hat{i}-\hat{j}+4\hat{k})}{2}=\frac{3\hat{i}+5\hat{k}}{2}.$$

The opposite $\hat{j}$ components cancel.

Step 2: Calculate and compare lengths The required length is the magnitude of the median vector, not that of its doubled vector. Thus

$$AM=\sqrt{(3/2)^2+0^2+(5/2)^2}=\frac{\sqrt{34}}{2}.$$

Its square is $34/4=17/2$. The squared lengths in (A), (B) and (D) are 8, 18 and 12, respectively, so all three differ from this value.

Step 3: Conclusion The median has length $\boxed{\dfrac{\sqrt{34}}{2}}$ units. Hence, the correct option is (C).

Quick Tip to Solve: Add the two vectors from A, halve the sum, and only then take its magnitude.

---

13. The function f defined by $f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$ is not continuous at: [1 mark]

(A) $x = 0$

(B) $x = 1$

(C) $x = 2$

(D) $x = 5$

Correct Answer: (B) $x = 1$

Solution:

Concept:

- Continuity at a requires $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.
- The functions x and the constant 5 are continuous within their respective branch intervals.

Step 1: Examine the joining point At $x = 1$, the first branch applies, so $f(1) = 1$. Approaching from below gives $\lim_{x \to 1^-} f(x) = 1$, whereas approaching from above gives $\lim_{x \to 1^+} f(x) = 5$. The unequal one-sided limits establish a jump discontinuity at this point.

Step 2: Check the remaining listed points In a neighborhood of $x = 0$, the rule is simply $f(x) = x$, which is continuous. Neighborhoods of $x = 2$ and $x = 5$ lie within the constant branch $f(x) = 5$, so continuity holds at both. Thus options (A), (C) and (D) cannot be the requested point of discontinuity.

Step 3: Conclusion The function is not continuous at $\boxed{x = 1}$. Hence, the correct option is (B).

Quick Tip to Solve: For a piecewise polynomial, check the branch boundary first by comparing one-sided limits.

---

14. $\int e^x(\cos x - \sin x)\, dx$ is equal to: [1 mark]

(A) $e^x \sin x + C$

(B) $-e^x \sin x + C$

(C) $-e^x \cos x + C$

(D) $e^x \cos x + C$

Correct Answer: (D) $e^x \cos x + C$

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x g(x)] = e^x[g(x) + g\prime(x)]$.
- An indefinite integral is verified by differentiating its proposed antiderivative; the constant has derivative zero.

Step 1: Identify the product derivative Choose $g(x) = \cos x$, for which $g\prime(x) = -\sin x$. Applying the product rule gives

$$\frac{d}{dx}(e^x \cos x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$$

Quick Tip to Solve: Look for the pattern e^x times a function plus its derivative before using integration by parts.

This reproduces both the terms and their signs in the integrand.

Step 2: Check the other options The derivatives of the expressions in (A), (B) and (C) are, respectively, $e^x(\sin x + \cos x)$, $-e^x(\sin x + \cos x)$, and $-e^x(\cos x - \sin x)$. None agrees with the given integrand. Adding an arbitrary constant C to the matching product supplies the full family of antiderivatives.

Step 3: Conclusion $\int e^x(\cos x - \sin x)\,dx = \boxed{e^x \cos x + C}$. Hence, the correct option is (D).

Quick Tip to Solve: Look for the pattern e^x times a function plus its derivative before using integration by parts.

15. The area of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 0$ and $x = 4$ and the x-axis is: [1 mark]

(A) $\frac{16}{9}$ sq. units

(B) $\frac{32}{9}$ sq. units

(C) $\frac{16}{3}$ sq. units

(D) $\frac{32}{3}$ sq. units

Correct Answer: (C) $\frac{16}{3}$ sq. units

Solution:

Concept:

- Area between a nonnegative curve $y = f(x)$ and the x-axis from a to b is $\int_a^b f(x)\,dx$.
- The power rule gives $\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$.

Step 1: Set up the enclosed area The graph $y = \sqrt{x}$ is on or above the x-axis throughout $[0, 4]$. The vertical lines provide the lower and upper integration limits, while the x-axis is the lower boundary. Therefore no sign reversal or split is needed:

$$A = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx.$$

Step 2: Evaluate and check the options Using the power rule,

$$A = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(4^{3/2}) - 0 = \frac{2}{3}(8) = \frac{16}{3}.$$

Here $4^{3/2} = (\sqrt{4})^3 = 8$, not 16. The official options (A), (B) and (D), namely $16/9$, $32/9$ and $32/3$, do not equal this area.

Step 3: Conclusion The area is $\boxed{\frac{16}{3}}$ square units. Hence, the correct option is (C).

Quick Tip to Solve: Use the curve as height and integrate from 0 to 4; compute $4^{3/2}$ as the cube of 2.

16. The integrating factor of the differential equation $\frac{dy}{dx} + y\tan x - \sec x = 0$ is: [1 mark]

(A) $-\cos x$

(B) $\sec x$

(C) $\log \sec x$

(D) $e^{\sec x}$

Correct Answer: (B) $\sec x$

Solution:

Concept:

- A linear equation $dy/dx + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$.
- Here the coefficient of y, rather than the right-hand side, determines the integrating factor.

Step 1: Put the equation in standard form Moving the last term to the right gives $dy/dx + (\tan x)y = \sec x$. Thus $P(x) = \tan x$ and $Q(x) = \sec x$. Choosing an integrating factor from Q would incorrectly lead toward option (D); it is the coefficient P that enters the formula.

Step 2: Integrate the coefficient Since $\int \tan x\,dx = \log|\sec x| + C$, an integrating factor is $\sec x$ on any interval where $\cos x \neq 0$; a nonzero constant multiple gives the same method. Indeed, $(\sec x)\prime = (\sec x)\tan x$, as required. Option (A) has the wrong logarithmic derivative, (C) is only an antiderivative of P, and (D) uses the wrong exponent.

Step 3: Conclusion The integrating factor is $\boxed{\sec x}$. Hence, the correct option is (B).

Quick Tip to Solve: Identify the coefficient of y first, integrate it, and then exponentiate the result.

17. The corner points of the feasible region of a Linear Programming Problem are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$ and $(0, 5)$. If $Z = ax + by;\ (a, b > 0)$ be the objective function, and maximum value of Z is obtained at $(0, 2)$ and $(3, 0)$, then the relation between a and b is: [1 mark]

(A) $a = b$

(B) $a = 3b$

(C) $b = 6a$

(D) $3a = 2b$

Correct Answer: (D) $3a = 2b$

Solution:

Concept:

- The objective function is $Z = ax + by$, where a, b are positive coefficients.
- Equal objective values at two specified points give a linear relation between the coefficients.

Step 1: Evaluate at the two stated points At $(0, 2)$, $Z = 2b$. At $(3, 0)$, $Z = 3a$. The question states that the same optimal value is attained at both points, so equating these values gives $2b = 3a$.

Step 2: Check the options and the printed premise Option (D) expresses precisely this equality. Option (A) would give $2a = 3a$, impossible for $a > 0$; option (B) would give $2b = 9b$; option (C)

Quick Tip to Solve: When two points share an objective value, substitute each into Z and equate before checking the optimality claim.

would give $12a = 3a$. There is a defect in the printed word "maximum": $(6, 8)$ gives $6a + 8b$, greater than either stated value for positive a, b. Thus these points cannot actually maximize the objective on the listed region. The requested equality of their values nevertheless determines the intended relation unambiguously.

Step 3: Conclusion $\boxed{3a = 2b}$. Hence, the intended correct option is (D).

Quick Tip to Solve: When two points share an objective value, substitute each into Z and equate before checking the optimality claim.

18. The value of $\int_0^1 \frac{1}{e^x + e^{-x}}\,dx$ is: [1 mark]

(A) $-\frac{\pi}{4}$

(B) $\frac{\pi}{4}$

(C) $\tan^{-1} e - \frac{\pi}{4}$

(D) $\tan^{-1} e$

Correct Answer: (C) $\tan^{-1} e - \frac{\pi}{4}$

Solution:

Concept:

- Multiplying numerator and denominator by e^x converts the integrand into a form suitable for substitution.
- Use $\int (1 + t^2)^{-1}\,dt = \tan^{-1} t + C$, with $t = e^x$.

Step 1: Transform the integral Write

$$I = \int_0^1 \frac{e^x}{1 + e^{2x}}\,dx.$$

Set $t = e^x$, so $dt = e^x\,dx$. The lower limit $x = 0$ becomes $t = 1$, and the upper limit $x = 1$ becomes $t = e$.

Step 2: Evaluate and compare the choices Hence

$$I = \int_1^e \frac{dt}{1 + t^2} = \left[\tan^{-1} t\right]_1^e = \tan^{-1} e - \frac{\pi}{4}.$$

Option (A) is negative although the integrand is positive. Option (B) is not the difference of the endpoint values. Option (D) omits the lower-limit contribution $\tan^{-1} 1 = \pi/4$.

Step 3: Conclusion $\boxed{\tan^{-1} e - \frac{\pi}{4}}$. Hence, the correct option is (C).

Quick Tip to Solve: Multiply by e^x/e^x and change both limits when setting $t = e^x$.

19. Assertion (A): If A and B are two events such that $P(A \cap B) = 0$, then A and B are independent events.

Reason (R): Two events are independent if the occurrence of one does not effect the occurrence of the other.

Select the correct answer from the codes given below: [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- Independence requires $P(A \cap B) = P(A)P(B)$.
- Zero intersection probability alone does not force the product of the individual probabilities to vanish.

Step 1: Judge the Assertion The Assertion is false. For a fair die, let $A = \{1\}$ and $B = \{2\}$. Then $P(A \cap B) = 0$, but $P(A)P(B) = 1/36 \neq 0$. Thus these events meet the stated condition and are not independent.

Step 2: Judge the Reason The Reason is true: independence means that occurrence of one event does not affect the probability of occurrence of the other. When $P(B) > 0$, this is expressed as $P(A \mid B) = P(A)$, which yields the product rule.

Step 3: Check the explanatory link A true definition of independence cannot explain the false universal Assertion. Options (A) and (B) require the Assertion to be true; option (C) also wrongly declares the Reason false. The required combination is false Assertion and true Reason.

Step 4: Conclusion The Assertion is false and the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Test independence with the product rule; disjoint events of positive probability are dependent.

20. Assertion (A): In a Linear Programming Problem, if the feasible region is empty, then the Linear Programming Problem has no solution.

Reason (R): A feasible region is defined as the region that satisfies all the constraints.

Select the correct answer from the codes given below: [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The feasible region consists of all points satisfying every constraint, including non-negativity restrictions.
- An optimal solution must first be a feasible point.

Step 1: Judge the Assertion The Assertion is true. An empty feasible region contains no permissible point (x, y), so there is no point at which the objective can attain a feasible maximum or minimum.

Step 2: Judge the Reason The Reason is true because it states the definition of the feasible region: the simultaneous solution set of all constraints. A point satisfying only some constraints is not feasible.

Step 3: Check the explanatory link The definition directly explains the Assertion. If the region is empty, no point satisfies all constraints, and hence no feasible solution exists. Options (B), (C) and (D) incorrectly reject, respectively, this explanatory link, the true Reason, or the true Assertion.

Step 4: Conclusion Both Assertion and Reason are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Check feasibility before optimizing: an empty intersection of all constraint regions gives no solution.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Using matrices and determinants, find the value(s) of k for which the pair of equations $5x - ky = 2;\ 7x - 5y = 3$ has a unique solution. [2 marks]

Solution:

Concept:

- A linear system can be written $AX = B$, where A is the coefficient matrix and X the column of unknowns.

Quick Tip to Solve: For two equations, form the coefficient determinant and impose nonzero determinant; do not set it equal to zero.

- A square coefficient matrix has an inverse exactly when $\det A \neq 0$; then $X = A^{-1}B$ is unique.

Step 1: Write the matrix equation The given pair is $5x - ky = 2$ and $7x - 5y = 3$. Therefore

$$\underbrace{\begin{pmatrix} 5 & -k \\ 7 & -5 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{B}.$$

The signs of both coefficients of y must be retained in the determinant.

Step 2: Apply the determinant condition Compute

$$\det A = 5(-5) - (-k)(7) = -25 + 7k.$$

For a unique solution the determinant must be nonzero, so $7k - 25 \neq 0$, equivalently $k \neq 25/7$. For such values, the inverse exists and multiplication by A^{-1} determines exactly one column X. At $k = 25/7$, multiplying the first equation by $7/5$ gives left side $7x - 5y$ but right side $14/5$, inconsistent with the second right side 3; thus the excluded value does not yield a unique solution.

Step 3: Conclusion The pair has a unique solution for every real $\boxed{k \neq \frac{25}{7}}$.

Quick Tip to Solve: For two equations, form the coefficient determinant and impose nonzero determinant; do not set it equal to zero.

22(a). **Simplify** $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$. [2 marks]

Solution:

Concept:

- The principal value of $\sin^{-1} u$ lies in $[-\pi/2, \pi/2]$.
- For real x, the principal angle $\theta = \tan^{-1} x$ lies in $(-\pi/2, \pi/2)$, where cosine is positive.

Step 1: Introduce a principal angle Let $\theta = \tan^{-1} x$, so $\tan\theta = x$ and $-\pi/2 < \theta < \pi/2$. The identity $1 + \tan^2\theta = \sec^2\theta$, together with $\cos\theta > 0$, gives

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}, \qquad \sin\theta = \tan\theta\cos\theta = \frac{x}{\sqrt{1+x^2}}.$$

This derivation also works for negative x because the sign is carried by $\tan\theta$.

Step 2: Use the principal range The expression is therefore $\sin^{-1}(\sin\theta)$. Since θ lies inside the principal range of the inverse sine, this equals θ, rather than any other angle with the same sine. Also, $1 + x^2 > 0$ and $x^2/(1+x^2) < 1$ for every real x, so the original expression is defined for all real inputs.

Step 3: Conclusion

$$\boxed{\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1} x, \quad x \in \mathbb{R}.}$$

Quick Tip to Solve: Choose $\theta = \tan^{-1} x$; its principal range makes $\sin^{-1}(\sin\theta) = \theta$ valid.

OR

22(b). Find the domain of $\sin^{-1}\sqrt{x-1}$. [2 marks]

Solution:

Concept:

- The real square root $\sqrt{x-1}$ requires its radicand to satisfy $x-1 \geq 0$.
- The inverse sine is real only for arguments in $[-1,1]$; a square-root argument is also non-negative.

Step 1: Impose the square-root restriction For $\sqrt{x-1}$ to be real, $x-1 \geq 0$, so $x \geq 1$. This condition alone is insufficient, since the resulting square root must also belong to the domain of inverse sine. The square root denotes the non-negative principal root.

Step 2: Impose the inverse-sine restriction We need $-1 \leq \sqrt{x-1} \leq 1$. The lower bound is automatic for a real square root. Thus $0 \leq \sqrt{x-1} \leq 1$. Squaring these non-negative quantities preserves both inequalities and gives $0 \leq x-1 \leq 1$, hence $1 \leq x \leq 2$. Both endpoints are included: at $x=1$ the argument is 0, and at $x=2$ it is 1, with inverse-sine values 0 and $\pi/2$, respectively. Values below 1 fail the square-root condition, while values above 2 give an argument exceeding 1.

Step 3: Conclusion The domain is $\boxed{[1,2]}$.

Quick Tip to Solve: Apply both restrictions: the radicand is non-negative and the square root is at most one.

---

23. Calculate the area of the region bounded by the curve $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the x-axis using integration. [2 marks]

Solution:

Concept:

- The area between a curve above the x-axis and that axis is $A=\int_a^b y\,dx$, where a, b are its intercepts.
- The ellipse is symmetric about the y-axis, so the area of its upper half is twice its first-quadrant area.

Step 1: Express the upper boundary and limits Putting $y=0$ in the given ellipse gives $x=\pm 3$. For the region above the axis,

$$y = 2\sqrt{1-\frac{x^2}{9}} = \frac{2}{3}\sqrt{9-x^2}.$$

Thus $A = 2\int_0^3 y\,dx = \frac{4}{3}\int_0^3 \sqrt{9-x^2}\,dx$. The lower bounded region has the same area by symmetry.

Step 2: Evaluate the integral Use $x = 3\sin\theta$, so $dx = 3\cos\theta\,d\theta$. The limits become 0 and $\pi/2$, and $\sqrt{9-x^2} = 3\cos\theta$ on this interval. Therefore

$$\begin{aligned} A &= 12\int_0^{\pi/2} \cos^2\theta\,d\theta \\ &= 6\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 3\pi. \end{aligned}$$

Quick Tip to Solve: Use symmetry to integrate only from zero to the positive intercept, then double for one half of the ellipse.

Step 3: Conclusion The area of either region bounded by the ellipse and the x-axis is $\boxed{3\pi}$ square units. The two regions together form the full ellipse, of area 6π square units.

Quick Tip to Solve: Use symmetry to integrate only from zero to the positive intercept, then double for one half of the ellipse.

24(a). Find the least value of 'a' so that $f(x) = 2x^2 - ax + 3$ is an increasing function on $[2, 4]$. [2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval if its derivative is nonnegative there.
- For $f(x) = 2x^2 - ax + 3$, $f\prime(x) = 4x - a$; this derivative increases as x increases.

Step 1: Apply the derivative condition Differentiate with respect to x:

$$f\prime(x) = 4x - a.$$

On $[2, 4]$, its minimum occurs at $x = 2$ and equals $8 - a$. Therefore it is nonnegative everywhere on the interval exactly when

$$8 - a \geq 0, \qquad a \leq 8.$$

For these values it is positive throughout the interior, so the function is strictly increasing.

Step 2: Examine the requested least value The condition is also necessary. If $a > 8$, then $f\prime(2) < 0$, and continuity makes the derivative negative on a short interval to the right of 2; the function decreases there. The permissible parameters therefore constitute $(-\infty, 8]$. This set has no smallest real number: for any permissible a, the smaller number $a - 1$ is permissible too. Thus the word least in the printed question cannot yield a finite value; 8 is the greatest permissible value.

Step 3: Conclusion As printed, there is $\boxed{\text{no least real value of } a}$. The exact condition is $\boxed{a \leq 8}$, whose greatest permissible value is 8.

Quick Tip to Solve: After solving the derivative inequality, inspect whether the admissible set actually has the requested least or greatest endpoint.

OR

24(b). If $f(x) = x + \dfrac{1}{x}$, $x \geq 1$, show that f is an increasing function. [2 marks]

Solution:

Concept:

- If $f\prime(x) > 0$ on the interior of an interval, the mean value theorem shows that f is strictly increasing.
- Differentiate x^{-1} as $-x^{-2}$. The domain here is $[1, \infty)$, so no zero denominator occurs.

Quick Tip to Solve: Check the derivative on the interior: an isolated zero at the endpoint does not invalidate strict increase.

Step 1: Differentiate and determine the sign For $f(x) = x + 1/x$,

$$f\prime(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}.$$

Since $x \geq 1$, the numerator is nonnegative and the denominator positive. Hence $f\prime(x) \geq 0$ on the specified domain, with equality only at the endpoint $x = 1$. In particular $f\prime(x) > 0$ for every $x > 1$.

Step 2: Establish increase including the endpoint Take arbitrary $1 \leq u < v$. The function is continuous on $[u, v]$ and differentiable on (u, v). By the mean value theorem, for some $c \in (u, v)$,

$$f(v) - f(u) = (v - u)f\prime(c).$$

Here $c > 1$, so $f\prime(c) > 0$, while $v - u > 0$. Therefore $f(v) > f(u)$. A zero derivative at the single endpoint does not prevent strict increase on the whole domain.

Step 3: Conclusion Thus $\boxed{f(x) = x + 1/x \text{ is strictly increasing on } [1, \infty)}$.

Quick Tip to Solve: Check the derivative on the interior: an isolated zero at the endpoint does not invalidate strict increase.

---

25. Find the local maxima and local minima of the function $f(x) = \frac{8}{3}x^3 - 12x^2 + 18x + 5$. [2 marks]

Solution:

Concept:

- Stationary points satisfy $f\prime(x) = 0$; a negative second derivative indicates a local maximum and a positive one a local minimum.
- The local maximum or minimum value is found by substituting its stationary abscissa into the original function.

Step 1: Locate and classify stationary points For the printed function $f(x) = \frac{8}{3}x^3 - 12x^2 + 18x + 5$,

$$f\prime(x) = 8x^2 - 24x + 18 = 2(2x - 3)^2.$$

The only stationary point is $x = 3/2$. Also $f\prime\prime(x) = 16x - 24$, which vanishes there, so the second derivative test is inconclusive; it cannot establish an extremum.

Step 2: Use the first derivative sign test The square $(2x - 3)^2$ is positive on both sides of $3/2$, and zero only at that point. Consequently $f\prime$ changes neither from positive to negative nor from negative to positive. The function increases through the stationary point rather than turning. Its value is

$$f(3/2) = 9 - 27 + 27 + 5 = 14.$$

Equivalently, $f(x) = \frac{8}{3}(x - \frac{3}{2})^3 + 14$, so the stationary point is an inflection point, not a local maximum or minimum.

Step 3: Conclusion The function has $\boxed{\text{no local maximum and no local minimum}}$. Its stationary point

Quick Tip to Solve: When the derivative has a repeated root, check its sign on both sides before claiming any maximum or minimum.

$(3/2, 14)$ is a point of inflection.

Quick Tip to Solve: When the derivative has a repeated root, check its sign on both sides before claiming any maximum or minimum.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26(a). Find the probability distribution of the number of boys in families having three children, assuming equal probability for a boy and a girl. [3 marks]

Solution:

Concept:

- Let X denote the number of boys among the three children; its possible values are $0, 1, 2, 3$.
- With independent births and equal probabilities $P(B) = P(G) = 1/2$, each ordered sequence of three children has probability $(1/2)^3 = 1/8$.
- The number of sequences containing exactly k boys is $\binom{3}{k}$, so $P(X = k) = \binom{3}{k}/8$.

Step 1: List the equally likely outcomes Record the children in birth order. The sample space is

$$S = \{GGG, BGG, GBG, GGB, BBG, BGB, GBB, BBB\}.$$

It consists of eight outcomes because each of the three positions independently has two possibilities. The events specified by different boy counts are disjoint, so their probabilities are obtained by adding the probabilities of the corresponding sequences. Keeping birth order distinguishes, for example, BGG from GBG, although both give the same value of X.

Step 2: Find each probability For $X = 0$, only GGG occurs, giving probability $1/8$. For $X = 1$, the sequences are BGG, GBG, GGB, giving $3/8$. For $X = 2$, they are BBG, BGB, GBB, also giving $3/8$. For $X = 3$, only BBB occurs, giving $1/8$. Thus the probability distribution is

x	0	1	2	3
$P(X = x)$	$1/8$	$3/8$	$3/8$	$1/8$

All entries are nonnegative and their sum is $(1 + 3 + 3 + 1)/8 = 1$, as required for a probability distribution.

Step 3: Conclusion The probabilities for $X = 0, 1, 2, 3$ are respectively $\boxed{1/8,\ 3/8,\ 3/8,\ 1/8}$. Equivalently, $X \sim \text{Bin}(3, 1/2)$.

Quick Tip to Solve: Use the binomial coefficients 1, 3, 3, 1 to count the birth-order sequences, then divide each count by eight.

OR

26(b). A coin is tossed twice. Let X be a random variable defined as number of heads minus number of tails. Obtain the probability distribution of X and also find its mean. [3 marks]

Solution:

Concept:

- A probability distribution lists every possible value of a random variable and its probability. For two independent tosses of a fair coin, the four ordered outcomes are equally likely.
- The mean of a discrete random variable is its expectation: $E(X) = \sum_x xP(X = x)$. Here $X = H - T$, where H and T count heads and tails.

Step 1: List the outcomes and compute the random variable The sample space is $S = \{HH, HT, TH, TT\}$. Each outcome has probability $1/4$. For HH, there are two heads and no tails, so $X = 2$. For each of HT and TH, one head and one tail give $X = 0$. For TT, no heads and two tails give $X = -2$. Thus the only possible values are $-2, 0, 2$; values -1 and 1 cannot occur because there are exactly two tosses.

Step 2: Obtain the probability distribution Group together all outcomes giving the same value. Thus

$$P(X = -2) = \frac{1}{4}, \qquad P(X = 0) = \frac{2}{4} = \frac{1}{2}, \qquad P(X = 2) = \frac{1}{4}.$$

These probabilities are non-negative and their sum is $1/4 + 1/2 + 1/4 = 1$, so the distribution includes every possible outcome.

Step 3: Calculate the mean Multiply each value by its probability and add:

$$E(X) = (-2)\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = -\frac{1}{2} + 0 + \frac{1}{2} = 0.$$

The negative and positive contributions cancel, consistent with the equal likelihood of heads and tails.

Step 4: Conclusion The distribution is $P(X = -2) = 1/4$, $P(X = 0) = 1/2$, $P(X = 2) = 1/4$, and the mean is $\boxed{E(X) = 0}$.

Quick Tip to Solve: Use $H + T = 2$ to write $X = 2H - 2$; this immediately restricts X to $-2, 0, 2$.

27(a). If $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_a x$ ($a > 0$ and $a \neq 1$) is defined, then prove that f is a bijection. ($\mathbb{R}^+$ is the set of all positive real numbers.) [3 marks]

Solution:

Concept:

- A function is bijective when it is both one-one (injective) and onto (surjective). Injectivity requires equal outputs to imply equal inputs.
- For $a > 0$, $a \neq 1$, the logarithm and exponential are inverse operations: $\log_a(a^y) = y$ and $a^{\log_a x} = x$ for $x > 0$.

Step 1: Check the domain and establish injectivity The logarithm $\log_a x$ is a real number for every

Quick Tip to Solve: To prove onto, start with an arbitrary output y and choose the positive input a^y.

$x > 0$, so the stated mapping into $\mathbb{R}$ is well-defined. Let $x_1, x_2 \in \mathbb{R}^+$ and suppose $f(x_1) = f(x_2)$. Then

$$\log_a x_1 = \log_a x_2.$$

Raising the positive base a to both sides gives

$$a^{\log_a x_1} = a^{\log_a x_2} \quad \Longrightarrow \quad x_1 = x_2.$$

Consequently no two distinct positive inputs have the same output, and f is one-one.

Step 2: Establish surjectivity onto the specified codomain Let $y \in \mathbb{R}$ be arbitrary. Choose $x = a^y$. Since $a > 0$, this number is positive for every real y, and hence lies in the domain $\mathbb{R}^+$. Now

$$f(x) = f(a^y) = \log_a(a^y) = y.$$

Thus each real number in the codomain has a preimage. This proves that the range is all of $\mathbb{R}$, so f is onto. The argument works for both $a > 1$ and $0 < a < 1$; it does not require the logarithm to be increasing.

Step 3: Conclusion Since f is both one-one and onto, $\boxed{f : \mathbb{R}^+ \to \mathbb{R} \text{ is a bijection}}$. Its inverse is $f^{-1}(y) = a^y$.

Quick Tip to Solve: To prove onto, start with an arbitrary output y and choose the positive input a^y.

OR

27(b). Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. A relation R from A to B is defined as $R = \{(x, y) : x + y = 6,\ x \in A,\ y \in B\}$.

(i) Write all elements of R.

(ii) Is R a function? Justify.

(iii) Determine the domain and range of R. [3 marks]

Solution:

Concept:

- A relation from A to B consists of ordered pairs in $A \times B$ that satisfy the stated condition. Both set-membership conditions must be checked.
- A function from A to B assigns exactly one image in B to every member of A. The domain and range of a relation are its sets of first and second components, respectively.

Step 1: (i) Find all elements of the relation The condition $x + y = 6$ gives $y = 6 - x$. Check each permissible first component: for $x = 1$, $y = 5 \in B$, giving $(1, 5)$; for $x = 2$, $y = 4 \in B$, giving $(2, 4)$; for $x = 3$, $y = 3 \notin B$, so no pair is included. Therefore

$$R = \{(1, 5), (2, 4)\}.$$

The pair $(3, 3)$ satisfies the equation but fails membership in $A \times B$, and must be excluded.

Quick Tip to Solve: After solving $y = 6 - x$, check that the resulting y belongs to B; an equation alone is insufficient.

Step 2: (ii) Check whether it is a function from A to B Although each of the inputs 1 and 2 has a unique image, $3 \in A$ has no image in B. A function on the specified source set must be defined at every source element. Hence R is not a function from A to B. The failure concerns the missing image of 3; the fact that $6 \in B$ is unused does not itself prevent a relation from being a function.

Step 3: (iii) Determine domain and range Taking first components of the pairs in R gives $\text{Dom}(R) = \{1, 2\}$. Taking second components gives $\text{Range}(R) = \{4, 5\}$. These are the actual components occurring in the relation, rather than automatically the entire source and target sets.

Step 4: Conclusion $\boxed{R = \{(1, 5), (2, 4)\}}$; it is not a function from A to B. Its domain is $\boxed{\{1, 2\}}$ and range is $\boxed{\{4, 5\}}$.

Quick Tip to Solve: After solving $y = 6 - x$, check that the resulting y belongs to B; an equation alone is insufficient.

---

28. Find: $\displaystyle\int \frac{\cos x}{1 + \cos x + \sin x}\, dx$. [3 marks]

Solution:

Concept:

- The half-angle identities are $1 + \cos x = 2\cos^2(x/2)$, $\sin x = 2\sin(x/2)\cos(x/2)$, and $\cos x = \cos^2(x/2) - \sin^2(x/2)$.
- For a nonzero differentiable function u, $\int u'/u\, dx = \ln|u| + C$. In particular, $\int \tan(x/2)\, dx = -2\ln|\cos(x/2)| + C$.

Step 1: Factor the numerator and denominator using half angles Put $c = \cos(x/2)$ and $s = \sin(x/2)$ temporarily. Then

$$1 + \cos x + \sin x = 2c^2 + 2sc = 2c(c + s), \qquad \cos x = c^2 - s^2 = (c - s)(c + s).$$

On any interval where the original integrand is defined, its denominator is nonzero. Thus both c and $c + s$ are nonzero, and cancellation of $c + s$ is valid. Consequently

$$\frac{\cos x}{1 + \cos x + \sin x} = \frac{c - s}{2c} = \frac{1}{2}\left(1 - \tan\frac{x}{2}\right).$$

Step 2: Integrate the simplified expression Using the last identity,

$$\begin{aligned} I &= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \tan\frac{x}{2}\, dx \\ &= \frac{x}{2} + \ln\left|\cos\frac{x}{2}\right| + C. \end{aligned}$$

To see the logarithmic term directly, set $u = \cos(x/2)$, so $du = -\frac{1}{2}\sin(x/2)\, dx$. Then $-\frac{1}{2}\tan(x/2)\, dx = du/u$, whose integral is $\ln|u|$. The constant is arbitrary on each interval of definition.

Quick Tip to Solve: Factor both numerator and denominator with half angles; the common factor $\cos(x/2) + \sin(x/2)$ cancels.

Step 3: Verify the antiderivative Differentiating the expression obtained gives

$$\frac{d}{dx}\left(\frac{x}{2}+\ln\left|\cos\frac{x}{2}\right|\right)=\frac{1}{2}-\frac{1}{2}\tan\frac{x}{2}=\frac{\cos x}{1+\cos x+\sin x}.$$

This verifies the sign and the half-angle factor; points where the original denominator vanishes remain excluded.

Step 4: Conclusion

$$\boxed{\int\frac{\cos x}{1+\cos x+\sin x}\,dx=\frac{x}{2}+\ln\left|\cos\frac{x}{2}\right|+C}$$

Quick Tip to Solve: Factor both numerator and denominator with half angles; the common factor $\cos(x/2)+\sin(x/2)$ cancels.

29(a). **Consider the experiment of tossing a coin. If the coin shows head, toss it again; but if it shows a tail, then throw a die. Find the conditional probability of the event A: 'the die shows a number greater than 3' given that B: 'there is at least one tail'.** [3 marks]

Solution:

Concept:

- Conditional probability is $P(A\mid B)=\dfrac{P(A\cap B)}{P(B)}$, provided $P(B)>0$.
- A fair coin gives head or tail with probability $\frac{1}{2}$, and each face of a fair die has probability $\frac{1}{6}$. Multiply probabilities along independent successive stages.

Step 1: Identify the branches of the experiment Let H and T denote head and tail. If the first toss is a head, a second coin toss gives HH or HT, each with probability $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$. If the first toss is a tail, the die is thrown, giving $T1,T2,\ldots,T6$, each with probability $\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$. These eight elementary outcomes are not equally likely; simply counting outcomes would give an incorrect conditional probability.

Step 2: Calculate the probability of at least one tail Event B consists of HT and all six outcomes beginning with T. These branches are disjoint, so

$$P(B)=P(HT)+P(T\text{ on the first toss})=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}.$$

Equivalently, the only outcome without any tail is HH, so $P(B)=1-P(HH)$.

Step 3: Find the intersection and apply the conditional formula A die result greater than 3 means $4,5$ or 6. The die is thrown only after a tail, so every outcome in A already belongs to B. Therefore $A\cap B=A=\{T4,T5,T6\}$, and

$$P(A\cap B)=\frac{1}{2}\times\frac{3}{6}=\frac{1}{4},\qquad P(A\mid B)=\frac{1/4}{3/4}.$$

Quick Tip to Solve: Assign branch probabilities first: the coin outcomes have probability $1/4$ each, but each tail-die outcome has probability $1/12$.

Step 4: Conclusion The required conditional probability is $\boxed{P(A \mid B) = \frac{1}{3}}$.

Quick Tip to Solve: Assign branch probabilities first: the coin outcomes have probability $1/4$ each, but each tail-die outcome has probability $1/12$.

OR

29(b). The probability distribution of a random variable X is given as:

X	1	2	3	2λ	3λ	4λ
$P(X)$	$\frac{11}{30}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{15}$	$\frac{1}{10}$

29(b)(i). Calculate λ, if $E(X) = 3.2$. [2 marks]

Solution:

Concept:

- For a discrete random variable, the expectation is $E(X) = \sum xP(X = x)$: multiply each possible value by its corresponding probability and add the products.
- The probabilities are already specified. The unknown λ occurs in the values of X, so the given expectation provides a linear equation in λ.

Step 1: Substitute the distribution into the expectation formula Read each probability directly below its corresponding value in the table. Thus

$$\begin{aligned} E(X) &= 1\left(\frac{11}{30}\right) + 2\left(\frac{1}{15}\right) + 3\left(\frac{1}{10}\right) \\ &\quad + (2\lambda)\left(\frac{3}{10}\right) + (3\lambda)\left(\frac{1}{15}\right) + (4\lambda)\left(\frac{1}{10}\right) \\ &= \frac{11+4+9}{30} + \frac{3\lambda}{5} + \frac{\lambda}{5} + \frac{2\lambda}{5} \\ &= \frac{4}{5} + \frac{6}{5}\lambda. \end{aligned}$$

Step 2: Use the prescribed mean and solve Since $3.2 = \frac{16}{5}$, equating the expectation to its given value gives

$$\frac{4}{5} + \frac{6}{5}\lambda = \frac{16}{5} \quad \Longrightarrow \quad 4 + 6\lambda = 16 \quad \Longrightarrow \quad \lambda = 2.$$

For this value, the last three outcomes are $4, 6, 8$; substituting into the weighted sum returns $\frac{4}{5} + \frac{12}{5} = \frac{16}{5}$, as required.

Step 3: Conclusion The required parameter is $\boxed{\lambda = 2}$.

Quick Tip to Solve: Combine the first three contributions and the coefficient of lambda separately before equating the mean to 3.2.

29(b)(ii). Find $P(X > 1)$. [1 mark]

Solution:

Concept:

- The probability of an event is the sum of the probabilities of all outcomes satisfying its condition.
- The complementary-event rule gives $P(X > 1) = 1 - P(X \leq 1)$; the strict inequality excludes the outcome $X = 1$.

Step 1: Identify the outcomes greater than one Using $\lambda = 2$ from part (i), the values of X are $1, 2, 3, 4, 6, 8$. All except 1 satisfy $X > 1$. There are no outcomes below 1, so $P(X \leq 1) = P(X = 1) = \frac{11}{30}$.

Step 2: Add the probabilities of the required outcomes The five included outcomes have total probability

$$\begin{aligned} P(X > 1) &= \frac{1}{15} + \frac{1}{10} + \frac{3}{10} + \frac{1}{15} + \frac{1}{10} \\ &= \frac{2+3+9+2+3}{30} = \frac{19}{30}. \end{aligned}$$

The complement gives the same result: $1 - \frac{11}{30} = \frac{19}{30}$.

Step 3: Conclusion The required probability is $\boxed{P(X > 1) = \frac{19}{30}}$.

Quick Tip to Solve: Substitute lambda first, then subtract the probability of the sole excluded outcome X=1 from one.

30. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$. [3 marks]

Solution:

Concept:

- Introduce separate parameters for two lines: a common ratio t in a symmetric equation converts the line to coordinate equations.
- At an intersection, all three coordinates agree. The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express both lines using independent parameters Put the common ratio of the first line equal to t. For the second line, put its common ratio equal to s; the final expression is z, so $z = s$. We obtain

$$\begin{aligned} L_1: &\quad x = 1 + 2t, \quad y = 2 + 3t, \quad z = 3 + 4t, \\ L_2: &\quad x = 4 + 5s, \quad y = 1 + 2s, \quad z = s. \end{aligned}$$

The parameters need not be equal, since they measure displacement from different starting points in different direction vectors.

Step 2: Solve for the common point and verify its coordinates Equating the x- and y-coordinates

Quick Tip to Solve: Use two coordinate equations to find the parameters, then verify the third before calculating the distance to the intersection.

gives

$$2t - 5s = 3, \qquad 3t - 2s = -1.$$

Multiply the first equation by 3, and the second by 2, then subtract:

$$(6t - 15s) - (6t - 4s) = 9 - (-2), \qquad -11s = 11.$$

Hence $s = -1$; substituting into $3t-2s = -1$ gives $3t+2 = -1$, so $t = -1$. The third coordinates agree as well, since $3+4(-1) = -1 = s$. Substituting into either line gives the intersection $Q = (-1, -1, -1)$. Checking the third coordinate is essential: solving only two coordinate equations does not establish an intersection in space.

Step 3: Substitute into the distance formula The given point is $P = (-1, -5, -10)$. The displacement from P to Q has components $0, 4, 9$. Therefore

$$PQ = \sqrt{[-1-(-1)]^2 + [-1-(-5)]^2 + [-1-(-10)]^2} = \sqrt{0^2 + 4^2 + 9^2}.$$

Step 4: Conclusion The lines intersect at $(-1, -1, -1)$, and the required distance is $\boxed{\sqrt{97} \text{ units}}$.

Quick Tip to Solve: Use two coordinate equations to find the parameters, then verify the third before calculating the distance to the intersection.

31. Solve the following Linear Programming Problem graphically:

Minimise $Z = 3x + 5y$, subject to the constraints

$x + 2y \geq 10$

$x + y \geq 6$

$3x + y \geq 8$

$x, y \geq 0$. [3 marks]

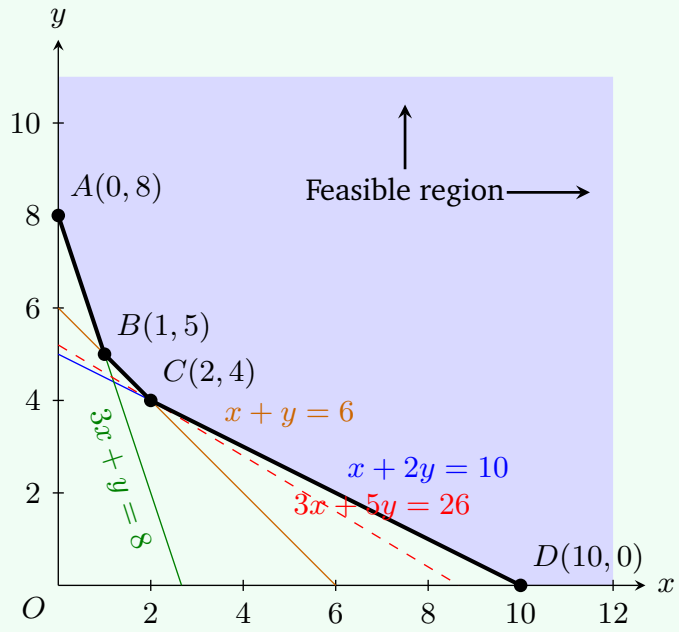
Solution:

Concept:

- The feasible region is the intersection of all the given half-planes in the first quadrant. Each boundary line is drawn by joining its two intercepts.
- Evaluate the linear objective $Z = 3x + 5y$ at the feasible corner points. For an unbounded region, also verify that the half-plane giving a smaller objective value has no feasible point.

Step 1: Plot the boundary lines and shade the feasible region The line $x + 2y = 10$ passes through $(10, 0)$ and $(0, 5)$; $x+y = 6$ passes through $(6, 0)$ and $(0, 6)$; and $3x+y = 8$ passes through $(8/3, 0)$ and $(0, 8)$. The origin fails each inequality, so take the side away from the origin for each line, restricted to $x \geq 0, y \geq 0$. The common region shown shaded is unbounded upwards and to the right.

Quick Tip to Solve: Write the objective as $2(x + 2y) + (x + y)$ to check the graphical minimum directly from two constraints.



Feasible region with vertices A, B, C, D; shading continues upwards and rightwards.

Step 2: Find the feasible corner points On the axes the limiting feasible points are $A = (0, 8)$ and $D = (10, 0)$. Solving $x + y = 6$ with $3x + y = 8$ gives $B = (1, 5)$; it satisfies $x + 2y = 11 \geq 10$. Solving $x + 2y = 10$ with $x + y = 6$ gives $C = (2, 4)$; it satisfies $3x + y = 10 \geq 8$. The remaining pair of lines meets at $(6/5, 22/5)$, where $x + y = 28/5 < 6$, so that point is infeasible.

Step 3: Compare objective values and verify the minimum At the feasible vertices,

$$Z(A) = 40, \quad Z(B) = 28,$$
$$Z(C) = 26, \quad Z(D) = 30.$$

The line $3x + 5y = 26$ touches the feasible region at C. The half-plane $3x + 5y < 26$ has no overlap with the shaded region. Algebraically, every feasible point satisfies $3x + 5y = 2(x + 2y) + (x + y) \geq 2(10) + 6 = 26$. Equality requires both boundary equalities and hence occurs at $(2, 4)$. This also establishes the minimum despite the region being unbounded.

Step 4: Conclusion The minimum value is $\boxed{Z_{\min} = 26}$, attained at $\boxed{x = 2,\ y = 4}$.

Quick Tip to Solve: Write the objective as $2(x + 2y) + (x + y)$ to check the graphical minimum directly from two constraints.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32. The relation between the height of the plant (y cm) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to sunlight.

(i) Find the rate of growth of the plant with respect to sunlight.

(ii) In how many days will the plant attain its maximum height? What is the maximum height?

[5 marks]

Solution:

Concept:

- If height y is expressed as a function of exposure time x, the instantaneous rate of growth is dy/dx. Here y is measured in centimetres and x in days, so the derivative has units cm/day.
- At an interior maximum of a differentiable function, its first derivative is zero. A negative second derivative at a stationary point establishes a local maximum; the sign of the first derivative on either side can verify the change from increasing to decreasing.

Step 1: Part (i): differentiate the height function The stated relation is $y = 4x - \frac{1}{2}x^2$. Treating exposure in days as the independent variable, differentiate term by term using $d(x^n)/dx = nx^{n-1}$:

$$\frac{dy}{dx} = \frac{d}{dx}(4x) - \frac{1}{2}\frac{d}{dx}(x^2) = 4 - \frac{1}{2}(2x) = 4 - x.$$

Thus the growth rate depends on the number of days already exposed. It is initially 4 cm/day, and decreases by 1 cm/day for each additional day. The rate is positive for $0 \le x < 4$, zero at $x = 4$, and negative for $x > 4$ under the given model. Height and rate have different units: differentiating centimetres with respect to days produces centimetres per day.

Step 2: Part (ii): locate and test the stationary point To find the time of maximum height, set the first derivative equal to zero:

$$4 - x = 0 \quad \Longrightarrow \quad x = 4.$$

This is an admissible exposure time because $x \ge 0$. Differentiating once more gives $d^2y/dx^2 = -1 < 0$, so the stationary point is a maximum. Equivalently, the plant's height increases before the fourth day and decreases after it, because the first derivative changes from positive to negative there. This test is needed: a zero derivative by itself only identifies a stationary point and does not distinguish a maximum from a minimum.

Step 3: Compute and confirm the maximum height Substitute $x = 4$ into the original height relation, rather than into the rate equation:

$$y(4) = 4(4) - \frac{1}{2}(4)^2 = 16 - 8 = 8\,\text{cm}.$$

For a direct global check, complete the square:

$$y = 4x - \frac{1}{2}x^2 = 8 - \frac{1}{2}(x-4)^2.$$

Quick Tip to Solve: Differentiate to find the time, then substitute that time into the height function to obtain the height; the derivative gives only the growth rate.

Since a square is non-negative, this expression never exceeds 8, and equality holds exactly when $x = 4$. Therefore the stationary point gives the greatest height on the non-negative exposure-time domain, as well as on the interval where the model gives a non-negative height.

Step 4: Conclusion (i) The rate of growth is $\boxed{dy/dx = 4 - x \text{ cm/day}}$. (ii) The plant attains its maximum height after $\boxed{4 \text{ days}}$, and this height is $\boxed{8 \text{ cm}}$.

Quick Tip to Solve: Differentiate to find the time, then substitute that time into the height function to obtain the height; the derivative gives only the growth rate.

---

33. If A is a 3×3 invertible matrix, show that for any scalar $k \neq 0$, $(kA)^{-1} = \frac{1}{k}A^{-1}$. Hence calculate $(3A)^{-1}$, where $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. [5 marks]

Solution:

Concept:

- The inverse of an invertible matrix satisfies $AA^{-1} = A^{-1}A = I_3$. To identify an inverse, verify multiplication in both orders.
- For $|A| \neq 0$, $A^{-1} = \text{adj } A/|A|$; the adjugate is the transpose of the signed cofactor matrix.
- A nonzero scalar commutes with matrix multiplication and can be cancelled against its reciprocal.

Step 1: Prove the scalar multiple formula Let $B = \frac{1}{k}A^{-1}$, which is defined since $k \neq 0$. Then

$$(kA)B = kA\left(\frac{1}{k}A^{-1}\right) = AA^{-1} = I_3,$$
$$B(kA) = \left(\frac{1}{k}A^{-1}\right)kA = A^{-1}A = I_3.$$

Thus B is the inverse of kA, by the definition and uniqueness of a matrix inverse. The nonzero condition is necessary: when k is zero, kA is the zero matrix and cannot have an inverse.

Step 2: Calculate the determinant Expand along the first row, using the signs positive, negative, positive:

$$\begin{aligned} |A| &= 2(2 \cdot 2 - (-1)(-1)) - (-1)((-1)2 - (-1)1) \\ &\quad + 1((-1)(-1) - 2 \cdot 1) \\ &= 2(3) + (-1) + (-1) = 4. \end{aligned}$$

The determinant is nonzero, so the inverse exists and can be calculated by the adjugate method. Keeping the middle first-row entry negative is essential in this expansion.

Step 3: Calculate the cofactors and inverse The diagonal cofactors are all 3. For the off-diagonal

Quick Tip to Solve: Find the inverse of A once and divide every entry by 3; check the result by multiplying it by 3A.

entries, the signed minors give

$$C_{12} = -[(-1)2 - (-1)1] = 1,$$
$$C_{13} = (-1)(-1) - 2(1) = -1,$$
$$C_{23} = -[2(-1) - (-1)1] = 1.$$

The remaining cofactors are $C_{21} = 1$, $C_{31} = -1$, $C_{32} = 1$. Hence the cofactor matrix is symmetric, and transposing it leaves it unchanged:

$$\operatorname{adj} A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \qquad A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Step 4: Use k equal to 3 and verify Putting $k = 3$ into the proved identity divides every entry of the inverse by 3. Write $M = \operatorname{adj} A$. A direct check gives

$$AM = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3.$$

Thus $(3A)(M/12) = (3/12)4I_3 = I_3$, which verifies the scale factor as well as the numerical entries. Both A and M are symmetric, so transposing this product also gives the reverse product.

Step 5: Conclusion Therefore $\boxed{(kA)^{-1} = \frac{1}{k}A^{-1}}$, and

$$\boxed{(3A)^{-1} = \frac{1}{12}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}}.$$

Quick Tip to Solve: Find the inverse of A once and divide every entry by 3; check the result by multiplying it by 3A.

34(a). Evaluate: $\displaystyle\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$. [5 marks]

Solution:

Concept:

- The substitution $t = \tan x$ uses $dx = dt/(1 + t^2)$, $\sin x = t/\sqrt{1 + t^2}$ and $\cos x = 1/\sqrt{1 + t^2}$.
- For a definite integral, change the limits at the same time as the variable. A second substitution $u = t^2$ gives $du = 2t\, dt$.
- The standard antiderivative $\int du/(1 + u^2) = \tan^{-1} u + C$ will evaluate the resulting integral.

Step 1: Check the integrand and choose a substitution Denote the required integral by I. On $0 \le x \le \pi/4$, cosine is positive, so $t = \tan x$ is defined and increases continuously. The denominator

Quick Tip to Solve: Use tangent to cancel the trigonometric denominators, then set u equal to t squared; change the limits after each substitution.

$\cos^4 x + \sin^4 x$ is positive throughout the interval; no singularity is crossed. Both fourth powers share the factor $(1+t^2)^{-2}$, which is why this substitution simplifies the ratio.

Step 2: Transform the numerator, denominator and limits Using the trigonometric identities associated with tangent,

$$\sin x \cos x = \frac{t}{1+t^2},$$
$$\cos^4 x + \sin^4 x = \frac{1+t^4}{(1+t^2)^2},$$
$$dx = \frac{dt}{1+t^2}.$$

Therefore

$$\frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx = \frac{t}{1+t^4}\,dt.$$

At the lower limit $x = 0$, $t = 0$, and at the upper limit $x = \pi/4$, $t = 1$. Thus the transformed integral is $I = \int_0^1 t/(1+t^4)\,dt$. All factors of $1+t^2$ cancel only after the differential is included.

Step 3: Reduce to the inverse tangent form Set $u = t^2$. Then $du = 2t\,dt$, so $t\,dt = du/2$ and $t^4 = u^2$. Since t ranges from 0 to 1, u also ranges from 0 to 1. Consequently,

$$\begin{aligned} I &= \frac{1}{2}\int_0^1 \frac{du}{1+u^2} \\ &= \frac{1}{2}[\tan^{-1} u]_0^1 \\ &= \frac{1}{2}\left(\frac{\pi}{4} - 0\right). \end{aligned}$$

The factor one-half comes from the differential in the second substitution; it must be retained when the endpoint values are subtracted.

Step 4: Verify the antiderivative As a check in the original variable, use $F(x) = \frac{1}{2}\tan^{-1}(\tan^2 x)$. Differentiating by the chain rule gives

$$F'(x) = \frac{\tan x \sec^2 x}{1+\tan^4 x} = \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}.$$

Also, $F(0) = 0$ and $F(\pi/4) = \frac{1}{2}\tan^{-1} 1$. This independently checks the transformed integrand and the limits used above.

Step 5: Conclusion

$$\boxed{\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx = \frac{\pi}{8}}.$$

Quick Tip to Solve: Use tangent to cancel the trigonometric denominators, then set u equal to t squared; change the limits after each substitution.

OR

34(b). Find: $\int \frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^2}\,dx$. [5 marks]

Solution:

Concept:

- For real logarithms in the printed integrand, $x>0$. Set $t=\sqrt{x^2+1}/x>1$; the logarithmic bracket becomes $2\log t$.
- The dilogarithm $\mathrm{Li}_2(z)$ is defined on the real range $z<1$ by $\mathrm{Li}_2(z)=-\int_0^z \frac{\log(1-s)}{s}\,ds$, with the integrand interpreted by its limit at zero. Its derivative is $-\log(1-z)/z$.

Step 1: Transform the algebraic and logarithmic factors Put $t=\sqrt{x^2+1}/x$. Since $x>0$, all the following logarithms are real. We have

$$t^2=1+\frac{1}{x^2}, \qquad \log(x^2+1)-2\log x=\log(t^2)=2\log t.$$

Differentiating t^2 gives $dt=-dx/(tx^3)$. Also $\sqrt{x^2+1}=tx$, so

$$\frac{\sqrt{x^2+1}}{x^2}\,dx=\frac{t}{x}\,dx=-t^2x^2\,dt=-\frac{t^2}{t^2-1}\,dt.$$

Step 2: Split the transformed integral Since $t^2/(t^2-1)=1+1/(t^2-1)$, and $2/(t^2-1)=1/(t-1)-1/(t+1)$, the integral becomes

$$I=-2\int\log t\,dt-\int\frac{\log t}{t-1}\,dt+\int\frac{\log t}{t+1}\,dt.$$

Integration by parts gives $\int\log t\,dt=t\log t-t$. The remaining terms use the dilogarithm; its arguments here are negative, so its real definition suffices.

Step 3: Identify why the usual elementary rules stop The substitution above removes the radical and combines the logarithms, but leaves logarithms divided by two different linear factors. Integration by parts makes the obstruction explicit. If

$$J_-=\int\frac{\log t}{t-1}\,dt, \qquad J_+=\int\frac{\log t}{t+1}\,dt,$$

then, for $t>1$,

$$J_-=\log t\log(t-1)-\int\frac{\log(t-1)}{t}\,dt,$$
$$J_+=\log t\log(t+1)-\int\frac{\log(t+1)}{t}\,dt.$$

A precise elementary-integrability test also confirms this obstruction. For a rational function $r(t)$, the rational-logarithm case of Liouville's criterion says that an elementary primitive of $r(t)\log t$ requires

$$r(t)=R'(t)+\frac{2c}{t},$$

Quick Tip to Solve: Combine the logarithms first, then use t = sqrt(x squared + 1)/x; check the transformed differential carefully before integrating.

where $R(t)$ is rational and c is constant. Here

$$r(t) = -\frac{2t^2}{t^2-1} = -2 - \frac{1}{t-1} + \frac{1}{t+1}.$$

A derivative of a rational function has zero coefficient of $1/(t-a)$ at every finite pole a: differentiating any Laurent power cannot produce that power. The term $2c/t$ contributes only at zero. Thus the nonzero coefficients at $a = 1$ and $a = -1$ rule out the required form. This criterion concerns the rational expression as a whole, even though our real interval is $t > 1$. Neither remainder has the matching logarithmic derivative needed for the rule $\int f'(t)/f(t)\,dt$, or for a power of a logarithm. Their difference still contains $\log((t-1)/(t+1))/t$; hence the two remainders do not cancel. Thus these standard substitution and integration-by-parts methods reduce the printed integral to logarithmic primitives of dilogarithm type. We evaluate those primitives using the function defined in the Concept block, rather than replacing the printed radical or denominator.

Step 4: Integrate the remaining terms and check The required identities follow directly by differentiation:

$$\frac{d}{dt}\operatorname{Li}_2(1-t) = -\frac{\log t}{t-1},$$

$$\frac{d}{dt}\left[\log t \log(1+t) + \operatorname{Li}_2(-t)\right] = \frac{\log t}{1+t}.$$

Combining these antiderivatives gives the result below. Both arguments, $1 - t < 0$ and $-t < -1$, lie within the real domain $z < 1$ of the definition; no complex branches are used. Denote the displayed final expression without its constant by $F(t)$. The derivatives of its logarithmic product and of $\operatorname{Li}_2(-t)$ cancel their $\log(1+t)/t$ terms. Therefore

$$\begin{aligned} F'(t) &= -2\log t - \frac{\log t}{t-1} + \frac{\log t}{t+1} \\ &= -\frac{2t^2 \log t}{t^2-1}. \end{aligned}$$

Apply the chain rule with $dt/dx = -1/(tx^3)$:

$$\begin{aligned} \frac{dF(t(x))}{dx} &= \frac{2t\log t}{x^3(t^2-1)} = \frac{2t\log t}{x} \\ &= \frac{\sqrt{x^2+1}}{x^2}[\log(x^2+1) - 2\log x]. \end{aligned}$$

Here the second equality uses $t^2 - 1 = 1/x^2$, and the last uses $2\log t = \log(x^2+1) - 2\log x$. This is an exact verification on the whole real domain $x > 0$.

Step 5: Conclusion With $t = \dfrac{\sqrt{x^2+1}}{x}$, the required antiderivative is

$$\boxed{-2t\log t + 2t + \operatorname{Li}_2(1-t) + \log t \log(1+t) + \operatorname{Li}_2(-t) + C}.$$

Quick Tip to Solve: Combine the logarithms first, then use t = sqrt(x squared + 1)/x; check the transformed differential carefully before integrating.

35(a). Show that the area of a parallelogram whose diagonals are represented by $\vec{a}$ and $\vec{b}$ is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$. Also find the area of a parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$. [5 marks]

Solution:

Concept:

- If adjacent sides are represented by vectors $\vec{u}, \vec{v}$, their parallelogram has area $|\vec{u} \times \vec{v}|$, the product of their lengths and the sine of the included angle.
- The diagonals can be represented by $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$. Reversing a diagonal changes the sign of a cross product but leaves its magnitude unchanged.
- The cross product is distributive and antisymmetric: $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$, while a vector crossed with itself is zero.

Step 1: Express the diagonals through adjacent sides Let the adjacent sides from one vertex of the parallelogram be $\vec{u}$ and $\vec{v}$. The opposite vertex has displacement $\vec{u} + \vec{v}$, so one diagonal is $\vec{a} = \vec{u} + \vec{v}$. Taking the other diagonal from the endpoint of $\vec{v}$ to the endpoint of $\vec{u}$, write $\vec{b} = \vec{u} - \vec{v}$. If the specified orientation of a diagonal is reversed, its cross product changes sign; the area formula using the magnitude is therefore unaffected.

Step 2: Relate the cross products Distribute the cross product term by term:

$$\begin{aligned}\vec{a} \times \vec{b} &= (\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) \\ &= \vec{u} \times \vec{u} - \vec{u} \times \vec{v} + \vec{v} \times \vec{u} - \vec{v} \times \vec{v} \\ &= -2(\vec{u} \times \vec{v}).\end{aligned}$$

Thus $|\vec{a} \times \vec{b}| = 2|\vec{u} \times \vec{v}|$. Since the area is $|\vec{u} \times \vec{v}|$, this proves that it is half the magnitude of the cross product of the diagonals. This also covers the zero-area case when the two side vectors are parallel.

Step 3: Calculate for the given diagonal vectors Substitute $\vec{a} = (2, -1, 1)$ and $\vec{b} = (1, 3, -1)$. Expanding the determinant with the minus sign on the middle cofactor gives

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ &= [1 - 3]\hat{i} - [-2 - 1]\hat{j} + [6 + 1]\hat{k} \\ &= -2\hat{i} + 3\hat{j} + 7\hat{k}.\end{aligned}$$

Its magnitude is $\sqrt{(-2)^2 + 3^2 + 7^2} = \sqrt{62}$. The diagonals, rather than adjacent sides, were supplied, so this magnitude must be divided by two to obtain the required area.

Step 4: Conclusion The area formula is $\boxed{A = \frac{1}{2}|\vec{a} \times \vec{b}|}$. For the given diagonals, the area is $\boxed{\frac{\sqrt{62}}{2}}$ square units.

Quick Tip to Solve: When diagonals are given, compute their cross product and halve its magnitude; retain the minus sign on the determinant's middle cofactor.

OR

35(b). Find the equation of a line in vector and cartesian form which passes through the point $(1, 2, -4)$ and is perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

[5 marks]

Solution:

Concept:

- Direction ratios of a line are the components of a vector parallel to it. In symmetric form, they are the three denominators; in vector form, they are the coefficients of the parameter.
- A vector perpendicular to two non-parallel vectors $\vec{a}$ and $\vec{b}$ is their cross product $\vec{a} \times \vec{b}$. Perpendicularity is checked using a zero dot product.
- A line through position vector $\vec{r}_0$, with nonzero direction vector $\vec{d}$, has equation $\vec{r} = \vec{r}_0 + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.

Step 1: Extract the two direction vectors The first given line has direction ratios $3, -16, 7$, so take $\vec{a} = 3\hat{i} - 16\hat{j} + 7\hat{k}$. From the coefficient of μ in the second line, take $\vec{b} = 3\hat{i} + 8\hat{j} - 5\hat{k}$. The constant position vectors on the given lines do not determine the required direction: perpendicularity depends on their direction vectors. The required line must pass through $P(1, 2, -4)$, whose position vector is $\vec{r}_0 = \hat{i} + 2\hat{j} - 4\hat{k}$.

Step 2: Calculate a perpendicular direction Form the cross product, taking care of the negative sign in the middle component:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= [80 - 56]\hat{i} - [-15 - 21]\hat{j} + [24 + 48]\hat{k} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}.\end{aligned}$$

Dividing by the common nonzero factor 12 gives the simpler direction vector $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Scaling a direction vector changes only the parameter, not the line.

Step 3: Verify both perpendicularities and form the line Check the dot products explicitly:

$$\vec{d} \cdot \vec{a} = 2(3) + 3(-16) + 6(7) = 6 - 48 + 42 = 0,$$

$$\vec{d} \cdot \vec{b} = 2(3) + 3(8) + 6(-5) = 6 + 24 - 30 = 0.$$

Thus the selected direction is perpendicular to both given lines. Substitution in the point-direction equation yields the component relations $x = 1 + 2\lambda$, $y = 2 + 3\lambda$, and $z = -4 + 6\lambda$. Eliminating the common parameter gives the Cartesian form. At $\lambda = 0$, the point is exactly $(1, 2, -4)$, which verifies the required passage through P.

Step 4: Conclusion The vector and Cartesian equations are, respectively,

$$\boxed{\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad \lambda \in \mathbb{R}}$$

Quick Tip to Solve: Read only the direction ratios of the given lines, cross them, and simplify the common factor before using the prescribed point.

and

$$\boxed{\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}}.$$

Quick Tip to Solve: Read only the direction ratios of the given lines, cross them, and simplify the common factor before using the prescribed point.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

Some students are having a misconception while comparing decimals. For example, a student may mention that $78.56 > 78.9$ as $7856 > 789$. In order to assess this concept, a decimal comparison test was administered to the students of class VI through the following question:

In the recently held Sports Day in the school, 5 students participated in a javelin throw competition. The distances to which they have thrown the javelin are shown below in the table:

Ajay: 47.7 m; Bijoy: 47.07 m; Kartik: 43.09 m; Dinesh: 43.9 m; Devesh: 45.2 m.

The students were asked to identify who has thrown the javelin the farthest.

Based on the test attempted by the students, the teacher concludes that 40% of the students have the misconception in the concept of decimal comparison and the rest do not have the misconception. 80% of the students having misconception answered Bijoy as the correct answer in the paper. 90% of the students who are identified with not having misconception, did not answer Bijoy as their answer.

On the basis of the above information, answer the following questions: [4 marks]

Name of student	Distance of javelin (in meters)
Ajay	47·7
Bijoy	47·07
Kartik	43·09
Dinesh	43·9
Devesh	45·2

36(i). What is the probability of a student not having misconception but still answers Bijoy in the test? [1 mark]

Solution:

Concept:

- For events A and B, the multiplication rule is $P(A \cap B) = P(A)P(B \mid A)$.
- Complementary conditional probabilities add to one when the conditioning event is unchanged.

Quick Tip to Solve: For "not having misconception but answers Bijoy", multiply the group's probability by the answering probability within that group.

Step 1: Identify the group without misconception Let M denote having the misconception and B denote answering Bijoy. Since $P(M) = 0.40$, the probability of not having the misconception is $P(M^c) = 1 - 0.40 = 0.60$. Among this group, 90% did not answer Bijoy, so $P(B \mid M^c) = 1 - 0.90 = 0.10$.

Step 2: Apply the multiplication rule to the joint event The word "but" asks for both properties together, namely $M^c \cap B$. Therefore,

$$P(M^c \cap B) = P(M^c)P(B \mid M^c) = 0.60 \times 0.10 = 0.06.$$

The value 0.10 alone describes the proportion answering Bijoy within the group without misconception; it is not the requested probability among all students.

Step 3: Conclusion The required probability is $\boxed{0.06 = \frac{3}{50}}$, or 6%.

Quick Tip to Solve: For "not having misconception but answers Bijoy", multiply the group's probability by the answering probability within that group.

36(ii). What is the probability that a randomly selected student answers Bijoy as his answer in the test? [1 mark]

Solution:

Concept:

- Let M denote having the misconception and B denote answering Bijoy. The complementary groups M and M^c partition all students.
- Total probability gives $P(B) = P(M)P(B \mid M) + P(M^c)P(B \mid M^c)$.

Step 1: Obtain the conditional probabilities The passage gives $P(M) = 0.40$, $P(M^c) = 0.60$ and $P(B \mid M) = 0.80$. Since 90 percent of students without the misconception do not answer Bijoy, $P(B \mid M^c) = 1 - 0.90 = 0.10$.

Step 2: Add the two disjoint contributions A Bijoy answer can come from either group, so both weighted contributions must be included:

$$P(B) = (0.40)(0.80) + (0.60)(0.10) = 0.32 + 0.06.$$

The group proportions must be used; simply averaging 0.80 and 0.10 would incorrectly treat the groups as equally large.

Step 3: Conclusion The required probability is $\boxed{0.38 = \frac{19}{50}}$.

Quick Tip to Solve: First turn the 90 percent non-Bijoy figure into 10 percent Bijoy, then weight each group's Bijoy rate by its share.

36(iii)(a). What is the probability that a student who answered as Bijoy is having misconception?

[2 marks]

Solution:

Concept:

- Let M denote having the misconception and B denote answering Bijoy. The question asks for the posterior probability $P(M \mid B)$, after the answer is known.
- Bayes' theorem gives $P(M \mid B) = \dfrac{P(M)P(B \mid M)}{P(B)}$, where the denominator includes Bijoy answers from both groups.

Step 1: Find the probability of a Bijoy answer The given information yields $P(M) = 0.40$, $P(M^c) = 0.60$ and $P(B \mid M) = 0.80$. Of those without the misconception, 90 percent do not answer Bijoy, so $P(B \mid M^c) = 0.10$. The two groups are disjoint and exhaustive. Thus

$$P(B) = (0.40)(0.80) + (0.60)(0.10) = 0.38.$$

Step 2: Apply Bayes' theorem to the misconception group The joint probability that a student has the misconception and answers Bijoy is $P(M \cap B) = 0.40 \times 0.80 = 0.32$. Conditioning restricts attention to the students who answered Bijoy, so divide this joint probability by the total probability of a Bijoy answer:

$$P(M \mid B) = \frac{0.32}{0.38} = \frac{32}{38} = \frac{16}{19}.$$

The given 0.80 is $P(B \mid M)$, which describes the reverse conditioning and therefore cannot directly answer this question.

Step 3: Conclusion The probability that a student who answered Bijoy has the misconception is $\boxed{\frac{16}{19}}$.

Quick Tip to Solve: For 'misconception given Bijoy', put the misconception-and-Bijoy contribution 0.32 over all Bijoy answers 0.38.

OR

36(iii)(b). What is the probability that a student who answered as Bijoy is amongst students who do not have the misconception?

[2 marks]

Solution:

Concept:

- Let M denote having the misconception, M^c its absence, and B answering Bijoy. We seek $P(M^c \mid B)$, the probability of no misconception among Bijoy answers.
- Bayes' theorem gives $P(M^c \mid B) = \dfrac{P(M^c)P(B \mid M^c)}{P(B)}$. Total probability supplies the denominator from both student groups.

Step 1: Calculate each group's Bijoy contribution Since 40 percent have the misconception, $P(M) =$

Quick Tip to Solve: Use 0.06 as the numerator for no misconception and Bijoy together; check that the two posterior probabilities sum to one.

0.40 and $P(M^c) = 0.60$. The passage gives $P(B \mid M) = 0.80$. Since 90 percent of those without the misconception do not answer Bijoy, $P(B \mid M^c) = 1 - 0.90 = 0.10$. Therefore

$$P(M \cap B) = 0.40 \times 0.80 = 0.32,$$
$$P(M^c \cap B) = 0.60 \times 0.10 = 0.06,$$
$$P(B) = 0.32 + 0.06 = 0.38.$$

Step 2: Condition on having answered Bijoy Within the Bijoy-answer group, the required students are exactly those in $M^c \cap B$. Divide their joint probability by the probability of the entire Bijoy group:

$$P(M^c \mid B) = \frac{0.06}{0.38} = \frac{6}{38} = \frac{3}{19}.$$

The value 0.10 instead conditions on having no misconception; reversing the given condition requires this Bayes calculation.

Step 3: Conclusion The required probability of having no misconception, given a Bijoy answer, is $\boxed{\frac{3}{19}}$.

Quick Tip to Solve: Use 0.06 as the numerator for no misconception and Bijoy together; check that the two posterior probabilities sum to one.

37. Case Study - 2

An engineer is designing a new metro rail network in a city.

Initially, two metro lines, Line A and Line B, each consisting of multiple stations are designed. The track for Line A is represented by $l_1 : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{4}$, while the track for Line B is represented by $l_2 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z+2}{-3}$.

Based on the above information, answer the following questions: [4 marks]

37(i). Find whether the two metro tracks are parallel. [1 mark]

Solution:

Concept:

- A line in symmetric form has direction ratios given by its three denominators.
- Two lines are parallel if their direction vectors are proportional; changing a point on a line does not change its direction.

Step 1: Read the direction vectors From the printed equations, Line A has direction vector $\mathbf{b}_1 = (3, -2, 4)$, and Line B has direction vector $\mathbf{b}_2 = (2, 1, -3)$. The numerators specify points on the tracks and are not direction ratios.

Step 2: Test proportionality The corresponding ratios are $3/2$, $-2/1 = -2$ and $4/(-3) = -4/3$. They are unequal, so there is no scalar λ with $\mathbf{b}_1 = \lambda \mathbf{b}_2$. Thus the parallelism criterion is not satisfied.

Step 3: Conclusion The two metro tracks are not parallel.

Quick Tip to Solve: Compare the denominators of the symmetric equations, not the constants in their numerators.

37(ii). Solar panels are to be installed on the rooftop of the metro stations. Determine the equation of the line representing the placement of solar panels on the rooftop of Line A's stations, given that panels are to be positioned parallel to Line A's track (l_1) and pass through the point $(1, -2, -3)$.

[1 mark]

Solution:

Concept:

- A line through $\mathbf{a}$, parallel to a nonzero vector $\mathbf{b}$, is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $t \in \mathbb{R}$.
- Parallel lines have proportional direction ratios, so the new line can use the direction ratios of Line A.

Step 1: Choose the point and direction The required line passes through $P = (1, -2, -3)$. From Line A, take direction vector $\mathbf{b} = (3, -2, 4)$. The new point replaces the original point of Line A while its direction remains the same.

Step 2: Form the line equation Using the point-direction form gives $\mathbf{r} = (1, -2, -3) + t(3, -2, 4)$. Hence $x = 1 + 3t$, $y = -2 - 2t$ and $z = -3 + 4t$. Eliminating t yields the symmetric form; at $t = 0$ the line passes through the prescribed point.

Step 3: Conclusion The solar-panel placement line is

$$\boxed{\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z+3}{4}.}$$

Quick Tip to Solve: Keep all three direction ratios of Line A and replace the point coordinates with the prescribed point.

37(iii)(a). To connect the stations, a pedestrian pathway perpendicular to the two metro lines is to be constructed which passes through point $(3, 2, 1)$. Determine the equation of the pedestrian walkway.

[2 marks]

Solution:

Concept:

- A vector perpendicular to two nonparallel direction vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ is their cross product $\mathbf{b}_1 \times \mathbf{b}_2$.
- A line through $P = (x_0, y_0, z_0)$ with direction ratios (a, b, c) has equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.

Step 1: Compute a perpendicular direction The track directions are $\mathbf{b}_1 = (3, -2, 4)$ and $\mathbf{b}_2 = (2, 1, -3)$. Thus

$$\begin{aligned}\mathbf{n} &= \mathbf{b}_1 \times \mathbf{b}_2 \\ &= \big((-2)(-3) - 4(1),\ 4(2) - 3(-3),\ 3(1) - (-2)(2)\big) \\ &= (2, 17, 7).\end{aligned}$$

This nonzero vector gives the required direction.

Quick Tip to Solve: Cross the two track direction vectors first, then use the given point in the point-direction equation.

Step 2: Use the prescribed point and verify Through $P = (3, 2, 1)$, the line is $\mathbf{r} = (3, 2, 1) + t(2, 17, 7)$. Its direction is perpendicular to both track directions since

$$(2, 17, 7) \cdot (3, -2, 4) = 6 - 34 + 28 = 0,$$
$$(2, 17, 7) \cdot (2, 1, -3) = 4 + 17 - 21 = 0.$$

At $t = 0$, it contains P, so both required directional and point conditions hold.

Step 3: Conclusion The pedestrian walkway is

$$\boxed{\frac{x-3}{2} = \frac{y-2}{17} = \frac{z-1}{7}.}$$

Quick Tip to Solve: Cross the two track direction vectors first, then use the given point in the point-direction equation.

OR

37(iii)(b). Find the shortest distance between Line A and Line B. [2 marks]

Solution:

Concept:

- For two nonparallel lines through points $\mathbf{a}_1, \mathbf{a}_2$ with directions $\mathbf{b}_1, \mathbf{b}_2$, their shortest distance is $d = |(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|/|\mathbf{b}_1 \times \mathbf{b}_2|$.
- This formula projects the separation vector onto a unit vector perpendicular to both lines; the absolute value makes the distance nonnegative.

Step 1: Identify points and a common normal From the symmetric equations, choose $\mathbf{a}_1 = (2, -1, 3)$, $\mathbf{a}_2 = (1, 3, -2)$, $\mathbf{b}_1 = (3, -2, 4)$ and $\mathbf{b}_2 = (2, 1, -3)$. Therefore

$$\mathbf{a}_2 - \mathbf{a}_1 = (-1, 4, -5), \qquad \mathbf{b}_1 \times \mathbf{b}_2 = (2, 17, 7).$$

The cross product is nonzero, so the nonparallel-lines distance formula applies.

Step 2: Evaluate the scalar projection The scalar triple product and normal magnitude are

$$(-1, 4, -5) \cdot (2, 17, 7) = -2 + 68 - 35 = 31,$$
$$|(2, 17, 7)| = \sqrt{2^2 + 17^2 + 7^2} = \sqrt{342}.$$

Consequently $d = |31|/\sqrt{342}$. The nonzero numerator also shows that these nonparallel lines are

Quick Tip to Solve: Use points from the numerators and directions from the denominators; take the absolute scalar triple product.

skew, rather than intersecting.

Step 3: Conclusion The shortest distance is

$$\boxed{d = \frac{31}{\sqrt{342}} = \frac{31}{3\sqrt{38}} \text{ units.}}$$

Quick Tip to Solve: Use points from the numerators and directions from the denominators; take the absolute scalar triple product.

38. Case Study - 3

During a heavy gaming session, the temperature of a student's laptop processor increases significantly. After the session, the processor begins to cool down, and the rate of cooling is proportional to the difference between the processor's temperature and the room temperature (25°C). Initially the processor's temperature is 85°C. The rate of cooling is defined by the equation $\frac{d}{dt}(T(t)) = -k(T(t)-25)$, where $T(t)$ represents the temperature of the processor at time t (in minutes) and k is a constant. Based on the above information, answer the following questions: [4 marks]

38(i). Find the expression for the temperature of the processor, $T(t)$, given that $T(0) = 85^\circ$C. [2 marks]

Solution:

Concept:

- Newton's law of cooling gives $\frac{dT}{dt} = -k(T - 25)$, where T is the processor temperature in degrees Celsius, t is time in minutes and $k > 0$ is the cooling constant.
- This is a separable differential equation. The excess temperature $T - 25$ decays exponentially, and the initial condition determines its coefficient.

Step 1: Separate the variables and integrate Initially, the excess temperature is positive, so along this cooling solution $T - 25 > 0$. Separating temperature and time gives

$$\frac{dT}{T - 25} = -k\, dt.$$

Integrating both sides,

$$\log_e(T - 25) = -kt + C.$$

Exponentiating gives $T - 25 = Ae^{-kt}$, where $A = e^C$ is a positive constant. This constant must be found from the given initial temperature; it is not the room temperature.

Step 2: Apply the initial condition and check the model At $t = 0$, the given temperature is 85°C. Hence

$$85 - 25 = Ae^0, \qquad A = 60.$$

Therefore the solution is $T(t) = 25 + 60e^{-kt}$. At zero time it gives $25 + 60 = 85$, as required. Its derivative is $-60ke^{-kt}$, which equals $-k(T-25)$, verifying the differential equation. As time increases, the excess temperature decreases and the temperature approaches the room temperature 25°C.

Quick Tip to Solve: Subtract the room temperature first: the initial excess is $85 - 25 = 60$, and this excess, rather than the total temperature, decays exponentially.

Step 3: Conclusion The processor temperature is $\boxed{T(t) = 25 + 60e^{-kt}}$, in degrees Celsius, with t measured in minutes.

Quick Tip to Solve: Subtract the room temperature first: the initial excess is $85 - 25 = 60$, and this excess, rather than the total temperature, decays exponentially.

38(ii). How long will it take for the processor's temperature to reach $40°C$? Given that $k = 0.03$, $\log_e 4 = 1.3863$. [2 marks]

Solution:

Concept:

- With room temperature $25°C$ and initial temperature $85°C$, Newton's cooling model is $T(t) - 25 = 60e^{-kt}$, where t is in minutes.
- To find the time for a specified temperature, compare its excess above room temperature with the initial excess, then use natural logarithms to isolate t.

Step 1: Substitute the desired temperature Put $k = 0.03$ and $T(t) = 40$ into the cooling expression from part (i):

$$40 - 25 = 60e^{-0.03t}.$$

The required excess above room temperature is $15°C$, whereas the initial excess was $60°C$. Thus

$$e^{-0.03t} = \frac{15}{60} = \frac{1}{4}.$$

The factor is $1/4$, not $40/85$, because the cooling law describes the difference between the processor and room temperatures.

Step 2: Take logarithms and calculate the time Taking natural logarithms on both sides gives

$$-0.03t = \log_e\left(\frac{1}{4}\right) = -\log_e 4.$$

Therefore

$$t = \frac{\log_e 4}{0.03} = \frac{1.3863}{0.03} = 46.21 \text{ minutes}.$$

Both negative signs cancel, so the time is positive. Since the model decreases strictly for $T > 25$, this is the unique time at which the temperature reaches $40°C$. Substitution makes the excess temperature one quarter of 60, giving $25 + 15 = 40$, as required.

Step 3: Conclusion The processor reaches $40°C$ after $\boxed{46.21 \text{ minutes}}$.

Quick Tip to Solve: Form the excess-temperature ratio $(40 - 25)/(85 - 25) = 1/4$; the given $\log_e 4$ then gives the time immediately as $(\log_e 4)/k$.