

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-6-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. Let both AB' and $B'A$ be defined for matrices A and B . If order of A is $n \times m$, then the order of B is: [1 mark]

- (A) $n \times n$
- (B) $n \times m$
- (C) $m \times m$
- (D) $m \times n$

Correct Answer: (B) $n \times m$

Solution:

Concept:

- For matrices of orders $r \times s$ and $s \times t$, multiplication is defined because the inner dimensions agree.
- Transposition interchanges rows and columns: if B has order $p \times q$, then B' has order $q \times p$.

Step 1: Apply compatibility to the first product Let the order of B be $p \times q$. Since A is $n \times m$, the product AB' has factors of orders $n \times m$ and $q \times p$. Its existence requires $m = q$.

Step 2: Apply compatibility to the second product In $B'A$, the factors have orders $q \times p$ and $n \times m$. Thus $p = n$. The two conditions determine the rows and columns of B . Options (A) and (C) incorrectly force a square order; (D) interchanges the required dimensions and works only in the special case $n = m$.

Step 3: Conclusion The order of B is $n \times m$. Hence, the correct option is (B).

Quick Tip to Solve: Give the unknown matrix two dimensions and impose the inner-dimension rule separately on both products.

2. If $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then A is a/an: [1 mark]

- (A) scalar matrix
- (B) identity matrix
- (C) symmetric matrix
- (D) skew-symmetric matrix

Correct Answer: (C) symmetric matrix

Solution:

Concept:

- A real square matrix is symmetric when $A' = A$, where the prime denotes transpose.
- A scalar matrix has equal diagonal entries, an identity matrix has all diagonal entries equal to 1,

Quick Tip to Solve: Every real diagonal matrix is symmetric; equal diagonal entries are needed only for the stronger scalar classification.

and a skew-symmetric matrix satisfies $A' = -A$.

Step 1: Compare the matrix with its transpose The given matrix is diagonal. Transposition leaves its diagonal entries $-1, 3, 5$ in place and merely interchanges zero off-diagonal entries. Therefore $A' = A$.

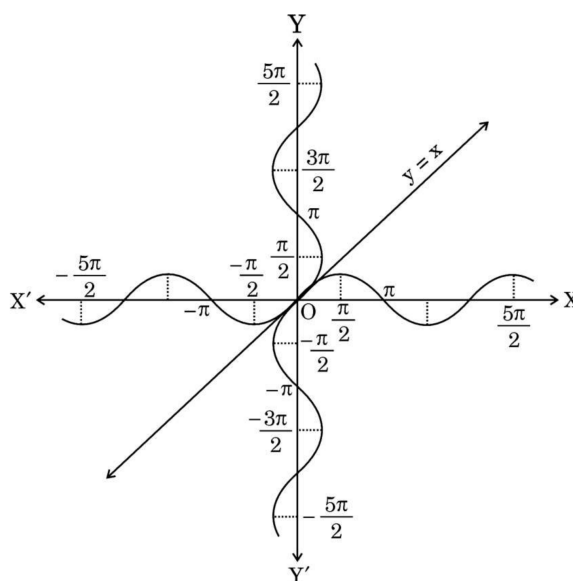
Step 2: Check the other classifications The unequal diagonal entries rule out a scalar matrix, and they are not all 1, so the matrix is not the identity. In a real skew-symmetric matrix each diagonal entry must satisfy $a_{ii} = -a_{ii}$, hence must be zero; the given nonzero diagonal entries rule this out.

Step 3: Conclusion The matrix A is symmetric. Hence, the correct option is (C).

Quick Tip to Solve: Every real diagonal matrix is symmetric; equal diagonal entries are needed only for the stronger scalar classification.

3. The following graph is a combination of :

[1 mark]



(A) $y = \sin^{-1} x$ and $y = \cos^{-1} x$

(B) $y = \cos^{-1} x$ and $y = \cos x$

(C) $y = \sin^{-1} x$ and $y = \sin x$

(D) $y = \cos^{-1} x$ and $y = \sin x$

Correct Answer: (C) $y = \sin^{-1} x$ and $y = \sin x$

Solution:

Concept:

- The sine curve passes through $(0, 0)$ and $(\pi/2, 1)$, and its reflection in $y = x$ interchanges the coordinates of every point.
- The principal inverse $y = \sin^{-1} x$ is the reflected branch with $-\pi/2 \leq y \leq \pi/2$. Reflecting the unrestricted sine curve gives the full inverse relation $x = \sin y$.

Step 1: Identify the horizontal wave The wave through the origin rises to its first positive maximum

Quick Tip to Solve: Use the origin and the first maximum to identify sine; then reflect across $y = x$ and restrict the inverse to its principal branch.

at $x = \pi/2$ and crosses the axis again at $x = \pi$. These features identify $y = \sin x$. The cosine curve would have value 1 at the origin, so option (B) does not fit.

Step 2: Identify the reflected curve The second curve is the reflection of the sine wave in the displayed line $y = x$. Its central branch is $y = \sin^{-1} x$, passing through the origin. The outer lobes depict the other branches of $x = \sin y$, rather than the principal inverse function alone. A principal cosine inverse has value $\pi/2$ at $x = 0$, so options (A) and (D) do not describe this reflected central branch.

Step 3: Conclusion The intended pair is $y = \sin^{-1} x$ and $y = \sin x$, with the inverse understood on its principal branch. Hence, the correct option is (C).

Quick Tip to Solve: Use the origin and the first maximum to identify sine; then reflect across $y = x$ and restrict the inverse to its principal branch.

4. Sum of two skew-symmetric matrices of same order is always a/an:

[1 mark]

- (A) skew-symmetric matrix
- (B) symmetric matrix
- (C) null matrix
- (D) identity matrix

Correct Answer: (A) skew-symmetric matrix

Solution:

Concept:

- A square matrix A is skew-symmetric when its transpose satisfies $A' = -A$.
- Transpose distributes over addition: $(A + B)' = A' + B'$, provided the matrices have the same order.

Step 1: Transpose the sum Let A and B be the two skew-symmetric matrices. Substituting $A' = -A$ and $B' = -B$ gives

$$(A + B)' = A' + B' = -A - B = -(A + B).$$

Step 2: Compare the defining properties The sum satisfies exactly the skew-symmetric identity. It need not be zero: for example, take both matrices equal to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This also gives a sum that is not symmetric. Every skew-symmetric matrix has zero diagonal, so its sum cannot be an identity matrix of positive order. Thus the other properties are not always true.

Step 3: Conclusion The sum is always skew-symmetric. Hence, the correct option is (A).

Quick Tip to Solve: Apply transpose to the whole expression: if it becomes the negative of itself, skew-symmetry is preserved.

5. $\sec^{-1}(-\sqrt{2}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is equal to:

[1 mark]

- (A) $\frac{11\pi}{12}$
- (B) $\frac{5\pi}{12}$
- (C) $-\frac{5\pi}{12}$
- (D) $\frac{7\pi}{12}$

Correct Answer: (D) $\frac{7\pi}{12}$

Solution:

Concept:

- The principal range of $\sec^{-1} x$ is $[0, \pi] \setminus \{\pi/2\}$; a negative input gives an angle in the second quadrant.
- The principal range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$, and $\tan(\pi/6) = 1/\sqrt{3}$.

Step 1: Evaluate the two principal angles Since $\cos(3\pi/4) = -1/\sqrt{2}$, its reciprocal is $-\sqrt{2}$. Therefore

$$\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}, \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Step 2: Subtract with a common denominator The required difference is

$$\frac{3\pi}{4} - \frac{\pi}{6} = \frac{9\pi - 2\pi}{12}.$$

Option (A) comes from adding the angles instead of subtracting. Options (B) and (C) do not follow from the principal angles; a negative acute angle is not the principal inverse secant of a negative input.

Step 3: Conclusion The value is $\frac{7\pi}{12}$. Hence, the correct option is (D).

Quick Tip to Solve: For a negative inverse-secant input, first select the second-quadrant principal angle before doing the subtraction.

6. If $f(x) = \begin{cases} \frac{\log(1+ax) + \log(1-bx)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is: [1 mark]

- (A) a
 (B) $a + b$
 (C) $a - b$
 (D) b

Correct Answer: (C) $a - b$

Solution:

Concept:

- Continuity at zero requires $f(0) = \lim_{x \rightarrow 0} f(x)$, so the assigned value k must equal the limiting value.
- For natural logarithms, $\lim_{x \rightarrow 0} \log(1+cx)/x = c$, for any fixed real constant c .

Step 1: Separate the two logarithmic limits Both logarithm arguments approach 1, so they are positive in a sufficiently small neighbourhood of zero. The standard limit gives

$$\lim_{x \rightarrow 0} \frac{\log(1+ax)}{x} = a, \quad \lim_{x \rightarrow 0} \frac{\log(1-bx)}{x} = -b.$$

These formulae also hold if either constant is zero.

Step 2: Impose continuity at zero Adding the limits yields $a + (-b)$, which must equal $f(0) = k$. Option (B) misses the negative coefficient in $1 - bx$. Options (A) and (D) omit a logarithm's contribution and therefore do not give the general limiting value.

Step 3: Conclusion Continuity requires $k = a - b$. Hence, the correct option is (C).

Quick Tip to Solve: Near zero, replace $\log(1+cx)$ by its first-order term cx , keeping the sign of c .

7. If $\tan^{-1}(x^2 - y^2) = a$, where a is a constant, then $\frac{dy}{dx}$ is: [1 mark]

- (A) $\frac{x}{y}$
 (B) $-\frac{x}{y}$
 (C) $\frac{a}{x}$
 (D) $\frac{a}{y}$

Correct Answer: (A) $\frac{x}{y}$

Solution:

Concept:

- Implicit differentiation treats y as a function of x , so $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.

Quick Tip to Solve: Alternatively take tangent first: $x^2 - y^2 = \tan a$ is constant, so its derivative vanishes immediately.

- The chain rule gives $\frac{d}{dx}(\tan^{-1} u) = \frac{u'}{1+u^2}$; the derivative of the constant a is zero.

Step 1: Differentiate both sides Apply the inverse-tangent derivative with $u = x^2 - y^2$:

$$\frac{2x - 2y \frac{dy}{dx}}{1 + (x^2 - y^2)^2} = 0.$$

The denominator is strictly positive for real x, y , so the numerator must vanish.

Step 2: Isolate the derivative Rearranging gives $2y \, dy/dx = 2x$; divide by $2y$ at points where $y \neq 0$. The positive sign follows because the negative y -term moves to the other side. Option (B) keeps that sign incorrectly. The constant a differentiates to zero, so neither a/x nor a/y follows from this equation.

Step 3: Conclusion Where $y \neq 0$, $\frac{dy}{dx} = \frac{x}{y}$. Hence, the correct option is (A).

Quick Tip to Solve: Alternatively take tangent first: $x^2 - y^2 = \tan a$ is constant, so its derivative vanishes immediately.

8. If $y = a \cos(\log x) + b \sin(\log x)$, then $x^2 y_2 + x y_1$ is:

[1 mark]

- (A) $\cot(\log x)$
 (B) y
 (C) $-y$
 (D) $\tan(\log x)$

Correct Answer: (C) $-y$

Solution:

Concept:

- For $x > 0$, the chain rule uses $\frac{d}{dx} \log x = 1/x$. Write $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$.
- The product rule gives $\frac{d}{dx}(xy_1) = y_1 + xy_2$, matching the expression in the question after multiplication by x .

Step 1: Differentiate once and multiply by x Differentiating the stated expression gives

$$y_1 = \frac{-a \sin(\log x) + b \cos(\log x)}{x}, \quad xy_1 = -a \sin(\log x) + b \cos(\log x).$$

Step 2: Differentiate the product Differentiating the last identity and multiplying by x gives

$$\begin{aligned} y_1 + xy_2 &= \frac{-a \cos(\log x) - b \sin(\log x)}{x}, \\ xy_1 + x^2 y_2 &= -a \cos(\log x) - b \sin(\log x). \end{aligned}$$

The right side is the negative of the original function, so option (B) has the wrong sign. Neither

Quick Tip to Solve: Differentiate xy_1 as a product to produce $xy_2 + y_1$, avoiding a longer quotient-rule calculation.

cotangent nor tangent retains the arbitrary linear coefficients a, b , ruling out (A) and (D).

Step 3: Conclusion $x^2y_2 + xy_1 = -y$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate xy_1 as a product to produce $xy_2 + y_1$, avoiding a longer quotient-rule calculation.

9. Let $f(x) = |x|$, $x \in \mathbb{R}$. Then, which of the following statements is incorrect?

[1 mark]

(A) f has a minimum value at $x = 0$.

(B) f has no maximum value in $\mathbb{R}$.

(C) f is continuous at $x = 0$.

(D) f is differentiable at $x = 0$.

Correct Answer: (D) f is differentiable at $x = 0$.

Solution:

Concept:

- The modulus function satisfies $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$.
- Differentiability at a point requires equal finite left-hand and right-hand derivatives; continuity alone does not ensure this.

Step 1: Check the extremum and continuity statements Since $|x| \geq 0$ with equality at $x = 0$, statement (A) is true. The values $|x|$ increase without bound as $x \rightarrow \infty$, so (B) is true. Also, $\lim_{x \rightarrow 0} |x| = 0 = f(0)$, making (C) true.

Step 2: Compare the derivatives at the origin Use the difference quotient at zero:

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1, \quad f'_+(0) = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1.$$

The unequal one-sided derivatives show that f is not differentiable at zero. Hence statement (D), which asserts differentiability, is incorrect.

Step 3: Conclusion The incorrect statement is that f is differentiable at $x = 0$. Hence, the correct option is (D).

Quick Tip to Solve: For a modulus function, test the corner by comparing one-sided slopes; remember that this question asks for the incorrect statement.

10. Let $f'(x) = 3(x^2 + 2x) - \frac{4}{x^3} + 5$, $f(1) = 0$. Then, $f(x)$ is:

[1 mark]

- (A) $x^3 + 3x^2 + \frac{2}{x^2} + 5x + 11$
(B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$
(C) $x^3 + 3x^2 - \frac{2}{x^2} + 5x - 11$
(D) $x^3 - 3x^2 - \frac{2}{x^2} + 5x - 11$

Correct Answer: (B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$

Solution:

Concept:

- An antiderivative of x^n is $x^{n+1}/(n+1)$ for $n \neq -1$.
- The condition $f(1) = 0$ determines the additive constant in the antiderivative on the interval containing 1.

Step 1: Integrate each term Expand the derivative as $3x^2 + 6x - 4x^{-3} + 5$, then integrate:

$$f(x) = x^3 + 3x^2 + 2x^{-2} + 5x + C.$$

The inverse-square term is positive because $-4/(-2) = 2$.

Step 2: Use the given function value and check the options Substitute $x = 1$:

$$0 = 1 + 3 + 2 + 5 + C = 11 + C, \quad C = -11.$$

Option (A) uses +11, so it fails $f(1) = 0$. Option (C) has the wrong sign on $2/x^2$, giving the wrong inverse-cube derivative. Option (D) additionally gives $-6x$ on differentiating its quadratic term.

Step 3: Conclusion $f(x) = x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$. Hence, the correct option is (B).

Quick Tip to Solve: After integrating a negative power, differentiate that term once to check its sign, then use the given value to fix the constant.

11. $\int \frac{x+5}{(x+6)^2} e^x dx$ is equal to:

[1 mark]

(A) $\log(x+6) + C$

(B) $e^x + C$

(C) $\frac{e^x}{x+6} + C$

(D) $-\frac{1}{(x+6)^2} + C$

Correct Answer: (C) $\frac{e^x}{x+6} + C$

Solution:

Concept:

- The product rule gives $\frac{d}{dx}(e^x g) = e^x(g + g')$.
- For $g(x) = 1/(x+6)$, we have $g'(x) = -1/(x+6)^2$, on intervals excluding $x = -6$.

Step 1: Recognise a product derivative Rewrite the rational factor as

$$\frac{x+5}{(x+6)^2} = \frac{1}{x+6} - \frac{1}{(x+6)^2}.$$

Thus the integrand is exactly $e^x(g + g')$ with $g = 1/(x+6)$.

Step 2: Verify the candidate and exclude the alternatives Direct differentiation gives

$$\frac{d}{dx} \left(\frac{e^x}{x+6} \right) = \frac{e^x(x+6) - e^x}{(x+6)^2} = \frac{e^x(x+5)}{(x+6)^2}.$$

Options (A), (B) and (D) differentiate to $1/(x+6)$, e^x and $2/(x+6)^3$, respectively; none matches the full integrand.

Step 3: Conclusion The integral is $\frac{e^x}{x+6} + C$. Hence, the correct option is (C).

Quick Tip to Solve: When an integrand contains e^x times a rational function, look for the pattern $e^x(g + g')$.

12. The order and degree of the following differential equation are, respectively:

$$-\frac{d^4y}{dx^4} + 2e^{dy/dx} + y^2 = 0$$

[1 mark]

- (A) -4, 1
- (B) 4, not defined
- (C) 1, 1
- (D) 4, 1

Correct Answer: (B) 4, not defined

Solution:

Concept:

- The order is the order of the highest derivative present, irrespective of its sign or coefficient.
- Degree is defined only when the differential equation is polynomial in its derivatives after removing permitted radicals and fractions involving derivatives.

Step 1: Identify the highest derivative The equation contains $-d^4y/dx^4$, the first derivative inside $e^{dy/dx}$, and y^2 . Its highest derivative is of order four. The minus sign multiplying that derivative does not make the order negative, and the term y^2 is not a derivative.

Step 2: Test whether a degree can be assigned The exponential $e^{dy/dx}$ is a non-polynomial function of a derivative. Consequently the equation is not polynomial in its derivatives, and its degree is not defined. Taking a logarithm would only introduce another non-polynomial dependence. Option (D) incorrectly reads the power of the highest derivative without checking this condition; (A) and (C) also give wrong orders.

Step 3: Conclusion The order is 4, and the degree is not defined. Hence, the correct option is (B).

Quick Tip to Solve: Before reading off a degree, check every derivative term: a derivative in an exponential makes the degree undefined.

13. The solution for the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ is:

[1 mark]

- (A) $3e^{4y} + 4e^{-3x} + C = 0$
- (B) $e^{3x+4y} + C = 0$
- (C) $3e^{-3y} + 4e^{4x} + 12C = 0$
- (D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Correct Answer: (D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Solution:

Concept:

- Exponentiating $\log(dy/dx) = 3x + 4y$ gives $dy/dx = e^{3x+4y} > 0$.
- A separable differential equation can be integrated after placing all factors involving y with dy , and all factors involving x with dx .

Quick Tip to Solve: In separable exponential equations, track the minus sign from integrating e^{-4y} ; check a proposed answer by implicit differentiation.

Step 1: Separate and integrate Write $e^{-4y} dy = e^{3x} dx$. Integrating gives

$$-\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + C.$$

Multiplication by 12 and rearrangement gives $3e^{-4y} + 4e^{3x} + 12C = 0$, where C is arbitrary.

Step 2: Check the exponent signs in each option Differentiating option (D) gives $-12e^{-4y}y' + 12e^{3x} = 0$, so $y' = e^{3x+4y}$, as required. Option (A) instead gives $y' = e^{-3x-4y}$. Option (B) gives $y' = -3/4$, incompatible with a real logarithm of y' . Option (C) gives $y' = (16/9)e^{4x+3y}$.

Step 3: Conclusion The solution is $3e^{-4y} + 4e^{3x} + 12C = 0$. Hence, the correct option is (D).

Quick Tip to Solve: In separable exponential equations, track the minus sign from integrating e^{-4y} ; check a proposed answer by implicit differentiation.

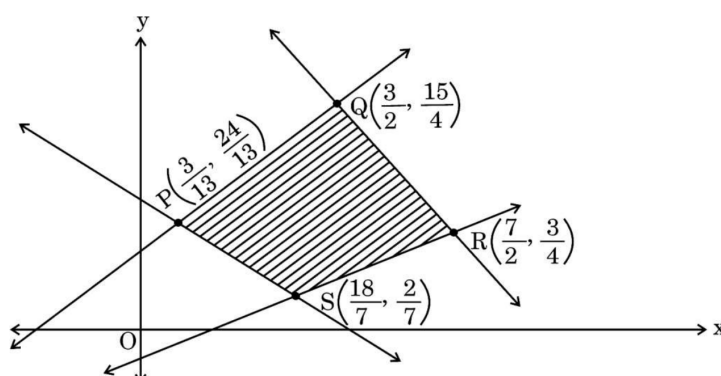
14. For a Linear Programming Problem (LPP), the given objective function is $Z = x + 2y$. The feasible region $PQRS$ determined by the set of constraints is shown as a shaded region in the graph.

(Note: The figure is not to scale)

$$P = \left(\frac{3}{13}, \frac{24}{13}\right), Q = \left(\frac{3}{2}, \frac{15}{4}\right), R = \left(\frac{7}{2}, \frac{3}{4}\right), S = \left(\frac{18}{7}, \frac{2}{7}\right)$$

Which of the following statements is correct?

[1 mark]



- (A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$
- (B) Z is maximum at $R\left(\frac{7}{2}, \frac{3}{4}\right)$
- (C) (Value of Z at P) > (Value of Z at Q)
- (D) (Value of Z at Q) < (Value of Z at R)

Correct Answer: (A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$

Solution:

Concept:

- A linear objective function on a bounded polygon attains its extrema at vertices.
- Evaluate $Z = x + 2y$ using the labelled coordinates, since the graph is not to scale.

Quick Tip to Solve: Ignore the apparent scale of the sketch: evaluate $x + 2y$ at each labelled vertex and compare the four exact values.

Step 1: Evaluate the objective at all four vertices Substitute all four vertices into $Z = x + 2y$:

$$Z(P) = \frac{3+48}{13} = \frac{51}{13}, \quad Z(Q) = \frac{3}{2} + \frac{15}{2} = 9,$$

$$Z(R) = \frac{7}{2} + \frac{3}{2} = 5, \quad Z(S) = \frac{18+4}{7} = \frac{22}{7}.$$

Step 2: Compare all four statements The values satisfy $22/7 < 51/13 < 5 < 9$. The minimum is at S and the maximum at Q , ruling out (B). Also, $Z(P) < Z(Q)$ and $Z(Q) > Z(R)$, so (C) and (D) reverse the correct inequalities.

Step 3: Conclusion The minimum value is $22/7$, attained at $S(18/7, 2/7)$. Hence, the correct option is (A).

Quick Tip to Solve: Ignore the apparent scale of the sketch: evaluate $x + 2y$ at each labelled vertex and compare the four exact values.

15. In a Linear Programming Problem (LPP), the objective function $Z = 2x + 5y$ is to be maximised under the following constraints:

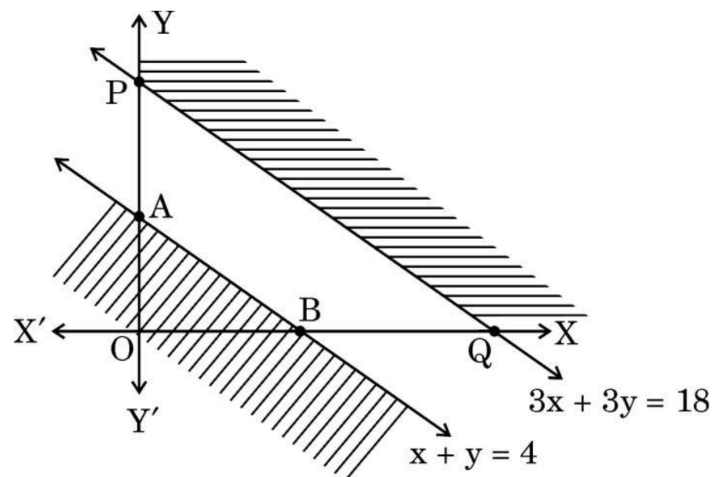
$$x + y \leq 4, \quad 3x + 3y \geq 18, \quad x, y \geq 0$$

Study the graph and select the correct option.

(Note: The figure is not to scale)

The solution of the given LPP:

[1 mark]



- (A) lies in the shaded unbounded region.
- (B) lies in $\triangle AOB$.
- (C) does not exist.
- (D) lies in the combined region of $\triangle AOB$ and unbounded shaded region.

Correct Answer: (C) does not exist.

Solution:**Concept:**

- The feasible region is the intersection of all constraint regions.
- Inconsistent constraints give no feasible solution, regardless of the objective function.

Step 1: Reduce the second constraint Dividing $3x + 3y \geq 18$ by the positive number 3 gives $x + y \geq 6$. But the first constraint is $x + y \leq 4$. Thus a feasible point would have to satisfy

$$6 \leq x + y \leq 4,$$

which is impossible.

Step 2: Interpret the graph and reject the proposed regions The triangle AOB satisfies $x + y \leq 4$; the unbounded shaded region satisfies $x + y \geq 6$. They do not overlap, so neither (A) nor (B) satisfies both constraints. Option (D) takes their union, but feasibility requires their intersection.

Step 3: Conclusion The empty feasible region gives no solution to this LPP. Hence, the correct option is (C).

Quick Tip to Solve: Before evaluating an objective function, simplify proportional constraint rows; contradictory bounds on $x + y$ immediately show infeasibility.

16. Let $|\vec{a}| = 5$ and $-2 \leq \lambda \leq 1$. Then, the range of $|\lambda\vec{a}|$ is:

[1 mark]

- (A) $[5, 10]$
 (B) $[-2, 5]$
 (C) $[-2, 1]$
 (D) $[-10, 5]$

Correct Answer: The correct range is $[0, 10]$; none of the printed options equals this interval.

Solution:**Concept:**

- The magnitude of a scalar multiple obeys $|\lambda\vec{a}| = |\lambda| |\vec{a}|$, and is always non-negative.
- When the interval of λ contains zero, the minimum possible magnitude is zero; the maximum uses the largest absolute endpoint.

Step 1: Find the range of the absolute scalar For $-2 \leq \lambda \leq 1$, zero is allowed and the greatest absolute value is 2, at $\lambda = -2$. Every value of $|\lambda|$ between 0 and 2 is attained, so

$$0 \leq |\lambda| \leq 2, \quad 0 \leq |\lambda\vec{a}| = 5|\lambda| \leq 10.$$

Step 2: Check each printed option The exact range is $[0, 10]$. Option (A), $[5, 10]$, omits the value zero and other attainable values below five. Options (B), (C) and (D) all include negative values, which a vector magnitude cannot take; they also do not give the required upper endpoint. Thus none of the four printed options equals the range.

Quick Tip to Solve: Apply the absolute value to the scalar before multiplying by the vector magnitude; check whether the allowed scalar interval includes zero.

Step 3: Conclusion The range is $[0, 10]$. None of the printed options is correct.

Quick Tip to Solve: Apply the absolute value to the scalar before multiplying by the vector magnitude; check whether the allowed scalar interval includes zero.

17. The area of the region bounded by the curve $y^2 = x$ between $x = 0$ and $x = 1$ is: [1 mark]

- (A) $\frac{3}{2}$ sq units
(B) $\frac{2}{3}$ sq units
(C) 3 sq units
(D) $\frac{4}{3}$ sq units

Correct Answer: (D) $\frac{4}{3}$ sq units

Solution:

Concept:

- For vertical strips, the area between two curves is $A = \int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$.
- The parabola $y^2 = x$ has the two branches $y = \sqrt{x}$ and $y = -\sqrt{x}$, symmetric about the x -axis.

Step 1: Identify the full strip height Between $x = 0$ and $x = 1$, both branches bound the region. Thus a vertical strip has height

$$\sqrt{x} - (-\sqrt{x}) = 2\sqrt{x}.$$

Taking just the upper branch would count only half of the required region.

Step 2: Integrate and compare the choices The required integral is

$$A = 2 \int_0^1 x^{1/2} dx = 2 \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{4}{3}.$$

Option (B), $2/3$, is the upper half only. Options (A), $3/2$, and (C), 3, do not equal the integral over both branches.

Step 3: Conclusion The area is $\boxed{\frac{4}{3} \text{ sq units}}$. Hence, the correct option is (D).

Quick Tip to Solve: For $y^2 = x$, include the negative branch: double the area under $y = \sqrt{x}$ on the given interval.

18. A box has 4 green, 8 blue and 3 red pens. A student picks up a pen at random, checks its colour and replaces it in the box. He repeats this process 3 times. The probability that at least one pen picked was red is: [1 mark]

- (A) $\frac{124}{125}$
(B) $\frac{1}{125}$
(C) $\frac{61}{125}$
(D) $\frac{64}{125}$

Correct Answer: (C) $\frac{61}{125}$

Solution:

Concept:

- Replacement keeps the colour probabilities unchanged and makes the three random picks independent.
- The complement of at least one red pen is no red pen: $P(\text{at least one red}) = 1 - P(\text{no red})$.

Step 1: Find the probability of a non-red pick There are $4 + 8 + 3 = 15$ pens. Of these, $4 + 8 = 12$ are not red. Hence

$$P(\text{not red}) = \frac{12}{15} = \frac{4}{5}.$$

Because the chosen pen is replaced, this probability is the same at each pick.

Step 2: Use the complement over three picks Independence gives

$$P(\text{no red in three picks}) = \left(\frac{4}{5}\right)^3 = \frac{64}{125}.$$

Therefore the desired probability is $1 - 64/125 = 61/125$. Option (D) gives the complement; option (B) is the probability of all three pens being red. Option (A) is the probability of not having all three red, a different event.

Step 3: Conclusion The required probability is $\frac{61}{125}$. Hence, the correct option is (C).

Quick Tip to Solve: With replacement, convert “at least one” to “one minus none” before multiplying trial probabilities.

19. Assertion (A): If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 256$ and $|\vec{b}| = 8$, then $|\vec{a}| = 2$.

Reason (R): $\sin^2 \theta + \cos^2 \theta = 1$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- For vectors with angle θ between them, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.
- Squaring and adding gives $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$, since $\sin^2 \theta + \cos^2 \theta = 1$.

Step 1: Test the assertion Applying the squared-magnitude identity to the given values gives

$$256 = |\vec{a}|^2 (8)^2, \quad |\vec{a}|^2 = 4.$$

A magnitude is nonnegative, so $|\vec{a}| = 2$. The assertion is true.

Step 2: Test the reason The stated sine and cosine identity is true. The stated cross-product magnitude and dot-product formulae are also the standard formulae for two vectors. Thus the reason is true.

Step 3: Check the explanatory connection Squaring the two product formulae in the reason, adding them, and using its trigonometric identity gives exactly the identity used to establish the assertion. The reason therefore explains the assertion, excluding (B); neither statement is false, excluding (C) and (D).

Step 4: Conclusion Both statements are true, and the reason correctly explains the assertion. Hence, the correct option is (A).

Quick Tip to Solve: Whenever squared dot and cross products are added, replace their sum directly by $|\vec{a}|^2 |\vec{b}|^2$.

20. Assertion (A): Let $f(x) = e^x$ and $g(x) = \log x$. Then $(f + g)(x) = e^x + \log x$, where domain of $(f + g)$ is $\mathbb{R}$.

Reason (R): $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g)$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A sum of two real functions is defined only where both summands are real and defined: $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g)$.
- The exponential e^x is defined for all real x , whereas the real logarithm $\log x$ requires $x > 0$.

Step 1: Test the assertion The algebraic expression $(f + g)(x) = e^x + \log x$ is correct on its domain. However,

$$\text{Dom}(f + g) = \mathbb{R} \cap (0, \infty) = (0, \infty).$$

At $x = 0$, for example, $\log x$ is undefined. Thus the assertion's claim that the domain is all of $\mathbb{R}$ makes the assertion false.

Step 2: Test the reason The stated intersection rule is true: forming a sum at an input requires a value from each of the two functions at that same input.

Step 3: Relate the reason to the assertion Applying the true reason gives a positive-real domain and exposes the error in the assertion. Options (A), (B) and (C) all require a true assertion, so none applies here.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: In any real expression containing $\log x$, impose $x > 0$ before considering the other terms' domains.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Find the domain of $f(x) = \sin^{-1}(-x^2)$.

[2 marks]

Solution:

Concept:

- The real inverse sine function $\sin^{-1} u$ is defined exactly for $-1 \leq u \leq 1$.
- For a composite function, impose the outer function's domain restriction on its entire argument. Every real square satisfies $x^2 \geq 0$.

Quick Tip to Solve: Apply $-1 \leq \text{argument} \leq 1$ to inverse sine; here the square reduces the check to $x^2 \leq 1$.

Step 1: Restrict the inverse-sine argument Here the input to inverse sine is $u = -x^2$. Hence the necessary and sufficient condition for a real value is

$$-1 \leq -x^2 \leq 1.$$

Multiplying the inequalities by -1 reverses their directions. Reordering them in increasing order gives

$$-1 \leq x^2 \leq 1.$$

Step 2: Solve the inequality and check the endpoints Since $x^2 \geq 0$ for every real x , the lower bound is automatic. The remaining restriction is

$$x^2 \leq 1 \iff |x| \leq 1 \iff -1 \leq x \leq 1.$$

At either endpoint $x = \pm 1$, the argument is -1 , and $\sin^{-1}(-1) = -\pi/2$ exists. Both endpoints are therefore included. For $|x| > 1$, the argument is less than -1 and has no real inverse sine.

Step 3: Conclusion The domain of the function is $[-1, 1]$.

Quick Tip to Solve: Apply $-1 \leq \text{argument} \leq 1$ to inverse sine; here the square reduces the check to $x^2 \leq 1$.

22(a). Differentiate $\sqrt{e^{\sqrt{2x}}}$ with respect to $e^{\sqrt{2x}}$, for $x > 0$.

[2 marks]

Solution:

Concept:

- To differentiate a function y with respect to another function u , express y in terms of u , then compute dy/du .
- The power rule gives $\frac{d}{du} u^{1/2} = \frac{1}{2} u^{-1/2}$ for $u > 0$. Equivalently, $\frac{dy}{du} = \frac{dy/dx}{du/dx}$ when $du/dx \neq 0$.

Step 1: Introduce the function used as the variable Put $u = e^{\sqrt{2x}}$ and $y = \sqrt{e^{\sqrt{2x}}}$. Since the exponential is positive, its square root is well defined. The two functions are related simply by

$$y = \sqrt{u} = u^{1/2}.$$

The requested derivative is with respect to u , not with respect to x .

Step 2: Apply the power rule and substitute back Differentiating the relation above gives

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{e^{\sqrt{2x}}}}.$$

This change of variable is valid for $x > 0$: the inner square root is differentiable and

$$\frac{du}{dx} = \frac{e^{\sqrt{2x}}}{\sqrt{2x}} > 0.$$

Quick Tip to Solve: When asked to differentiate $\sqrt{u(x)}$ with respect to $u(x)$, use $1/(2\sqrt{u})$ directly.

Thus the ratio-of-derivatives method would cancel the common inner derivative and give the same expression.

Step 3: Conclusion The required derivative is $\boxed{\frac{1}{2e^{\sqrt{2x}/2}}}$, for $x > 0$.

Quick Tip to Solve: When asked to differentiate $\sqrt{u(x)}$ with respect to $u(x)$, use $1/(2\sqrt{u})$ directly.

OR

22(b). If $(x)^y = (y)^x$, then find $\frac{dy}{dx}$.

[2 marks]

Solution:

Concept:

- Logarithmic differentiation converts a variable exponent into a product: $\ln(a^b) = b \ln a$, for $a > 0$.
- Treat y as a function of x , so $\frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}$; apply the product rule to both sides.

Step 1: Take logarithms of the relation On the positive real domain, take natural logarithms of $x^y = y^x$. This removes the variable exponents without treating either exponent as a constant:

$$y \ln x = x \ln y.$$

Step 2: Differentiate and collect the derivative terms Write $y' = \frac{dy}{dx}$. Applying the product rule on each side gives

$$\begin{aligned} y' \ln x + \frac{y}{x} &= \ln y + \frac{x}{y} y', \\ y' \left(\ln x - \frac{x}{y} \right) &= \ln y - \frac{y}{x}. \end{aligned}$$

Divide by the coefficient of y' wherever it is nonzero; multiplying numerator and denominator by xy gives an equivalent fraction without nested quotients.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{\ln y - y/x}{\ln x - x/y} = \frac{y(x \ln y - y)}{x(y \ln x - x)}}.$$

The formula applies where its denominator is nonzero on a differentiable positive branch.

Quick Tip to Solve: When both the base and exponent depend on the variables, take logarithms first and collect all terms containing dy/dx on one side.

23. Determine the values of x for which $f(x) = \frac{x-4}{x+1}$, $x \neq -1$ is an increasing or a decreasing function. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where $f'(x) > 0$, and strictly decreasing where $f'(x) < 0$.
- For $f = u/v$, the quotient rule gives $f' = (vu' - uv')/v^2$. A point excluded from the domain separates the intervals to be tested.

Step 1: Differentiate using the quotient rule Here $u = x - 4$, $v = x + 1$, and both u' and v' equal 1. Therefore,

$$\begin{aligned} f'(x) &= \frac{(x+1) - (x-4)}{(x+1)^2} \\ &= \frac{5}{(x+1)^2}, \quad x \neq -1. \end{aligned}$$

Step 2: Determine the derivative's sign on the domain intervals The numerator is positive, and $(x+1)^2 > 0$ for every allowed value of x . Thus $f'(x) > 0$ on each of $(-\infty, -1)$ and $(-1, \infty)$. There is no interval in the domain where the derivative is negative. Since $x = -1$ is excluded, it cannot be included in either interval or used to join them into one interval of increase.

Step 3: Conclusion The function is strictly increasing separately on $(-\infty, -1)$ and $(-1, \infty)$. It has no decreasing interval.

Quick Tip to Solve: For this quotient, the derivative numerator simplifies to the constant 5; inspect the squared denominator and split the domain at $x = -1$.

24(a). If $\vec{a}$ and $\vec{b}$ are position vectors of point A and point B respectively, find the position vector of point C on BA produced such that $BC = 3BA$. [2 marks]

Solution:

Concept:

- If $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$, then the displacement from B to A is $\vec{BA} = \vec{a} - \vec{b}$.
- Vectors along the same directed line are positive scalar multiples. The position vector of an endpoint equals the starting position vector plus the displacement.

Step 1: Express the directed displacement from B to C Since C lies on BA produced, the order of the points is B, A, C . Thus $\vec{BC}$ points in the same direction as $\vec{BA}$. Using the given length relation,

$$\vec{BC} = 3\vec{BA} = 3(\vec{a} - \vec{b}).$$

The positive factor 3 is essential: the extension is beyond A , starting from B .

Step 2: Add the displacement to the starting position vector Travel from the origin O to B , then from

Quick Tip to Solve: Start at B : write its position vector plus three times the displacement from B to A . This avoids reversing the extension direction.

B to C . The triangle law gives

$$\begin{aligned}\vec{OC} &= \vec{OB} + \vec{BC} \\ &= \vec{b} + 3(\vec{a} - \vec{b}) \\ &= 3\vec{a} - 2\vec{b}.\end{aligned}$$

Step 3: Conclusion The position vector of C is $\boxed{3\vec{a} - 2\vec{b}}$.

Quick Tip to Solve: Start at B : write its position vector plus three times the displacement from B to A . This avoids reversing the extension direction.

OR

24(b). Vector $\vec{r}$ is inclined at equal angles to the three axes x , y and z . If magnitude of $\vec{r}$ is $5\sqrt{3}$ units, then find $\vec{r}$. [2 marks]

Solution:

Concept:

- The direction cosines l, m, n of a nonzero vector are the cosines of its angles with the positive coordinate axes and obey $l^2 + m^2 + n^2 = 1$.
- A vector with magnitude r and direction cosines l, m, n is $\vec{r} = r(l\hat{i} + m\hat{j} + n\hat{k})$.

Step 1: Use the equality of the three angles Let the common angle with the three axes be θ . Equal angles have equal cosines, so $l = m = n = \cos \theta$. The direction-cosine identity becomes

$$3 \cos^2 \theta = 1, \quad \cos \theta = \pm \frac{1}{\sqrt{3}}.$$

All three direction cosines must have the same sign; their signs cannot be chosen independently because the angles are equal.

Step 2: Multiply the direction cosines by the magnitude Using $|\vec{r}| = 5\sqrt{3}$, the three components are equal to

$$5\sqrt{3} \left(\pm \frac{1}{\sqrt{3}} \right) = \pm 5.$$

Hence the possible component triples are $(5, 5, 5)$ and $(-5, -5, -5)$. Both have magnitude $\sqrt{25 + 25 + 25} = 5\sqrt{3}$, and each makes three equal angles with the positive axes.

Step 3: Conclusion $\vec{r} = \pm 5(\hat{i} + \hat{j} + \hat{k})$, with one common sign. The positive choice applies when the common angle is acute; the question does not impose that restriction.

Quick Tip to Solve: Equal angles force equal direction cosines, including their signs. Use $3l^2 = 1$, then multiply by $5\sqrt{3}$ to obtain the components.

25. Determine if the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect with each other. [2 marks]

Solution:

Concept:

- Two lines intersect if one value of each line's parameter produces the same position vector, so all three corresponding coordinates must agree simultaneously.
- In $\vec{r} = \vec{a} + t\vec{b}$, $\vec{a}$ gives a point on the line and $\vec{b}$ gives its direction. Parameters for different lines are independent.

Step 1: Write and equate the coordinate forms A general point on the first line is $(1 + 3\lambda, 1 - \lambda, -1)$, and one on the second is $(4 + 2\mu, 0, -1 + 3\mu)$. At an intersection,

$$\begin{aligned}1 + 3\lambda &= 4 + 2\mu, \\1 - \lambda &= 0, \\-1 &= -1 + 3\mu.\end{aligned}$$

Step 2: Solve for the parameters and verify the remaining coordinate The second equation gives $\lambda = 1$, while the third gives $\mu = 0$. Substituting these into the first equation gives $1 + 3(1) = 4 + 2(0) = 4$, so it too is satisfied. The same pair of parameter values therefore makes all three coordinates equal. Substitution into either line yields the position vector $4\hat{i} - \hat{k}$.

Step 3: Conclusion Yes, the lines intersect at $(4, 0, -1)$, corresponding to $\lambda = 1$ and $\mu = 0$.

Quick Tip to Solve: Use the y - and z -coordinates first: each immediately fixes one parameter. Then check the x -coordinate to confirm a true intersection.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. Let $A = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4 & 2 \\ 12 & 16 & 8 \\ -6 & -8 & -4 \end{pmatrix}$ be two matrices. Then, find the matrix B if $AB = C$.

[3 marks]

Solution:

Concept:

- If an $m \times n$ matrix is multiplied by an $n \times p$ matrix, the product has order $m \times p$. Each entry is the dot product of a row of the first matrix and a column of the second.
- Two matrices are equal when they have the same order and all corresponding entries are equal. Multiplying a column by a row produces rows that are scalar multiples of that row.

Step 1: Determine the order of the unknown matrix Here A has order 3×1 , while C has order 3×3 . For AB to be defined, B must have one row; to give three columns in the product, it must have three

Quick Tip to Solve: When the first entry of a column matrix is 1, the first row of its product with a row matrix is exactly that row matrix; verify the other rows by scaling.

columns. Therefore, write

$$B = \begin{pmatrix} a & b & c \end{pmatrix}.$$

This uses the order of the product and does not require an inverse of the non-square matrix A .

Step 2: Multiply and compare corresponding entries The product of the given column and the unknown row is

$$AB = \begin{pmatrix} a & b & c \\ 4a & 4b & 4c \\ -2a & -2b & -2c \end{pmatrix} = \begin{pmatrix} 3 & 4 & 2 \\ 12 & 16 & 8 \\ -6 & -8 & -4 \end{pmatrix}.$$

Comparing the first rows gives $a = 3$, $b = 4$ and $c = 2$. These values also satisfy every remaining entry: four times $(3, 4, 2)$ is $(12, 16, 8)$, and negative twice $(3, 4, 2)$ is $(-6, -8, -4)$. Thus the first-row values are consistent with the whole matrix equation and determine the unknown row uniquely.

Step 3: Conclusion The required matrix is $B = \begin{pmatrix} 3 & 4 & 2 \end{pmatrix}$, of order 1×3 .

Quick Tip to Solve: When the first entry of a column matrix is 1, the first row of its product with a row matrix is exactly that row matrix; verify the other rows by scaling.

27(a). Differentiate $y = \sin^{-1}(3x - 4x^3)$ **w.r.t. x , if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.**

[3 marks]

Solution:

Concept:

- The triple-angle identity is $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$. It can simplify an inverse trigonometric expression before differentiation.
- The principal range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$; consequently $\sin^{-1}(\sin t) = t$ only when t lies in this range.
- For $-1 < x < 1$, $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

Step 1: Choose a substitution and check its range Put $x = \sin \theta$, with $\theta = \sin^{-1} x$. Since the given interval is $-\frac{1}{2} \leq x \leq \frac{1}{2}$, monotonicity of the principal inverse sine gives

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}, \quad -\frac{\pi}{2} \leq 3\theta \leq \frac{\pi}{2}.$$

The second interval is precisely what is needed to cancel the inverse sine and sine without changing the branch.

Step 2: Simplify the function and differentiate Apply the triple-angle identity and then the checked principal-range rule:

$$\begin{aligned} y &= \sin^{-1}(3\sin \theta - 4\sin^3 \theta) \\ &= \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x. \end{aligned}$$

Quick Tip to Solve: Before replacing an inverse sine of a triple-angle sine by the angle, multiply the substitution's entire interval by 3 and check the principal range.

Differentiating this equivalent expression on the interior of the stated interval gives

$$\frac{dy}{dx} = 3 \frac{d}{dx}(\sin^{-1} x) = \frac{3}{\sqrt{1-x^2}}.$$

At either endpoint the corresponding derivative from within the given interval is $3/\sqrt{1-1/4} = 2\sqrt{3}$; ordinary two-sided differentiation on a closed restricted domain is understood only at interior points.

Step 3: Conclusion $\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$ for $-\frac{1}{2} < x < \frac{1}{2}$. The one-sided derivatives at the endpoints of the given interval are both $2\sqrt{3}$.

Quick Tip to Solve: Before replacing an inverse sine of a triple-angle sine by the angle, multiply the substitution's entire interval by 3 and check the principal range.

OR

27(b). Differentiate $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ **with respect to** x , **when** $x \in (0, 1)$.

[3 marks]

Solution:

Concept:

- The double-angle identity in terms of tangent is $\cos 2\theta = (1 - \tan^2 \theta)/(1 + \tan^2 \theta)$.
- The principal range of $\cos^{-1}$ is $[0, \pi]$, so $\cos^{-1}(\cos t) = t$ when $0 \leq t \leq \pi$.
- For a real variable x , $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.

Step 1: Express the argument as a double-angle cosine Take $x = \tan \theta$, where $\theta = \tan^{-1} x$. The assumption $0 < x < 1$ implies $0 < \theta < \pi/4$. Therefore,

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta.$$

In this interval the numerator and denominator are positive, and the fraction lies between 0 and 1, as expected for the cosine of an acute angle.

Step 2: Check the inverse branch and differentiate Since $0 < 2\theta < \pi/2$, the angle 2θ is in the principal range of inverse cosine. Thus

$$y = \cos^{-1}(\cos 2\theta) = 2\theta = 2 \tan^{-1} x.$$

The interval check is essential: inverse cosine of a cosine cannot be cancelled for an arbitrary real angle. Here it permits us to differentiate a simple expression:

$$\frac{dy}{dx} = 2 \frac{d}{dx}(\tan^{-1} x) = \frac{2}{1+x^2}.$$

The derivative is positive, consistent with the argument of inverse cosine decreasing while the inverse cosine function itself is decreasing.

Quick Tip to Solve: Recognise $(1-x^2)/(1+x^2)$ as $\cos(2 \tan^{-1} x)$; use the given positive interval to select the correct inverse-cosine branch before differentiating.

Step 3: Conclusion For $x \in (0, 1)$, the required derivative is $\boxed{\frac{dy}{dx} = \frac{2}{1+x^2}}$.

Quick Tip to Solve: Recognise $(1-x^2)/(1+x^2)$ as $\cos(2 \tan^{-1} x)$; use the given positive interval to select the correct inverse-cosine branch before differentiating.

28(a). A student wants to pair up natural numbers in such a way that they satisfy the equation $2x + y = 41$, $x, y \in \mathbb{N}$. Find the domain and range of the relation. Check if the relation thus formed is reflexive, symmetric and transitive. Hence, state whether it is an equivalence relation or not.

[3 marks]

Solution:

Concept:

- For a relation $R \subseteq \mathbb{N} \times \mathbb{N}$, its domain is the set of first coordinates and its range is the set of second coordinates of its ordered pairs. Here $\mathbb{N} = \{1, 2, 3, \dots\}$.
- Reflexivity requires $(a, a) \in R$ for every natural number a . Symmetry requires $(b, a) \in R$ whenever $(a, b) \in R$.
- Transitivity requires $(a, c) \in R$ whenever both (a, b) and (b, c) belong to R . An equivalence relation must satisfy all three properties.

Step 1: List the possible first and second coordinates Rearrange the condition as $y = 41 - 2x$. Since both variables are positive integers, $x \geq 1$ and $41 - 2x \geq 1$, giving $1 \leq x \leq 20$. Each integer in this interval is allowed. Thus

$$R = \{(1, 39), (2, 37), (3, 35), \dots, (20, 1)\}.$$

The first coordinates run from 1 to 20, whereas the second coordinates are exactly the positive odd integers from 1 to 39. Consequently,

$$\text{Dom}(R) = \{1, 2, \dots, 20\}, \quad \text{Ran}(R) = \{1, 3, 5, \dots, 39\}.$$

Step 2: Test reflexivity and symmetry using counterexamples For a diagonal pair (a, a) to be in R , we would need $3a = 41$, which has no natural-number solution. In particular, $(1, 1) \notin R$, so R is not reflexive. Next, $(1, 39) \in R$ because $2(1) + 39 = 41$, but $(39, 1) \notin R$ because $2(39) + 1 = 79 \neq 41$. Hence the implication required for symmetry fails.

Step 3: Test transitivity with two composable pairs We have $(19, 3) \in R$ and $(3, 35) \in R$, as

$$2(19) + 3 = 41, \quad 2(3) + 35 = 41.$$

Transitivity would then require $(19, 35) \in R$. However, $2(19) + 35 = 73 \neq 41$, so this pair is absent. This supplies a counterexample to transitivity using two pairs whose middle coordinate agrees.

Step 4: Conclusion The domain is $\{1, 2, \dots, 20\}$ and the range is $\{1, 3, 5, \dots, 39\}$. The relation is neither reflexive nor symmetric nor transitive, and therefore is not an equivalence relation.

Quick Tip to Solve: For transitivity, choose an allowed pair whose second coordinate also occurs in the domain; then form a second pair starting with that coordinate and test the composed pair.

28(b). Show that the function $f : \mathbb{N} \rightarrow \mathbb{N}$, where $\mathbb{N}$ is a set of natural numbers, given by $f(n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$ is a bijection. [3 marks]

Solution:

Concept:

- A function is one-one (injective) if equal images imply equal inputs, and onto (surjective) if every element of the codomain has a preimage.
- The given function interchanges each pair of consecutive natural numbers: $1 \leftrightarrow 2, 3 \leftrightarrow 4$, and so on. Applying it twice returns the original number.

Step 1: Check the function and compute its composition Here $\mathbb{N} = \{1, 2, 3, \dots\}$. If n is even, then $n \geq 2$, so $f(n) = n - 1$ is a positive odd integer. If n is odd, then $f(n) = n + 1$ is a positive even integer. Thus every value lies in the stated codomain. Applying the other branch of the definition on the second application gives

$$f(f(n)) = \begin{cases} f(n-1) = (n-1) + 1 = n, & n \text{ even,} \\ f(n+1) = (n+1) - 1 = n, & n \text{ odd.} \end{cases}$$

Hence $f \circ f$ is the identity function on $\mathbb{N}$.

Step 2: Prove that the function is one-one Let $a, b \in \mathbb{N}$ and suppose $f(a) = f(b)$. Apply f to both sides. Using the identity just proved,

$$f(f(a)) = f(f(b)) \implies a = b.$$

Therefore no distinct inputs have the same image, so f is injective.

Step 3: Prove that the function is onto Take any $m \in \mathbb{N}$. Choose $n = f(m)$, which belongs to $\mathbb{N}$ by the first step. Then

$$f(n) = f(f(m)) = m.$$

Explicitly, the preimage is $m + 1$ when m is odd and $m - 1$ when m is even. Both choices are positive natural numbers. Every codomain element is therefore attained, so f is surjective.

Step 4: Conclusion The function f is both one-one and onto; hence it is a bijection. In fact, $f^{-1} = f$.

Quick Tip to Solve: When a piecewise function swaps adjacent odd and even numbers, compute $f(f(n))$. The identity $f \circ f = I$ proves both injectivity and surjectivity efficiently.

29. Consider the Linear Programming Problem, where the objective function $Z = (x + 4y)$ needs to be minimized subject to constraints

$$2x + y \geq 1000$$

$$x + 2y \geq 800$$

$$x, y \geq 0.$$

Draw a neat graph of the feasible region and find the minimum value of Z .

[3 marks]

Solution:

Concept:

- Each linear inequality represents a half-plane. The feasible region is their intersection with the first quadrant $x, y \geq 0$.
- A finite optimum of a linear objective occurs at a corner point of the feasible region. For an unbounded region, also verify that the proposed value is a lower bound throughout the region.

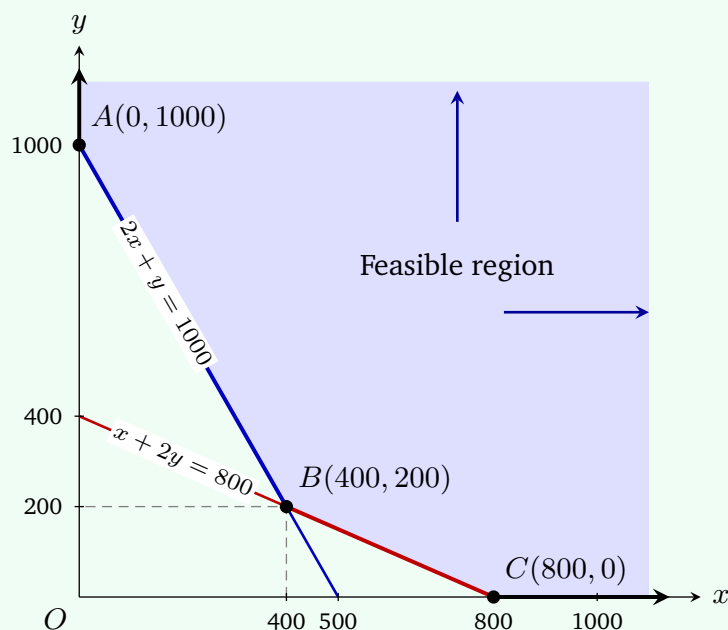
Step 1: Plot the boundary lines and select the half-planes The line $2x + y = 1000$ passes through $(500, 0)$ and $(0, 1000)$. The line $x + 2y = 800$ passes through $(800, 0)$ and $(0, 400)$. Draw both as solid lines because equality is permitted. The origin fails both inequalities, so choose the side away from the origin for each line. Keep only the first quadrant. Their common shaded region extends indefinitely upwards and to the right; the top and right edges of the picture are only the plotting window.

Step 2: Find the corner points of the feasible region Solve the two boundary equations simultaneously:

$$\begin{aligned} 2x + y &= 1000, & x + 2y &= 800, \\ 4x + 2y &= 2000, & 3x &= 1200. \end{aligned}$$

Thus $x = 400$ and $y = 200$. On the y -axis both inequalities require $y \geq 1000$; on the x -axis they require $x \geq 800$. The corner points are therefore $A(0, 1000)$, $B(400, 200)$, and $C(800, 0)$. The intercepts $(500, 0)$ and $(0, 400)$ are not feasible because each violates the other inequality.

Quick Tip to Solve: For this unbounded region, write $x + 4y = (x + 2y) + 2y$. The constraint and $y \geq 0$ immediately give the lower bound 800, which the corner $(800, 0)$ attains.



Feasible region shaded; arrows indicate its unbounded extension.

Step 3: Evaluate the objective and check the unbounded region Evaluate $Z = x + 4y$ at each feasible corner:

Corner point	$Z = x + 4y$
(0, 1000)	4000
(400, 200)	1200
(800, 0)	800

To ensure that the unbounded part gives no smaller value, use the constraints directly:

$$Z = (x + 2y) + 2y \geq 800 + 0 = 800.$$

Equality requires $x + 2y = 800$ and $y = 0$, giving $x = 800$. This point also satisfies $2x + y \geq 1000$, so the lower bound is attained.

Step 4: Conclusion The minimum value is $Z_{\min} = 800$, attained uniquely at $(x, y) = (800, 0)$. The shaded feasible region is unbounded.

Quick Tip to Solve: For this unbounded region, write $x + 4y = (x + 2y) + 2y$. The constraint and $y \geq 0$ immediately give the lower bound 800, which the corner (800, 0) attains.

30(a). Find the distance of the point $P(2, 4, -1)$ from the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$.

[3 marks]

Solution:

Concept:

- A line in symmetric form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ passes through $A(x_0, y_0, z_0)$ and has direction vector $\mathbf{d} = (a, b, c)$.
- The shortest distance from a point to a line is the length of its perpendicular to the line. If $H = A + t\mathbf{d}$ is the foot, then $(\overrightarrow{AP} - t\mathbf{d}) \cdot \mathbf{d} = 0$.

Step 1: Identify a point and the direction of the line Equate each ratio to the same parameter t :

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = t.$$

Therefore a general point on the line is $(-5+t, -3+4t, 6-9t)$. At $t = 0$, the line passes through $A(-5, -3, 6)$, with direction vector $\mathbf{d} = (1, 4, -9)$. For $P(2, 4, -1)$,

$$\overrightarrow{AP} = P - A = (7, 7, -7).$$

The signs of the point coordinates come from setting the numerators equal to zero; the direction ratios come from the denominators.

Step 2: Locate the foot of the perpendicular Let $H = A + t\mathbf{d}$ be the foot of the perpendicular from P . Then $\overrightarrow{HP} = \overrightarrow{AP} - t\mathbf{d}$ must be perpendicular to $\mathbf{d}$, so

$$\begin{aligned} (\overrightarrow{AP} - t\mathbf{d}) \cdot \mathbf{d} &= 0, \\ t &= \frac{\overrightarrow{AP} \cdot \mathbf{d}}{\mathbf{d} \cdot \mathbf{d}} = \frac{7 + 28 + 63}{1 + 16 + 81} = 1. \end{aligned}$$

Substitution into the parametric coordinates gives $H = (-4, 1, -3)$. Indeed, $\overrightarrow{HP} = (6, 3, 2)$ and $\overrightarrow{HP} \cdot \mathbf{d} = 6 + 12 - 18 = 0$, confirming perpendicularity.

Step 3: Calculate the perpendicular length Use the distance formula between P and the foot H :

$$PH = \sqrt{(2+4)^2 + (4-1)^2 + (-1+3)^2} = \sqrt{36 + 9 + 4} = 7.$$

Any other point on the line forms a right triangle with P and H , whose hypotenuse is longer than PH . Thus this is the shortest distance to the line.

Step 4: Conclusion The distance of $P(2, 4, -1)$ from the given line is 7 units.

Quick Tip to Solve: Here $\overrightarrow{AP} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} = 98$, so the perpendicular foot occurs at parameter $t = 1$. This avoids expanding a cross product.

30(b). Let the position vectors of the points A , B and C be $3\hat{i} - \hat{j} - 2\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$ and $\hat{i} + 5\hat{j} + 3\hat{k}$ respectively. Find the vector and cartesian equations of the line passing through A and parallel to line BC . [3 marks]

Solution:

Concept:

- If the position vectors of two points are $\vec{b}$ and $\vec{c}$, the vector $\overrightarrow{BC} = \vec{c} - \vec{b}$ gives a direction of the line joining them.
- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{d}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- Cartesian equations follow by equating components and eliminating the parameter. A zero direction component means that the corresponding coordinate is constant; division by that zero component must be avoided.

Step 1: Calculate a direction vector parallel to BC The given points have coordinates $A(3, -1, -2)$, $B(1, 2, -1)$ and $C(1, 5, 3)$. Subtract the coordinates of B from those of C :

$$\overrightarrow{BC} = (1 - 1)\hat{i} + (5 - 2)\hat{j} + \{3 - (-1)\}\hat{k} = 3\hat{j} + 4\hat{k}.$$

Thus the required line has direction ratios 0, 3, 4. Its direction vector is nonzero even though its first component vanishes.

Step 2: Use point A to obtain the vector form Substitute the position vector of A and the direction vector just found into the point-direction form:

$$\vec{r} = (3\hat{i} - \hat{j} - 2\hat{k}) + \lambda(3\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}.$$

At $\lambda = 0$, the point is A , and the coefficient of λ is exactly $\overrightarrow{BC}$, verifying both requirements.

Step 3: Eliminate the parameter carefully Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ gives

$$x = 3, \quad y = -1 + 3\lambda, \quad z = -2 + 4\lambda.$$

Hence $\lambda = (y + 1)/3 = (z + 2)/4$. The first coordinate remains fixed at 3 for every point of the line, so it is stated as a separate Cartesian equation.

Step 4: Conclusion The required vector equation is

$$\vec{r} = (3\hat{i} - \hat{j} - 2\hat{k}) + \lambda(3\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}.$$

Its Cartesian equations are $x = 3, \quad \frac{y + 1}{3} = \frac{z + 2}{4}.$

Quick Tip to Solve: Since B and C have the same x -coordinate, the parallel line has constant x ; use $x=3$ through A and form ratios only for y and z .

31. A person is Head of two independent selection committees I and II. If the probability of making a wrong selection in committee I is 0.03 and that in committee II is 0.01, then find the probability that the person makes the correct decision of selection:

(i) in both committees

(ii) in only one committee

[3 marks]

Solution:

Concept:

- The probabilities of a correct and a wrong decision in the same committee add to 1. For any event E , $P(E^c) = 1 - P(E)$, where E^c is its complement.
- For independent events E and F , $P(E \cap F) = P(E)P(F)$. Their complementary events are independent as well.
- Exactly one correct decision consists of two mutually exclusive possibilities: correct in I and wrong in II, or wrong in I and correct in II. Add the probabilities of these two possibilities.

Step 1: Define the correct-decision events Let E be the event of a correct selection in committee I and F the event of a correct selection in committee II. The given wrong-selection probabilities are $P(E^c) = 0.03$ and $P(F^c) = 0.01$. Therefore,

$$P(E) = 1 - 0.03 = 0.97, \quad P(F) = 1 - 0.01 = 0.99.$$

The question specifies independent committees, so multiplication is valid when combining decisions from the two committees.

Step 2: Find the probability of correct selection in both committees Part (i) asks for $E \cap F$. By independence,

$$P(E \cap F) = P(E)P(F) = 0.97 \times 0.99 = 0.9603.$$

Both factors here represent correct decisions; the given wrong-decision probabilities first had to be complemented.

Step 3: Count the two ways of being correct in only one committee For part (ii), the required event is $(E \cap F^c) \cup (E^c \cap F)$. These cases cannot occur together. Thus,

$$\begin{aligned} P(\text{exactly one correct}) &= P(E)P(F^c) + P(E^c)P(F) \\ &= 0.97 \times 0.01 + 0.03 \times 0.99 \\ &= 0.0097 + 0.0297 = 0.0394. \end{aligned}$$

As a check, the probability of both wrong is $0.03 \times 0.01 = 0.0003$, and $0.9603 + 0.0394 + 0.0003 = 1$. These three cases account for all outcomes.

Step 4: Conclusion The probability of a correct selection is (i) 0.9603 in both committees, and (ii) 0.0394 in only one committee.

Quick Tip to Solve: For exactly one correct decision, pair each correct probability with the other committee's wrong probability, then add the two products.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Find: $\int \frac{x^2 + 1}{(x - 1)^2(x + 3)} dx$

[5 marks]

Solution:

Concept:

- A repeated linear factor $(x - a)^2$ requires both terms $A/(x - a) + B/(x - a)^2$ in partial fractions. Determine the constants by clearing denominators and equating polynomials.
- Use $\int (x - a)^{-1} dx = \ln |x - a| + C$ and the power rule $\int (x - a)^{-2} dx = -(x - a)^{-1} + C$, where C is an arbitrary constant.

Step 1: Set up the partial fractions The numerator degree is less than the denominator degree, so polynomial division is unnecessary. Including both powers of the repeated factor, write

$$\frac{x^2 + 1}{(x - 1)^2(x + 3)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{D}{x + 3}.$$

Clear denominators to obtain the polynomial identity

$$x^2 + 1 = A(x - 1)(x + 3) + B(x + 3) + D(x - 1)^2.$$

The constants A, B and D must make this identity hold for all x.

Step 2: Determine the constants Substitute $x = 1$ and $x = -3$ into the polynomial identity, then compare the coefficients of x^2 :

$$\begin{aligned} x = 1 : \quad 2 &= 4B \quad \Rightarrow \quad B = \frac{1}{2}, \\ x = -3 : \quad 10 &= 16D \quad \Rightarrow \quad D = \frac{5}{8}, \\ [x^2] : \quad 1 &= A + D \quad \Rightarrow \quad A = \frac{3}{8}. \end{aligned}$$

Substituting the excluded points is valid in this polynomial identity, even though the original fractions are undefined there. The other coefficients check: $2A + B - 2D = 0$ and $-3A + 3B + D = 1$.

Step 3: Integrate the three terms Substitute the constants and integrate term by term:

$$\begin{aligned} I &= \frac{3}{8} \int \frac{dx}{x - 1} + \frac{1}{2} \int (x - 1)^{-2} dx + \frac{5}{8} \int \frac{dx}{x + 3} \\ &= \frac{3}{8} \ln |x - 1| - \frac{1}{2(x - 1)} + \frac{5}{8} \ln |x + 3| + C. \end{aligned}$$

The squared denominator gives a negative reciprocal, while the simple factors give logarithms. Each linear expression has derivative 1, so no further substitution factor is required.

Quick Tip to Solve: Include both powers of $x - 1$, then substitute $x = 1$ and $x = -3$ after clearing denominators to isolate two constants immediately.

Step 4: Conclusion

$$I = \frac{3}{8} \ln |x - 1| - \frac{1}{2(x - 1)} + \frac{5}{8} \ln |x + 3| + C.$$

This antiderivative holds on each interval avoiding $x = 1, -3$.

Quick Tip to Solve: Include both powers of $x-1$, then substitute $x=1$ and $x=-3$ after clearing denominators to isolate two constants immediately.

OR

32(b). Evaluate: $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

[5 marks]

Solution:**Concept:**

- Reflection property of a definite integral: $\int_0^a f(x) dx = \int_0^a f(a - x) dx$. Here the reflection about $\pi/4$ interchanges sine and cosine, leaving their sum unchanged.
- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ to convert the remaining integral into a secant integral. Its standard antiderivative is $\int \sec u du = \ln |\sec u + \tan u| + C$.

Step 1: Reflect the original integral Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. The denominator is positive on the interval. Substitute $x = \pi/2 - t$, so $dx = -dt$; reverse the limits and use $\sin(\pi/2 - t) = \cos t$ and $\cos(\pi/2 - t) = \sin t$. This gives

$$I = \int_0^{\pi/2} \frac{\pi/2 - t}{\cos t + \sin t} dt = \int_0^{\pi/2} \frac{\pi/2 - x}{\sin x + \cos x} dx.$$

Step 2: Add the two representations Adding the original and reflected forms cancels the variable in the numerator:

$$2I = \int_0^{\pi/2} \frac{x + (\pi/2 - x)}{\sin x + \cos x} dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

Denoting the integral on the right by J , we have $I = (\pi/4)J$. The factor $\pi/4$ includes division by two because the addition produced $2I$.

Step 3: Evaluate the secant integral Substitute $u = x - \pi/4$. The limits become $-\pi/4$ and $\pi/4$, and the denominator is $\sqrt{2} \cos u$. Therefore

$$\begin{aligned} J &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec u du \\ &= \frac{1}{\sqrt{2}} [\ln |\sec u + \tan u|]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\sqrt{2}} [\ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1)]. \end{aligned}$$

Both logarithm arguments are positive. Since $(\sqrt{2} + 1)(\sqrt{2} - 1) = 1$, their ratio is $(\sqrt{2} + 1)^2$. It follows

Quick Tip to Solve: When the denominator is unchanged by $x \mapsto \pi/2 - x$, add the reflected integral first: the numerator x is effectively replaced by its midpoint value $\pi/4$.

that

$$J = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)^2 = \sqrt{2} \ln(1 + \sqrt{2}).$$

Step 4: Conclusion Substituting this value of J into $I = (\pi/4)J$, the required integral is

$$I = \frac{\pi}{2\sqrt{2}} \ln(1 + \sqrt{2}).$$

Quick Tip to Solve: When the denominator is unchanged by $x \mapsto \pi/2 - x$, add the reflected integral first: the numerator x is effectively replaced by its midpoint value $\pi/4$.

33. Draw a rough sketch for the curve $y = 2 + |x + 1|$. Using integration, find the area of the region bounded by the curve $y = 2 + |x + 1|$, $x = -4$, $x = 3$ and $y = 0$. [5 marks]

Solution:

Concept:

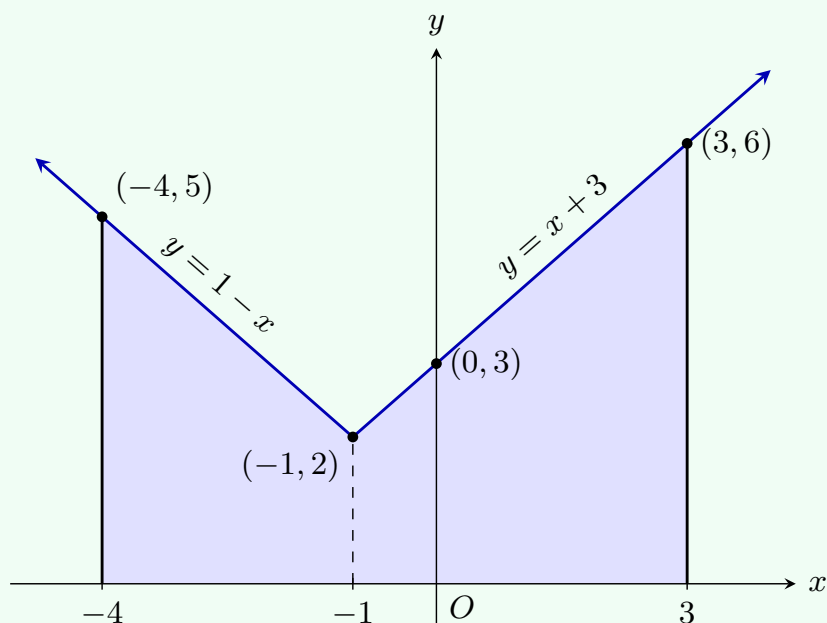
- Resolve the absolute value at its zero: $|x + 1| = -(x + 1)$ for $x < -1$, and $|x + 1| = x + 1$ for $x \geq -1$.
- The area between a curve $y = f(x) \geq 0$, the x -axis and the lines $x = a$, $x = b$ is $A = \int_a^b f(x) dx$. Here A denotes area in square units.
- When the formula for the curve changes within the integration interval, split the integral at that point and add the two positive areas.

Step 1: Resolve the modulus and sketch the curve The expression inside the modulus changes sign at $x = -1$. Therefore,

$$y = 2 + |x + 1| = \begin{cases} 1 - x, & x < -1, \\ x + 3, & x \geq -1. \end{cases}$$

The graph consists of two straight rays of slopes -1 and 1 , meeting at the vertex $(-1, 2)$. It passes through $(-4, 5)$, $(0, 3)$ and $(3, 6)$. Draw the vertical boundaries $x = -4$ and $x = 3$, and shade the region below the graph and above $y = 0$. The minimum ordinate is 2 , so the curve never meets the x -axis.

Quick Tip to Solve: Split at $x = -1$, where $x + 1 = 0$, rather than at the origin; use the two trapezia in the sketch to check your integrals.



The shaded region lies between the V-shaped curve, the two vertical lines and the x -axis.

Step 2: Set up the required area integral A vertical strip of width dx has height $y - 0$. Since the vertex lies between -4 and 3 , one linear formula cannot be used over the whole interval. The required region is the union of the portions on the two sides of $x = -1$, with no overlap of positive area. Thus,

$$\begin{aligned} A &= \int_{-4}^3 (2 + |x + 1|) dx \\ &= \int_{-4}^{-1} (1 - x) dx + \int_{-1}^3 (x + 3) dx. \end{aligned}$$

Both integrands are positive on their respective intervals, so their integrals give geometric areas directly.

Step 3: Evaluate the area to the left of the vertex Integrating $1 - x$ and substituting the upper and lower limits separately gives

$$\begin{aligned} A_1 &= \left[x - \frac{x^2}{2} \right]_{-4}^{-1} \\ &= \left(-1 - \frac{1}{2} \right) - \left(-4 - \frac{16}{2} \right) \\ &= -\frac{3}{2} + 12 = \frac{21}{2}. \end{aligned}$$

This is the area under the descending branch from $x = -4$ to $x = -1$.

Step 4: Evaluate the area to the right of the vertex For the ascending branch, integrate $x + 3$ between

Quick Tip to Solve: Split at $x = -1$, where $x + 1 = 0$, rather than at the origin; use the two trapezia in the sketch to check your integrals.

-1 and 3:

$$\begin{aligned} A_2 &= \left[\frac{x^2}{2} + 3x \right]_{-1}^3 \\ &= \left(\frac{9}{2} + 9 \right) - \left(\frac{1}{2} - 3 \right) \\ &= \frac{27}{2} + \frac{5}{2} = 16. \end{aligned}$$

As a check on the sketch, the left trapezium has width 3 and parallel sides 5, 2; the right one has width 4 and parallel sides 2, 6. Their trapezium areas agree with the two integrals.

Step 5: Conclusion The required area bounded by the curve and the three given lines is

$$A = A_1 + A_2 = \frac{21}{2} + 16 = \frac{53}{2} \text{ square units.}$$

Quick Tip to Solve: Split at $x = -1$, where $x + 1 = 0$, rather than at the origin; use the two trapezia in the sketch to check your integrals.

34(a). Solve the differential equation: $x^2y \, dx - (x^3 + y^3) \, dy = 0$.

[5 marks]

Solution:

Concept:

- A differential equation is homogeneous when its differential coefficients are homogeneous expressions of the same degree. Here both x^2y and $x^3 + y^3$ have degree three.
- Taking x as a function of y , use $x = vy$, where $v = x/y$. The product rule gives $dx/dy = v + y \, dv/dy$, converting the homogeneous equation into a separable equation.
- Integrate separated variables using $\int v^2 \, dv = v^3/3$ and $\int dy/y = \ln |y|$; any solution excluded by division must be checked in the original equation.

Step 1: Express the derivative in terms of the ratio On a region where $y \neq 0$, the given equation can be arranged as

$$x^2y \frac{dx}{dy} = x^3 + y^3.$$

The matching total degrees suggest using x/y . This choice is particularly convenient because the x^3 term cancels after substitution, leaving just one power of the new variable to integrate.

Step 2: Substitute and separate the variables Put $x = vy$, so $dx/dy = v + y \, dv/dy$. Substituting without dividing by v gives

$$\begin{aligned} v^2y^3 \left(v + y \frac{dv}{dy} \right) &= (v^3 + 1)y^3, \\ v^3 + v^2y \frac{dv}{dy} &= v^3 + 1, \\ v^2 \, dv &= \frac{dy}{y}. \end{aligned}$$

Quick Tip to Solve: Use $x = vy$, rather than $y = vx$, here: it makes the two cubic terms cancel and immediately leaves $v^2 \, dv = dy/y$.

The cancellation of the equal cubic terms is essential; the remaining right-hand side is positive.

Step 3: Integrate and restore the original variables Integrating both sides introduces a single arbitrary constant:

$$\frac{v^3}{3} = \ln |y| + C_1.$$

Substitution of $v = x/y$ and multiplication by three give

$$\frac{x^3}{y^3} - 3 \ln |y| = 3C_1.$$

The factor three multiplying the arbitrary constant can be absorbed into a new constant C . The absolute value allows the formula on intervals with either positive or negative y .

Step 4: Verify the relation and check the excluded solution Differentiating the implicit relation gives

$$\frac{3x^2}{y^3} dx - \left(\frac{3x^3}{y^4} + \frac{3}{y} \right) dy = 0.$$

Multiplication by $y^4/3$ recovers exactly $x^2y dx - (x^3 + y^3) dy = 0$. Since the derivation assumed $y \neq 0$, check $y = 0$ separately: then $dy = 0$, and both terms in the original differential equation vanish.

Step 5: Conclusion The general implicit solution for $y \neq 0$ is

$$\boxed{\frac{x^3}{y^3} - 3 \ln |y| = C}.$$

The constant solution $\boxed{y = 0}$ also satisfies the original equation.

Quick Tip to Solve: Use $x = vy$, rather than $y = vx$, here: it makes the two cubic terms cancel and immediately leaves $v^2 dv = dy/y$.

OR

34(b). Solve the differential equation $(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$ **subject to initial condition** $y(0) = 0$.

[5 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $dy/dx + P(x)y = Q(x)$. Its integrating factor is $\text{IF} = e^{\int P(x) dx}$.
- Multiplication by the integrating factor converts the left-hand side into a product derivative: $\frac{d}{dx}(y \text{ IF}) = Q(x) \text{ IF}$. Integrate first, then determine the constant from the initial condition.

Step 1: Put the equation into linear form Move the term independent of y to the right and divide by

Quick Tip to Solve: Spot $(1 + x^2)y' + 2xy$ as the derivative of $(1 + x^2)y$; integrate it directly and then apply $y(0) = 0$.

$1 + x^2$:

$$(1 + x^2) \frac{dy}{dx} + 2xy = 4x^2,$$
$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}.$$

Here $P(x) = 2x/(1 + x^2)$ and $Q(x) = 4x^2/(1 + x^2)$. This division is valid for every real x , since $1 + x^2 > 0$; in particular the equation is regular at the initial point $x = 0$.

Step 2: Calculate and apply the integrating factor The numerator $2x$ is the derivative of the denominator $1 + x^2$, so

$$\text{IF} = e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = 1 + x^2.$$

An arbitrary nonzero constant factor in the integrating factor is unnecessary. Multiplying the linear equation by this factor yields

$$(1 + x^2) \frac{dy}{dx} + 2xy = 4x^2.$$

By the product rule, the left side is $\frac{d}{dx}[(1 + x^2)y]$. Thus the original left side already contains the exact product derivative needed for integration.

Step 3: Integrate and use the initial condition Integrate the product derivative with respect to x :

$$\frac{d}{dx}[(1 + x^2)y] = 4x^2,$$
$$(1 + x^2)y = \frac{4x^3}{3} + C.$$

Now substitute $x = 0$ and $y(0) = 0$. The equation becomes $(1 + 0) \cdot 0 = 0 + C$, giving $C = 0$. The initial condition therefore selects one member of the general family, with no arbitrary constant remaining.

Step 4: Check the differential equation For the selected function, the quotient rule gives

$$y' = \frac{4x^2(3 + x^2)}{3(1 + x^2)^2}.$$

Substitution into the two terms on the left gives

$$(1 + x^2)y' + 2xy = \frac{4x^2(3 + x^2) + 8x^4}{3(1 + x^2)}$$
$$= \frac{12x^2(1 + x^2)}{3(1 + x^2)} = 4x^2.$$

Hence the differential equation holds, and substitution of zero into the function also verifies the prescribed initial value.

Step 5: Conclusion The required particular solution is

$$y = \frac{4x^3}{3(1 + x^2)}, \quad x \in \mathbb{R}.$$

Quick Tip to Solve: Spot $(1 + x^2)y' + 2xy$ as the derivative of $(1 + x^2)y$; integrate it directly and then apply $y(0) = 0$.

35. Let the polished side of the mirror be along the line $\frac{x}{1} = \frac{1-y}{-2} = \frac{2z-4}{6}$. A point $P(1, 6, 3)$, some distance away from the mirror, has its image formed behind the mirror. Find the coordinates of the image point and the distance between the point P and its image. [5 marks]

Solution:

Concept:

- For reflection of a point P in a line, the foot H of the perpendicular from P to the line is the midpoint of the segment joining P and its image P' .
- If the line has direction vector $\mathbf{d}$, perpendicularity gives $\overrightarrow{PH} \cdot \mathbf{d} = 0$. Once H is known, $P' = 2H - P$ coordinate by coordinate.

Step 1: Express a general point on the mirror line Put each printed ratio equal to the same parameter t :

$$\frac{x}{1} = \frac{1-y}{-2} = \frac{2z-4}{6} = t.$$

Therefore $x = t$, $1 - y = -2t$, and $2z - 4 = 6t$. A general point on the line is $H(t, 1 + 2t, 2 + 3t)$, with direction vector $\mathbf{d} = (1, 2, 3)$. Divide the third numerator and denominator by two to obtain the correct direction ratio for the third coordinate. Also retain the sign in the numerator of the second ratio.

Step 2: Locate the perpendicular foot For the given point $P(1, 6, 3)$, the vector from P to this general point is $\overrightarrow{PH} = (t - 1, 2t - 5, 3t - 1)$. The required foot lies on the line and has PH perpendicular to its direction. Thus

$$\begin{aligned}(t - 1) + 2(2t - 5) + 3(3t - 1) &= 0, \\ 14t - 14 &= 0, \\ t &= 1.\end{aligned}$$

Substitution gives $H = (1, 3, 5)$.

Step 3: Use the midpoint relation to obtain the image Write the image as $P'(a, b, c)$. Reflection puts the image the same perpendicular distance on the other side of the line, so

$$\left(\frac{1+a}{2}, \frac{6+b}{2}, \frac{3+c}{2} \right) = (1, 3, 5).$$

Hence $a = 1$, $b = 0$ and $c = 7$.

Step 4: Calculate the separation Using the coordinate differences between P and P' ,

$$PP' = \sqrt{(1-1)^2 + (0-6)^2 + (7-3)^2} = \sqrt{52} = 2\sqrt{13}.$$

Step 5: Conclusion The image point is $P'(1, 0, 7)$, and the distance between the point and its image is $2\sqrt{13}$ units.

Quick Tip to Solve: First rewrite the mirror line as $(x, y, z) = (0, 1, 2) + t(1, 2, 3)$; then find the perpendicular foot and double its coordinates before subtracting those of P .

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

Three students, Neha, Rani and Sam go to a market to purchase stationery items. Neha buys 4 pens, 3 notepads and 2 erasers and pays Rs. 60. Rani buys 2 pens, 4 notepads and 6 erasers for Rs. 90. Sam pays Rs. 70 for 6 pens, 2 notepads and 3 erasers.

Based upon the above information, answer the following questions:

[4 marks]

36(i). Form the equations required to solve the problem of finding the price of each item, and express it in the matrix form $AX = B$.

[1 mark]

Solution:

Concept:

- The total bill is the sum of quantity times unit price for each type of item.
- In $AX = B$, rows of A contain the quantities, X contains the unknown unit prices, and B contains the corresponding bills in the same order.

Step 1: Define the prices and form the equations Let x, y, z be the prices in rupees of one pen, one notepad and one eraser, respectively. Using Neha's, Rani's and Sam's purchases in that order gives

$$4x + 3y + 2z = 60,$$

$$2x + 4y + 6z = 90,$$

$$6x + 2y + 3z = 70.$$

Step 2: Collect coefficients in a matrix Keep the columns in the order pen, notepad, eraser, and set

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Each row of the product AX reproduces the left-hand side of its corresponding purchase equation.

Step 3: Conclusion The required matrix equation is

$$\begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Quick Tip to Solve: Keep one fixed item order across all rows: pens, notepads, erasers. Put each student's total bill beside that student's row.

36(ii). Find $|A|$ and confirm if it is possible to find A^{-1} .

[1 mark]

Solution:

Concept:

- A square matrix has an inverse if and only if its determinant is nonzero.
- Expansion of a determinant along its first row uses the cofactor signs $+, -, +$, with each entry multiplied by its corresponding minor.

Step 1: Expand the coefficient determinant The purchase equations give

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}.$$

Expanding along the first row,

$$\begin{aligned} |A| &= 4(4 \cdot 3 - 6 \cdot 2) - 3(2 \cdot 3 - 6 \cdot 6) + 2(2 \cdot 2 - 4 \cdot 6) \\ &= 4(0) - 3(-30) + 2(-20) \\ &= 90 - 40 = 50. \end{aligned}$$

Step 2: Apply the inverse criterion Since $|A| = 50 \neq 0$, the matrix is nonsingular. Thus division by $|A|$ in the inverse formula $A^{-1} = \text{adj}(A)/|A|$ is valid, and the three purchase equations determine a unique price vector.

Step 3: Conclusion $|A| = 50$. Since the determinant is nonzero, A^{-1} exists.

Quick Tip to Solve: The first minor is zero here, so only the last two first-row terms contribute; preserve the minus sign on the middle cofactor.

36(iii)(a). Find A^{-1} , if possible, and write the formula to find X .

[2 marks]

Solution:

Concept:

- A square matrix with nonzero determinant has inverse $A^{-1} = \text{adj}(A)/|A|$, where the adjugate is the transpose of the cofactor matrix.
- Multiplying $AX = B$ on the left by A^{-1} gives $X = A^{-1}B$, since $A^{-1}A = I$.

Step 1: Check invertibility and calculate cofactors For the purchase coefficient matrix,

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad |A| = 50 \neq 0.$$

Using $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j , the cofactor

Quick Tip to Solve: After calculating the signed cofactors, transpose them before dividing by 50; forgetting this transpose gives the wrong inverse.

matrix is

$$C = \begin{pmatrix} 12 - 12 & -(6 - 36) & 4 - 24 \\ -(9 - 4) & 12 - 12 & -(8 - 18) \\ 18 - 8 & -(24 - 4) & 16 - 6 \end{pmatrix} = \begin{pmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{pmatrix}.$$

Step 2: Transpose the cofactors and divide by the determinant Transposing interchanges rows and columns. Therefore

$$\text{adj}(A) = C^T = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}.$$

Dividing every entry by 50 gives

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{pmatrix}.$$

To isolate the unknown price vector in $AX = B$, the inverse must multiply on the left; matrix multiplication cannot be reordered freely.

Step 3: Conclusion

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{pmatrix}, \quad \boxed{X = A^{-1}B}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Quick Tip to Solve: After calculating the signed cofactors, transpose them before dividing by 50; forgetting this transpose gives the wrong inverse.

OR

36(iii)(b). Find $A^2 - 8I$, where I is an identity matrix.

[2 marks]

Solution:

Concept:

- The square of a matrix means the matrix product $A^2 = A \cdot A$, not the square of each individual entry. Its (i, j) entry is row i dotted with column j .
- The identity matrix I has ones on the main diagonal and zeros elsewhere. Thus subtracting $8I$ decreases only the diagonal entries by 8.

Step 1: Multiply the coefficient matrix by itself The purchase information gives

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}.$$

Quick Tip to Solve: First calculate all nine row-by-column products in A^2 . Then subtract 8 only from the top-left, centre and bottom-right entries.

Applying row-by-column multiplication,

$$A^2 = \begin{pmatrix} 16 + 6 + 12 & 12 + 12 + 4 & 8 + 18 + 6 \\ 8 + 8 + 36 & 6 + 16 + 12 & 4 + 24 + 18 \\ 24 + 4 + 18 & 18 + 8 + 6 & 12 + 12 + 9 \end{pmatrix} = \begin{pmatrix} 34 & 28 & 32 \\ 52 & 34 & 46 \\ 46 & 32 & 33 \end{pmatrix}.$$

For example, the first-row, third-column entry is $4 \cdot 2 + 3 \cdot 6 + 2 \cdot 3 = 32$. Each entry involves three products because A has order 3×3 .

Step 2: Subtract the scalar multiple of the identity The identity must have the same order as A^2 . Thus

$$8I = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Subtraction is entry by entry: the three diagonal entries become $34 - 8 = 26$, $34 - 8 = 26$, and $33 - 8 = 25$. Every off-diagonal entry stays unchanged because the corresponding entry of $8I$ is zero.

Step 3: Conclusion

$$A^2 - 8I = \begin{pmatrix} 26 & 28 & 32 \\ 52 & 26 & 46 \\ 46 & 32 & 25 \end{pmatrix}.$$

Quick Tip to Solve: First calculate all nine row-by-column products in A^2 . Then subtract 8 only from the top-left, centre and bottom-right entries.

37. Case Study - 2

A ladder of fixed length ' h ' is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions:

[4 marks]

37(i). Express the distance (y) between the wall and foot of the ladder in terms of ' h ' and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer.

[1 mark]

Solution:

Concept:

- For a right triangle, Pythagoras' theorem gives $x^2 + y^2 = h^2$, where x is the height, y the horizontal distance and $h > 0$ the fixed ladder length.
- The area is $A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}xy$.

Step 1: Express the horizontal distance The ladder is the hypotenuse, while the wall and ground give the perpendicular sides. Rearranging the fixed-length relation,

$$y^2 = h^2 - x^2.$$

Quick Tip to Solve: Treat the ladder length as the hypotenuse, and take the positive square root for the horizontal distance.

Since a distance cannot be negative, choose the nonnegative square root: $y = \sqrt{h^2 - x^2}$, with $0 \leq x \leq h$.

Step 2: Substitute into the area formula The base is the distance y along the ground and the altitude is x along the wall. Thus

$$A = \frac{1}{2}xy = \frac{x}{2}\sqrt{h^2 - x^2}.$$

For a nondegenerate triangle, $0 < x < h$; the endpoint positions give zero area.

Step 3: Conclusion

$$y = \sqrt{h^2 - x^2}, \quad A = \frac{x}{2}\sqrt{h^2 - x^2}.$$

Quick Tip to Solve: Treat the ladder length as the hypotenuse, and take the positive square root for the horizontal distance.

37(ii). Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point. [1 mark]

Solution:

Concept:

- The area is $A(x) = \frac{1}{2}x(h^2 - x^2)^{1/2}$, with $h > 0$ fixed and $0 < x < h$.
- Use the product and chain rules; an interior critical point satisfies $A'(x) = 0$ when the derivative exists.

Step 1: Differentiate the area function Keeping the ladder length h constant,

$$\begin{aligned} \frac{dA}{dx} &= \frac{1}{2} \left[\sqrt{h^2 - x^2} + x \frac{-2x}{2\sqrt{h^2 - x^2}} \right] \\ &= \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}. \end{aligned}$$

The denominator is positive throughout the interior physical domain.

Step 2: Locate the interior critical point Therefore the derivative vanishes precisely when its numerator is zero:

$$h^2 - 2x^2 = 0 \implies x^2 = \frac{h^2}{2}.$$

Reject the negative root because x is a height. The positive root lies strictly between 0 and h .

Step 3: Conclusion

$$\frac{dA}{dx} = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}, \quad x = \frac{h}{\sqrt{2}}.$$

The interior critical point on the area curve is $(h/\sqrt{2}, h^2/4)$.

Quick Tip to Solve: After differentiating, set only the numerator to zero: the denominator is positive for $0 < x < h$.

37(iii)(a). Show that the area (A) of the right triangle is maximum at the critical point.

[2 marks]

Solution:

Concept:

- For a ladder of fixed length $h > 0$, the area is $A(x) = \frac{x}{2}\sqrt{h^2 - x^2}$, for $0 \leq x \leq h$.
- By the first derivative test, a change of $A'(x)$ from positive to negative establishes a maximum. Endpoint values determine whether this is the maximum on the full physical interval.

Step 1: Determine the sign of the derivative From the product and chain rules,

$$A'(x) = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}, \quad 0 < x < h.$$

Its denominator is positive. If $0 < x < h/\sqrt{2}$, then $2x^2 < h^2$, so $A'(x) > 0$. If $h/\sqrt{2} < x < h$, then $2x^2 > h^2$, so $A'(x) < 0$.

Step 2: Apply the maximum test and check the boundary The area increases up to $x = h/\sqrt{2}$ and decreases afterwards. Thus this critical point is a strict maximum. Also $A(0) = A(h) = 0$, so neither endpoint gives a larger value. At the critical point,

$$y = \sqrt{h^2 - \frac{h^2}{2}} = \frac{h}{\sqrt{2}}, \quad A = \frac{1}{2} \frac{h}{\sqrt{2}} \frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

The maximizing triangle therefore has equal perpendicular sides.

Step 3: Conclusion The area is maximum at $x = h/\sqrt{2}$, and its maximum value is $A_{\max} = h^2/4$.

Quick Tip to Solve: For this derivative, the denominator stays positive; compare $2x^2$ with h^2 on either side of the critical height.

OR

37(iii)(b). If the foot of the ladder whose length is 5 m, is being pulled towards the wall such that the rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the ladder is 3 m away from the wall?

[2 marks]

Solution:

Concept:

- The fixed ladder length imposes $x^2 + y^2 = 25$, where x is the height and y the horizontal distance, both in metres.
- For related rates, differentiate with respect to time t . Motion towards the wall means $dy/dt = -2$ m/s, since the distance decreases.

Step 1: Find the height at the given instant When the foot is 3 m from the wall, substitute $y = 3$ in the fixed-length equation:

$$x^2 + 3^2 = 5^2 \implies x^2 = 16.$$

Quick Tip to Solve: Enter a decreasing distance as a negative rate before substitution, and differentiate the fixed-length equation before inserting instantaneous lengths.

The physical height is the positive root, so $x = 4$ m. This value describes the specified instant, not a constant height throughout the motion.

Step 2: Differentiate the constraint and substitute As both distances depend on time while the ladder length remains fixed,

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}.$$

At the given instant,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2} \text{ m/s}.$$

The positive sign confirms that the top moves upward as the foot moves towards the wall.

Step 3: Conclusion The height on the wall is increasing at $\boxed{\frac{3}{2} \text{ m/s}}$, or 1.5 m/s.

Quick Tip to Solve: Enter a decreasing distance as a negative rate before substitution, and differentiate the fixed-length equation before inserting instantaneous lengths.

38. Case Study - 3

A shop selling electronic items sells smartphones of only three reputed companies A, B and C because chances of their manufacturing a defective smartphone are only 5%, 4% and 2% respectively. In his inventory he has 25% smartphones from company A, 35% smartphones from company B and 40% smartphones from company C.

A person buys a smartphone from this shop.

[4 marks]

38(i). Find the probability that it was defective.

[2 marks]

Solution:

Concept:

- Let E_A, E_B, E_C denote the company supplying the smartphone, and let D denote the event that it is defective.
- The manufacturers form mutually exclusive and exhaustive events. The total probability law gives $P(D) = \sum_{j=A,B,C} P(E_j)P(D | E_j)$.

Step 1: Write the inventory and conditional probabilities The inventory shares give

$$P(E_A) = 0.25, \quad P(E_B) = 0.35, \quad P(E_C) = 0.40.$$

The stated defect rates are conditional on the manufacturer:

$$P(D | E_A) = 0.05, \quad P(D | E_B) = 0.04, \quad P(D | E_C) = 0.02.$$

The inventory probabilities add to one, so these three cases cover every possible purchase.

Step 2: Weight each defect rate by its inventory share A defective phone can come from any one of

Quick Tip to Solve: Multiply each company's stock share by its own defect rate, then add the three contributions.

the three companies. Add the probabilities of these disjoint possibilities:

$$\begin{aligned}P(D) &= (0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02) \\&= 0.0125 + 0.0140 + 0.0080 \\&= 0.0345.\end{aligned}$$

The differing inventory shares are why the defect rates must be weighted, rather than simply averaged.

Step 3: Conclusion The probability of buying a defective smartphone is $0.0345 = \frac{69}{2000}$, equivalent to 3.45%.

Quick Tip to Solve: Multiply each company's stock share by its own defect rate, then add the three contributions.

38(ii). What is the probability that this defective smartphone was manufactured by company B?

[2 marks]

Solution:

Concept:

- Let D denote a defective smartphone and E_B denote a smartphone from company B. The required probability is $P(E_B | D)$.
- Bayes' theorem gives $P(E_B | D) = P(E_B)P(D | E_B)/P(D)$. The denominator includes defective phones from all three companies.

Step 1: Find the joint and total defect probabilities The probability that a purchased phone is both from B and defective is

$$P(E_B \cap D) = P(E_B)P(D | E_B) = 0.35 \times 0.04 = 0.014.$$

Using the inventory shares and all three conditional defect rates,

$$\begin{aligned}P(D) &= 0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02 \\&= 0.0125 + 0.014 + 0.008 = 0.0345.\end{aligned}$$

These contributions represent disjoint sources of a defective phone.

Step 2: Condition on the phone being defective We now restrict attention to the defective phones. The share attributable to B is its defective-phone contribution divided by the total:

$$P(E_B | D) = \frac{0.014}{0.0345} = \frac{140}{345} = \frac{28}{69}.$$

This is different from $P(D | E_B) = 0.04$: the latter starts with B phones, while the question starts with defective phones.

Step 3: Conclusion The probability that the defective smartphone was manufactured by company B

Quick Tip to Solve: For a manufacturer given a defect, divide that manufacturer's weighted defect contribution by the total defect probability.

is $\frac{28}{69}$, approximately 0.4058.

Quick Tip to Solve: For a manufacturer given a defect, divide that manufacturer's weighted defect contribution by the total defect probability.