

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. Domain of $\sin^{-1}(2x^2 - 3)$ is:

[1 mark]

- (A) $(-1, 0) \cup (1, \sqrt{2})$
- (B) $(-\sqrt{2}, -1) \cup (0, 1)$
- (C) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$
- (D) $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

Correct Answer: (C) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

Solution:

Concept:

- The real inverse sine $\sin^{-1} t$ is defined precisely when $-1 \leq t \leq 1$, including both endpoints.
- An inequality involving x^2 must account for both positive and negative values of x .

Step 1: Restrict the argument of inverse sine Apply the real-domain condition to the complete argument $2x^2 - 3$:

$$-1 \leq 2x^2 - 3 \leq 1 \implies 2 \leq 2x^2 \leq 4.$$

Dividing throughout by the positive number 2 preserves the inequalities, giving $1 \leq x^2 \leq 2$.

Step 2: Separate the positive and negative intervals Taking the nonnegative square root gives $1 \leq |x| \leq \sqrt{2}$. Thus the negative values run from $-\sqrt{2}$ to -1 , and the positive values from 1 to $\sqrt{2}$. All four endpoints are included. Options (A) and (B) wrongly admit values with $|x| < 1$; option (D) wrongly excludes the endpoints.

Step 3: Conclusion The domain is $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$. Hence, the correct option is (C).

Quick Tip to Solve: For inverse sine of a quadratic, first bound the entire argument between -1 and 1, then solve using absolute value so the negative interval is retained.

2. The matrix $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -7 \\ 2 & 7 & 0 \end{pmatrix}$ is a:

[1 mark]

- (A) diagonal matrix
- (B) symmetric matrix
- (C) skew symmetric matrix
- (D) scalar matrix

Correct Answer: (C) skew symmetric matrix

Solution:

Concept:

- A square matrix M is skew symmetric when its transpose satisfies $M^T = -M$, equivalently $m_{ij} = -m_{ji}$.
- A symmetric matrix satisfies $M^T = M$; a diagonal matrix has every off-diagonal entry zero, and a scalar matrix is a special diagonal matrix.

Step 1: Compare corresponding off-diagonal entries Denote the given matrix by M . Its off-diagonal pairs are $(1, -1)$, $(-2, 2)$ and $(-7, 7)$. Each pair consists of opposite numbers, and all diagonal entries are zero, as required for a real skew symmetric matrix.

Step 2: Verify the transpose and exclude the other types Interchanging rows and columns gives

$$M^T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix} = -M.$$

The matrix is not symmetric since $m_{12} = 1 \neq -1 = m_{21}$. Its nonzero off-diagonal entries rule out both a diagonal matrix and a scalar matrix, so options (A), (B) and (D) do not apply.

Step 3: Conclusion The given matrix is skew symmetric. Hence, the correct option is (C).

Quick Tip to Solve: Check the three pairs across the main diagonal: opposite entries together with zero diagonal entries establish skew symmetry immediately.

3. If $f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 + ax, & 1 < x < 2 \end{cases}$ is continuous for $x \in (0, 2)$, then a is equal to: [1 mark]

(A) -4

(B) $-\frac{7}{2}$

(C) -2

(D) -1

Correct Answer: (D) -1

Solution:

Concept:

- Continuity at $x = c$ requires the left-hand limit, the right-hand limit and the value $f(c)$ to be equal.
- Polynomial expressions are continuous on their own intervals, so only the joining point $x = 1$ needs a condition on a .

Step 1: Evaluate the value and the left-hand limit The printed first branch includes $x = 1$. Therefore

$$f(1) = 3(1) - 2 = 1, \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x - 2) = 1.$$

The endpoints 0 and 2 are outside the required interval, so no continuity conditions are imposed there.

Step 2: Match the right-hand limit Approaching 1 from the right uses the second branch:

$$\lim_{x \rightarrow 1^+} f(x) = 2(1)^2 + a(1) = 2 + a.$$

Continuity requires $2 + a = 1$. The candidate values $-4, -7/2, -2$ give right-hand limits $-2, -3/2, 0$, respectively, none equal to 1; only $a = -1$ matches the value and both limits.

Step 3: Conclusion The required value is $a = -1$. Hence, the correct option is (D).

Quick Tip to Solve: At a piecewise join, first identify which branch contains the equality sign; it determines the actual function value to which both limits must agree.

4. If $y = \log_{2x}(\sqrt{2x})$, then $\frac{dy}{dx}$ is equal to:

[1 mark]

- (A) 0
- (B) 1
- (C) $\frac{1}{x}$
- (D) $\frac{1}{\sqrt{2x}}$

Correct Answer: (A) 0

Solution:

Concept:

- For a valid real logarithm, the base b satisfies $b > 0$, $b \neq 1$, and the argument is positive.
- The identity $\log_b(b^r) = r$ makes a logarithm constant when its argument is a fixed power of its base; the derivative of a constant is zero.

Step 1: Identify the domain and simplify the logarithm Here the base is $2x$, so $x > 0$ and $x \neq 1/2$. On this domain, $\sqrt{2x} = (2x)^{1/2}$, and the change-of-base formula gives

$$y = \frac{\ln((2x)^{1/2})}{\ln(2x)} = \frac{\frac{1}{2} \ln(2x)}{\ln(2x)} = \frac{1}{2}.$$

The cancellation is valid because the excluded base 1 is precisely where $\ln(2x) = 0$.

Step 2: Differentiate the simplified expression The function has the constant value $1/2$ on each interval of its domain, so its rate of change is zero. Option (B) gives a nonzero constant derivative, while (C) and (D) result from treating only part of the expression as variable; none matches a constant function.

Step 3: Conclusion $\frac{dy}{dx} = 0$ wherever the function is defined. Hence, the correct option is (A).

Quick Tip to Solve: Before differentiating a logarithm with a variable base, check whether the argument is a fixed power of that same base.

5. If $f : \mathbb{N} \rightarrow W$ is defined as $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$, then f is:

[1 mark]

- (A) injective only
- (B) surjective only
- (C) a bijection
- (D) neither surjective nor injective

Correct Answer: (B) surjective only

Solution:

Concept:

- A function is injective when distinct inputs always have distinct images; two different inputs with the same image disprove injectivity.
- A function is surjective when every element of its codomain has at least one preimage. Here $W = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$.

Step 1: Test injectivity using odd inputs The rule sends every odd natural number to zero. In particular,

$$f(1) = 0 = f(3), \quad 1 \neq 3.$$

Thus distinct inputs have the same image, and f is not injective. This immediately rules out both option (A) and a bijection in option (C).

Step 2: Find a preimage of every whole number The codomain element 0 has preimage 1. For any positive whole number k , choose the even natural number $n = 2k$; then

$$f(2k) = \frac{2k}{2} = k.$$

Therefore every member of W occurs as an output. The function is surjective, so option (D), which denies surjectivity, is also excluded.

Step 3: Conclusion The function is surjective only. Hence, the correct option is (B).

Quick Tip to Solve: Use two odd inputs to disprove injectivity, then use the even input $2k$ to reach each positive whole number k ; check zero separately.

6. The coordinates of the foot of the perpendicular drawn from the point $A(-2, 3, 5)$ on the y -axis is: [1 mark]

- (A) $(0, 0, 5)$
- (B) $(0, 3, 0)$
- (C) $(-2, 0, 5)$
- (D) $(-2, 0, 0)$

Correct Answer: (B) $(0, 3, 0)$

Solution:

Concept:

- Every point on the y -axis has coordinates $(0, t, 0)$, where t is real.
- The perpendicular from a point to an axis has zero component along that axis; equivalently its direction vector has zero dot product with the axis direction.

Step 1: Represent the foot on the axis Let the foot be $P = (0, t, 0)$. Since $A = (-2, 3, 5)$, the direction vector from A to P is

$$\overrightarrow{AP} = (2, t - 3, -5).$$

A direction vector along the y -axis is $(0, 1, 0)$.

Step 2: Apply perpendicularity and check the choices Their dot product must vanish:

$$(2, t - 3, -5) \cdot (0, 1, 0) = t - 3 = 0.$$

Thus the foot retains the given point's y -coordinate and has the other two coordinates zero. Options (A), (C) and (D) each have a nonzero x - or z -coordinate, so they are not even points on the y -axis.

Step 3: Conclusion The foot of the perpendicular is $(0, 3, 0)$. Hence, the correct option is (B).

Quick Tip to Solve: To project a point onto a coordinate axis, keep the coordinate named by that axis and set the other two coordinates equal to zero.

7. If A and B are invertible matrices of order 3×3 such that $\det(A) = 4$ and $\det[(AB)^{-1}] = \frac{1}{20}$, then $\det(B)$ is equal to: [1 mark]

- (A) $\frac{1}{20}$
- (B) $\frac{1}{5}$
- (C) 20
- (D) 5

Correct Answer: (D) 5

Solution:

Concept:

- For an invertible square matrix M , $\det(M^{-1}) = 1/\det(M)$.

Quick Tip to Solve: First take the reciprocal to remove the inverse, then divide by $\det(A)$; do not multiply by the matrix order.

- For square matrices of the same order, $\det(AB) = \det(A) \det(B)$; the order does not introduce any additional factor in this identity.

Step 1: Remove the inverse from the determinant The printed information is $\det[(AB)^{-1}] = 1/20$. Since A and B are invertible, their product is invertible as well. Consequently,

$$\frac{1}{\det(AB)} = \frac{1}{20} \implies \det(AB) = 20.$$

Step 2: Use the determinant of a product Substitute the known value $\det(A) = 4$:

$$\det(A) \det(B) = 20 \implies 4 \det(B) = 20.$$

Option (C) gives the determinant of the product, not that of B . Option (A) merely repeats the given inverse determinant, while (B) is the reciprocal of the required value. Dividing the product determinant by 4 identifies the correct choice.

Step 3: Conclusion The determinant of matrix B is $\boxed{5}$. Hence, the correct option is (D).

Quick Tip to Solve: First take the reciprocal to remove the inverse, then divide by $\det(A)$; do not multiply by the matrix order.

8. For real x , let $f(x) = x^3 + 5x + 1$. Then:

[1 mark]

- (A) f is one-one but not onto on $\mathbb{R}$
- (B) f is onto on $\mathbb{R}$ but not one-one
- (C) f is one-one and onto on $\mathbb{R}$
- (D) f is neither one-one nor onto on $\mathbb{R}$

Correct Answer: (C) f is one-one and onto on $\mathbb{R}$

Solution:

Concept:

- A strictly increasing real function is one-one: different arguments cannot give the same value.
- A continuous real function whose values tend to $-\infty$ and $+\infty$ at opposite ends of the real line takes every real value, by the intermediate value theorem.

Step 1: Show strict increase on the real line Differentiate the given polynomial:

$$f'(x) = 3x^2 + 5 \geq 5 > 0 \quad \text{for every } x \in \mathbb{R}.$$

Thus f is strictly increasing throughout $\mathbb{R}$, which establishes injectivity and excludes options (B) and (D).

Step 2: Show that every real output occurs The cubic leading term controls the end behaviour:

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty.$$

Quick Tip to Solve: For this cubic, use the everywhere-positive derivative for one-one behaviour, and continuity plus opposite infinite end limits for onto behaviour.

For any prescribed real value k , choose inputs far enough to the left and right that their function values lie below and above k . Polynomial continuity then guarantees an intervening input with output k . Therefore the range is all of $\mathbb{R}$, excluding option (A).

Step 3: Conclusion The function is both one-one and onto on $\mathbb{R}$. Hence, the correct option is (C).

Quick Tip to Solve: For this cubic, use the everywhere-positive derivative for one-one behaviour, and continuity plus opposite infinite end limits for onto behaviour.

9. The values of λ so that $f(x) = \sin x - \cos x - \lambda x + C$ decreases for all real values of x are: [1 mark]

(A) $1 < \lambda < \sqrt{2}$

(B) $\lambda \geq 1$

(C) $\lambda \geq \sqrt{2}$

(D) $\lambda < 1$

Correct Answer: (C) $\lambda \geq \sqrt{2}$

Solution:

Concept:

- A differentiable function is non-increasing where its derivative is non-positive.
- The identity $\sin x + \cos x = \sqrt{2} \sin(x + \pi/4)$ gives its greatest value as $\sqrt{2}$.

Step 1: Differentiate the function The constant C disappears on differentiation, while $-\lambda x$ contributes $-\lambda$. Thus

$$f'(x) = \cos x + \sin x - \lambda.$$

To prevent any interval of increase, this expression must be non-positive for every real x .

Step 2: Test the greatest derivative Since the trigonometric sum reaches $\sqrt{2}$, the greatest derivative is $\sqrt{2} - \lambda$. Thus we require $\sqrt{2} - \lambda \leq 0$. At equality the derivative vanishes only at isolated points, so the function still decreases strictly over every nonzero interval. Options (A), (B) and (D) all allow values below $\sqrt{2}$, for which the derivative is positive near $x = \pi/4$.

Step 3: Conclusion The required values are $\lambda \geq \sqrt{2}$. Hence, the correct option is (C).

Quick Tip to Solve: For a sine-cosine sum minus a linear term, compare the slope parameter with the amplitude of the derivative's trigonometric part.

10. If A and B are square matrices of same order such that $AB = A$ and $BA = B$, then $A^2 + B^2$ is equal to: [1 mark]

- (A) $A + B$
- (B) BA
- (C) $2(A + B)$
- (D) $2BA$

Correct Answer: (A) $A + B$

Solution:

Concept:

- Matrix multiplication is associative: $A(BA) = (AB)A$; changing the order of factors is not allowed in general.
- An idempotent matrix M satisfies $M^2 = M$.

Step 1: Use associativity to find the squares Substitute $BA = B$ and $AB = A$ without interchanging any factors:

$$A^2 = (AB)A = A(BA) = AB = A.$$

Similarly, the second square satisfies

$$B^2 = (BA)B = B(AB) = BA = B.$$

Step 2: Compare the proposed expressions Both matrices are therefore idempotent. Their squares contribute one copy of each matrix to the required sum. The expression BA is only B , whereas $2BA$ is $2B$; neither is necessarily the desired sum. Multiplying $A + B$ by two also introduces an unjustified factor. For example, taking $A = B = I$ immediately rejects options (B) and (C); option (D) need not hold when $A \neq B$.

Step 3: Conclusion $A^2 + B^2 = A + B$. Hence, the correct option is (A).

Quick Tip to Solve: Insert the given products into a square and regroup the three factors; never assume that the two matrices commute.

11. The area of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 0$ and $x = 4$ and the x -axis is: [1 mark]

- (A) $\frac{16}{9}$ sq. units
(B) $\frac{32}{9}$ sq. units
(C) $\frac{16}{3}$ sq. units
(D) $\frac{32}{3}$ sq. units

Correct Answer: (C) $\frac{16}{3}$ sq. units

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$ and the x -axis over $[a, b]$ is $\int_a^b f(x) dx$.
- The power rule is $\int x^n dx = x^{n+1}/(n+1) + C$, for $n \neq -1$.

Step 1: Set up the area integral On the full interval $0 \leq x \leq 4$, the curve $y = \sqrt{x}$ lies on or above the x -axis. Therefore no absolute-value split or subtraction of regions is needed. The vertical boundary lines give

$$A = \int_0^4 x^{1/2} dx.$$

Step 2: Evaluate at both limits Increase the exponent by one and divide by the new exponent:

$$A = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{2}{3}(8 - 0).$$

The value $4^{3/2}$ is $(\sqrt{4})^3 = 8$. Options (A) and (B) have an erroneous denominator of nine; option (D) doubles the required area, although only the region above the axis is enclosed.

Step 3: Conclusion The area is $\frac{16}{3}$ square units. Hence, the correct option is (C).

Quick Tip to Solve: Evaluate $4^{3/2}$ as $(\sqrt{4})^3$, and do not double an area unless a second symmetric region is actually included.

12. The value of $\int_0^1 \frac{1}{e^x + e^{-x}} dx$ is:

[1 mark]

(A) $-\frac{\pi}{4}$

(B) $\frac{\pi}{4}$

(C) $\tan^{-1} e - \frac{\pi}{4}$

(D) $\tan^{-1} e$

Correct Answer: (C) $\tan^{-1} e - \frac{\pi}{4}$

Solution:

Concept:

- Multiplying numerator and denominator by e^x converts the integrand into $e^x/(1 + e^{2x})$.
- The standard integral $\int dt/(1 + t^2) = \tan^{-1} t + C$ applies after a suitable substitution.

Step 1: Substitute the exponential variable Put $t = e^x$, so that $dt = e^x dx$. The lower limit $x = 0$ becomes $t = 1$, while $x = 1$ becomes $t = e$. Hence

$$I = \int_0^1 \frac{e^x}{1 + e^{2x}} dx = \int_1^e \frac{dt}{1 + t^2}.$$

Step 2: Apply the new limits The antiderivative is the inverse tangent. Evaluating the upper and lower endpoints gives

$$I = \tan^{-1} e - \tan^{-1} 1.$$

The lower-end contribution is $\pi/4$, so option (D) omits it. Option (A) is negative although the integrand is positive. Also $\pi/4 < \tan^{-1} e < \pi/2$, making the integral less than $\pi/4$, which rejects option (B).

Step 3: Conclusion $I = \tan^{-1} e - \frac{\pi}{4}$. Hence, the correct option is (C).

Quick Tip to Solve: Multiply by e^x/e^x first; after substituting e^x , remember that the new lower limit is one, not zero.

13. The corner points of the feasible region of a Linear Programming Problem are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$ and $(0, 5)$. If $Z = ax + by$; $(a, b > 0)$ be the objective function, and maximum value of Z is obtained at $(0, 2)$ and $(3, 0)$, then the relation between a and b is: [1 mark]

- (A) $a = b$
- (B) $a = 3b$
- (C) $b = 6a$
- (D) $3a = 2b$

Correct Answer: No option satisfies the printed maximum condition for positive coefficients. Option (D) is the intended relation obtained by equating the two stated values.

Solution:

Concept:

- If two points have the same optimum value of an objective function, their objective values must be equal.
- For $Z = ax + by$, substitute the coordinates into the same expression before comparing the values.

Step 1: Equate the stated objective values The values at the two named points are

$$Z(0, 2) = 2b, \quad Z(3, 0) = 3a.$$

Their stated equality requires $2b = 3a$. None of (A), (B) or (C) gives this ratio for positive a, b ; they respectively give $a/b = 1, 3, 1/6$, whereas equality requires $a/b = 2/3$.

Step 2: Check the printed maximum claim There is an inconsistency in calling these values the maximum: the listed feasible vertex $(6, 0)$ gives $Z = 6a > 3a$ because $a > 0$. Thus no positive coefficients actually make the named points maxima for this region. The equality relation asked for remains the intended relation in option (D); under that relation the named points would instead give the minimum among the listed vertices.

Step 3: Conclusion No positive coefficients satisfy the printed maximum condition, so there is no valid option as stated. Equating the two named values gives $3a = 2b$, the intended relation (D).

Quick Tip to Solve: First equate the objective values at tied vertices; then check another listed vertex before accepting a claimed maximum.

14. If $\int e^{-3 \log x} dx = f(x) + C$, then $f(x)$ is:

[1 mark]

(A) $e^{-3 \log x}$

(B) $e^{\log\left(\frac{1}{x^3}\right)}$

(C) $\frac{-1}{2x^2}$

(D) $\frac{-1}{4x^4}$

Correct Answer: (C) $\frac{-1}{2x^2}$

Solution:

Concept:

- For $x > 0$, the exponential and natural logarithm are inverse functions, giving $e^{k \log x} = x^k$.
- The power rule for integration is $\int x^n dx = x^{n+1}/(n+1) + C$, with $n \neq -1$.

Step 1: Simplify the integrand The logarithm restricts the given expression to $x > 0$. Use its exponent to write

$$e^{-3 \log x} = x^{-3}.$$

This is now an ordinary power integral; integration by parts is unnecessary.

Step 2: Integrate and check the candidates Adding one to the exponent gives -2 , so

$$\int x^{-3} dx = \frac{x^{-2}}{-2} + C.$$

Differentiating $-1/(2x^2)$ gives x^{-3} , as required. Options (A) and (B) both equal the original integrand x^{-3} , whose derivative is $-3x^{-4}$. Option (D) has derivative x^{-5} , so it cannot be an antiderivative of the given function.

Step 3: Conclusion $f(x) = -\frac{1}{2x^2}$. Hence, the correct option is (C).

Quick Tip to Solve: Collapse $e^{k \log x}$ to x^k before integrating, and differentiate a proposed answer to check its exponent and sign.

15. The function f defined by $f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$ is not continuous at: [1 mark]

- (A) $x = 0$
- (B) $x = 1$
- (C) $x = 2$
- (D) $x = 5$

Correct Answer: (B) $x = 1$

Solution:

Concept:

- Continuity at $x = c$ requires the left-hand limit, right-hand limit and function value to agree.
- A linear function and a constant function are each continuous in the interior of their respective domains.

Step 1: Examine the joining point The formula changes at $x = 1$. The left branch includes this point, so $f(1) = 1$. The one-sided limits are

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x = 1, \quad \lim_{x \rightarrow 1^+} f(x) = 5.$$

These limits differ, so the two-sided limit does not exist at the join.

Step 2: Check the other proposed points Around $x = 0$, the same formula $f(x) = x$ holds on both sides and is continuous. Around $x = 2$ and $x = 5$, the function is identically five and is also continuous. Thus options (A), (C) and (D) lie inside continuous branches; the value five in the definition does not itself create a discontinuity at $x = 5$.

Step 3: Conclusion The function is not continuous at $x = 1$. Hence, the correct option is (B).

Quick Tip to Solve: Test the point where a piecewise formula changes; compare both one-sided limits before comparing either with the function value.

16. The solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ represents family of: [1 mark]

- (A) parabolas
- (B) circles
- (C) ellipses
- (D) hyperbolas

Correct Answer: (B) circles

Solution:

Concept:

- In a separable differential equation, collect the y -terms with dy and the x -terms with dx , then integrate.

Quick Tip to Solve: After separation, look for the sign and coefficients of the squared terms: equal positive coefficients identify a circle.

- The equation $x^2 + y^2 = R^2$, with $R > 0$, represents a circle of radius R centred at the origin.

Step 1: Separate and integrate The differential equation is defined where $y \neq 0$. Multiplying by $y \, dx$ gives $y \, dy = -x \, dx$. Integrating both sides yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + C.$$

Hence the sum of the two squared coordinates is constant.

Step 2: Identify the family of curves Rearranging gives $x^2 + y^2 = 2C$. For a real nontrivial solution, $2C = R^2 > 0$, which is the standard circle equation. Solution graphs are local arcs away from $y = 0$, where the given slope is undefined. The positive equal coefficients exclude a hyperbola and identify circles specifically, rather than general ellipses; a parabola has only one squared coordinate in its standard equation.

Step 3: Conclusion The solutions lie on the family of concentric circles $x^2 + y^2 = R^2$. Hence, the correct option is (B).

Quick Tip to Solve: After separation, look for the sign and coefficients of the squared terms: equal positive coefficients identify a circle.

17. If the sides AB and AC of $\triangle ABC$ are represented by vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ respectively, then the length of the median through A on BC is: [1 mark]

- (A) $2\sqrt{2}$ units
- (B) $\sqrt{18}$ units
- (C) $\frac{\sqrt{34}}{2}$ units
- (D) $\frac{\sqrt{48}}{2}$ units

Correct Answer: (C) $\frac{\sqrt{34}}{2}$ units

Solution:

Concept:

- If M is the midpoint of BC , the median vector is $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$.
- The magnitude of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$; a median's length is its vector's magnitude.

Step 1: Calculate the median vector Both given vectors start at A , so their average ends at the midpoint of BC . The $\hat{j}$ components cancel:

$$\overrightarrow{AM} = \frac{(\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k})}{2} = \frac{3\hat{i} + 5\hat{k}}{2}.$$

Step 2: Compare the magnitudes The squared length is $|\overrightarrow{AM}|^2 = (3/2)^2 + (5/2)^2 = 34/4$. Squaring the proposed lengths in (A), (B), (C) and (D) gives 8, 18, 34/4, 12, respectively. Only (C) agrees; using the sum without halving would give the wrong median length.

Quick Tip to Solve: Average the two side vectors first, then take the magnitude; do not average their lengths.

Step 3: Conclusion The median has length $\frac{\sqrt{34}}{2}$ units. Hence, the correct option is (C).

Quick Tip to Solve: Average the two side vectors first, then take the magnitude; do not average their lengths.

18. If $f(x) = 2x + \cos x$, then $f(x)$:

[1 mark]

- (A) has a maxima at $x = \pi$
- (B) has a minima at $x = \pi$
- (C) is an increasing function
- (D) is a decreasing function

Correct Answer: (C) is an increasing function

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- For all real x , $-1 \leq \sin x \leq 1$; this bounds the derivative without solving a trigonometric equation.

Step 1: Differentiate and bound the derivative Differentiating the given function term by term gives

$$f'(x) = 2 - \sin x, \quad 1 \leq 2 - \sin x \leq 3.$$

Thus the derivative stays strictly positive for every real x .

Step 2: Check the alternatives Since $f'(x) > 0$ everywhere, the function increases throughout its domain, so it cannot be decreasing as in (D). Also $f'(\pi) = 2 \neq 0$; the differentiable function has neither a local maximum nor a local minimum at π . Hence (A) and (B) are ruled out.

Step 3: Conclusion $f(x) = 2x + \cos x$ is strictly increasing on $\mathbb{R}$. Hence, the correct option is (C).

Quick Tip to Solve: Use the bound $\sin x \leq 1$ to see immediately that $2 - \sin x \geq 1 > 0$.

19. Assertion (A): If A and B are two events such that $P(A \cap B) = 0$, then A and B are independent events.

Reason (R): Two events are independent if the occurrence of one does not effect the occurrence of the other.

Select the correct answer from the codes given below:

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- Independent events satisfy $P(A \cap B) = P(A)P(B)$; zero intersection probability alone does not imply independence.
- Independence means knowledge that one event occurred does not change the probability of the other, whenever the corresponding conditional probability is defined.

Step 1: Evaluate the assertion The assertion is false in general. For example, for one fair die let $A = \{1\}$ and $B = \{2\}$. Then $P(A \cap B) = 0$, whereas $P(A)P(B) = 1/36 \neq 0$. These events are not independent. If either event has zero probability, independence can hold, but that special case does not justify the given general claim.

Step 2: Evaluate the reason The reason states the standard meaning of independent events: the occurrence of one does not affect the probability of the other. It is therefore true.

Step 3: Test the explanatory link A true definition cannot establish the false assertion. Options (A) and (B) incorrectly call the assertion true, while (C) also calls the reason false. The required combination is a false assertion and a true reason.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Test independence with $P(A \cap B) = P(A)P(B)$; do not confuse it with a zero intersection.

20. Assertion (A): In a Linear Programming Problem, if the feasible region is empty, then the Linear Programming Problem has no solution.

Reason (R): A feasible region is defined as the region that satisfies all the constraints.

Select the correct answer from the codes given below:

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The feasible region is the set of points satisfying all the constraints of a linear programming problem, including non-negativity restrictions where given.
- A solution that optimizes the objective function must belong to the feasible region; an empty feasible set contains no candidate solution.

Step 1: Evaluate the assertion If the feasible region is empty, no point satisfies all the restrictions simultaneously. No admissible point can therefore maximize or minimize the objective function. The assertion is true.

Step 2: Evaluate the reason The reason correctly defines the feasible region as the common region satisfying every constraint. Each feasible point must satisfy all the restrictions, rather than merely one or some of them. Thus the reason is true.

Step 3: Check whether the reason explains the assertion The definition directly explains the assertion: an empty common region means there is no feasible point, so there is no solution to the problem. Option (B) wrongly denies this explanatory link, while (C) and (D) each wrongly declare one statement false.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Check whether the constraints have a common point before attempting to optimize the objective.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. If $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $A^2 = kA$, then find the value of k .

[2 marks]

Solution:

Concept:

- The square $A^2 = A \cdot A$ is obtained by row-by-column multiplication, not by squaring entries separately.
- Equal matrices have equal corresponding entries. A scalar multiple kA multiplies every entry of A by the same scalar k .

Step 1: Multiply the matrix by itself Each entry of the product is the dot product of a row of the first matrix and a column of the second. Thus

$$\begin{aligned} A^2 &= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2(2) + (-2)(-2) & 2(-2) + (-2)(2) \\ (-2)(2) + 2(-2) & (-2)(-2) + 2(2) \end{pmatrix} \\ &= \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}. \end{aligned}$$

Step 2: Compare corresponding entries The stipulated equation has the form

$$\begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} = \begin{pmatrix} 2k & -2k \\ -2k & 2k \end{pmatrix}.$$

Comparing the first diagonal entry gives $2k = 8$. The off-diagonal entries also require $-2k = -8$, giving the same scalar. All four entries therefore satisfy the matrix equation. Direct entry comparison avoids any need to divide by, or invert, the matrix.

Step 3: Conclusion Since $A^2 = 4A$, the required value is $k = 4$.

Quick Tip to Solve: Compute one diagonal and one off-diagonal product entry, then check that both have the same ratio to the original entries.

22(a). Simplify $\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$.

[2 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} x$ lies in $(-\pi/2, \pi/2)$, where cosine is positive.

Quick Tip to Solve: Set $x = \tan \theta$ with θ in the principal arctangent interval; this also handles negative x correctly.

- The identity $\sin^{-1}(\sin \theta) = \theta$ holds when $\theta \in [-\pi/2, \pi/2]$, the principal-value interval of inverse sine.

Step 1: Introduce a principal angle Put $\theta = \tan^{-1} x$, so $\tan \theta = x$ and $-\pi/2 < \theta < \pi/2$. Using $1 + \tan^2 \theta = \sec^2 \theta$ and the fact that $\cos \theta > 0$, we obtain

$$\cos \theta = \frac{1}{\sqrt{1+x^2}}, \quad \sin \theta = \tan \theta \cos \theta = \frac{x}{\sqrt{1+x^2}}.$$

The positive square root is essential because the angle belongs to the interval where cosine is positive; the sine still has the sign of x .

Step 2: Apply inverse sine on its principal interval Since θ already lies in the principal interval of inverse sine,

$$\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) = \sin^{-1}(\sin \theta) = \theta.$$

For every real x , $1+x^2 > 0$ and $x^2/(1+x^2) < 1$, so the original inverse sine is defined. No restriction such as $x \geq 0$ is needed.

Step 3: Conclusion $\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) = \boxed{\tan^{-1} x}$ for all $x \in \mathbb{R}$.

Quick Tip to Solve: Set $x = \tan \theta$ with θ in the principal arctangent interval; this also handles negative x correctly.

OR

22(b). Find the domain of $\sin^{-1} \sqrt{x-1}$.

[2 marks]

Solution:

Concept:

- A real square root $\sqrt{u}$ is defined only when $u \geq 0$, and its value is non-negative.
- The real function $\sin^{-1} t$ is defined exactly for $-1 \leq t \leq 1$. A composite expression must satisfy both its inner and outer domain conditions.

Step 1: Impose the square-root restriction For $\sqrt{x-1}$ to be real, its radicand must satisfy

$$x-1 \geq 0 \implies x \geq 1.$$

For these values, $\sqrt{x-1} \geq 0$, so the lower bound -1 for the inverse sine input is automatically satisfied.

Step 2: Impose the inverse-sine restriction The remaining restriction on the input is

$$0 \leq \sqrt{x-1} \leq 1.$$

Both sides of the upper inequality are non-negative, so squaring preserves it and gives $x-1 \leq 1$, or $x \leq 2$. Combining the restrictions yields $1 \leq x \leq 2$. The endpoints are allowed: at $x=1$ the expression equals $\sin^{-1} 0 = 0$, and at $x=2$ it equals $\sin^{-1} 1 = \pi/2$. Values below 1 make the square

Quick Tip to Solve: For an inverse sine of a square root, impose $0 \leq \text{radicand} \leq 1$ and retain both endpoints.

root non-real, while values above 2 make its value exceed 1.

Step 3: Conclusion The domain of $\sin^{-1} \sqrt{x-1}$ is $[1, 2]$.

Quick Tip to Solve: For an inverse sine of a square root, impose $0 \leq \text{radicand} \leq 1$ and retain both endpoints.

23. Calculate the area of the region bounded by the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the x -axis using integration.

[2 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x) \geq 0$ and the x -axis from $x = a$ to $x = b$ is $\int_a^b f(x) dx$.
- The ellipse is symmetric about both axes. Its upper and lower halves each bound an equal region with the x -axis.

Step 1: Identify the intercepts and the upper boundary Setting $y = 0$ gives $x = \pm 3$. The upper half of the ellipse is

$$y = 2\sqrt{1 - \frac{x^2}{9}} = \frac{2}{3}\sqrt{9 - x^2}.$$

Thus the region above the axis extends from -3 to 3 . By symmetry about the y -axis, its area is twice the area in the first quadrant.

Step 2: Evaluate the area integral Write

$$A = 2 \int_0^3 \frac{2}{3} \sqrt{9 - x^2} dx.$$

Use $x = 3 \sin \theta$, so $dx = 3 \cos \theta d\theta$, and the limits become 0 and $\pi/2$. Hence

$$\begin{aligned} A &= 12 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 6 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 3\pi. \end{aligned}$$

Step 3: Conclusion The area bounded by the ellipse and the x -axis on either side of the axis is 3π square units. The two equal regions together form the whole ellipse, of area 6π square units.

Quick Tip to Solve: Use the x -axis to identify a half-ellipse; double the first-quadrant integral, not quadruple it, for one such region.

24(a). Find the least value of 'a' so that $f(x) = 2x^2 - ax + 3$ is an increasing function on $[2, 4]$. [2 marks]

Solution:

Concept:

- For a differentiable function to be increasing on an interval, check that its derivative is nonnegative throughout that interval.
- A condition required for every $x \in [2, 4]$ must be tested at the point where the derivative takes its smallest value.

Step 1: Differentiate and impose the monotonicity condition Treating a as a real constant, differentiate the given polynomial:

$$f'(x) = 4x - a.$$

Thus we require $4x - a \geq 0$ throughout $[2, 4]$. Since $4x - a$ increases with x , its smallest value on this interval is at $x = 2$. The condition is therefore

$$8 - a \geq 0 \iff a \leq 8.$$

For $a = 8$, the derivative vanishes only at the left endpoint and is positive inside the interval; the function is still strictly increasing.

Step 2: Interpret the requested least value The full set of admissible parameters is $(-\infty, 8]$. Every admissible value a has a smaller admissible value, for example $a - 1$. Consequently this set has no least real member. In particular, 8 is its greatest member, not its least. The printed request for a least value cannot yield a finite real answer without an additional restriction on a .

Step 3: Conclusion f is increasing on $[2, 4]$ exactly when $a \leq 8$. As printed, there is no least real value of a ; the greatest admissible value is 8.

Quick Tip to Solve: After finding $a \leq 8$, inspect the direction of the inequality: an upper bound gives a greatest value and does not supply a least value.

OR

24(b). If $f(x) = x + \frac{1}{x}$, $x \geq 1$, show that f is an increasing function. [2 marks]

Solution:

Concept:

- A differentiable function whose derivative is positive throughout the interior of an interval is strictly increasing on that interval.
- The power rule gives $\frac{d}{dx}(x^{-1}) = -x^{-2}$. Check the sign of the derivative using the given domain.

Step 1: Differentiate the function Since $x \geq 1$, the reciprocal term is defined and the function is

Quick Tip to Solve: Factor the derivative as $(x^2 - 1)/x^2$ and use $x \geq 1$; a zero derivative only at the endpoint does not spoil increase.

continuous on its domain. Differentiating for positive x gives

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}.$$

The denominator is positive, so the sign is determined by $x^2 - 1$.

Step 2: Use the domain to establish the sign For $x \geq 1$, we have $x^2 \geq 1$, and hence $f'(x) \geq 0$. Moreover, $f'(x) > 0$ for every $x > 1$, with equality only at the endpoint $x = 1$. For any two points $1 \leq u < v$, the mean value theorem gives

$$f(v) - f(u) = f'(c)(v - u) > 0$$

for some $c \in (u, v)$, because $c > 1$. Thus the zero derivative at the single endpoint does not prevent strict increase.

Step 3: Conclusion Therefore $f(x) = x + \frac{1}{x}$ is increasing on $[1, \infty)$.

Quick Tip to Solve: Factor the derivative as $(x^2 - 1)/x^2$ and use $x \geq 1$; a zero derivative only at the endpoint does not spoil increase.

25. A cylindrical water container has developed a leak at the bottom. The water is leaking at the rate of $5 \text{ cm}^3/\text{s}$ from the leak. If the radius of the container is 15 cm, find the rate at which the height of water is decreasing inside the container, when the height of water is 2 metres. [2 marks]

Solution:

Concept:

- The water in a vertical cylindrical container has volume $V = \pi r^2 h$, where r is the fixed radius and h is the water height.
- Differentiate with respect to time t to relate the volume rate to the height rate: $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$. A decreasing quantity has a negative derivative.

Step 1: Express the volume using consistent units Measure lengths in centimetres and time in seconds. Since $r = 15 \text{ cm}$,

$$V = \pi(15)^2 h = 225\pi h.$$

Water leaves the container at $5 \text{ cm}^3/\text{s}$, so its volume changes at $dV/dt = -5 \text{ cm}^3/\text{s}$. The given water height 2 metres is 200 cm.

Step 2: Differentiate the volume relation The radius remains constant as the container empties. Differentiating gives

$$\begin{aligned}\frac{dV}{dt} &= 225\pi \frac{dh}{dt}, \\ -5 &= 225\pi \frac{dh}{dt}, \\ \frac{dh}{dt} &= -\frac{1}{45\pi} \text{ cm/s}.\end{aligned}$$

Quick Tip to Solve: Divide the outflow rate by the constant base area 225π ; report a positive magnitude when asked for the rate of decrease.

This expression contains no h , since the cylinder has the same cross-sectional area at every height. It therefore applies at $h = 200$ cm as well.

Step 3: Conclusion The water height is decreasing at $\frac{1}{45\pi}$ cm/s, approximately 0.00707 cm/s. Its signed rate of change is $-1/(45\pi)$ cm/s.

Quick Tip to Solve: Divide the outflow rate by the constant base area 225π ; report a positive magnitude when asked for the rate of decrease.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. Find: $\int \frac{\sqrt{x}}{1 + \sqrt{x^{3/2}}} dx.$

[3 marks]

Solution:

Concept:

- For real $x \geq 0$, the radical in the denominator satisfies $\sqrt{x^{3/2}} = x^{3/4}$. Retain both the square root and the printed exponent.
- Substitution rewrites an integral in a new variable together with its differential. The identity $t/(1+t) = 1 - 1/(1+t)$ then reduces the integrand to elementary terms.

Step 1: Rewrite the powers and choose a substitution Let the required integral be I . Using the positive real powers of x ,

$$I = \int \frac{x^{1/2}}{1 + x^{3/4}} dx.$$

Set $t = x^{3/4}$. For $x > 0$,

$$dt = \frac{3}{4}x^{-1/4} dx, \quad dx = \frac{4}{3}x^{1/4} dt.$$

The numerator together with the differential consequently becomes

$$x^{1/2} dx = \frac{4}{3}x^{3/4} dt = \frac{4}{3}t dt.$$

Step 2: Integrate in the new variable Substitute in the whole integral, including the differential:

$$\begin{aligned} I &= \frac{4}{3} \int \frac{t}{1+t} dt \\ &= \frac{4}{3} \int \left(1 - \frac{1}{1+t}\right) dt \\ &= \frac{4}{3} (t - \ln|1+t|) + C. \end{aligned}$$

Here $t \geq 0$, so $1+t$ is positive and the absolute-value signs may be omitted on the real domain.

Quick Tip to Solve: First combine the nested powers: $\sqrt{x^{3/2}} = x^{3/4}$. Choosing this as t turns the numerator and differential into $(4/3)t dt$.

Step 3: Substitute back and verify Restoring $t = x^{3/4}$, differentiate the resulting expression for $x > 0$:

$$\begin{aligned}\frac{d}{dx} \left[\frac{4}{3} \{x^{3/4} - \ln(1 + x^{3/4})\} \right] &= x^{-1/4} \left(1 - \frac{1}{1 + x^{3/4}} \right) \\ &= \frac{x^{1/2}}{1 + x^{3/4}}.\end{aligned}$$

This equals the original integrand, confirming the substitution and its factor.

Step 4: Conclusion

$$I = \frac{4}{3} \left[x^{3/4} - \ln(1 + x^{3/4}) \right] + C.$$

Quick Tip to Solve: First combine the nested powers: $\sqrt{x^{3/2}} = x^{3/4}$. Choosing this as t turns the numerator and differential into $(4/3)t \, dt$.

27. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$. [3 marks]

Solution:

Concept:

- A point on a line in symmetric form is obtained by setting all its ratios equal to a parameter. Use separate parameters for separate lines.
- Intersecting lines have one common point satisfying all three coordinate equations. The distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express the lines parametrically Set the ratios of the first line equal to t and those of the second equal to s . Their general points are

$$(x, y, z) = (1 + 2t, 2 + 3t, 3 + 4t), \quad (x, y, z) = (4 + 5s, 1 + 2s, s).$$

In the second line, the last expression is $z = s$; there is no added constant in its third coordinate.

Step 2: Solve for the common point Equate the first two coordinates:

$$2t - 5s = 3, \quad 3t - 2s = -1.$$

Multiplying these equations by 3 and 2, respectively, gives

$$6t - 15s = 9, \quad 6t - 4s = -2.$$

Subtraction yields $-11s = 11$, so $s = -1$, and substitution gives $t = -1$. The third coordinates also agree, since $3 + 4(-1) = -1 = s$. Thus the common point is $Q = (-1, -1, -1)$; checking this last coordinate rules out a false intersection obtained from only two equations.

Quick Tip to Solve: Solve two coordinate equations for the two line parameters, then always verify the third coordinate before using the distance formula.

Step 3: Calculate the required distance With the given point $P = (-1, -5, -10)$, the coordinate differences from P to Q are 0, 4, 9. Hence

$$PQ = \sqrt{(-1+1)^2 + (-1+5)^2 + (-1+10)^2} = \sqrt{0+16+81}.$$

Step 4: Conclusion The required distance is $\boxed{\sqrt{97} \text{ units}}$.

Quick Tip to Solve: Solve two coordinate equations for the two line parameters, then always verify the third coordinate before using the distance formula.

28(a). If $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) = \log_a x$ ($a > 0$ and $a \neq 1$) is defined, then prove that f is a bijection.

($\mathbb{R}^+$ is the set of all positive real numbers.)

[3 marks]

Solution:

Concept:

- A function is one-one if equal outputs imply equal inputs: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
- A function is onto if every element of its stated codomain has a preimage in its domain. A function that is both one-one and onto is a bijection.
- For $a > 0$, $a \neq 1$, logarithms and exponentials are inverse operations: $a^{\log_a x} = x$ for $x > 0$, and $\log_a(a^y) = y$ for every real y .

Step 1: Establish that the function is well-defined Since the base satisfies $a > 0$ and $a \neq 1$, $\log_a x$ exists as a unique real number for each positive real x . Thus the rule does define a function from $\mathbb{R}^+$ to $\mathbb{R}$, including when $0 < a < 1$.

Step 2: Prove the function is one-one Let $x_1, x_2 \in \mathbb{R}^+$ and suppose $f(x_1) = f(x_2)$. Then

$$\log_a x_1 = \log_a x_2.$$

Applying the exponential function with base a to both sides gives

$$a^{\log_a x_1} = a^{\log_a x_2} \Rightarrow x_1 = x_2.$$

Hence no two distinct positive inputs have the same image, so f is one-one.

Step 3: Prove the function is onto Take an arbitrary $y \in \mathbb{R}$. Choose $x = a^y$. The positivity of the base ensures $a^y > 0$, so this chosen input belongs to the domain. Now

$$f(x) = f(a^y) = \log_a(a^y) = y.$$

Every real number therefore has a preimage in $\mathbb{R}^+$, establishing that f is onto the stated codomain.

Step 4: Conclusion The function $f(x) = \log_a x$ is both one-one and onto; therefore $\boxed{f \text{ is a bijection}}$.

Quick Tip to Solve: For surjectivity of a logarithm, start with an arbitrary real output y and choose the positive input a^y ; this also handles bases between 0 and 1.

28(b). Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. A relation R from A to B is defined as $R = \{(x, y) : x + y = 6, x \in A, y \in B\}$.

(i) Write all elements of R .

(ii) Is R a function? Justify.

(iii) Determine the domain and range of R .

[3 marks]

Solution:

Concept:

- A relation from A to B is a subset of $A \times B$. An ordered pair is included only when both its coordinates lie in the specified sets and satisfy the defining condition.
- A function from A to B must assign exactly one image in B to every element of A .
- The domain of a relation is the set of first coordinates of its pairs; its range is the set of second coordinates actually attained.

Step 1: List the ordered pairs satisfying the condition The defining equation $x + y = 6$ gives $y = 6 - x$. Test each permitted value of x in $A = \{1, 2, 3\}$:

$$x = 1 \Rightarrow y = 5 \in B,$$

$$x = 2 \Rightarrow y = 4 \in B,$$

$$x = 3 \Rightarrow y = 3 \notin B.$$

Thus the first two choices give the pairs $(1, 5)$ and $(2, 4)$, while $(3, 3)$ is excluded because its second coordinate is not in $B = \{4, 5, 6\}$. No other first coordinate is allowed, so $R = \{(1, 5), (2, 4)\}$.

Step 2: Test whether the relation is a function from A to B The inputs 1 and 2 each have a single image, but the element $3 \in A$ has no image in B . A function from the stated set A must be defined at every member of A . Therefore R is not a function from A to B ; uniqueness for only the inputs that occur is insufficient.

Step 3: Read the domain and range from the pairs The first coordinates present in R are 1 and 2, whereas the second coordinates are 5 and 4. Consequently

$$\text{Dom}(R) = \{1, 2\}, \quad \text{Range}(R) = \{4, 5\}.$$

The element 6 belongs to the codomain B but not to the range, since no pair of the relation has second coordinate 6.

Step 4: Conclusion (i) $R = \{(1, 5), (2, 4)\}$. (ii) R is not a function from A to B , because 3 has no image. (iii) The domain is $\{1, 2\}$ and the range is $\{4, 5\}$.

Quick Tip to Solve: After solving $y = 6 - x$, check membership in B ; then compare the first coordinates obtained with every element of A before claiming a function.

29(a). The probability distribution of a random variable X is given by:

X	0	1	2	3
$P(X)$	p	$\frac{p}{3}$	$\frac{p}{6}$	$\frac{p}{12}$

(i) Determine the value of p .

(ii) Calculate $P(X \geq 1)$.

(iii) Calculate expectation of X , i.e. $E(X)$.

[3 marks]

Solution:

Concept:

- The probabilities of all possible values of a discrete random variable are nonnegative and sum to 1.
- The probability of an event is the sum of the probabilities of the mutually exclusive values satisfying that event.
- The expectation is the probability-weighted mean $E(X) = \sum_x xP(X = x)$, where the sum extends over all possible values of X .

Step 1: Determine the unknown probability p The four possible values of X are 0, 1, 2 and 3. Their probabilities exhaust the distribution, so

$$p + \frac{p}{3} + \frac{p}{6} + \frac{p}{12} = 1 \Rightarrow \frac{19p}{12} = 1.$$

Therefore $p = 12/19$. Substitution gives probabilities $12/19, 4/19, 2/19, 1/19$, respectively. All are nonnegative and their numerators sum to 19, confirming a valid distribution.

Step 2: Add the probabilities for X at least 1 The event $X \geq 1$ includes the values 1, 2 and 3. These outcomes are mutually exclusive, so

$$P(X \geq 1) = \frac{p}{3} + \frac{p}{6} + \frac{p}{12} = \frac{4 + 2 + 1}{19} = \frac{7}{19}.$$

The same value follows from the complement $1 - P(X = 0) = 1 - 12/19$. In particular, the equality sign means that the probability at $X = 1$ must be included.

Step 3: Calculate the probability-weighted mean Multiply every possible value by its own probability, including the zero-valued outcome, and add:

$$\begin{aligned} E(X) &= 0 \left(\frac{12}{19} \right) + 1 \left(\frac{4}{19} \right) + 2 \left(\frac{2}{19} \right) + 3 \left(\frac{1}{19} \right) \\ &= \frac{0 + 4 + 4 + 3}{19} = \frac{11}{19}. \end{aligned}$$

The mean lies between 0 and 3, as required.

Step 4: Conclusion (i) $p = \frac{12}{19}$, (ii) $P(X \geq 1) = \frac{7}{19}$, and (iii) $E(X) = \frac{11}{19}$.

Quick Tip to Solve: Once p is found, write the probabilities as $12 : 4 : 2 : 1$ over 19; use the complement of $X = 0$ for $X \geq 1$.

29(b). In a city, a survey was conducted among residents about their preferred mode of commuting. It was found that 50% people preferred using public transport, 35% preferred using a bicycle and 20% use both public transport and a bicycle. If a person is selected at random, find the probability that:

- (i) The person uses only public transport.
- (ii) The person uses a bicycle, given that they also use the public transport.
- (iii) The person uses neither public transport nor a bicycle.

[3 marks]

Solution:

Concept:

- For events P and B , the probability of only P is $P(P \cap B') = P(P) - P(P \cap B)$, where the prime denotes the complementary event.
- Conditional probability restricts the sample space: $P(B | P) = \frac{P(B \cap P)}{P(P)}$, provided $P(P) > 0$.
- The addition rule gives $P(P \cup B) = P(P) + P(B) - P(P \cap B)$. Neither event has probability $1 - P(P \cup B)$.

Step 1: Calculate the probability of only public transport Let P denote using public transport and B denote using a bicycle. The data give $P(P) = 0.50$, $P(B) = 0.35$ and $P(P \cap B) = 0.20$. For part (i), remove the people using both modes from the public-transport group:

$$P(P \cap B') = 0.50 - 0.20 = 0.30.$$

Step 2: Restrict the sample space to public-transport users In part (ii), the condition is that the selected person uses public transport. Hence the denominator is $P(P)$, while the numerator counts those who also use a bicycle:

$$P(B | P) = \frac{0.20}{0.50} = 0.40.$$

Dividing by 0.35 would reverse the condition and answer a different question.

Step 3: Use the complement to find neither mode For part (iii), first count those using at least one mode, subtracting the overlap once to avoid double counting:

$$P(P \cup B) = 0.50 + 0.35 - 0.20 = 0.65.$$

Therefore, the probability of neither mode is

$$P(P' \cap B') = 1 - 0.65 = 0.35.$$

Step 4: Conclusion The required probabilities are (i) 0.30, (ii) 0.40, (iii) 0.35.

Quick Tip to Solve: For “only”, subtract the overlap; for “given public transport”, divide by 0.50; for “neither”, complement the union.

30(a). Find k so that $f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x + 1}, & x \neq -1 \\ k, & x = -1 \end{cases}$ is continuous at $x = -1$.

[3 marks]

Solution:

Concept:

- A function is continuous at $x = a$ when its left-hand limit, right-hand limit and assigned value $f(a)$ are equal.
- A common factor can be cancelled for $x \neq a$ while evaluating a limit. This simplifies nearby values without determining the separately assigned value at a .

Step 1: State the condition at the specified point The second branch defines $f(-1) = k$. To make the function continuous at $x = -1$, its value must equal the limit of the first branch as x approaches -1 :

$$k = \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x + 1}.$$

Direct substitution in this quotient produces $0/0$, an indeterminate form, so simplify before evaluating the limit.

Step 2: Factor the numerator and cancel away from the point The two numbers with product -3 and sum -2 are -3 and 1 . Thus

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

For every $x \neq -1$, the factor $x + 1$ is nonzero, giving

$$\frac{(x - 3)(x + 1)}{x + 1} = x - 3.$$

Cancellation is legitimate in the limiting neighbourhood because a limit uses nearby points, not the value at the point itself.

Step 3: Evaluate both one-sided limits The simplified expression is a polynomial, so both one-sided limits follow by substitution:

$$\begin{aligned} \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} (x - 3) = -4, \\ \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} (x - 3) = -4. \end{aligned}$$

These equal limits establish that the two-sided limit exists. The continuity condition now requires $f(-1) = k = -4$; any other assigned value would differ from the common limiting value.

Step 4: Conclusion The required value is $k = -4$. With this choice, $\lim_{x \rightarrow -1} f(x) = f(-1) = -4$, so the function is continuous at $x = -1$.

Quick Tip to Solve: When substitution gives $0/0$, factor first; choose the separately assigned value to fill the removable hole with the simplified limit.

30(b). Check the differentiability of the function $f(x) = x|x|$ at $x = 0$.

[3 marks]

Solution:

Concept:

- The derivative at $x = 0$ is $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$, if this finite limit exists.
- Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$, expressions containing an absolute value can be checked separately on the two sides.

Step 1: Express the function on each side of zero Multiplying the piecewise values of $|x|$ by x gives

$$f(x) = x|x| = \begin{cases} -x^2, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

Also, $f(0) = 0$. Both branches approach zero at the origin, so the function is continuous there. However, continuity alone does not establish differentiability; the difference quotients must also agree.

Step 2: Calculate the left-hand derivative For a negative increment h , use $f(h) = -h^2$. Then

$$\begin{aligned} f'_-(0) &= \lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h^2}{h} = \lim_{h \rightarrow 0^-} (-h) = 0. \end{aligned}$$

The denominator remains h , including its negative sign; it is not replaced by $|h|$.

Step 3: Calculate the right-hand derivative and compare For a positive increment h , use $f(h) = h^2$. Thus

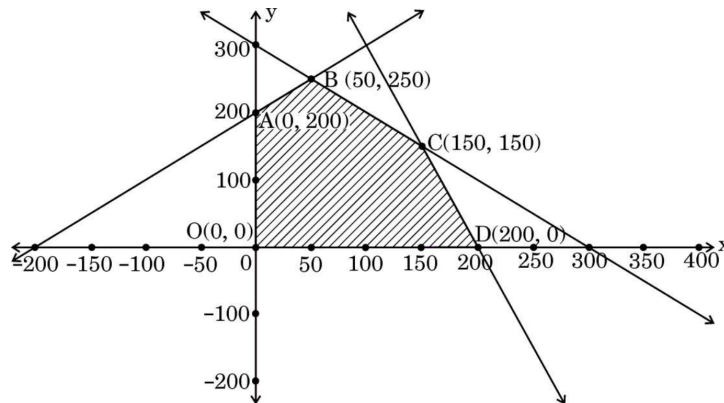
$$\begin{aligned} f'_+(0) &= \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h^2}{h} = \lim_{h \rightarrow 0^+} h = 0. \end{aligned}$$

Both one-sided derivatives exist and are finite and equal. Equivalently, for $h \neq 0$, the full difference quotient reduces to $h|h|/h = |h|$, which tends to zero from either side.

Step 4: Conclusion The function $f(x) = x|x|$ is differentiable at $x = 0$, and $f'(0) = 0$.

Quick Tip to Solve: At the origin, use the definition directly: $(h|h| - 0)/h = |h| \rightarrow 0$. The extra factor x removes the slope mismatch of $|x|$.

31. For the given graph of a Linear Programming Problem, write all the constraints satisfying the given feasible region. [3 marks]



Solution:

Concept:

- Find each boundary line from two labelled points, then test a shaded point to choose its inequality. Solid boundaries are included, so use non-strict inequalities.
- A region in the first quadrant satisfies $x \geq 0$ and $y \geq 0$. Its other constraints come from the sloping edges.

Step 1: Find the constraint along AB Through $A(0, 200)$ and $B(50, 250)$, the slope is $(250 - 200)/(50 - 0) = 1$. Thus the line is $y = x + 200$. The shaded point $(50, 50)$ satisfies $y - x = 0 < 200$, so the required side is

$$y - x \leq 200.$$

Step 2: Find the constraint along BC Through $B(50, 250)$ and $C(150, 150)$, the slope is $(150 - 250)/(150 - 50) = -1$. Substitution of either point gives $x + y = 300$. At the shaded point $(50, 50)$, $x + y = 100 < 300$, giving

$$x + y \leq 300.$$

Step 3: Find the constraint along CD Through $C(150, 150)$ and $D(200, 0)$, the slope is $(0 - 150)/(200 - 150) = -3$. Thus $y = -3(x - 200)$, or $3x + y = 600$. At $(50, 50)$, $3x + y = 200 < 600$, so

$$3x + y \leq 600.$$

Step 4: Include both coordinate-axis boundaries The shaded polygon $OABCD$ lies on or to the right of the y -axis and on or above the x -axis. Hence $x \geq 0$ and $y \geq 0$ are necessary. These five half-planes intersect in exactly the given region, including its solid boundary segments.

Step 5: Conclusion All the constraints are

$$x \geq 0, \quad y \geq 0, \quad y - x \leq 200, \quad x + y \leq 300, \quad 3x + y \leq 600.$$

Quick Tip to Solve: Obtain each line from its labelled vertices, then test the shaded point $(50, 50)$ to choose the sign; include both axis constraints.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32. The relation between the height of the plant (y cm) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to sunlight. [5 marks]

32(i). Find the rate of growth of the plant with respect to sunlight. [2 marks]

Solution:

Concept:

- When height y is a function of elapsed time x , the instantaneous rate of growth is the derivative $\frac{dy}{dx}$.
- The power rule gives $\frac{d}{dx}(x^n) = nx^{n-1}$. Since height is measured in centimetres and time in days, the derivative has units cm day^{-1} .

Step 1: Identify the quantity to differentiate The independent variable x measures the number of days of exposure to sunlight. Therefore, the requested growth rate is the rate of change of height with respect to x , rather than the height itself or the average ratio y/x . The given relation is

$$y = 4x - \frac{1}{2}x^2.$$

Step 2: Differentiate the height function Apply the power rule separately to the linear and quadratic terms, keeping the negative sign of the quadratic term:

$$\frac{dy}{dx} = 4 - \frac{1}{2}(2x) = 4 - x.$$

This gives the growth rate at any specified day x ; no particular day is supplied in this part. Its sign describes whether the modelled height is increasing or decreasing.

Step 3: Conclusion The rate of growth after x days of exposure is $\frac{dy}{dx} = 4 - x \text{ cm day}^{-1}$.

Quick Tip to Solve: Differentiate height with respect to the number of days; do not use height divided by time for an instantaneous growth rate.

32(ii). In how many days will the plant attain its maximum height? What is the maximum height? [3 marks]

Solution:

Concept:

- A differentiable height function can have an interior maximum at a stationary point, where its first derivative is zero.
- The second derivative test confirms a strict local maximum at $x = a$ when $y'(a) = 0$ and $y''(a) < 0$.

Quick Tip to Solve: First set the growth rate to zero and confirm a maximum; then substitute the day into the height function to obtain the height.

For a quadratic with a negative leading coefficient, this vertex is also its global maximum.

Step 1: Find the stationary day The height, in centimetres, after x days is $y = 4x - \frac{1}{2}x^2$. Differentiating with respect to the number of days gives the instantaneous growth rate. At a candidate maximum this rate must vanish:

$$\begin{aligned}\frac{dy}{dx} &= 4 - x, \\ 4 - x &= 0 \implies x = 4.\end{aligned}$$

The candidate occurs at a non-negative time, so it is admissible for the given model.

Step 2: Verify that the height is greatest there Differentiate again to test the stationary point:

$$\frac{d^2y}{dx^2} = -1 < 0.$$

Hence the stationary point is a strict maximum. In addition, $4 - x > 0$ for $0 \leq x < 4$ and $4 - x < 0$ for $x > 4$, so height increases up to this day and decreases after it. This establishes the greatest height for the non-negative times described by the quadratic model.

Step 3: Evaluate the height at the maximum Substitute the stationary value into the original height function, rather than into its derivative:

$$y(4) = 4(4) - \frac{1}{2}(4)^2 = 16 - 8 = 8 \text{ cm}.$$

As an algebraic check, completing the square gives $y = 8 - \frac{1}{2}(x-4)^2$. The squared term is non-negative, so this expression cannot exceed its value when $x = 4$.

Step 4: Conclusion The plant attains its maximum height after 4 days, and the maximum height is 8 cm.

Quick Tip to Solve: First set the growth rate to zero and confirm a maximum; then substitute the day into the height function to obtain the height.

33(a). Show that the area of a parallelogram whose diagonals are represented by $\vec{a}$ and $\vec{b}$ is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$. Also find the area of a parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$. [5 marks]

Solution:

Concept:

- For adjacent side vectors $\vec{u}$ and $\vec{v}$, the area of a parallelogram is $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, where θ is their included angle.
- The diagonals of a parallelogram can be represented by the sum and difference of its adjacent side vectors. Reversing a diagonal changes its vector's sign but leaves the required magnitude unchanged.
- The cross product is distributive and anti-commutative: $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$, while the cross product of a vector with itself is zero.

Quick Tip to Solve: When the given vectors are diagonals, halve the magnitude of their cross product. Keep the alternating cofactor signs $+, -, +$ when expanding the determinant.

Step 1: Express the diagonals using adjacent sides Let $ABCD$ be a parallelogram with adjacent side vectors $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \vec{v}$. By the triangle law of vector addition, its diagonals may be oriented so that

$$\vec{a} = \overrightarrow{AC} = \vec{u} + \vec{v}, \quad \vec{b} = \overrightarrow{BD} = \vec{v} - \vec{u}.$$

These identities connect the given diagonal vectors to the side vectors that occur in the standard area formula. If either given diagonal has the reverse orientation, the magnitude of the final cross product is unaffected.

Step 2: Derive the diagonal formula for area Expand the cross product using distributivity:

$$\begin{aligned} \vec{a} \times \vec{b} &= (\vec{u} + \vec{v}) \times (\vec{v} - \vec{u}) \\ &= \vec{u} \times \vec{v} - \vec{u} \times \vec{u} + \vec{v} \times \vec{v} - \vec{v} \times \vec{u} \\ &= \vec{u} \times \vec{v} + \vec{u} \times \vec{v} \\ &= 2(\vec{u} \times \vec{v}). \end{aligned}$$

Taking magnitudes gives $|\vec{a} \times \vec{b}| = 2|\vec{u} \times \vec{v}|$. Since $|\vec{u} \times \vec{v}|$ is the parallelogram's area, division by two establishes the required formula in terms of its diagonals.

Step 3: Calculate the cross product of the given diagonals For $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$, expand the determinant along its first row:

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ &= \hat{i}[(-1)(-1) - 1(3)] - \hat{j}[2(-1) - 1(1)] \\ &\quad + \hat{k}[2(3) - (-1)(1)] \\ &= -2\hat{i} + 3\hat{j} + 7\hat{k}. \end{aligned}$$

In particular, the middle component is positive because the negative cofactor sign multiplies -3 . Its magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 3^2 + 7^2} = \sqrt{62}.$$

The vectors represent diagonals, so the area uses half this magnitude; the full magnitude would apply to adjacent side vectors instead.

Step 4: Conclusion The area of a parallelogram with diagonal vectors $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$. For the given diagonals, the area is $\frac{\sqrt{62}}{2}$ square units.

Quick Tip to Solve: When the given vectors are diagonals, halve the magnitude of their cross product. Keep the alternating cofactor signs $+, -, +$ when expanding the determinant.

33(b). Find the equation of a line in vector and cartesian form which passes through the point $(1, 2, -4)$ and is perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

[5 marks]

Solution:**Concept:**

- In the symmetric equation of a line, the denominators give its direction ratios. In vector form $\vec{r} = \vec{a} + \mu\vec{b}$, the coefficient vector $\vec{b}$ gives its direction.
- A direction perpendicular to two nonparallel vectors $\vec{b}_1$ and $\vec{b}_2$ is their cross product $\vec{b}_1 \times \vec{b}_2$. Multiplying a direction vector by any nonzero scalar represents the same line direction.
- A line through a point with position vector $\vec{a}$, in direction $\vec{d}$, has equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.

Step 1: Identify the two given directions From the denominators of the first line and the coefficient of μ in the second line, take

$$\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}, \quad \vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}.$$

The constants locating these given lines do not change their directions. We need a direction orthogonal to both vectors and then use the prescribed point to locate our line.

Step 2: Calculate a perpendicular direction Expand the cross product, taking care of the minus sign in the middle cofactor:

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= (80 - 56)\hat{i} - (-15 - 21)\hat{j} + (24 + 48)\hat{k} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}. \end{aligned}$$

Dividing all components by 12, use the simpler direction vector $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. The cross product is nonzero, so this determines the perpendicular direction uniquely up to a nonzero factor.

Step 3: Verify orthogonality and use the given point Check the direction using scalar products:

$$\begin{aligned} \vec{d} \cdot \vec{b}_1 &= 2(3) + 3(-16) + 6(7) = 6 - 48 + 42 = 0, \\ \vec{d} \cdot \vec{b}_2 &= 2(3) + 3(8) + 6(-5) = 6 + 24 - 30 = 0. \end{aligned}$$

Thus it makes a right angle with both given directions. The point $(1, 2, -4)$ has position vector $\hat{i} + 2\hat{j} - 4\hat{k}$. Using this point and $\vec{d}$, the parametric coordinates are

$$x = 1 + 2\lambda, \quad y = 2 + 3\lambda, \quad z = -4 + 6\lambda.$$

Setting $\lambda = 0$ recovers the required point. Eliminate λ by dividing the three coordinate differences by their respective direction ratios.

Quick Tip to Solve: Use only the direction ratios when taking the cross product; reduce their common factor, then check both dot products before inserting the given point.

Step 4: Conclusion The required vector and Cartesian equations, respectively, are

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad \lambda \in \mathbb{R}$$

and

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

Quick Tip to Solve: Use only the direction ratios when taking the cross product; reduce their common factor, then check both dot products before inserting the given point.

34(a). Evaluate: $\int_0^{3/2} |x \cos \pi x| dx.$

[5 marks]

Solution:

Concept:

- A modulus satisfies $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$. Split a definite integral at every interior point where the expression changes sign.
- Integration by parts gives $\int u dv = uv - \int v du$. For $x \cos(\pi x)$, choose $u = x$ so that differentiating it leaves a constant.

Step 1: Determine the sign inside the modulus Write $I = \int_0^{3/2} |x \cos(\pi x)| dx$. On this interval $x \geq 0$, so the sign of the product is determined by $\cos(\pi x)$. Its zeros satisfy

$$\pi x = \frac{\pi}{2} + n\pi, \quad x = \frac{1}{2} + n,$$

with integer n . The only interior zero is $x = \frac{1}{2}$; $x = \frac{3}{2}$ is the upper endpoint. Thus $\cos(\pi x) \geq 0$ from 0 to $\frac{1}{2}$, and it is nonpositive from $\frac{1}{2}$ to $\frac{3}{2}$. Consequently,

$$I = \int_0^{1/2} x \cos(\pi x) dx - \int_{1/2}^{3/2} x \cos(\pi x) dx.$$

The second minus sign reverses the negative signed contribution, as required by the modulus.

Step 2: Find an antiderivative by parts Take $u = x$ and $dv = \cos(\pi x) dx$. Then $du = dx$ and $v = \sin(\pi x)/\pi$. Hence

$$\begin{aligned} \int x \cos(\pi x) dx &= \frac{x \sin(\pi x)}{\pi} - \frac{1}{\pi} \int \sin(\pi x) dx \\ &= \frac{x \sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi^2} + C. \end{aligned}$$

Let $F(x) = x \sin(\pi x)/\pi + \cos(\pi x)/\pi^2$. Differentiating checks the result: the terms $\sin(\pi x)/\pi$ and $-\sin(\pi x)/\pi$ cancel, leaving $F'(x) = x \cos(\pi x)$.

Step 3: Evaluate the two signed contributions Use $\sin 0 = 0$, $\cos 0 = 1$, $\sin(\pi/2) = 1$, $\sin(3\pi/2) = -1$,

Quick Tip to Solve: Since $x \geq 0$, split only where the cosine changes sign, at $x = 1/2$; subtract the whole second definite integral before simplifying its limits.

and the zero cosine values at both odd half multiples of π . These give

$$F(0) = \frac{1}{\pi^2}, \quad F\left(\frac{1}{2}\right) = \frac{1}{2\pi}, \quad F\left(\frac{3}{2}\right) = -\frac{3}{2\pi}.$$

Therefore the first integral contributes $\frac{1}{2\pi} - \frac{1}{\pi^2}$. The signed second integral equals $-\frac{3}{2\pi} - \frac{1}{2\pi} = -\frac{2}{\pi}$, so subtracting it contributes $\frac{2}{\pi}$. Combining the two nonnegative contributions gives

$$I = \left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) + \frac{2}{\pi}.$$

Step 4: Conclusion The value of the integral is

$$\int_0^{3/2} |x \cos(\pi x)| dx = \frac{5}{2\pi} - \frac{1}{\pi^2} = \frac{5\pi - 2}{2\pi^2}.$$

Quick Tip to Solve: Since $x \geq 0$, split only where the cosine changes sign, at $x = 1/2$; subtract the whole second definite integral before simplifying its limits.

OR

34(b). Find: $\int \frac{dx}{\sin x + \sin 2x}.$

[5 marks]

Solution:

Concept:

- The double-angle identity $\sin 2x = 2 \sin x \cos x$ factors the denominator into $\sin x(1 + 2 \cos x)$.
- The substitution $u = \cos x$, with $du = -\sin x dx$, converts the integral to a rational function by using $\sin^2 x = 1 - u^2$.
- Resolve a rational function with distinct linear factors into partial fractions, and use $\int dt/t = \ln |t| + C$. Absolute values keep the logarithms valid wherever their arguments are nonzero.

Step 1: Factor the denominator and choose a substitution Write the requested integral as I . Applying the double-angle identity gives

$$I = \int \frac{dx}{\sin x(1 + 2 \cos x)} = \int \frac{\sin x dx}{\sin^2 x(1 + 2 \cos x)}.$$

Multiplying the numerator and denominator by $\sin x$ introduces the differential needed for $u = \cos x$. This is permitted on every interval where the original integrand is defined, since its denominator already requires $\sin x \neq 0$.

$$I = - \int \frac{du}{(1 - u^2)(1 + 2u)} = - \int \frac{du}{(1 - u)(1 + u)(1 + 2u)}.$$

Quick Tip to Solve: Factor $\sin x + \sin 2x$ first, then multiply by $\sin x / \sin x$ to use $u = \cos x$. Keep both the substitution minus sign and the factor $1/2$ from integrating $1/(1 + 2u)$.

Step 2: Determine the partial-fraction coefficients Set

$$\frac{1}{(1-u)(1+u)(1+2u)} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{D}{1+2u}.$$

Clearing the denominators gives the polynomial identity

$$1 = A(1+u)(1+2u) + B(1-u)(1+2u) + D(1-u^2).$$

Evaluate this identity at the roots of the three factors. At $u = 1$, it gives $1 = 6A$; at $u = -1$, it gives $1 = -2B$; and at $u = -1/2$, it gives $1 = 3D/4$. Therefore

$$A = \frac{1}{6}, \quad B = -\frac{1}{2}, \quad D = \frac{4}{3}.$$

These substitutions determine coefficients in an identity; they do not require the integrand itself to be defined at these values.

Step 3: Integrate each linear factor Substitute the coefficients and retain the minus sign from $du = -\sin x \, dx$:

$$I = -\frac{1}{6} \int \frac{du}{1-u} + \frac{1}{2} \int \frac{du}{1+u} - \frac{4}{3} \int \frac{du}{1+2u}.$$

The derivatives of $1-u$ and $1+2u$ contribute factors of -1 and 2 , respectively. Hence

$$I = \frac{1}{6} \ln |1-u| + \frac{1}{2} \ln |1+u| - \frac{2}{3} \ln |1+2u| + C.$$

Step 4: Restore the original variable and check the derivative Replace u by $\cos x$. To check the signs and coefficients, differentiate the logarithmic expression with respect to u . The result is

$$-\frac{1}{6(1-u)} + \frac{1}{2(1+u)} - \frac{4}{3(1+2u)} = -\frac{1}{(1-u^2)(1+2u)}.$$

Multiplication by $du/dx = -\sin x$ gives $1/[\sin x(1+2\cos x)]$, which is precisely the original integrand. The antiderivative applies on each interval avoiding its singularities.

Step 5: Conclusion

$$I = \frac{1}{6} \ln |1 - \cos x| + \frac{1}{2} \ln |1 + \cos x| - \frac{2}{3} \ln |1 + 2 \cos x| + C.$$

Quick Tip to Solve: Factor $\sin x + \sin 2x$ first, then multiply by $\sin x / \sin x$ to use $u = \cos x$. Keep both the substitution minus sign and the factor $1/2$ from integrating $1/(1+2u)$.

35. If A is a 3×3 invertible matrix, show that for any scalar $k \neq 0$, $(kA)^{-1} = \frac{1}{k}A^{-1}$. Hence calculate $(3A)^{-1}$, where $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. [5 marks]

Solution:

Concept:

- An inverse A^{-1} satisfies $AA^{-1} = A^{-1}A = I_3$, where I_3 is the identity matrix of order three. To prove a proposed inverse, check both products.
- A scalar commutes with matrix multiplication. For a nonzero scalar k , its reciprocal $1/k$ exists and cancels the scalar factor in kA .
- For a nonsingular matrix, $A^{-1} = \text{adj}(A)/|A|$. The adjugate is the transpose of the cofactor matrix, whose entries are $C_{ij} = (-1)^{i+j}M_{ij}$, with M_{ij} the corresponding minor.

Step 1: Prove the inverse formula for a scalar multiple Since A is invertible and $k \neq 0$, the matrix $B = (1/k)A^{-1}$ is well defined. Compute its products with kA in both orders:

$$\begin{aligned}(kA)B &= (kA) \left(\frac{1}{k}A^{-1} \right) = \frac{k}{k}AA^{-1} = I_3, \\ B(kA) &= \left(\frac{1}{k}A^{-1} \right) (kA) = \frac{k}{k}A^{-1}A = I_3.\end{aligned}$$

Thus B is the inverse of kA , proving

$$(kA)^{-1} = \frac{1}{k}A^{-1}.$$

The condition $k \neq 0$ is essential: when $k = 0$, the scalar multiple is the zero matrix and has no inverse.

Step 2: Evaluate the determinant of the given matrix Expand along the first row of

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Including the alternating signs of this expansion, we obtain

$$\begin{aligned}|A| &= 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \\ &= 2(4 - 1) + (-2 + 1) + (1 - 2) \\ &= 6 - 1 - 1 = 4.\end{aligned}$$

In particular, the determinant is nonzero, so the adjugate formula can be used for this matrix.

Step 3: Find the cofactors and the adjugate Delete the row and column of each entry, evaluate the

Quick Tip to Solve: Prove the scalar rule by multiplying the proposed inverse on both sides. Then invert A only once and divide by 3; there is no need to recompute all cofactors of $3A$.

resulting determinant, and apply the sign $(-1)^{i+j}$. This gives

$$\begin{aligned} C_{11} &= 4 - 1 = 3, & C_{12} &= -(-2 + 1) = 1, & C_{13} &= 1 - 2 = -1, \\ C_{21} &= -(-2 + 1) = 1, & C_{22} &= 4 - 1 = 3, & C_{23} &= -(-2 + 1) = 1, \\ C_{31} &= 1 - 2 = -1, & C_{32} &= -(-2 + 1) = 1, & C_{33} &= 4 - 1 = 3. \end{aligned}$$

The cofactor matrix is symmetric, so transposing it leaves it unchanged:

$$\text{adj}(A) = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad A^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Step 4: Apply the result with the required scalar Taking $k = 3$ in the proved identity, multiply every entry of A^{-1} by $1/3$. The common factor is consequently $(1/3)(1/4) = 1/12$. As an arithmetic check,

$$A \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3.$$

Therefore multiplying $3A$ by one twelfth of this adjugate gives $(3/12)(4I_3) = I_3$, as required.

Step 5: Conclusion For every nonzero scalar k , $(kA)^{-1} = \frac{1}{k}A^{-1}$. In particular,

$$(3A)^{-1} = \frac{1}{12} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Quick Tip to Solve: Prove the scalar rule by multiplying the proposed inverse on both sides. Then invert A only once and divide by 3; there is no need to recompute all cofactors of $3A$.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

Some students are having a misconception while comparing decimals. For example, a student may mention that $78.56 > 78.9$ as $7856 > 789$. In order to assess this concept, a decimal comparison test was administered to the students of class VI through the following question:

In the recently held Sports Day in the school, 5 students participated in a javelin throw competition. The distances to which they have thrown the javelin are shown below in the table:

Name of student	Distance of javelin (in meters)
Ajay	47.7
Bijoy	47.07
Kartik	43.09
Dinesh	43.9
Devesh	45.2

The students were asked to identify who has thrown the javelin the farthest.

Based on the test attempted by the students, the teacher concludes that 40% of the students have the misconception in the concept of decimal comparison and the rest do not have the misconception. 80% of the students having misconception answered Bijoy as the correct answer in the paper. 90% of the students who are identified with not having misconception, did not answer Bijoy as their answer.

On the basis of the above information, answer the following questions:

[4 marks]

36(i). What is the probability of a student not having misconception but still answers Bijoy in the test?

[1 mark]

Solution:

Concept:

- Let M denote having the misconception and B denote answering Bijoy. The complement M' denotes having no misconception.
- The multiplication rule gives $P(M' \cap B) = P(M')P(B | M')$. A joint probability requires both stated events to occur.

Step 1: Find the complementary probabilities Since 40% have the misconception, $P(M') = 1 - 0.40 = 0.60$. Among students without it, 90% do not answer Bijoy. Therefore

$$P(B | M') = 1 - 0.90 = 0.10.$$

Step 2: Multiply the group probability by its response probability We need a randomly selected student to belong to the group without misconception and still answer Bijoy. Thus

$$P(M' \cap B) = (0.60)(0.10) = 0.06.$$

The value 0.10 concerns only the students within that group; multiplication by 0.60 expresses the

Quick Tip to Solve: Take the complement of 90% within the no-misconception group, then multiply by the 60% share of that group.

required fraction of all students.

Step 3: Conclusion The required probability is $0.06 = \frac{3}{50}$.

Quick Tip to Solve: Take the complement of 90% within the no-misconception group, then multiply by the 60% share of that group.

36(ii). What is the probability that a randomly selected student answers Bijoy as his answer in the test? [1 mark]

Solution:

Concept:

- Let M denote having the misconception, M' having no misconception, and B answering Bijoy.
- The disjoint, exhaustive groups M and M' give the total probability formula $P(B) = P(M)P(B | M) + P(M')P(B | M')$.

Step 1: Identify the group and conditional probabilities The given data imply $P(M) = 0.40$, $P(M') = 0.60$, and $P(B | M) = 0.80$. Of students without misconception, 90% do not choose Bijoy, so $P(B | M') = 0.10$.

Step 2: Add the two contributions A student can answer Bijoy from either group. These possibilities are disjoint, and their joint probabilities are $0.40 \times 0.80 = 0.32$ and $0.60 \times 0.10 = 0.06$. Therefore

$$P(B) = 0.32 + 0.06 = 0.38.$$

Adding 0.80 and 0.10 directly would ignore the different group sizes; each conditional probability must first be weighted by its group's probability.

Step 3: Conclusion The probability that a randomly selected student answers Bijoy is $0.38 = \frac{19}{50}$.

Quick Tip to Solve: Count Bijoy answers in both groups: out of 100 students, 32 with misconception and 6 without it give a total of 38.

36(iii)(a). What is the probability that a student who answered as Bijoy is having misconception? [2 marks]

Solution:

Concept:

- Let M denote the event that a student has the misconception, M' its complement, and B the event that the student answers Bijoy.
- Bayes' theorem reverses the conditioning: $P(M | B) = P(M)P(B | M)/P(B)$. The denominator is found using the law of total probability.

Step 1: Translate the percentages into probabilities We have $P(M) = 0.40$, $P(M') = 0.60$, and

Quick Tip to Solve: Among 100 students, 32 with misconception and 6 without it answer Bijoy; condition on these 38 students to get 32/38.

$P(B | M) = 0.80$. Since 90% of the students without misconception do not answer Bijoy, the probability that they do is

$$P(B | M') = 1 - 0.90 = 0.10.$$

Step 2: Find the total probability of answering Bijoy The two misconception groups partition the entire class. Adding their contributions to the answer Bijoy gives

$$P(B) = (0.40)(0.80) + (0.60)(0.10) = 0.32 + 0.06 = 0.38.$$

The event of both having the misconception and answering Bijoy has probability $P(M \cap B) = 0.32$.

Step 3: Condition on the observed answer Only the students who answered Bijoy are now under consideration. Hence divide the joint probability by their total probability:

$$P(M | B) = \frac{P(M \cap B)}{P(B)} = \frac{0.32}{0.38} = \frac{16}{19}.$$

This is different from $P(B | M) = 0.80$, which starts by restricting the group to students already known to have the misconception.

Step 4: Conclusion The probability that a student who answered Bijoy has the misconception is $\boxed{\frac{16}{19}}$.

Quick Tip to Solve: Among 100 students, 32 with misconception and 6 without it answer Bijoy; condition on these 38 students to get 32/38.

OR

36(iii)(b). What is the probability that a student who answered as Bijoy is amongst students who do not have the misconception? [2 marks]

Solution:

Concept:

- Let M be the event that a student has the misconception, and B the event that the student answers Bijoy. The complement M' means no misconception.
- Bayes' theorem gives $P(M' | B) = P(M')P(B | M')/P(B)$, where $P(B)$ includes contributions from both groups.

Step 1: Translate the given percentages We have $P(M) = 0.40$, so $P(M') = 0.60$. Among students with the misconception, $P(B | M) = 0.80$. Since 90% of those without it did not answer Bijoy, the complementary probability is

$$P(B | M') = 1 - 0.90 = 0.10.$$

Step 2: Find the probability of answering Bijoy The events M and M' partition the students into disjoint groups. Thus the total probability formula gives

$$P(B) = (0.40)(0.80) + (0.60)(0.10) = 0.32 + 0.06 = 0.38.$$

Quick Tip to Solve: In a hypothetical group of 100 students, 6 without the misconception and 32 with it answer Bijoy; the desired fraction is therefore 6 out of 38.

The joint probability of having no misconception and answering Bijoy is $P(M' \cap B) = 0.06$. The question conditions on the answer Bijoy, so we divide this joint probability by $P(B)$, not by $P(M')$.

Step 3: Apply Bayes' theorem Among all students who answer Bijoy, the required fraction belongs to the group without the misconception:

$$P(M' | B) = \frac{0.06}{0.38} = \frac{6}{38} = \frac{3}{19}.$$

Step 4: Conclusion The required probability is $\boxed{\frac{3}{19}}$.

Quick Tip to Solve: In a hypothetical group of 100 students, 6 without the misconception and 32 with it answer Bijoy; the desired fraction is therefore 6 out of 38.

37. Case Study - 2

An engineer is designing a new metro rail network in a city.

Initially, two metro lines, Line A and Line B, each consisting of multiple stations are designed. The track for Line A is represented by $l_1 : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{4}$, while the track for Line B is represented by $l_2 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z+2}{-3}$.

Based on the above information, answer the following questions:

[4 marks]

37(i). Find whether the two metro tracks are parallel.

[1 mark]

Solution:

Concept:

- The denominators of a line in symmetric form give a set of direction ratios for that line.
- Two lines are parallel if and only if their direction vectors are scalar multiples, so all corresponding direction ratios must be proportional.

Step 1: Read the direction ratios From the equations of the two tracks, direction vectors are

$$\vec{d}_1 = 3\hat{i} - 2\hat{j} + 4\hat{k}, \quad \vec{d}_2 = 2\hat{i} + \hat{j} - 3\hat{k}.$$

The coordinates of the fixed points in the numerators do not determine whether the tracks are parallel.

Step 2: Test proportionality Compare the ratios of corresponding components:

$$\frac{3}{2}, \quad \frac{-2}{1} = -2, \quad \frac{4}{-3} = -\frac{4}{3}.$$

These three ratios are unequal. No single nonzero scalar takes $\vec{d}_2$ to $\vec{d}_1$, so the criterion for parallel lines is not satisfied.

Step 3: Conclusion The two metro tracks are $\boxed{\text{not parallel}}$.

Quick Tip to Solve: Compare the denominators in the symmetric equations; two unequal component ratios already rule out parallel tracks.

37(ii). Solar panels are to be installed on the rooftop of the metro stations. Determine the equation of the line representing the placement of solar panels on the rooftop of Line A's stations, given that panels are to be positioned parallel to Line A's track (l_1) and pass through the point $(1, -2, -3)$.

[1 mark]

Solution:

Concept:

- A line parallel to a given line can use the same direction ratios as that line.
- A line through (x_0, y_0, z_0) with nonzero direction ratios (a, b, c) has equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.

Step 1: Choose the point and direction Line A has direction ratios $(3, -2, 4)$. The solar-panel line must be parallel to Line A, so it uses these same ratios and passes through $P(1, -2, -3)$.

Step 2: Substitute into the point-direction form Putting the coordinates of P and the direction ratios into the symmetric equation gives

$$\frac{x - 1}{3} = \frac{y - (-2)}{-2} = \frac{z - (-3)}{4}.$$

Equivalently, the parametrisation is $x = 1 + 3t$, $y = -2 - 2t$, $z = -3 + 4t$. At $t = 0$ it gives the required point, while its direction vector agrees with Line A's direction.

Step 3: Conclusion The required line is $\boxed{\frac{x - 1}{3} = \frac{y + 2}{-2} = \frac{z + 3}{4}}.$

Quick Tip to Solve: Keep Line A's denominators and replace the fixed point in the numerators; subtracting the negative coordinates gives $y + 2$ and $z + 3$.

37(iii)(a). To connect the stations, a pedestrian pathway perpendicular to the two metro lines is to be constructed which passes through point $(3, 2, 1)$. Determine the equation of the pedestrian walkway.

[2 marks]

Solution:

Concept:

- A line's direction ratios are the denominators of its symmetric equation. The metro lines have direction vectors $\vec{b}_1 = (3, -2, 4)$ and $\vec{b}_2 = (2, 1, -3)$.
- The cross product $\vec{b}_1 \times \vec{b}_2$ is perpendicular to both direction vectors. A line through position vector $\vec{a}$ with direction $\vec{d}$ is $\vec{r} = \vec{a} + \lambda \vec{d}$, where $\lambda \in \mathbb{R}$.

Quick Tip to Solve: Take the cross product of the two metro directions, then use the given point in the numerators; verify both dot products are zero.

Step 1: Find the perpendicular direction Use the cross product of the two given directions:

$$\begin{aligned}\vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ 2 & 1 & -3 \end{vmatrix} \\ &= ((-2)(-3) - 4)\hat{i} - (3(-3) - 8)\hat{j} + (3 + 4)\hat{k} \\ &= 2\hat{i} + 17\hat{j} + 7\hat{k}.\end{aligned}$$

Its dot products with the original directions are $6 - 34 + 28 = 0$ and $4 + 17 - 21 = 0$, confirming perpendicularity to both.

Step 2: Use the specified point The pathway passes through $P = (3, 2, 1)$. Using direction ratios $(2, 17, 7)$, its parametric equations are $x = 3 + 2\lambda$, $y = 2 + 17\lambda$, and $z = 1 + 7\lambda$. Eliminate the common parameter to obtain its symmetric form. Multiplying all three direction ratios by the same nonzero constant would describe the same line.

Step 3: Conclusion The required pedestrian walkway has equation

$$\boxed{\frac{x-3}{2} = \frac{y-2}{17} = \frac{z-1}{7}}.$$

Quick Tip to Solve: Take the cross product of the two metro directions, then use the given point in the numerators; verify both dot products are zero.

OR

37(iii)(b). Find the shortest distance between Line A and Line B.

[2 marks]

Solution:

Concept:

- For nonparallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, their shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$.
- This formula measures the absolute projection of the vector joining points on the lines onto their common perpendicular direction; the denominator makes that direction a unit vector.

Step 1: Identify points and directions From the symmetric equations, take $\vec{a}_1 = (2, -1, 3)$, $\vec{a}_2 = (1, 3, -2)$, $\vec{b}_1 = (3, -2, 4)$, and $\vec{b}_2 = (2, 1, -3)$. Thus

$$\vec{a}_2 - \vec{a}_1 = (-1, 4, -5).$$

Quick Tip to Solve: Use the absolute scalar triple product in the numerator and the cross-product magnitude in the denominator; reversing the connecting vector does not change the distance.

The direction perpendicular to both lines is

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ 2 & 1 & -3 \end{vmatrix} = 2\hat{i} + 17\hat{j} + 7\hat{k}.$$

This vector is nonzero, so the nonparallel-line distance formula applies.

Step 2: Evaluate the projection Substitute the connecting vector and cross product:

$$\begin{aligned} |(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)| &= |-2 + 68 - 35| = 31, \\ |\vec{b}_1 \times \vec{b}_2| &= \sqrt{2^2 + 17^2 + 7^2} = \sqrt{342}. \end{aligned}$$

Since the scalar triple product is nonzero, these two nonparallel lines are skew, consistent with a positive shortest distance.

Step 3: Conclusion The shortest distance between Line A and Line B is

$$d = \frac{31}{\sqrt{342}} = \frac{31}{3\sqrt{38}} \text{ units}.$$

Quick Tip to Solve: Use the absolute scalar triple product in the numerator and the cross-product magnitude in the denominator; reversing the connecting vector does not change the distance.

38. Case Study - 3

During a heavy gaming session, the temperature of a student's laptop processor increases significantly. After the session, the processor begins to cool down, and the rate of cooling is proportional to the difference between the processor's temperature and the room temperature (25°C). Initially the processor's temperature is 85°C . The rate of cooling is defined by the equation $\frac{dT}{dt}(T(t)) = -k(T(t) - 25)$, where $T(t)$ represents the temperature of the processor at time t (in minutes) and k is a constant.

Based on the above information, answer the following questions:

[4 marks]

38(i). Find the expression for the temperature of the processor, $T(t)$, given that $T(0) = 85^\circ\text{C}$. [2 marks]

Solution:

Concept:

- Newton's law of cooling gives $\frac{dT}{dt} = -k(T - 25)$, where T is the processor temperature in degrees Celsius, t is time in minutes, and $k > 0$ is the cooling constant.
- This equation is separable: collect the temperature terms with dT and the time terms with dt . The initial condition determines the integration constant.

Step 1: Separate variables and integrate Initially the processor is hotter than the surroundings, so its

Quick Tip to Solve: Exponential decay applies to the temperature excess above the room: start with $T - 25$, whose initial value is $85 - 25 = 60$.

excess temperature $T - 25$ is positive. Separating the given differential equation yields

$$\frac{dT}{T - 25} = -k dt.$$

Integrating both sides,

$$\ln |T - 25| = -kt + C.$$

For the cooling solution above room temperature, exponentiation gives $T - 25 = Ae^{-kt}$, where A is a positive constant representing the initial temperature excess.

Step 2: Apply the initial temperature At $t = 0$, the given temperature is 85°C . Therefore,

$$85 - 25 = Ae^0 \Rightarrow A = 60.$$

Substituting the constant gives $T - 25 = 60e^{-kt}$. As a check, differentiating this expression gives $\frac{dT}{dt} = -60ke^{-kt} = -k(T - 25)$, so it satisfies the original cooling equation as well as the initial condition.

Step 3: Conclusion The temperature at time t minutes is

$$T(t) = 25 + 60e^{-kt} \quad (^\circ\text{C}).$$

For $k > 0$, it decreases from 85°C towards the room temperature 25°C .

Quick Tip to Solve: Exponential decay applies to the temperature excess above the room: start with $T - 25$, whose initial value is $85 - 25 = 60$.

38(ii). How long will it take for the processor's temperature to reach 40°C ? Given that $k = 0.03$, $\log_e 4 = 1.3863$. [2 marks]

Solution:

Concept:

- Newton's law of cooling makes the excess temperature above the room temperature decay exponentially: $T(t) - 25 = (T(0) - 25)e^{-kt}$, where t is measured in minutes.
- To find the time for a specified temperature, isolate the exponential and take natural logarithms, using $\ln(e^x) = x$.

Step 1: Substitute the initial temperature and cooling constant The processor begins at 85°C , while the room remains at 25°C . Its initial excess temperature is therefore $85 - 25 = 60^\circ\text{C}$. With $k = 0.03$, the solution of the cooling equation gives

$$T(t) = 25 + 60e^{-0.03t}.$$

This formula satisfies both the initial condition and the given negative rate of cooling.

Step 2: Set the temperature equal to the target value At the required time, the excess temperature is

Quick Tip to Solve: Form the ratio of excess temperatures above the room temperature: $(40 - 25)/(85 - 25) = 1/4$, so the time is immediately $\ln 4/k$.

$40 - 25 = 15^\circ\text{C}$. Hence

$$\begin{aligned}40 &= 25 + 60e^{-0.03t}, \\e^{-0.03t} &= \frac{15}{60} = \frac{1}{4}, \\-0.03t &= \ln\left(\frac{1}{4}\right) = -\ln 4, \\t &= \frac{\ln 4}{0.03} = \frac{1.3863}{0.03} = 46.21.\end{aligned}$$

The positive time is consistent with cooling from the initial temperature towards the room temperature.

Step 3: Conclusion The processor reaches 40°C after 46.21 minutes.

Quick Tip to Solve: Form the ratio of excess temperatures above the room temperature: $(40 - 25)/(85 - 25) = 1/4$, so the time is immediately $\ln 4/k$.