

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-5-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section B, **3** questions in Section C, **2** questions in Section D and **2** questions in Section E.
- (ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is:

[1 mark]

- (A) $-\frac{\pi}{3}$
- (B) $-\frac{2\pi}{3}$
- (C) $\frac{\pi}{3}$
- (D) $\frac{2\pi}{3}$

Correct Answer: (D) $\frac{2\pi}{3}$

Solution:

Concept:

- The principal-value range of $\cot^{-1} x$ is $(0, \pi)$. A principal value must lie in this interval.
- Cotangent is negative in the second quadrant and $\cot(\pi - \theta) = -\cot \theta$.

Step 1: Find the reference angle Since $\cot(\pi/3) = 1/\sqrt{3}$, the reference angle for the given negative cotangent is $\pi/3$. We must select an angle in the principal range rather than any angle having this cotangent.

Step 2: Check the options against the range and sign The second-quadrant angle is

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Both negative angles in (A) and (B) lie outside the principal range. The positive angle in (C) gives positive cotangent, whereas the input is negative. Thus only (D) meets both conditions.

Step 3: Conclusion The principal value is $\boxed{2\pi/3}$. Hence, the correct option is (D).

Quick Tip to Solve: For a negative inverse cotangent, subtract the acute reference angle from π ; do not use a negative angle.

2. If $A = [a_{ij}]$ is a 3×3 diagonal matrix such that $a_{11} = 1$, $a_{22} = 5$ and $a_{33} = -2$, then $|A|$ is: [1 mark]

- (A) 0
- (B) -10
- (C) 10
- (D) 1

Correct Answer: (B) -10

Solution:

Concept:

- A diagonal matrix has zero entries outside its main diagonal.
- The determinant of a diagonal matrix is the product of all its diagonal entries: $|A| = a_{11}a_{22}a_{33}$.

Step 1: Write the diagonal matrix The given entries give

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Expanding along its first row leaves the determinant of the diagonal block containing 5 and -2 .

Step 2: Multiply the diagonal entries Therefore

$$|A| = 1(5(-2) - 0 \cdot 0) = -10.$$

The zero off-diagonal entries do not make the determinant zero, ruling out (A). Option (C) loses the negative sign, and (D) retains only the first diagonal entry instead of multiplying all three.

Step 3: Conclusion $|A| = -10$. Hence, the correct option is (B).

Quick Tip to Solve: For a diagonal determinant, multiply the diagonal entries and keep their signs; zeros elsewhere do not imply a zero determinant.

3. If $A = kB$, where A and B are two square matrices of order n and k is a scalar, then: [1 mark]

- (A) $|A| = k|B|$
- (B) $|A| = k^n|B|$
- (C) $|A| = k + |B|$
- (D) $|A| = |B|^k$

Correct Answer: (B) $|A| = k^n|B|$

Solution:

Concept:

- Multiplying one row of a determinant by k multiplies its value by k .
- For an $n \times n$ matrix B , the matrix kB has each of its n rows multiplied by the same scalar k .

Quick Tip to Solve: Count the matrix order before scaling a determinant: multiplying the whole matrix by k contributes one factor per row.

Step 1: Apply the row-scaling property The relation $A = kB$ scales every entry, hence every row, of B by k . Factoring k from the first row contributes one factor of k , and each subsequent row contributes another.

Step 2: Count all the factors There are n rows, so

$$|A| = |kB| = \underbrace{k \cdot k \cdots k}_{n \text{ factors}} |B| = k^n |B|.$$

Option (A) accounts for only one scaled row. Neither adding k to a determinant as in (C), nor raising $|B|$ to the power k as in (D), is the row-scaling rule. Special coincidences do not establish a general identity.

Step 3: Conclusion $|A| = k^n |B|$. Hence, the correct option is (B).

Quick Tip to Solve: Count the matrix order before scaling a determinant: multiplying the whole matrix by k contributes one factor per row.

4. If $f(x) = \begin{cases} \frac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of a is: [1 mark]

- (A) 1
- (B) -1
- (C) ± 1
- (D) 0

Correct Answer: (C) ± 1

Solution:

Concept:

- Continuity at zero requires $\lim_{x \rightarrow 0} f(x) = f(0) = 1$.
- The standard limit $\lim_{t \rightarrow 0} \sin t/t = 1$ uses angles in radians.

Step 1: Evaluate the limit at zero For $a \neq 0$, rewrite the expression as

$$\frac{\sin^2(ax)}{x^2} = a^2 \left(\frac{\sin(ax)}{ax} \right)^2.$$

Thus its limit is a^2 . When $a = 0$, the expression is identically zero for $x \neq 0$, so its limit is also $a^2 = 0$.

Step 2: Impose the continuity condition The limit must equal the assigned value at zero, giving

$$a^2 = 1.$$

Both $a = 1$ and $a = -1$ satisfy this condition because the sine is squared. Options (A) and (B) each list only one admissible value, while (D) gives limit zero and fails continuity.

Step 3: Conclusion $a = \pm 1$. Hence, the correct option is (C).

Quick Tip to Solve: When the sine expression is squared, the continuity condition determines a^2 ; retain both signs when taking its square root.

5. If $\begin{vmatrix} 2x & 5 \\ 12 & x \end{vmatrix} = \begin{vmatrix} 6 & -5 \\ 4 & 3 \end{vmatrix}$, then the value of x is:

[1 mark]

- (A) 3
(B) 7
(C) ± 7
(D) ± 3

Correct Answer: (C) ± 7

Solution:

Concept:

- For a 2×2 matrix, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- Equality of the two determinants gives an algebraic equation in x ; both real roots must be considered.

Step 1: Expand both determinants The left determinant is

$$2x \cdot x - 5 \cdot 12 = 2x^2 - 60.$$

The right determinant is

$$6 \cdot 3 - (-5) \cdot 4 = 18 + 20 = 38.$$

In particular, the negative entry -5 makes the subtracted off-diagonal product negative.

Step 2: Solve and check the roots Equating the determinants gives

$$\begin{aligned} 2x^2 - 60 &= 38, \\ 2x^2 &= 98, \\ x^2 &= 49. \end{aligned}$$

Both 7 and -7 give left determinant 38. Option (B) misses the negative root. Values ± 3 give -42 , so (A) and (D) fail the equality.

Step 3: Conclusion $x = \pm 7$. Hence, the correct option is (C).

Quick Tip to Solve: Expand the numerical determinant first, carefully subtracting its signed off-diagonal product, then retain both roots of x^2 .

6. If $P(A \cup B) = 0.9$ and $P(A \cap B) = 0.4$, then $P(\overline{A}) + P(\overline{B})$ is:

[1 mark]

- (A) 0.3
- (B) 1
- (C) 1.3
- (D) 0.7

Correct Answer: (D) 0.7

Solution:

Concept:

- The addition rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- For any event E , its complement has probability $P(\overline{E}) = 1 - P(E)$.

Step 1: Find the sum of the event probabilities Rearranging the addition rule gives

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) = 0.9 + 0.4 = 1.3.$$

This is a sum of two probabilities, so exceeding one is possible because common outcomes are counted twice.

Step 2: Convert to the sum of complements The requested sum is

$$\begin{aligned} P(\overline{A}) + P(\overline{B}) &= 2 - [P(A) + P(B)] \\ &= 2 - 1.3 = 0.7. \end{aligned}$$

Option (C) is the sum before taking complements. Using 1 instead of 2 would wrongly treat two complements as one. Thus neither (A) nor (B) follows the required complement rule.

Step 3: Conclusion $P(\overline{A}) + P(\overline{B}) = 0.7$. Hence, the correct option is (D).

Quick Tip to Solve: For the sum of two complement probabilities, use $2 - P(A \cup B) - P(A \cap B)$, with the initial constant equal to two.

7. If $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then A^3 is:

[1 mark]

(A) $3 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(B) $\begin{bmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

(D) $\begin{bmatrix} 5^3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Correct Answer: (B) $\begin{bmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

Solution:

Concept:

- The identity matrix I_3 satisfies $I_3^2 = I_3$ and $I_3^3 = I_3$.
- A scalar diagonal matrix $A = cI_3$ has $A^m = c^m I_3$ for every positive integer m .

Step 1: Express the matrix using the identity The three diagonal entries are all 5, and all other entries are zero. Thus $A = 5I_3$. Matrix powers mean repeated matrix multiplication, so

$$A^2 = (5I_3)(5I_3) = 25I_3.$$

Step 2: Multiply once more and compare Continuing the multiplication gives

$$A^3 = (25I_3)(5I_3) = 125I_3.$$

Every diagonal entry is therefore 125. Options (A) and (C) give $3A = 15I_3$, confusing cubing with multiplying by three. Option (D) cubes only the first diagonal entry and leaves the other two unchanged, contrary to the repeated product.

Step 3: Conclusion

$$A^3 = \begin{bmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{bmatrix}. \text{ Hence, the correct option is (B).}$$

Quick Tip to Solve: For $A = cI$, raise the scalar c to the required power and keep the identity matrix; cube every diagonal entry here.

8. Let A and B be two matrices of suitable orders. Then, which of the following is not correct?

[1 mark]

- (A) $(A')' = A$
- (B) $(kA)' = kA'$, k is a scalar
- (C) $(A' + B')' = A + B$
- (D) $(AB)' = A'B'$

Correct Answer: (D) $(AB)' = A'B'$

Solution:

Concept:

- Transposition interchanges rows and columns, and $(A')' = A$.
- Transposition preserves scalar multiplication and addition, but reverses the order of a product: $(AB)' = B'A'$.

Step 1: Verify the first three identities Transposing twice restores each original entry, so (A) is true. A scalar factors out of transposition, making (B) true. Using additivity and double transposition,

$$(A' + B')' = (A')' + (B')' = A + B,$$

so (C) is true whenever the sum is defined.

Step 2: Check the order in the product rule For A of order $m \times n$ and B of order $n \times p$, $(AB)'$ has order $p \times m$. The product $B'A'$ has exactly that order. The proposed $A'B'$ may not even be defined; for square matrices it is generally different because matrix multiplication is not commutative. Hence (D) is not a valid general identity.

Step 3: Conclusion The incorrect identity is $(AB)' = A'B'$; the correct order is $B'A'$. Hence, the correct option is (D).

Quick Tip to Solve: When transposing a product, reverse the order of its factors as well as transposing each one; addition does not require this reversal.

9. The area of the region enclosed between the curve $y = x|x|$, x -axis, $x = -2$ and $x = 2$ is: [1 mark]

- (A) $\frac{8}{3}$
- (B) $\frac{16}{3}$
- (C) 0
- (D) 8

Correct Answer: (B) $\frac{16}{3}$

Solution:

Concept:

- Geometrical area between a curve and the x -axis is $A = \int_a^b |y| dx$; negative ordinates do not give negative area.
- For $x < 0$, $x|x| = -x^2$, whereas for $x \geq 0$, $x|x| = x^2$.

Step 1: Split at the change of sign The curve lies below the axis on $[-2, 0]$ and above it on $[0, 2]$. Therefore use the magnitudes of the two contributions:

$$A = \int_{-2}^0 x^2 dx + \int_0^2 x^2 dx.$$

The signed integral of the original odd function would cancel, but that does not measure the enclosed area.

Step 2: Evaluate both positive contributions Each region has the same area by symmetry. Thus

$$A = 2 \left[\frac{x^3}{3} \right]_0^2 = 2 \cdot \frac{8}{3} = \frac{16}{3}.$$

Option (A) counts only one side; option (C) is the signed integral; option (D) is not the result of integrating the quadratic ordinate.

Step 3: Conclusion The enclosed area is $\boxed{\frac{16}{3}}$ square units. Hence, the correct option is (B).

Quick Tip to Solve: For an odd curve crossing the axis, double the positive-side area; never cancel geometrical areas.

10. If $f(x) = \{[x], x \in \mathbb{R}\}$ is the greatest integer function, then the correct statement is: [1 mark]

- (A) f is continuous but not differentiable at $x = 2$.
- (B) f is neither continuous nor differentiable at $x = 2$.
- (C) f is continuous as well as differentiable at $x = 2$.
- (D) f is not continuous but differentiable at $x = 2$.

Correct Answer: (B) f is neither continuous nor differentiable at $x = 2$.

Solution:

Concept:

- The greatest integer function $[x]$ is the largest integer not exceeding x .
- Continuity at a requires both one-sided limits to equal $f(a)$. Differentiability at a point implies continuity there.

Step 1: Compute the limits at the integer For $1 < x < 2$, $[x] = 1$; for $2 \leq x < 3$, $[x] = 2$. Consequently,

$$\lim_{x \rightarrow 2^-} f(x) = 1, \quad \lim_{x \rightarrow 2^+} f(x) = 2 = f(2).$$

The one-sided limits differ, so there is a jump discontinuity at $x = 2$.

Step 2: Test differentiability and the statements Because differentiability requires continuity, the discontinuous function cannot be differentiable at 2. Directly, the left difference quotient is $(1 - 2)/h = -1/h$ for small $h < 0$, so it has no finite limit. Statements (A) and (C) incorrectly assert continuity; (D) incorrectly asserts differentiability despite the jump.

Step 3: Conclusion At $x = 2$, f is neither continuous nor differentiable. Hence, the correct option is (B).

Quick Tip to Solve: At an integer n , $\text{floor}(x)$ approaches $n-1$ from the left and n from the right; this immediately shows a jump.

11. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to: [1 mark]

- (A) $2(\sin x + x \cos \theta) + C$
- (B) $2(\sin x - x \cos \theta) + C$
- (C) $2(\sin x + \sin \theta) + C$
- (D) $2(\sin x - x \sin \theta) + C$

Correct Answer: (A) $2(\sin x + x \cos \theta) + C$

Solution:

Concept:

- Use $\cos 2u = 2 \cos^2 u - 1$ and the factorisation $a^2 - b^2 = (a - b)(a + b)$.
- Treat θ as constant while integrating with respect to x .

Quick Tip to Solve: Whenever a difference of double-angle cosines sits over a difference of cosines, factor the numerator first.

Step 1: Factor and cancel the common factor On an interval where the denominator is nonzero,

$$\begin{aligned}\cos 2x - \cos 2\theta &= 2(\cos^2 x - \cos^2 \theta) \\ &= 2(\cos x - \cos \theta)(\cos x + \cos \theta).\end{aligned}$$

Hence the integrand simplifies to $2(\cos x + \cos \theta)$. Cancellation is valid on the domain of the original expression.

Step 2: Integrate and compare the options Integrating the two terms gives

$$\int 2(\cos x + \cos \theta) dx = 2 \sin x + 2x \cos \theta + C.$$

Differentiating (A) recovers the simplified integrand. Option (B) has the wrong sign for the constant term; (C) differentiates to only $2 \cos x$; (D) produces $2 \cos x - 2 \sin \theta$, which is not the required expression.

Step 3: Conclusion The integral is $2(\sin x + x \cos \theta) + C$. Hence, the correct option is (A).

Quick Tip to Solve: Whenever a difference of double-angle cosines sits over a difference of cosines, factor the numerator first.

12. $\int_0^1 \frac{2x}{5x^2 + 1} dx$ is equal to:

[1 mark]

- (A) $\frac{1}{5} \log 6$
- (B) $\frac{1}{5} \log 5$
- (C) $\frac{1}{2} \log 6$
- (D) $\frac{1}{2} \log 5$

Correct Answer: (A) $\frac{1}{5} \log 6$

Solution:

Concept:

- For a nonzero differentiable $g(x)$, $\int g'(x)/g(x) dx = \log |g(x)| + C$.
- In this question $g(x) = 5x^2 + 1$ stays positive, and $g'(x) = 10x$.

Step 1: Substitute the denominator Put $u = 5x^2 + 1$, so $du = 10x dx$ and $2x dx = du/5$. When $x = 0$, $u = 1$; when $x = 1$, $u = 6$. The definite integral becomes

$$\frac{1}{5} \int_1^6 \frac{du}{u}.$$

Changing both endpoints avoids mixing the variables.

Quick Tip to Solve: Match the numerator with the derivative of the denominator and evaluate that denominator at both bounds.

Step 2: Evaluate the logarithm and eliminate distractors Therefore

$$\frac{1}{5}[\log u]_1^6 = \frac{1}{5}(\log 6 - \log 1) = \frac{1}{5} \log 6.$$

The factor is $1/5$, not $1/2$, because the numerator is one fifth of the denominator derivative. Options (B) and (D) incorrectly use 5 instead of the endpoint value $5(1)^2 + 1 = 6$; option (C) has the wrong factor.

Step 3: Conclusion The value is $\frac{1}{5} \log 6$. Hence, the correct option is (A).

Quick Tip to Solve: Match the numerator with the derivative of the denominator and evaluate that denominator at both bounds.

13. The slope of the curve $y = -x^3 + 3x^2 + 8x - 20$ is maximum at:

[1 mark]

- (A) (1, -10)
- (B) (1, 10)
- (C) (10, 1)
- (D) (-10, 1)

Correct Answer: (A) (1, -10)

Solution:

Concept:

- The tangent slope at abscissa x is $m(x) = dy/dx$. To maximise slope, maximise this derivative function.
- A quadratic with negative leading coefficient has a global maximum at its vertex.

Step 1: Differentiate the curve and maximise its slope For $y = -x^3 + 3x^2 + 8x - 20$,

$$m(x) = -3x^2 + 6x + 8 = 11 - 3(x - 1)^2.$$

Since $(x - 1)^2 \geq 0$, the largest possible slope is 11, attained only at $x = 1$. Equivalently, $m'(x) = -6x + 6 = 0$ at 1, with $m''(x) = -6 < 0$.

Step 2: Find the point on the original curve At this abscissa,

$$y(1) = -1 + 3 + 8 - 20 = -10.$$

Thus the required point has coordinates (1, -10). Option (B) has the correct abscissa but the wrong ordinate. Options (C) and (D) have abscissae 10 and -10, where the squared term makes the slope smaller than 11.

Step 3: Conclusion The slope is maximum at (1, -10). Hence, the correct option is (A).

Quick Tip to Solve: Maximum slope means maximise dy/dx , then substitute the resulting x into y to get the point.

14. The integrating factor of the differential equation $\frac{dx}{dy} = \frac{-(1 + \sin x)}{x + y \cos x}$ is:

[1 mark]

- (A) $\log \cos x$
- (B) $1 + \sin x$
- (C) $e^{(1+\sin x)}$
- (D) $e^{\log \cos x}$

Correct Answer: (B) $1 + \sin x$

Solution:

Concept:

- A linear equation $dy/dx + P(x)y = Q(x)$ has integrating factor I.F. = $e^{\int P(x) dx}$.
- Here the printed equation must first be rewritten with y dependent on x , rather than x dependent on y .

Step 1: Rewrite in standard linear form Taking the reciprocal relation gives

$$\frac{dy}{dx} = -\frac{x + y \cos x}{1 + \sin x}.$$

On an interval where $1 + \sin x \neq 0$, bring the term involving y to the left:

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x} y = -\frac{x}{1 + \sin x}.$$

Thus $P(x) = \cos x / (1 + \sin x)$.

Step 2: Integrate the coefficient With $u = 1 + \sin x$ and $du = \cos x dx$,

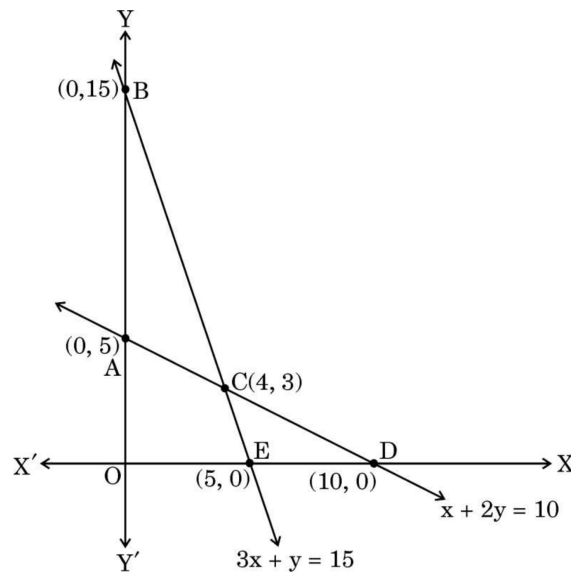
$$\text{I.F.} = e^{\int \frac{\cos x}{1 + \sin x} dx} = e^{\log |1 + \sin x|} = 1 + \sin x.$$

On the intervals in use this quantity is positive. Option (A) is not the exponential of the required integral. Option (C) exponentiates $1 + \sin x$ instead of its logarithm; (D) gives $\cos x$, corresponding to a different coefficient.

Step 3: Conclusion The integrating factor is $1 + \sin x$. Hence, the correct option is (B).

Quick Tip to Solve: If dx/dy contains $x+y$ times a function of x , invert it first and collect the y coefficient.

15. For a Linear Programming Problem (LPP), the given objective function $Z = 3x + 2y$ is subject to constraints: $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$. The correct feasible region is: [1 mark]



- (A) ABC
 (B) $AOEC$
 (C) CED
 (D) Open unbounded region BCD

Correct Answer: (B) $AOEC$

Solution:

Concept:

- The feasible region is the common intersection of all constraint half-planes, including $x \geq 0$ and $y \geq 0$.
- Since $(0, 0)$ satisfies both inequalities, select the side of each boundary line that contains the origin.

Step 1: Determine the boundaries in the first quadrant The line $x + 2y = 10$ meets the axes at $A(0, 5)$ and $D(10, 0)$, while $3x + y = 15$ meets them at $B(0, 15)$ and $E(5, 0)$. Solving the two equations gives

$$x + 2(15 - 3x) = 10 \implies x = 4, \quad y = 3,$$

so their intersection is $C(4, 3)$.

Step 2: Intersect the allowed half-planes On the positive y -axis, the more restrictive bound is $y \leq 5$, and on the positive x -axis it is $x \leq 5$. Therefore the common region is enclosed by OA , AC , CE and EO . All four vertices satisfy both inequalities. Option (A) includes B , which violates $x + 2y \leq 10$; option (C) includes D , which violates $3x + y \leq 15$. Option (D) cannot be feasible because both inequalities bound the region in the first quadrant. The objective function does not change the constraints' common intersection.

Step 3: Conclusion The feasible region is $AOEC$. Hence, the correct option is (B).

Quick Tip to Solve: Test the origin to select each half-plane, then retain the nearer intercept on each positive axis.

16. The sum of the order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$ is: [1 mark]

- (A) 2
- (B) $\frac{5}{2}$
- (C) 3
- (D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The order is the highest order of derivative occurring in the differential equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The equation is

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}.$$

It contains a first derivative and a second derivative, but no derivative of higher order. Its order is therefore 2. The exponent outside the bracket does not specify the order.

Step 2: Determine its degree and add Writing $p = dy/dx$ and $q = d^2y/dx^2$, the equation is

$$1 + 3p^2 + 3p^4 + p^6 - q = 0.$$

This is already polynomial in the derivatives. The highest-order derivative q occurs to power 1, so the degree is 1. Hence the requested sum is $2 + 1 = 3$. Option (A) counts only order, (B) cannot be the integer sum here, and (D) misidentifies the degree.

Step 3: Conclusion The sum of order and degree is $\boxed{3}$. Hence, the correct option is (C).

Quick Tip to Solve: Find the highest-order derivative first; degree is its power, not the largest power of any derivative.

17. The respective values of $|\vec{a}|$ and $|\vec{b}|$, if given $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 512$ and $|\vec{a}| = 3|\vec{b}|$, are: [1 mark]

- (A) 48 and 16
- (B) 3 and 1
- (C) 24 and 8
- (D) 6 and 2

Correct Answer: (C) 24 and 8

Solution:

Concept:

- The dot product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. Also, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
- A magnitude is non-negative; the given relation is $|\vec{a}| = 3|\vec{b}|$.

Step 1: Expand the dot product Using distributivity, the mixed products cancel:

$$(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 512.$$

This identity does not require the two vectors to be perpendicular.

Step 2: Use the magnitude ratio and check the options Put $|\vec{b}| = t \geq 0$, so $|\vec{a}| = 3t$. Then $9t^2 - t^2 = 512$, giving $t^2 = 64$ and $t = 8$. Hence the larger magnitude is 24. All four listed pairs have ratio 3 : 1, but their squared differences are respectively 2048, 8, 512, 32. Only the third pair also satisfies the dot-product condition.

Step 3: Conclusion $|\vec{a}| = 24, |\vec{b}| = 8$. The correct option is (C).

Quick Tip to Solve: Cancel the mixed dot products first; the ratio alone cannot distinguish these four options.

18. Let $\vec{a}$ be a position vector whose tip is the point $(2, -3)$. If $\overrightarrow{AB} = \vec{a}$, where coordinates of A are $(-4, 5)$, then the coordinates of B are: [1 mark]

- (A) $(-2, -2)$
- (B) $(2, -2)$
- (C) $(-2, 2)$
- (D) $(2, 2)$

Correct Answer: (C) $(-2, 2)$

Solution:

Concept:

- A position vector to (u, v) has components (u, v) relative to the origin.
- For points $A(x_A, y_A)$, $B(x_B, y_B)$, the displacement is $\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$.

Step 1: Identify the displacement vector The vector with tip $(2, -3)$ and initial point at the origin is $\vec{a} = (2, -3)$. This is the displacement from A to B, so it must be added to the coordinates of A, not treated as the coordinates of B.

Quick Tip to Solve: For a displacement from A to B, use $B = A + \text{displacement}$ and verify by subtracting A.

Step 2: Add the displacement to the initial point Let $B = (x, y)$. Then

$$(x + 4, y - 5) = (2, -3), \quad x = -2, \quad y = 2.$$

Subtracting $A = (-4, 5)$, the option displacements are $(2, -7), (6, -7), (2, -3), (6, -3)$, respectively. Only option (C) gives the specified vector.

Step 3: Conclusion The coordinates are $B = (-2, 2)$. The correct option is (C).

Quick Tip to Solve: For a displacement from A to B, use $B = A + \text{displacement}$ and verify by subtracting A.

19. Assertion (A): The shaded portion of the graph represents the feasible region for the given Linear Programming Problem (LPP).

Min $Z = 50x + 70y$

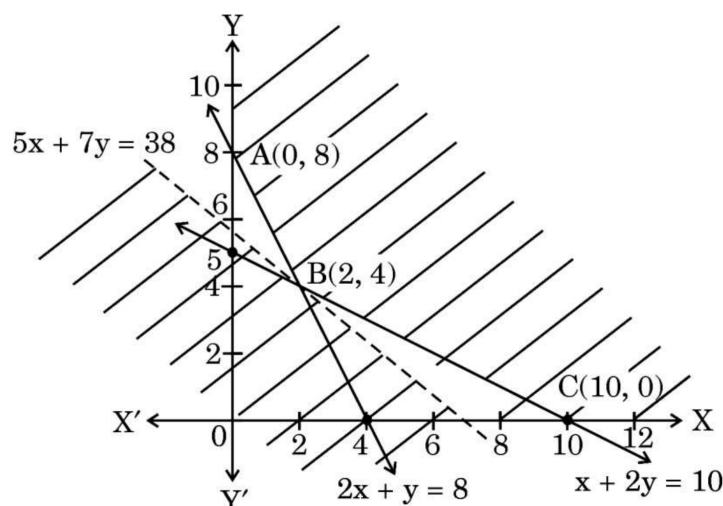
subject to constraints

$$2x + y \geq 8, \quad x + 2y \geq 10, \quad x, y \geq 0$$

$$Z = 50x + 70y \text{ has a minimum value} = 380 \text{ at } B(2, 4).$$

Reason (R): The region representing $50x + 70y < 380$ does not have any point common with the feasible region.

[1 mark]



- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

Solution:

Concept:

- Feasible points satisfy every constraint, including $x, y \geq 0$. For an unbounded feasible region, a corner value is a minimum only if no feasible point gives a smaller value.
- A non-negative linear combination of the constraints can provide a lower bound for the objective function.

Step 1: Judge the assertion The two boundary lines intersect at $B = (2, 4)$. The feasible region lies on the side of both lines away from the origin in the first quadrant, as shaded. The other corners are $A = (0, 8)$ and $C = (10, 0)$. Their objective values are 560, 380, 500, respectively. Thus B gives the least corner value.

Step 2: Judge the reason and the unbounded region For any feasible point,

$$Z = 50x + 70y = 10(2x + y) + 30(x + 2y) \geq 10(8) + 30(10) = 380.$$

Equality holds at B , where both constraints are equalities. Consequently the region $50x + 70y < 380$ has no point in common with the feasible region. The reason is true.

Step 3: Check the explanatory connection The reason rules out every feasible value below 380; together with the feasible point B attaining 380, it establishes the asserted minimum even though the region is unbounded. Hence it explains the assertion. Options (B), (C) and (D) respectively deny this explanation or incorrectly make one of the true statements false.

Step 4: Conclusion Both Assertion (A) and Reason (R) are true, and R correctly explains A. The correct option is (A).

Quick Tip to Solve: For an unbounded LPP, prove a lower bound by a positive combination of constraints instead of relying only on corner values.

20. Assertion (A): Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. If $f : A \rightarrow A$ is defined as $f(x) = x^2$, then f is not an onto function.

Reason (R): If $y = -1 \in A$, then $x = \pm\sqrt{-1} \notin A$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

Solution:

Concept:

- A function is onto if every element of its codomain has at least one preimage in its domain.

Quick Tip to Solve: To disprove onto, choose a negative codomain value and show that a real square cannot equal it.

- For real x , $x^2 \geq 0$; on $[-1, 1]$, the range of x^2 is $[0, 1]$.

Step 1: Judge the assertion Here the domain and codomain are both $A = [-1, 1]$, while the output $f(x) = x^2$ always belongs to $[0, 1]$. Negative members of the codomain are never outputs, so the range is a proper subset of the codomain. The assertion that f is not onto is true.

Step 2: Judge the reason The element -1 belongs to the codomain A . A preimage would have to satisfy $x^2 = -1$, which has no real solution. The formal values $x = \pm\sqrt{-1}$ are not real and therefore cannot lie in A . Thus the reason is true.

Step 3: Check whether the reason explains the assertion One codomain element with no preimage is sufficient to disprove surjectivity. The reason supplies exactly such an element, so it correctly explains the assertion. Option (B) wrongly denies the connection; options (C) and (D) wrongly declare a true statement false.

Step 4: Conclusion Both A and R are true, and R is the correct explanation of A. The correct option is (A).

Quick Tip to Solve: To disprove onto, choose a negative codomain value and show that a real square cannot equal it.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Find the domain of $\sin^{-1}(x^2 - 3)$.

[2 marks]

Solution:

Concept:

- The real inverse sine $\sin^{-1} u$ is defined precisely when $-1 \leq u \leq 1$.
- When solving an inequality involving x^2 , include both positive and negative values of x and retain equality at allowed endpoints.

Step 1: Restrict the argument of the inverse sine For $\sin^{-1}(x^2 - 3)$ to be real, its argument must be between -1 and 1 , inclusive. Therefore

$$-1 \leq x^2 - 3 \leq 1.$$

Adding 3 to all three members preserves both inequalities and gives $2 \leq x^2 \leq 4$.

Step 2: Solve the quadratic double inequality The upper restriction $x^2 \leq 4$ gives $-2 \leq x \leq 2$. The lower restriction $x^2 \geq 2$ gives $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$. Intersecting these conditions produces the two intervals $[-2, -\sqrt{2}]$ and $[\sqrt{2}, 2]$. The negative interval cannot be discarded because squaring removes the sign of x .

Step 3: Check the boundary values At $x = \pm\sqrt{2}$, the argument is -1 ; at $x = \pm 2$, it is 1 . Both values are permitted by inverse sine. Values with $|x| < \sqrt{2}$ give an argument below -1 , while values with $|x| > 2$ give an argument above 1 , so there are no additional domain points.

Quick Tip to Solve: First confine the whole inverse-sine argument to $[-1, 1]$, then solve both bounds on the square.

Step 4: Conclusion The domain is $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$.

Quick Tip to Solve: First confine the whole inverse-sine argument to $[-1, 1]$, then solve both bounds on the square.

22. Let the volume of a metallic hollow sphere be constant. If the inner radius increases at the rate of 2 cm/s, find the rate of increase of the outer radius when the radii are 2 cm and 4 cm respectively.

[2 marks]

Solution:

Concept:

- For inner radius r and outer radius R , the volume of metal is $V = \frac{4\pi}{3}(R^3 - r^3)$.
- Both radii depend on time t . Differentiate using the chain rule, and use $dV/dt = 0$ because the volume of metal is constant.

Step 1: Write the constant-volume condition The hollow sphere consists of the material between its outer spherical surface and the inner spherical cavity. Hence its metal volume is the outer volume minus the cavity volume:

$$V = \frac{4\pi}{3}(R^3 - r^3) = \text{constant.}$$

It is this difference, rather than either individual spherical volume, that remains fixed.

Step 2: Differentiate with respect to time Applying the chain rule gives

$$0 = \frac{dV}{dt} = 4\pi \left(R^2 \frac{dR}{dt} - r^2 \frac{dr}{dt} \right).$$

Therefore $R^2 \frac{dR}{dt} = r^2 \frac{dr}{dt}$, or $\frac{dR}{dt} = \frac{r^2}{R^2} \frac{dr}{dt}$. Both derivatives refer to the same instant.

Step 3: Substitute the instantaneous radii and rate The inner radius is $r = 2$ cm, the outer radius is $R = 4$ cm, and $dr/dt = 2$ cm/s. Thus

$$\frac{dR}{dt} = \frac{2^2}{4^2} \times 2 = \frac{1}{2} \text{ cm/s.}$$

The positive sign shows that the outer radius also increases. Its rate is smaller because its larger surface area needs a smaller radial change for the same volume change.

Step 4: Conclusion The outer radius increases at $\frac{1}{2}$ cm/s.

Quick Tip to Solve: Use the volume of metal as outer sphere minus inner sphere; the rate ratio equals the squared radius ratio.

23. A man needs to hang two lanterns on a straight wire whose end points have coordinates $A(4, 1, -2)$ and $B(6, 2, -3)$. Find the coordinates of the points where he hangs the lanterns such that these points trisect the wire AB . [2 marks]

Solution:

Concept:

- The trisection points divide a segment into three equal lengths. The point nearer A divides AB internally in the ratio $1 : 2$, and the other in the ratio $2 : 1$.
- If $AP : PB = m : n$, the section formula is $P = (nA + mB)/(m + n)$, applied separately to the three coordinates.

Step 1: Locate the first lantern Let P be the trisection point nearer $A(4, 1, -2)$. Since $AP : PB = 1 : 2$, the weight of A is 2 and that of $B(6, 2, -3)$ is 1. Thus

$$P = \left(\frac{2(4) + 6}{3}, \frac{2(1) + 2}{3}, \frac{2(-2) + (-3)}{3} \right) = \left(\frac{14}{3}, \frac{4}{3}, -\frac{7}{3} \right).$$

Step 2: Locate the second lantern and check the spacing Let Q be the point nearer B . Here $AQ : QB = 2 : 1$, so

$$Q = \left(\frac{4 + 2(6)}{3}, \frac{1 + 2(2)}{3}, \frac{-2 + 2(-3)}{3} \right) = \left(\frac{16}{3}, \frac{5}{3}, -\frac{8}{3} \right).$$

As a check, each of the displacement vectors $P - A$, $Q - P$ and $B - Q$ equals $(2/3, 1/3, -1/3)$. The two computed points therefore lie on the wire and divide its length equally.

Step 3: Conclusion The lanterns should be hung at $\left(\frac{14}{3}, \frac{4}{3}, -\frac{7}{3} \right)$ and $\left(\frac{16}{3}, \frac{5}{3}, -\frac{8}{3} \right)$.

Quick Tip to Solve: For trisection, add one-third and two-thirds of the displacement vector $B - A$ to A .

24(a). Differentiate $\frac{\sin x}{\sqrt{\cos x}}$ with respect to x . [2 marks]

Solution:

Concept:

- For a product $u(x)v(x)$, the derivative is $u'v + v'u$.
- The chain rule gives $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. The real-valued expression here requires $\cos x > 0$.

Step 1: Rewrite the expression and differentiate its factors Write $y = \sin x (\cos x)^{-1/2}$. Differentiating the first factor gives $\cos x$. For the second factor, the chain rule gives

$$\frac{d}{dx}(\cos x)^{-1/2} = -\frac{1}{2}(\cos x)^{-3/2}(-\sin x) = \frac{\sin x}{2(\cos x)^{3/2}}.$$

The two negative signs cancel; retaining this sign is essential.

Quick Tip to Solve: Rewrite the square-root denominator as $(\cos x)^{-1/2}$; the chain-rule derivative of this factor is positive.

Step 2: Apply the product rule and simplify Therefore

$$\begin{aligned}\frac{dy}{dx} &= \cos x (\cos x)^{-1/2} + \sin x \frac{\sin x}{2(\cos x)^{3/2}} \\ &= \frac{2 \cos^2 x + \sin^2 x}{2(\cos x)^{3/2}} \\ &= \frac{1 + \cos^2 x}{2(\cos x)^{3/2}}.\end{aligned}$$

In the second line both terms have been put over the common denominator, and the final line uses $\sin^2 x + \cos^2 x = 1$. The formula applies on each interval where the original denominator is real and nonzero.

Step 3: Conclusion $\boxed{\frac{d}{dx} \left(\frac{\sin x}{\sqrt{\cos x}} \right) = \frac{1 + \cos^2 x}{2(\cos x)^{3/2}}}.$

Quick Tip to Solve: Rewrite the square-root denominator as $(\cos x)^{-1/2}$; the chain-rule derivative of this factor is positive.

OR

24(b). If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2 y}{dx^2} + y = 0$.

[2 marks]

Solution:

Concept:

- The trigonometric derivative rules are $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$.
- Differentiation is linear: constant coefficients remain unchanged, and the derivative of a sum is the sum of its derivatives.

Step 1: Find the first derivative The given function is $y = 5 \cos x - 3 \sin x$. Differentiating both terms with respect to x , while retaining their constant multipliers, gives

$$\frac{dy}{dx} = 5(-\sin x) - 3(\cos x) = -5 \sin x - 3 \cos x.$$

In particular, differentiating $-3 \sin x$ gives $-3 \cos x$; the negative coefficient is retained.

Step 2: Find the second derivative and substitute Differentiate once again. The sine term differentiates to a cosine term, while differentiating the negative cosine term produces a positive sine term:

$$\frac{d^2 y}{dx^2} = -5 \cos x + 3 \sin x = -(5 \cos x - 3 \sin x) = -y.$$

Substitution into the required expression now gives

$$\frac{d^2 y}{dx^2} + y = (-5 \cos x + 3 \sin x) + (5 \cos x - 3 \sin x) = 0.$$

Quick Tip to Solve: Every linear combination of $\sin x$ and $\cos x$ has its second derivative equal to the negative of itself.

Both the cosine terms and the sine terms cancel separately, establishing the identity for every real x .

Step 3: Conclusion Hence $\frac{d^2y}{dx^2} + y = 0$, as required.

Quick Tip to Solve: Every linear combination of $\sin x$ and $\cos x$ has its second derivative equal to the negative of itself.

25(a). Find a vector of magnitude 5 which is perpendicular to both the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $4\hat{i} + 3\hat{j} - 2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product of two nonparallel vectors is perpendicular to each. If $\vec{n} \neq \vec{0}$, then $\vec{n}/|\vec{n}|$ is a unit vector in its direction.
- A vector of prescribed magnitude m in that direction is $m\vec{n}/|\vec{n}|$. Its negative has the same magnitude and perpendicularity.

Step 1: Find a common perpendicular direction Put $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - 2\hat{k}$. Then

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix} \\ &= \hat{i}(4 - 3) - \hat{j}(-6 - 4) + \hat{k}(9 + 8) \\ &= \hat{i} + 10\hat{j} + 17\hat{k}.\end{aligned}$$

This nonzero vector is perpendicular to both given vectors.

Step 2: Normalize and scale the vector The magnitude of this cross product is

$$\sqrt{1^2 + 10^2 + 17^2} = \sqrt{390}.$$

Hence a unit perpendicular vector is $(\hat{i} + 10\hat{j} + 17\hat{k})/\sqrt{390}$; multiplication by 5 gives the required magnitude. To check the direction explicitly, the dot products with the two given vectors have numerators $3 - 20 + 17 = 0$ and $4 + 30 - 34 = 0$, respectively. The magnitude after scaling is $5\sqrt{390}/\sqrt{390} = 5$.

Step 3: Conclusion One required vector is $\frac{5}{\sqrt{390}}(\hat{i} + 10\hat{j} + 17\hat{k})$. Its negative is also a valid answer.

Quick Tip to Solve: Compute the cross product first, then divide by its magnitude before multiplying by 5; the raw cross product has the wrong length.

25(b). Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$. Show that $\vec{b} = \vec{c}$. [2 marks]

Solution:

Concept:

- Dot and cross products distribute over subtraction. Set $\vec{d} = \vec{b} - \vec{c}$ to compare the two given equalities.
- For real vectors, Lagrange's identity states $|\vec{a} \times \vec{d}|^2 = |\vec{a}|^2 |\vec{d}|^2 - (\vec{a} \cdot \vec{d})^2$. A vector has zero magnitude exactly when it is the zero vector.

Step 1: Express the hypotheses using the difference vector Let $\vec{d} = \vec{b} - \vec{c}$. Subtracting the two sides of the dot-product equality gives

$$\vec{a} \cdot \vec{d} = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0.$$

Likewise the cross-product equality gives

$$\vec{a} \times \vec{d} = \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}.$$

Thus the same difference vector has both its dot product and its cross product with $\vec{a}$ equal to zero.

Step 2: Use the magnitude identity and the nonzero hypothesis Apply Lagrange's identity to $\vec{a}$ and $\vec{d}$. The preceding equalities imply

$$0 = |\vec{a} \times \vec{d}|^2 = |\vec{a}|^2 |\vec{d}|^2 - (\vec{a} \cdot \vec{d})^2 = |\vec{a}|^2 |\vec{d}|^2.$$

Because the question specifies $\vec{a} \neq \vec{0}$, its squared magnitude is strictly positive. Dividing by it gives $|\vec{d}|^2 = 0$, hence $\vec{d} = \vec{0}$. The nonzero hypothesis is essential: if $\vec{a}$ were zero, both given equalities would hold for arbitrary $\vec{b}$ and $\vec{c}$.

Step 3: Conclusion It follows that $\vec{b} - \vec{c} = \vec{0}$, so $\boxed{\vec{b} = \vec{c}}$.

Quick Tip to Solve: Subtract the vectors first: the difference has zero dot and cross products with a nonzero vector, forcing its magnitude to vanish.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. Find the interval/intervals in which the function $f(x) = \sin 3x - \cos 3x$, $0 < x < \frac{\pi}{2}$, is strictly increasing. [3 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval when its derivative is positive throughout the interior of that interval.

Quick Tip to Solve: Combine $\sin 3x + \cos 3x$ into one shifted sine, then intersect its positive-angle range with the given domain before dividing by 3.

- The chain rule gives $\frac{d}{dx} \sin 3x = 3 \cos 3x$ and $\frac{d}{dx} \cos 3x = -3 \sin 3x$. Also, $\sin t + \cos t = \sqrt{2} \sin(t + \pi/4)$.

Step 1: Differentiate the function We differentiate both trigonometric terms, keeping the minus sign before the cosine term. Thus

$$f'(x) = 3 \cos 3x + 3 \sin 3x = 3\sqrt{2} \sin\left(3x + \frac{\pi}{4}\right).$$

The factor $3\sqrt{2}$ is positive, so the sign of the derivative is determined entirely by the sine factor.

Step 2: Apply the given domain to the sign condition For strict increase, solve $f'(x) > 0$ within the prescribed domain. Since $0 < x < \pi/2$, the angle satisfies

$$\frac{\pi}{4} < 3x + \frac{\pi}{4} < \frac{7\pi}{4}.$$

In this range sine is positive precisely before the angle reaches π . Therefore

$$\begin{aligned} \frac{\pi}{4} < 3x + \frac{\pi}{4} < \pi &\implies 0 < 3x < \frac{3\pi}{4} \\ &\implies 0 < x < \frac{\pi}{4}. \end{aligned}$$

On $\pi/4 < x < \pi/2$, the angle lies between π and $7\pi/4$, so the derivative is negative. At $x = \pi/4$ it is zero and the sign changes from positive to negative.

Step 3: Conclusion The required interval of strict increase is $(0, \pi/4)$. The stationary boundary $x = \pi/4$ can also be included as the right endpoint of the increasing restriction, giving $(0, \pi/4]$; the derivative-positive interval is the open interval stated above.

Quick Tip to Solve: Combine $\sin 3x + \cos 3x$ into one shifted sine, then intersect its positive-angle range with the given domain before dividing by 3.

27(a). If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ such that $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$. [3 marks]

Solution:

Concept:

- For vectors $\vec{a}$ and $\vec{b}$, their scalar product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$, where $0 \leq \theta \leq \pi$ is the angle between their directions.
- The squared magnitude of a sum is obtained by dotting it with itself: $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$. Changing a vector's sign does not change its magnitude.

Step 1: Express the sum using the third vector The given vector equation implies $\vec{a} + \vec{b} = -\vec{c}$. Take the squared magnitude of both sides so that the known length of $\vec{c}$ can be used:

$$|\vec{a} + \vec{b}|^2 = |-\vec{c}|^2 = |\vec{c}|^2 = 49.$$

Squaring a magnitude is a scalar operation; it does not mean separately squaring the vector compo-

Quick Tip to Solve: Move the third vector to the other side and square the magnitude of the sum. The cross term has a plus sign and directly gives the angle between the original vectors.

nents without their cross terms.

Step 2: Expand the scalar product and substitute the lengths Expanding the left side, and using $|\vec{a}| = 3$ and $|\vec{b}| = 5$, gives

$$\begin{aligned}49 &= 9 + 25 + 2(3)(5) \cos \theta, \\30 \cos \theta &= 49 - 34 = 15, \\ \cos \theta &= \frac{1}{2}.\end{aligned}$$

The angle between two nonzero vectors lies in $[0, \pi]$. On this interval the cosine value $1/2$ uniquely corresponds to $\pi/3$, so no supplementary angle is required. In particular, using a minus cross term would find a triangle's supplementary interior angle rather than the angle between the original vectors.

Step 3: Conclusion The angle between $\vec{a}$ and $\vec{b}$ is $\theta = \pi/3 = 60^\circ$.

Quick Tip to Solve: Move the third vector to the other side and square the magnitude of the sum. The cross term has a plus sign and directly gives the angle between the original vectors.

OR

27(b). If $\vec{a}$ and $\vec{b}$ are unit vectors inclined with each other at an angle θ , then prove that $\frac{1}{2}|\vec{a} - \vec{b}| = \sin \frac{\theta}{2}$.
[3 marks]

Solution:

Concept:

- Unit vectors have magnitude 1. If their angle is θ , their dot product is $\vec{a} \cdot \vec{b} = \cos \theta$, with $0 \leq \theta \leq \pi$.
- The squared magnitude of their difference is $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$. The half-angle identity is $1 - \cos \theta = 2 \sin^2(\theta/2)$.

Step 1: Expand the squared magnitude of the difference Form the scalar product of the vector difference with itself. The two cross terms are equal because the scalar product is commutative. Therefore

$$\begin{aligned}|\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\&= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} \\&= 1 + 1 - 2 \cos \theta = 2(1 - \cos \theta).\end{aligned}$$

The unit-vector hypothesis is essential in replacing both squared magnitudes by 1 and the scalar product by $\cos \theta$.

Step 2: Apply the half-angle identity and take the square root Replacing $1 - \cos \theta$ with $2 \sin^2(\theta/2)$, we obtain

$$|\vec{a} - \vec{b}|^2 = 4 \sin^2 \frac{\theta}{2}.$$

A magnitude is nonnegative, so its square root is $2|\sin(\theta/2)|$, rather than an arbitrary signed value. Since the angle between vectors satisfies $0 \leq \theta \leq \pi$, we have $0 \leq \theta/2 \leq \pi/2$, where sine is nonnegative.

Quick Tip to Solve: Square the vector magnitude first, use $1 - \cos \theta = 2 \sin^2(\theta/2)$, and justify removing the absolute value by the range of the angle between vectors.

Thus

$$|\vec{a} - \vec{b}| = 2 \sin \frac{\theta}{2}.$$

This also covers the limiting cases: equal unit vectors give zero difference, while opposite unit vectors give magnitude 2.

Step 3: Conclusion Dividing by 2 proves the required identity: $\frac{1}{2}|\vec{a} - \vec{b}| = \sin \frac{\theta}{2}$.

Quick Tip to Solve: Square the vector magnitude first, use $1 - \cos \theta = 2 \sin^2(\theta/2)$, and justify removing the absolute value by the range of the angle between vectors.

28. Solve the differential equation $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$.

[3 marks]

Solution:

Concept:

- A first-order equation with dy/dx expressed as a function of y/x is homogeneous. Set $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$.
- Separate the variables before integrating. The standard integrals used here are $\int \cos v dv = \sin v$ and $\int dx/x = \ln|x|$, where $x \neq 0$.

Step 1: Recognise the homogeneous form The original equation contains y/x , so $x \neq 0$. Dividing by $x \cos(y/x)$ wherever a differentiable solution exists gives

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right).$$

The right side depends only on the ratio y/x , which suggests the substitution $v = y/x$. There is no omitted solution where $\cos(y/x) = 0$: the original equation would then require $0 = x$, contrary to its domain.

Step 2: Substitute and separate the variables Write $y = vx$, differentiate with the product rule, and substitute directly into the original equation:

$$\begin{aligned} x \cos v \left(v + x \frac{dv}{dx} \right) &= vx \cos v + x, \\ x^2 \cos v \frac{dv}{dx} &= x, \\ \cos v dv &= \frac{dx}{x}. \end{aligned}$$

The terms $vx \cos v$ cancel. Keeping the factor x contributed by differentiating vx is necessary to obtain the correct separated equation.

Step 3: Integrate and restore the original variables Integrate both sides and absorb the integration constants into a single constant C :

$$\sin v = \ln|x| + C.$$

Quick Tip to Solve: When y/x appears inside a trigonometric function, try $y = vx$. Substitute into the original equation first so the $vx \cos v$ terms cancel immediately.

Replacing v by y/x yields the implicit general solution. As a check, differentiating it gives

$$\cos\left(\frac{y}{x}\right) \frac{xy' - y}{x^2} = \frac{1}{x},$$

which rearranges to exactly the stated differential equation. The solution applies on intervals with $x \neq 0$ and $\cos(y/x) \neq 0$, where it defines a differentiable branch.

Step 4: Conclusion The general solution is $\sin\left(\frac{y}{x}\right) = \ln|x| + C$, with C an arbitrary constant, on its differentiable branches.

Quick Tip to Solve: When y/x appears inside a trigonometric function, try $y = vx$. Substitute into the original equation first so the $vx \cos v$ terms cancel immediately.

29(a). The probability that a student buys a colouring book is 0.7 and that she buys a box of colours is 0.2. The probability that she buys a colouring book, given that she buys a box of colours, is 0.3. Find the probability that the student:

(i) Buys both the colouring book and the box of colours.

(ii) Buys a box of colours given that she buys the colouring book.

[3 marks]

Solution:

Concept:

- Let B denote buying a colouring book and C denote buying a box of colours. Conditional probability restricts the sample space to the event named after the word given.
- For $P(C) > 0$, $P(B | C) = P(B \cap C)/P(C)$. Similarly, $P(C | B) = P(B \cap C)/P(B)$ when $P(B) > 0$.

Step 1: Identify the given probabilities The data are $P(B) = 0.7$, $P(C) = 0.2$, and $P(B | C) = 0.3$. The last probability refers to buying a book among students who buy colours. It is not the reverse conditional probability requested in part (ii). No assumption of independence is needed or justified by the question.

Step 2: (i) Find the probability of buying both Buying both items is the intersection $B \cap C$. Apply the multiplication rule with the given conditioning event:

$$P(B \cap C) = P(B | C)P(C) = 0.3 \times 0.2 = 0.06.$$

This probability is smaller than either individual probability, as an intersection must be. Multiplying 0.7 and 0.2 would incorrectly treat the two events as independent.

Step 3: (ii) Reverse the conditioning event We now restrict attention to students who buy a colouring book, so the denominator is $P(B)$. Using the intersection already calculated,

$$P(C | B) = \frac{P(B \cap C)}{P(B)} = \frac{0.06}{0.7} = \frac{6}{70} = \frac{3}{35}.$$

Both conditional probabilities have the same numerator, but different denominators. Therefore they

Quick Tip to Solve: First obtain the intersection using the given conditional probability; then divide by the probability of the new given event.

need not be equal.

Step 4: Conclusion (i) The probability of buying both is $\boxed{0.06}$. (ii) The probability of buying colours given buying a book is $\boxed{3/35}$.

Quick Tip to Solve: First obtain the intersection using the given conditional probability; then divide by the probability of the new given event.

OR

29(b). A person has a fruit box that contains 6 apples and 4 oranges. He picks out a fruit three times, one after the other, after replacing the previous one in the box. Find:

(i) The probability distribution of the number of oranges he draws.

(ii) The expectation of the random variable (number of oranges).

[3 marks]

Solution:

Concept:

- With replacement, the box composition stays unchanged after every draw. The draws are independent, with orange probability $p = 4/10 = 2/5$ and apple probability $q = 6/10 = 3/5$.
- The number X of oranges in n independent draws has binomial probabilities $P(X = r) = \binom{n}{r} p^r q^{n-r}$, for $r = 0, 1, \dots, n$.
- The expectation of a discrete random variable is $E(X) = \sum_r rP(X = r)$; for a binomial variable it is also np .

Step 1: Identify the possible values of the random variable Define X to be the number of oranges in the three draws. It can take the values 0, 1, 2, 3. Because each fruit is replaced before the next draw, the probability of orange remains $2/5$ on all three trials. Thus X follows a binomial distribution with $n = 3$ and $p = 2/5$.

Step 2: (i) Calculate the complete probability distribution For exactly r oranges, there are $\binom{3}{r}$ choices of their positions. Each such sequence has probability $(2/5)^r (3/5)^{3-r}$. Therefore

x	0	1	2	3
$P(X = x)$	$27/125$	$54/125$	$36/125$	$8/125$

For example, $P(X = 1) = 3(2/5)(3/5)^2 = 54/125$ and $P(X = 2) = 3(2/5)^2(3/5) = 36/125$. The probabilities sum to $(27 + 54 + 36 + 8)/125 = 1$, confirming a complete distribution.

Step 3: (ii) Compute the expectation from the distribution Multiply each possible number of oranges by its probability and add:

$$E(X) = \frac{0(27) + 1(54) + 2(36) + 3(8)}{125} = \frac{150}{125} = \frac{6}{5}.$$

This agrees with the binomial check $np = 3(2/5) = 6/5$. An expectation is a long-run average, so it need not be one of the integer values that occur in an individual experiment.

Quick Tip to Solve: Replacement makes the orange probability constant; use the binomial coefficients 1, 3, 3, 1 for three draws.

Step 4: Conclusion The distribution for $X = 0, 1, 2, 3$ is respectively $\boxed{27/125, 54/125, 36/125, 8/125}$, and $\boxed{E(X) = 6/5 = 1.2}$ oranges.

Quick Tip to Solve: Replacement makes the orange probability constant; use the binomial coefficients 1, 3, 3, 1 for three draws.

30(a). Find: $\int \frac{2x}{(x^2 + 3)(x^2 - 5)} dx$.

[3 marks]

Solution:

Concept:

- A numerator containing $2x$ suggests substituting $t = x^2$, because $dt = 2x dx$. This converts both quadratic factors into linear factors in t .
- A rational function with distinct linear denominator factors can be decomposed into partial fractions. The integral $\int du/u = \ln |u| + C$ requires an absolute value when the sign may vary.

Step 1: Reduce the integral by substitution Denote the requested integral by I . Set $t = x^2$, so $dt = 2x dx$, and substitute consistently in both denominator factors:

$$I = \int \frac{dt}{(t + 3)(t - 5)}.$$

The substitution removes the numerator and leaves a rational function with two distinct linear factors. The original integrand is defined wherever $x^2 \neq 5$.

Step 2: Determine the partial fraction coefficients Write

$$\frac{1}{(t + 3)(t - 5)} = \frac{A}{t + 3} + \frac{B}{t - 5}.$$

Multiplication by the common denominator gives $1 = A(t - 5) + B(t + 3)$. Setting $t = -3$ gives $A = -1/8$, and setting $t = 5$ gives $B = 1/8$. Hence the integrand equals $\frac{1}{8}(\frac{1}{t-5} - \frac{1}{t+3})$.

Step 3: Integrate and restore the original variable Integrating each simple fraction yields

$$I = \frac{1}{8}(\ln |t - 5| - \ln |t + 3|) + C.$$

Substitute $t = x^2$. Since $x^2 + 3 > 0$ for every real x , the logarithms can be combined as $\frac{1}{8} \ln |(x^2 - 5)/(x^2 + 3)| + C$. As a check, differentiation gives

$$\frac{1}{8} \left(\frac{2x}{x^2 - 5} - \frac{2x}{x^2 + 3} \right) = \frac{2x}{(x^2 + 3)(x^2 - 5)},$$

which recovers the given integrand on each interval of its domain.

Quick Tip to Solve: Substitute x^2 first; the coefficient $1/8$ comes from the difference between the two constants 3 and -5 .

Step 4: Conclusion

$$\int \frac{2x}{(x^2 + 3)(x^2 - 5)} dx = \frac{1}{8} \ln \left| \frac{x^2 - 5}{x^2 + 3} \right| + C.$$

Quick Tip to Solve: Substitute x^2 first; the coefficient $1/8$ comes from the difference between the two constants 3 and -5 .

OR

30(b). Evaluate: $\int_1^4 (|x - 2| + |x - 4|) dx$.

[3 marks]

Solution:**Concept:**

- For a real quantity u , $|u| = u$ when $u \geq 0$ and $|u| = -u$ when $u < 0$. Break the integration interval at any interior zero of a modulus argument.
- Definite integrals add over adjacent intervals: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, for $a < c < b$.

Step 1: Locate the changes of sign On the interval $[1, 4]$, $x - 2$ changes sign at the interior point $x = 2$. The quantity $x - 4$ is non-positive throughout the interval, becoming zero only at its upper endpoint. Thus $|x - 4| = 4 - x$ throughout, and only the split at 2 is needed.

Step 2: Remove the modulus signs on each interval For $1 \leq x \leq 2$, both arguments are non-positive, giving

$$|x - 2| + |x - 4| = (2 - x) + (4 - x) = 6 - 2x.$$

For $2 \leq x \leq 4$, the first argument is non-negative while the second remains non-positive:

$$|x - 2| + |x - 4| = (x - 2) + (4 - x) = 2.$$

Both formulas give 2 at $x = 2$, consistent with continuity there. Removing the modulus signs using one formula over the entire interval would give an incorrect contribution on one side of 2.

Step 3: Integrate the two pieces Apply interval additivity and evaluate the antiderivatives at their own limits:

$$\begin{aligned} I &= \int_1^2 (6 - 2x) dx + \int_2^4 2 dx \\ &= [6x - x^2]_1^2 + [2x]_2^4 \\ &= (12 - 4) - (6 - 1) + (8 - 4) \\ &= 3 + 4 = 7. \end{aligned}$$

The first piece is positive and decreases from 4 to 2, while the second is the constant 2; the positive total is consistent with integrating a sum of absolute values.

Quick Tip to Solve: Split at the interior zero $x = 2$; $x - 4$ never becomes positive before the upper limit, so its modulus is always $4 - x$.

Step 4: Conclusion

$$\int_1^4 (|x-2| + |x-4|) dx = 7.$$

Quick Tip to Solve: Split at the interior zero $x = 2$; $x - 4$ never becomes positive before the upper limit, so its modulus is always $4 - x$.

31. In the Linear Programming Problem (LPP), find the point/points giving maximum value for $Z = 5x + 10y$ subject to constraints $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$. [3 marks]

Solution:**Concept:**

- A feasible point satisfies all the constraints simultaneously. For a bounded polygonal feasible region, a linear objective function attains its maximum at a vertex or along an entire edge.
- Here $Z = 5x + 10y = 5(x + 2y)$. Thus the constraint $x + 2y \leq 120$ gives an upper bound for Z ; equality is possible only on the boundary $x + 2y = 120$.

Step 1: Find the feasible vertices In the first quadrant the conditions are $x + 2y \leq 120$, $x + y \geq 60$ and $x \geq 2y$. The intersections of the relevant boundary lines are found as follows:

$$\begin{aligned} x + y = 60, x = 2y &\Rightarrow (x, y) = (40, 20), \\ x + 2y = 120, x = 2y &\Rightarrow (x, y) = (60, 30), \\ x + 2y = 120, y = 0 &\Rightarrow (x, y) = (120, 0), \\ x + y = 60, y = 0 &\Rightarrow (x, y) = (60, 0). \end{aligned}$$

Each of these four points satisfies all the inequalities. The feasible region is the quadrilateral with these vertices; the intersection $(0, 60)$ of the first two boundary lines is excluded because it violates $x \geq 2y$.

Step 2: Evaluate the objective and identify the optimal edge At the feasible vertices, the objective has the values

$$\begin{aligned} Z(40, 20) &= 5(40) + 10(20) = 400, \\ Z(60, 30) &= 5(60) + 10(30) = 600, \\ Z(120, 0) &= 5(120) + 10(0) = 600, \\ Z(60, 0) &= 5(60) + 10(0) = 300. \end{aligned}$$

Since the objective is linear and the two adjacent vertices $(60, 30)$ and $(120, 0)$ give the same largest value, every point on the edge joining them also gives that value. To verify the full range explicitly, on $x + 2y = 120$ write $x = 120 - 2y$. The condition $x \geq 2y$ becomes $y \leq 30$, while non-negativity requires $y \geq 0$. The remaining condition is $x + y = 120 - y \geq 60$, which holds throughout this range. Hence the whole closed segment is feasible and attains the upper bound.

Step 3: Conclusion The maximum value is $Z_{\max} = 600$, attained at every point on the line segment joining $(60, 30)$ and $(120, 0)$, including both endpoints. Equivalently, all maximizing points are

Quick Tip to Solve: Factor the objective as $5(x + 2y)$ first. A matching constraint gives its upper bound immediately; then check which part of the equality line is feasible to find all maximizing points.

$$(x, y) = (120 - 2t, t), 0 \leq t \leq 30.$$

Quick Tip to Solve: Factor the objective as $5(x + 2y)$ first. A matching constraint gives its upper bound immediately; then check which part of the equality line is feasible to find all maximizing points.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32. In a rough sketch, mark the region bounded by $y = 1 + |x + 1|$, $x = -2$, $x = 2$ and $y = 0$. Using integration, find the area of the marked region. [5 marks]

Solution:

Concept:

- For a curve $y = f(x) \geq 0$, the area between the curve and the x -axis from $x = a$ to $x = b$ is $A = \int_a^b f(x) dx$.
- The expression inside $|x + 1|$ changes sign at $x = -1$. Resolve the absolute value on either side of this point and split the integral there.

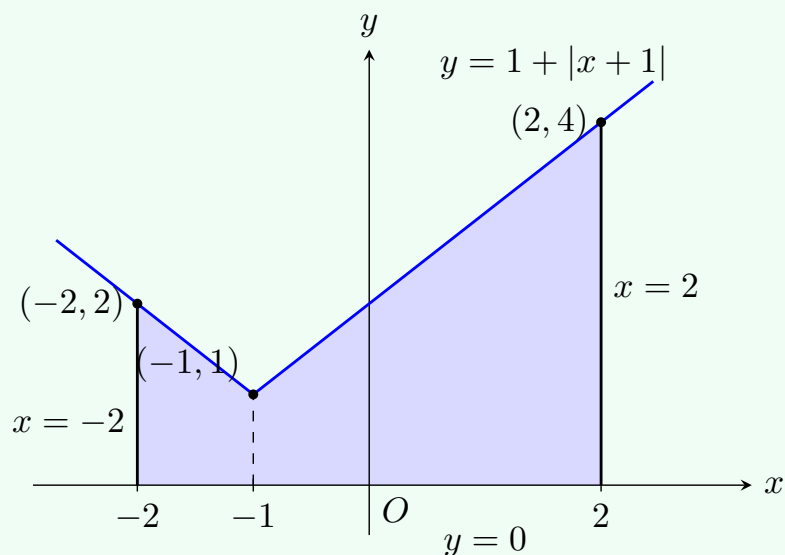
Step 1: Resolve the absolute value and locate the boundary Since $x + 1 \leq 0$ for $x \leq -1$ and $x + 1 \geq 0$ for $x \geq -1$, the curve becomes

$$y = 1 + |x + 1| = \begin{cases} -x, & x \leq -1, \\ x + 2, & x \geq -1. \end{cases}$$

The two straight branches meet at $(-1, 1)$, the lowest point of the curve. On the specified interval the boundary points on the curve are $(-2, 2)$ and $(2, 4)$. The curve always lies above the x -axis, so no cancellation of signed areas occurs in the integration.

Step 2: Sketch and mark the required region Draw the left branch from $(-2, 2)$ to $(-1, 1)$ and the right branch from $(-1, 1)$ to $(2, 4)$. Draw the vertical boundaries $x = -2$ and $x = 2$, and use $y = 0$ as the lower boundary. Shade the region below these branches and above the x -axis, between the two vertical lines. The sketch shows the joining point $x = -1$, where the integration must be split.

Quick Tip to Solve: Split an absolute-value area integral where the expression inside the modulus vanishes. Here the split is at $x = -1$, giving the two simple heights $-x$ and $x + 2$.



Shaded region below $y = 1 + |x + 1|$ between $x = -2$ and $x = 2$.

Step 3: Integrate the left portion Between $x = -2$ and $x = -1$, the height above the x -axis is $-x$. Thus

$$\begin{aligned} A_1 &= \int_{-2}^{-1} (-x) dx \\ &= \left[-\frac{x^2}{2} \right]_{-2}^{-1} \\ &= -\frac{1}{2} - \left(-\frac{4}{2} \right) = \frac{3}{2}. \end{aligned}$$

Step 4: Integrate the right portion and add the areas Between $x = -1$ and $x = 2$, the height is $x + 2$. Therefore

$$\begin{aligned} A_2 &= \int_{-1}^2 (x + 2) dx \\ &= \left[\frac{x^2}{2} + 2x \right]_{-1}^2 \\ &= (2 + 4) - \left(\frac{1}{2} - 2 \right) = \frac{15}{2}, \\ A &= A_1 + A_2 = \frac{3}{2} + \frac{15}{2} = 9. \end{aligned}$$

As a check, the left portion is a trapezium of width 1 with endpoint heights 2 and 1, while the right portion has width 3 and heights 1 and 4. Their trapezium areas agree with the two integral values.

Step 5: Conclusion The shaded region bounded by $y = 1 + |x + 1|$, $x = -2$, $x = 2$ and $y = 0$ has area 9 square units.

Quick Tip to Solve: Split an absolute-value area integral where the expression inside the modulus vanishes. Here the split is at $x = -1$, giving the two simple heights $-x$ and $x + 2$.

33. Three students run on a racing track such that their speeds add up to 6 km/h. However, double the speed of the third runner added to the speed of the first results in 7 km/h. If thrice the speed of the first runner is added to the original speeds of the other two, the result is 12 km/h. Using matrix method, find the original speed of each runner. [5 marks]

Solution:

Concept:

- Translate each condition into a linear equation in the three unknown speeds. A system of three equations can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of constants.
- If $|A| \neq 0$, the system has the unique solution $X = A^{-1}B$, with $A^{-1} = \frac{\text{adj } A}{|A|}$. The adjugate is the transpose of the cofactor matrix.

Step 1: Form the equations from the runner conditions Let the speeds of the first, second and third runners be x, y, z , respectively, in km/h. The sum of their speeds is 6, so $x + y + z = 6$. The first runner's speed plus twice the third runner's speed is 7, giving $x + 2z = 7$. The last condition gives $3x + y + z = 12$. Thus the matrix equation is

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 6 \\ 7 \\ 12 \end{pmatrix}}_B.$$

The order of the columns is first, second, third runner throughout, so the second equation has coefficient zero for y .

Step 2: Calculate the determinant and adjugate Expand the determinant along the first row:

$$|A| = 1(0 - 2) - 1(1 - 6) + 1(1 - 0) = -2 + 5 + 1 = 4.$$

Since the determinant is non-zero, the inverse exists. The cofactors of A , including the alternating signs, give the cofactor matrix

$$C = \begin{pmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{pmatrix}.$$

For example, $C_{12} = -(1 - 6) = 5$ and $C_{23} = -(1 - 3) = 2$. Transposing the cofactor matrix, rather than using it directly, gives

$$\text{adj } A = C^T = \begin{pmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}, \quad A^{-1} = \frac{1}{4} \begin{pmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Quick Tip to Solve: Keep the runner order fixed in every matrix column; transpose the cofactor matrix before multiplying by the constants.

Step 3: Multiply the inverse by the constants and verify Premultiplying $AX = B$ by A^{-1} yields

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{4} \begin{pmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 12 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} -12 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}. \end{aligned}$$

Check all three original conditions: $3 + 1 + 2 = 6$, $3 + 2(2) = 7$, and $3(3) + 1 + 2 = 12$. All speeds are positive and satisfy the stated relations, so the solution is physically meaningful.

Step 4: Conclusion The original speeds of the first, second and third runners are, respectively, 3 km/h, 1 km/h, 2 km/h.

Quick Tip to Solve: Keep the runner order fixed in every matrix column; transpose the cofactor matrix before multiplying by the constants.

34(a). For a positive constant a , differentiate $a^{t+\frac{1}{t}}$ with respect to $\left(t + \frac{1}{t}\right)^a$, where t is a non-zero real number. [5 marks]

Solution:

Concept:

- For two differentiable functions $y(t)$ and $x(t)$, differentiation of y with respect to x gives $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ wherever $dx/dt \neq 0$.
- The exponential and power rules are $\frac{d}{du}a^u = a^u \log a$ for $a > 0$, and $\frac{d}{du}u^a = au^{a-1}$ wherever the real power is defined and differentiable. Here $\log$ denotes the natural logarithm.
- A shared inner expression can be used as an intermediate variable. This makes the common chain-rule factor visible and simplifies the derivative.

Step 1: Introduce the common inner expression Put $u = t + \frac{1}{t}$, with $t \neq 0$, and write

$$y = a^u = a^{t+1/t}, \quad x = u^a = \left(t + \frac{1}{t}\right)^a.$$

The question asks for dy/dx , not merely dy/dt . The parameter a is a positive constant, so it is held fixed during differentiation. First find the derivative of the common inner expression:

$$\frac{du}{dt} = 1 - \frac{1}{t^2}.$$

Quick Tip to Solve: Set the repeated expression equal to u and divide dy/du by dx/du ; its derivative with respect to t cancels automatically.

Step 2: Differentiate both functions with respect to the parameter By the chain rule for the exponential function,

$$\frac{dy}{dt} = a^{t+1/t} \log a \left(1 - \frac{1}{t^2}\right).$$

Likewise the power rule and chain rule give

$$\frac{dx}{dt} = a \left(t + \frac{1}{t}\right)^{a-1} \left(1 - \frac{1}{t^2}\right).$$

The logarithmic factor belongs to the exponential a^u ; the factor a and reduction of the exponent by one belong to the power u^a . Keeping these two rules separate prevents interchange of their derivatives.

Step 3: Take the ratio and simplify At parameter values where $dx/dt \neq 0$, divide the two derivatives:

$$\begin{aligned} \frac{dy}{dx} &= \frac{a^{t+1/t} \log a (1 - 1/t^2)}{a(t + 1/t)^{a-1} (1 - 1/t^2)} \\ &= a^{t+1/t-1} \left(t + \frac{1}{t}\right)^{1-a} \log a. \end{aligned}$$

Both derivatives contain the same inner derivative, which cancels. The same result follows directly from $(dy/du)/(dx/du)$, using $dy/du = a^u \log a$ and $dx/du = au^{a-1}$. This also explains the derivative of the relation between y and x at $t = 1$ (and at $t = -1$ when a real differentiable branch exists), although the parameter ratio there initially has the form $0/0$. For an arbitrary positive real a , the real-power expression is considered on its real differentiable domain; in particular $t > 0$ makes $u > 0$. If $a = 1$, then $y = 1$ is constant and the formula correctly gives zero since $\log 1 = 0$.

Step 4: Conclusion Hence, on the real differentiable domain,

$$\frac{d(a^{t+1/t})}{d((t + 1/t)^a)} = a^{t+1/t-1} \left(t + \frac{1}{t}\right)^{1-a} \log a.$$

Quick Tip to Solve: Set the repeated expression equal to u and divide dy/du by dx/du ; its derivative with respect to t cancels automatically.

OR

34(b). Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$, where a and b are constants.

[5 marks]

Solution:

Concept:

- In implicit differentiation, y depends on x ; write $y' = \frac{dy}{dx}$ and apply the chain rule.
- A variable power is handled by writing $u^v = e^{v \ln u}$. For a positive base u , its derivative is

Quick Tip to Solve: Differentiate each power separately using $e^{v \ln u}$, then collect all terms containing y' . Do not take the logarithm of the sum as though it were a product.

$u^v \left(v' \ln u + v \frac{u'}{u} \right)$, where both u and v may depend on x .

- The quantity a^b is constant because both a and b are constants. Its derivative with respect to x is therefore zero.

Step 1: Differentiate the term with base y For $x > 0$ and $y > 0$, set $u = y^x$. From $\ln u = x \ln y$, the product and chain rules give $u'/u = \ln y + (x/y)y'$. Therefore,

$$\frac{d}{dx}(y^x) = y^x \ln y + xy^{x-1}y'.$$

Step 2: Differentiate the remaining powers For $v = x^y$, differentiating $\ln v = y \ln x$ gives $v'/v = y' \ln x + y/x$. Consequently,

$$\frac{d}{dx}(x^y) = x^y \ln x y' + yx^{y-1}.$$

For $w = x^x$, differentiate $\ln w = x \ln x$ to obtain $w'/w = \ln x + 1$. Thus

$$\frac{d}{dx}(x^x) = x^x(\ln x + 1).$$

Step 3: Collect the terms multiplying y' Differentiate the original equation using these three derivatives:

$$y^x \ln y + xy^{x-1}y' + x^y \ln x y' + yx^{y-1} + x^x(\ln x + 1) = 0.$$

Collect the two terms containing y' on the left and transfer the others to the right:

$$(xy^{x-1} + x^y \ln x)y' = -y^x \ln y - yx^{y-1} - x^x(\ln x + 1).$$

The coefficient of y' receives one contribution from the variable base in y^x and another from the variable exponent in x^y . The term x^x contributes only to the remaining terms. Divide by the coefficient of y' wherever it is nonzero.

Step 4: Conclusion The required derivative is

$$\frac{dy}{dx} = -\frac{y^x \ln y + yx^{y-1} + x^x(1 + \ln x)}{xy^{x-1} + x^y \ln x}.$$

Quick Tip to Solve: Differentiate each power separately using $e^{v \ln u}$, then collect all terms containing y' . Do not take the logarithm of the sum as though it were a product.

35(a). Find the foot of the perpendicular drawn from the point $(1, 1, 4)$ on the line $\frac{x+2}{5} = \frac{y+1}{2} = \frac{-z+4}{-3}$. [5 marks]

Solution:

Concept:

- A line through a point A with nonzero direction vector $\mathbf{d}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$. The parameter t locates a general point on the line.
- Two vectors are perpendicular exactly when their scalar product is zero. If H is the foot from a point P onto the line, then $(H - P) \cdot \mathbf{d} = 0$.
- The foot H must satisfy both the equation of the line and the perpendicularity condition. Since $\mathbf{d} \neq \mathbf{0}$, these conditions determine a unique point.

Step 1: Write a general point on the given line Let the common value of the three ratios be t :

$$\frac{x+2}{5} = \frac{y+1}{2} = \frac{-z+4}{-3} = t.$$

Solving these equations individually gives

$$x = -2 + 5t, \quad y = -1 + 2t, \quad z = 4 + 3t.$$

In the last equation, $-z + 4 = -3t$ becomes $z = 4 + 3t$. Thus the direction in the z -coordinate is positive. The line passes through $A = (-2, -1, 4)$ and has direction vector $\mathbf{d} = (5, 2, 3)$. Write the candidate foot as

$$H = (-2 + 5t, -1 + 2t, 4 + 3t).$$

Step 2: Impose the perpendicularity condition Subtract the coordinates of $P = (1, 1, 4)$ from those of H :

$$\overrightarrow{PH} = (-3 + 5t, -2 + 2t, 3t).$$

Since PH is perpendicular to the line, it is perpendicular to its direction vector. Therefore,

$$\begin{aligned} 5(-3 + 5t) + 2(-2 + 2t) + 3(3t) &= 0, \\ 38t - 19 &= 0. \end{aligned}$$

This linear equation has the unique solution $t = \frac{19}{38} = \frac{1}{2}$, since the squared length of the direction vector is $25 + 4 + 9 = 38 > 0$.

Step 3: Substitute the parameter and verify the foot Substitution of $t = \frac{1}{2}$ gives

$$H = \left(-2 + \frac{5}{2}, -1 + 1, 4 + \frac{3}{2}\right) = \left(\frac{1}{2}, 0, \frac{11}{2}\right).$$

Each ratio in the original line equals $\frac{1}{2}$ at this point, so it lies on the line. At this point the perpendicular

Quick Tip to Solve: First rewrite $\frac{-z+4}{-3}$ as $\frac{z-4}{3}$, so the direction ratios are $(5, 2, 3)$. Then use $(H - P) \cdot (5, 2, 3) = 0$ to obtain the line parameter directly.

vector is $\overrightarrow{PH} = (-\frac{1}{2}, -1, \frac{3}{2})$, and its scalar product with $(5, 2, 3)$ equals

$$-\frac{5}{2} - 2 + \frac{9}{2} = 0.$$

Thus both defining conditions for the perpendicular foot are satisfied.

Step 4: Conclusion The foot of the perpendicular from $(1, 1, 4)$ onto the given line is

$$H = \left(\frac{1}{2}, 0, \frac{11}{2}\right).$$

Quick Tip to Solve: First rewrite $\frac{-z+4}{-3}$ as $\frac{z-4}{3}$, so the direction ratios are $(5, 2, 3)$. Then use $(H - P) \cdot (5, 2, 3) = 0$ to obtain the line parameter directly.

OR

35(b). Find the point on the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3}$ at a distance of $2\sqrt{2}$ units from the point $(-1, -1, 2)$. [5 marks]

Solution:

Concept:

- Equate the three symmetric ratios to a parameter t to obtain every point on the line; its direction vector is $(3, 2, 3)$.
- The squared distance from $P(x, y, z)$ to $Q(a, b, c)$ is $(x-a)^2 + (y-b)^2 + (z-c)^2$. Solve the parameter equation and check all real roots.

Step 1: Express the general point on the printed line Put

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3} = t.$$

Then $x = 1 + 3t$, $y = -1 + 2t$ and $z = 4 + 3t$. Thus every point on the line has the form $P = (1 + 3t, -1 + 2t, 4 + 3t)$, with $t \in \mathbb{R}$. The fixed point from which the distance is measured is $Q = (-1, -1, 2)$.

Step 2: Impose the distance condition The coordinate differences from Q are $2 + 3t$, $2t$ and $2 + 3t$. Since $PQ = 2\sqrt{2}$, we obtain

$$\begin{aligned}(2 + 3t)^2 + (2t)^2 + (2 + 3t)^2 &= (2\sqrt{2})^2, \\ 2(4 + 12t + 9t^2) + 4t^2 &= 8, \\ 22t^2 + 24t + 8 &= 8, \\ 2t(11t + 12) &= 0.\end{aligned}$$

Therefore the possible parameters are $t = 0$ and $t = -12/11$.

Quick Tip to Solve: Use one common parameter for all three ratios, square the distance, and retain every real root before substituting into the coordinates.

Step 3: Find both corresponding points For $t = 0$, the coordinates are $(1, -1, 4)$. For $t = -12/11$, substitution gives $x = 1 - 36/11 = -25/11$, $y = -1 - 24/11 = -35/11$, and $z = 4 - 36/11 = 8/11$. Although the question uses the singular word point, the distance equation has two distinct roots, so both points must be retained.

Step 4: Verify the line and the distance For the second point, all three line ratios equal $-12/11$; for the first they equal zero. The first squared distance from Q is $2^2 + 0^2 + 2^2 = 8$. For the second, the difference vector is $(-14/11, -24/11, -14/11)$, giving $PQ^2 = (196 + 576 + 196)/121 = 968/121 = 8$. Both candidates satisfy the line and distance conditions. There are no others since the parameter equation has exactly these two roots.

Step 5: Conclusion The required points are $(1, -1, 4)$ and $\left(-\frac{25}{11}, -\frac{35}{11}, \frac{8}{11}\right)$. Each is at a distance $2\sqrt{2}$ units from $(-1, -1, 2)$.

Quick Tip to Solve: Use one common parameter for all three ratios, square the distance, and retain every real root before substituting into the coordinates.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Let A be the set of 30 students of class XII in a school. Let $f : A \rightarrow \mathbb{N}$, $\mathbb{N}$ is a set of natural numbers such that function $f(x) = \text{Roll Number of student } x$.

On the basis of the given information, answer the following:

[4 marks]

36(i). Is f a bijective function?

[1 mark]

Solution:

Concept:

- A bijection is a function that is both one-one (injective) and onto (surjective).
- Onto means every element of the stated codomain has a preimage in the domain.

Step 1: Identify the domain and codomain The domain A contains exactly 30 students, while the codomain is the entire infinite set $\mathbb{N}$. The image $f(A)$ consists only of the roll numbers assigned to these students.

Step 2: Check the necessary onto condition A function from a set of 30 elements can have at most 30 distinct outputs. Therefore it cannot have every natural number as an output. Hence $f(A) \neq \mathbb{N}$, so f is not onto. Failure of this necessary condition is sufficient to rule out a bijection.

Step 3: Conclusion No, f is not bijective, because it is not onto $\mathbb{N}$.

Quick Tip to Solve: Compare the finite domain with the stated infinite codomain before testing injectivity.

36(ii). Give reasons to support your answer to (i).

[1 mark]

Solution:

Concept:

- A function is injective if $f(x) = f(y)$ implies $x = y$. Distinct students in a class have distinct roll numbers.
- A function is surjective only when its image equals its entire stated codomain.

Step 1: Check one-one behaviour Each student receives a unique roll number. If two students $x, y \in A$ satisfy $f(x) = f(y)$, their roll numbers are the same and therefore they are the same student. Thus f is one-one.

Step 2: Check onto behaviour and combine the tests The image contains the 30 class roll numbers, whereas $\mathbb{N}$ contains infinitely many natural numbers. Natural numbers outside this finite collection have no student preimage. Thus f is not onto the specified codomain. A bijection requires both properties simultaneously; being one-one alone does not suffice.

Step 3: Conclusion f is one-one but not onto $\mathbb{N}$; hence f is not bijective.

Quick Tip to Solve: Keep the codomain distinct from the image: unique roll numbers establish injectivity, not surjectivity onto all natural numbers.

36(iii)(a). Let R be a relation defined by the teacher to plan the seating arrangement of students in pairs, where $R = \{(x, y) : x, y \text{ are Roll Numbers of students such that } y = 3x\}$. List the elements of R . Is the relation R reflexive, symmetric and transitive? Justify your answer.

[2 marks]

Solution:

Concept:

- The class roll-number set is $S = \{1, 2, \dots, 30\}$. A pair in R must have both its coordinates in S .
- Reflexivity requires $(x, x) \in R$ for every $x \in S$; symmetry requires each pair to have its reverse; transitivity requires (x, z) whenever $(x, y), (y, z) \in R$.

Step 1: List all ordered pairs From $y = 3x$ and $1 \leq y \leq 30$, we obtain $1 \leq x \leq 10$. Therefore

$$R = \{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15), \\ (6, 18), (7, 21), (8, 24), (9, 27), (10, 30)\}.$$

Every listed coordinate is a roll number, and a larger positive x would give $y > 30$.

Step 2: Test reflexivity and symmetry The pair $(1, 1)$ is not in R , since $1 \neq 3 \cdot 1$. Hence R is not reflexive. Also $(1, 3) \in R$, but $(3, 1) \notin R$, since $1 \neq 3 \cdot 3$. Therefore R is not symmetric.

Step 3: Test transitivity with a counterexample Both $(1, 3)$ and $(3, 9)$ belong to R . Transitivity would then require $(1, 9) \in R$, but $9 \neq 3 \cdot 1$, so this pair is absent. This single chain disproves transitivity; checking every possible triple is unnecessary.

Step 4: Conclusion The listed relation R is neither reflexive nor symmetric nor transitive.

Quick Tip to Solve: For a relation $y=3x$, test the chain $(1,3),(3,9)$: it immediately exposes the failure of transitivity.

36(iii)(b). Let R be a relation defined by $R = \{(x, y) : x, y \text{ are Roll Numbers of students such that } y = x^3\}$. List the elements of R . Is R a function? Justify your answer. [2 marks]

Solution:

Concept:

- The class roll-number set is $S = \{1, 2, \dots, 30\}$, and both entries of a pair in R must belong to S .
- A relation is a function on S only if every element of S has exactly one image. Uniqueness without existence for all inputs is insufficient.

Step 1: List all pairs satisfying the equation We need $y = x^3$ with $x, y \in S$. Since $3^3 = 27 \leq 30$ and $4^3 = 64 > 30$, the only possible positive integer inputs are 1, 2, 3. Their outputs are 1, 8, 27, respectively. Thus

$$R = \{(1, 1), (2, 8), (3, 27)\}.$$

No further roll number yields a cube in the class roll-number set.

Step 2: Test whether the relation defines a function on the roll-number set Each of the inputs 1, 2, 3 has a unique output, but the other roll numbers have none. In particular, the input $4 \in S$ would require output 64, which is not a class roll number; hence no pair $(4, y)$ lies in R . Therefore the requirement that every element of S have an image is violated. If the domain were restricted to $\{1, 2, 3\}$, these same pairs would define a function, but the question concerns the class roll-number set.

Step 3: Conclusion $R = \{(1, 1), (2, 8), (3, 27)\}$. As a relation on all 30 class roll numbers, R is not a function, because some inputs have no image.

Quick Tip to Solve: After listing the pairs, compare their first coordinates with the full input set; missing inputs prevent a relation from being a function on that set.

37. Case Study – 2. A gardener wanted to plant vegetables in his garden. Hence he bought 10 seeds of brinjal plant, 12 seeds of cabbage plant and 8 seeds of radish plant. The shopkeeper assured him of germination probabilities of brinjal, cabbage and radish to be 25%, 35% and 40% respectively. But before he could plant the seeds, they got mixed up in the bag and he had to sow them randomly.

Based upon the above information, answer the following questions:

[4 marks]

37(i). Calculate the probability of a randomly chosen seed to germinate.

[2 marks]

Solution:

Concept:

- The law of total probability gives $P(G) = \sum P(E_i)P(G | E_i)$, where G denotes germination and the events E_i partition the seed types.
- Random selection gives each seed equal chance; hence the probability of its type is the number of that type divided by the total number of seeds.

Quick Tip to Solve: Multiply each germination probability by its seed count, add, then divide by 30; the seed types are not equally likely.

Step 1: Find the probabilities of the seed types Let B, C, R denote selecting a brinjal, cabbage and radish seed respectively. The total number is $10 + 12 + 8 = 30$. These events are mutually exclusive and exhaustive. Thus

$$P(B) = \frac{10}{30}, \quad P(C) = \frac{12}{30}, \quad P(R) = \frac{8}{30}.$$

Step 2: Weight the germination probabilities The given conditional probabilities are $P(G | B) = 1/4$, $P(G | C) = 7/20$ and $P(G | R) = 2/5$. The type counts are unequal, so their germination probabilities must be weighted, rather than simply averaged.

$$\begin{aligned} P(G) &= \frac{10}{30} \cdot \frac{1}{4} + \frac{12}{30} \cdot \frac{7}{20} + \frac{8}{30} \cdot \frac{2}{5} \\ &= \frac{50 + 84 + 64}{600} = \frac{198}{600} = \frac{33}{100}. \end{aligned}$$

Step 3: Conclusion The required probability of germination is $\boxed{33/100}$, or 0.33.

Quick Tip to Solve: Multiply each germination probability by its seed count, add, then divide by 30; the seed types are not equally likely.

37(ii). What is the probability that it is a cabbage seed, given that the chosen seed germinates?

[2 marks]

Solution:

Concept:

- Bayes' theorem gives $P(C | G) = P(C)P(G | C)/P(G)$, where C means cabbage and G means germination.
- The denominator is the total germination probability, obtained by adding the joint probabilities for all three seed types.

Step 1: Calculate the joint and total probabilities Among 30 seeds, 12 are cabbage seeds, so $P(C) = 12/30$. Their germination probability is $35\% = 7/20$. Consequently

$$P(C \cap G) = \frac{12}{30} \cdot \frac{7}{20} = \frac{7}{50}.$$

The total probability, including brinjal and radish, is

$$P(G) = \frac{10}{30} \cdot \frac{1}{4} + \frac{12}{30} \cdot \frac{7}{20} + \frac{8}{30} \cdot \frac{2}{5} = \frac{33}{100}.$$

Step 2: Condition on germination The information that the chosen seed germinates restricts the sample space to germinating seeds. Divide the cabbage-and-germination probability by the probability of this condition:

$$P(C | G) = \frac{P(C \cap G)}{P(G)} = \frac{7/50}{33/100} = \frac{14}{33}.$$

This differs from the original cabbage probability $12/30$, because the three types have different chances

Quick Tip to Solve: For a seed known to germinate, use cabbage contribution $12(0.35)$ over total contribution $10(0.25) + 12(0.35) + 8(0.40)$.

of germination.

Step 3: Conclusion The probability that a germinating seed is cabbage is $\boxed{14/33}$.

Quick Tip to Solve: For a seed known to germinate, use cabbage contribution $12(0.35)$ over total contribution $10(0.25) + 12(0.35) + 8(0.40)$.

38. Case Study – 3. A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.

On the basis of the above information, answer the following questions:

[4 marks]

38(i). Taking length = breadth = x m and height = y m, express the surface area (S) of the box in terms of x and its volume (V), which is constant.

[1 mark]

Solution:

Concept:

- A closed cuboid has surface area $S = 2(lb + bh + hl)$, where l, b, h are its length, breadth and height.
- For a square base of edge x and height y , the fixed volume is $V = x^2y$, with $x, y > 0$.

Step 1: Eliminate the height The volume condition relates the two dimensions:

$$V = x^2y \implies y = \frac{V}{x^2}.$$

Since the base edge is positive, division by x^2 is valid.

Step 2: Substitute into the surface area The top and bottom contribute $2x^2$; the four side faces contribute $4xy$. Therefore

$$S = 2x^2 + 4xy = 2x^2 + 4x \frac{V}{x^2} = 2x^2 + \frac{4V}{x}.$$

The inclusion of both square faces uses the fact that the box is closed.

Step 3: Conclusion $\boxed{S(x) = 2x^2 + 4V/x}$, for $x > 0$, with S in square metres.

Quick Tip to Solve: For a closed square-base box, count two square faces and four rectangular faces before replacing y by V/x^2 .

38(ii). Find $\frac{dS}{dx}$.

[1 mark]

Solution:

Concept:

- Using the fixed-volume constraint, the surface area is $S(x) = 2x^2 + 4V/x$, with V constant and $x > 0$.

Quick Tip to Solve: First replace $1/x$ by x^{-1} ; its derivative is $-x^{-2}$, while the fixed volume V stays constant.

- The power rule is $d(x^n)/dx = nx^{n-1}$. A fixed V is treated as a constant coefficient when differentiating.

Step 1: Write the reciprocal as a power For differentiation, rewrite the surface-area expression in powers of the single variable:

$$S(x) = 2x^2 + 4Vx^{-1}.$$

The height has already been eliminated using $y = V/x^2$; it is not held fixed as x changes.

Step 2: Differentiate term by term Apply the power rule to each term, keeping the volume constant:

$$\frac{dS}{dx} = 2(2x) + 4V(-x^{-2}) = 4x - \frac{4V}{x^2}.$$

The negative sign in the second term comes from differentiating x^{-1} .

Step 3: Conclusion $\frac{dS}{dx} = 4x - \frac{4V}{x^2}.$

Quick Tip to Solve: First replace $1/x$ by x^{-1} ; its derivative is $-x^{-2}$, while the fixed volume V stays constant.

38(iii)(a). Find a relation between x and y such that the surface area (S) is minimum.

[2 marks]

Solution:

Concept:

- For fixed volume $V = x^2y > 0$, the surface area is $S(x) = 2x^2 + 4V/x$, where $x > 0$.
- At an interior minimum, $S'(x) = 0$; a positive second derivative establishes that the stationary value is a minimum.

Step 1: Find the stationary base edge Differentiate the surface area after eliminating y with the volume condition. Then

$$S'(x) = 4x - \frac{4V}{x^2}.$$

Setting this to zero and multiplying by the positive quantity $x^2/4$ gives

$$x^3 = V.$$

Since also $V = x^2y$, the stationary condition becomes $x^3 = x^2y$, and division by $x^2 > 0$ yields $y = x$.

Step 2: Verify that the surface area is minimum Differentiate again while holding V fixed:

$$S''(x) = 4 + \frac{8V}{x^3}.$$

This is positive for every admissible $x > 0$, and equals 12 at $x^3 = V$. Moreover, $S(x)$ tends to infinity as x tends to zero from above or to infinity. Thus the unique positive stationary point is the global minimum.

Step 3: Conclusion The required relation is $x = y$; the minimum-area box is a cube.

Quick Tip to Solve: Use the fixed volume to remove y before differentiating; the condition $x^3 = V = x^2y$ immediately gives $x = y$.

38(iii)(b). If surface area (S) is constant, the volume (V) = $\frac{1}{4}(Sx - 2x^3)$, x being the edge of base.

Show that volume (V) is maximum for $x = \sqrt{\frac{S}{6}}$.

[2 marks]

Solution:

Concept:

- With surface area $S > 0$ fixed, the printed volume function is $V(x) = \frac{1}{4}(Sx - 2x^3)$, with $x > 0$.
- At a stationary maximum, $V'(x) = 0$ and $V''(x) < 0$. A positive-volume box also requires $x < \sqrt{S/2}$.

Step 1: Find the stationary value of the edge Treat S as a constant while applying the power rule:

$$V'(x) = \frac{1}{4}(S - 6x^2).$$

The stationary equation is $S - 6x^2 = 0$, so $x^2 = S/6$. Only the positive root is an edge length, giving $x = \sqrt{S/6}$. This lies inside the admissible positive-volume interval $0 < x < \sqrt{S/2}$.

Step 2: Establish the maximum A second differentiation gives

$$V''(x) = -3x.$$

At $x = \sqrt{S/6} > 0$, this is negative, so the stationary value is a maximum. In addition, $V'(x) > 0$ for $0 < x < \sqrt{S/6}$ and $V'(x) < 0$ for $\sqrt{S/6} < x < \sqrt{S/2}$. Thus the volume increases up to this edge length and decreases thereafter throughout the admissible domain.

Step 3: Conclusion Hence the volume is maximum for $x = \sqrt{S/6}$, as required.

Quick Tip to Solve: Keep S constant, solve $S - 6x^2 = 0$, take the positive root, then use $V'' = -3x < 0$ to confirm the maximum.