CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-6-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. Sum of two skew-symmetric matrices of same order is always a/an: [1 mark]

(A) skew-symmetric matrix

(B) symmetric matrix

(C) null matrix

(D) identity matrix

Correct Answer: (A) skew-symmetric matrix

Solution:

Concept:

- A square matrix A is skew-symmetric when $A^T = -A$.
- Transposition distributes over addition: $(A+B)^T = A^T + B^T$.

Step 1: Use the defining identities Let the two matrices be A and B, of the same order. Since both are skew-symmetric, $A^T = -A$ and $B^T = -B$. Thus

$$(A+B)^T = A^T + B^T = -A - B = -(A+B).$$

This is precisely the defining condition for their sum to be skew-symmetric.

Step 2: Check the remaining alternatives The sum may be zero for the special choice $B = -A$, but it need not be zero. For example, adding $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to itself gives nonzero opposite off-diagonal entries, so the sum is neither symmetric nor the identity. Therefore those alternatives are not always true.

Step 3: Conclusion The sum is always a skew-symmetric matrix. Hence, the correct option is (A).

Quick Tip to Solve: Transpose the sum and use the two minus signs to test skew-symmetry directly.

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2. If $A = \begin{pmatrix} 0 & -3 & 8 \\ 3 & 0 & 5 \\ -8 & -5 & 0 \end{pmatrix}$, then A is a: [1 mark]

(A) null matrix

(B) symmetric matrix

(C) skew-symmetric matrix

(D) diagonal matrix

Correct Answer: (C) skew-symmetric matrix

Solution:

Concept:

- Skew-symmetry requires $A^T = -A$; symmetry requires $A^T = A$.

Quick Tip to Solve: For a skew-symmetric matrix, check zero diagonal entries and opposite signs across the diagonal.

- A diagonal matrix has zero off-diagonal entries, whereas a null matrix has every entry zero.

Step 1: Compare opposite entries The diagonal entries are all zero. The off-diagonal pairs are $(-3, 3)$, $(8, -8)$ and $(5, -5)$, so every pair satisfies $a_{ij} = -a_{ji}$. Explicitly,

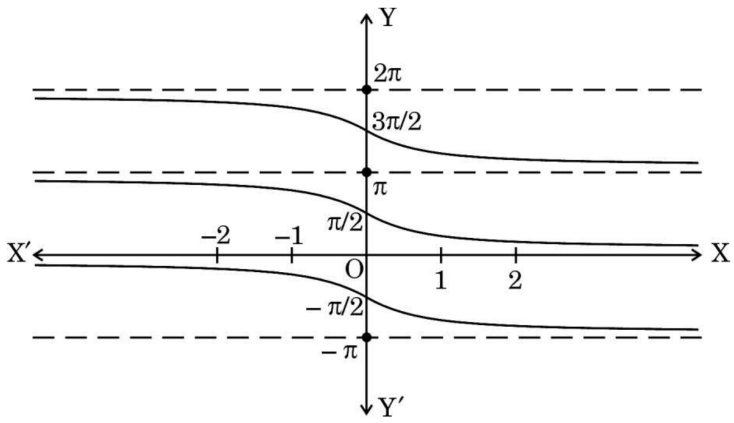
$$A^T = \begin{pmatrix} 0 & 3 & -8 \\ -3 & 0 & -5 \\ 8 & 5 & 0 \end{pmatrix} = -A.$$

Step 2: Eliminate the other matrix types The nonzero entries rule out the null matrix. Since entries such as $a_{12} = -3$ and $a_{21} = 3$ differ, the matrix is not symmetric. Its nonzero off-diagonal entries also rule out a diagonal matrix. The negative transpose condition therefore uniquely identifies its type.

Step 3: Conclusion A is skew-symmetric. Hence, the correct option is (C).

Quick Tip to Solve: For a skew-symmetric matrix, check zero diagonal entries and opposite signs across the diagonal.

3. The graph shown below depicts : [1 mark]



(A) $y = \cot x$

(B) $y = \cot^{-1} x$

(C) $y = \tan x$

(D) $y = \tan^{-1} x$

Correct Answer: (B) $y = \cot^{-1} x$

Solution:

Concept:

- The principal branch of $\cot^{-1} x$ has domain $\mathbb{R}$, range $(0, \pi)$, and value $\pi/2$ at $x = 0$.
- Its derivative is $-1/(1 + x^2)$, so this branch decreases everywhere.

Step 1: Read the central branch The displayed central branch decreases continuously through $(0, \pi/2)$. It approaches π to the left and 0 to the right. These are the characteristic features of the principal branch $y = \cot^{-1} x$. The additional branches are vertical translates by integer multiples of π,

Quick Tip to Solve: Identify the branch through (0, pi/2): decreasing with horizontal limits 0 and pi means inverse cotangent.

illustrating the inverse cotangent relation.

Step 2: Compare the other candidates $y = \tan^{-1} x$ is increasing and passes through the origin, unlike the central curve. The direct functions $\tan x$ and $\cot x$ have vertical asymptotes and repeat along the horizontal axis; the displayed curves instead have horizontal limiting levels. Thus neither direct trigonometric function fits the graph.

Step 3: Conclusion The graph depicts inverse cotangent. Hence, the correct option is (B).

Quick Tip to Solve: Identify the branch through (0, pi/2): decreasing with horizontal limits 0 and pi means inverse cotangent.

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4. Let both AB' and $B'A$ be defined for matrices A and B. If order of A is $n \times m$, then the order of B is: [1 mark]

(A) $n \times n$

(B) $n \times m$

(C) $m \times m$

(D) $m \times n$

Correct Answer: (B) $n \times m$

Solution:

Concept:

- A product is defined when the first matrix column count equals the second matrix row count.
- If B has order $p \times q$, then its transpose B' has order $q \times p$.

Step 1: Apply the condition to the first product Let B be $p \times q$, so B' is $q \times p$. Since A is $n \times m$, the product AB' exists only if $m = q$. This determines the number of columns of B.

Step 2: Apply the second condition The product $B'A$ has factors of orders $q \times p$ and $n \times m$. Its existence requires $p = n$. Combining the two conditions gives B order $p \times q = n \times m$. The other listed orders do not guarantee both products for general n, m; square special cases do not change the general result.

Step 3: Conclusion The order of B is $n \times m$. Hence, the correct option is (B).

Quick Tip to Solve: Assign two unknown dimensions to B, transpose them, and match the inner dimensions twice.

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5. If $f(x) = \begin{cases} \dfrac{\log(1+ax)+\log(1-bx)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ **is continuous at** $x = 0$**, then the value of** k **is:** [1 mark]

(A) a

(B) $a + b$

(C) $a - b$

(D) b

Correct Answer: (C) $a - b$

Solution:

Concept:

- Continuity at zero requires $f(0) = \lim_{x\to 0} f(x)$.
- The standard logarithmic limit is $\lim_{t\to 0} \log(1+t)/t = 1$.

Step 1: Evaluate each logarithmic term For fixed constants a, b, the logarithms are defined in a sufficiently small neighbourhood of zero. The standard limit gives

$$\lim_{x\to 0} \frac{\log(1+ax)}{x} = a, \qquad \lim_{x\to 0} \frac{\log(1-bx)}{x} = -b.$$

These statements also hold when the corresponding constant is zero, because that numerator is then identically zero.

Step 2: Enforce continuity Adding the limits gives $\lim_{x\to 0} f(x) = a - b$. Since the defined value at zero is k, continuity requires $k = a - b$. The plus sign in $a + b$ misses the negative coefficient inside the second logarithm; a and b alone omit one of the two contributions.

Step 3: Conclusion $k = a - b$. Hence, the correct option is (C).

Quick Tip to Solve: The coefficient of x inside each log is its contribution to the limit; preserve the minus sign.

6. If $y = a\cos(\log x) + b\sin(\log x)$**, then** $x^2y_2 + xy_1$ **is:** [1 mark]

(A) $\cot(\log x)$

(B) y

(C) $-y$

(D) $\tan(\log x)$

Correct Answer: (C) $-y$

Solution:

Concept:

- Write $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$, with $x > 0$ because of $\log x$.
- Use the chain rule $d(\log x)/dx = 1/x$ and the product rule.

Step 1: Differentiate once and multiply by x Differentiating the given expression yields

$$y_1 = \frac{-a\sin(\log x) + b\cos(\log x)}{x}.$$

Quick Tip to Solve: Differentiate the simpler product xy_1 instead of expanding a lengthy second derivative.

Hence $xy_1 = -a\sin(\log x) + b\cos(\log x)$. Keeping this product intact makes the second differentiation shorter.

Step 2: Differentiate the product By the product and chain rules,

$$y_1 + xy_2 = \frac{-a\cos(\log x) - b\sin(\log x)}{x} = -\frac{y}{x}.$$

Multiplication by x gives $xy_1 + x^2y_2 = -y$. The sign rules out y, and neither $\cot(\log x)$ nor $\tan(\log x)$ can equal this expression for arbitrary constants a, b.

Step 3: Conclusion $x^2y_2 + xy_1 = -y$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate the simpler product xy_1 instead of expanding a lengthy second derivative.

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7. $\sec^{-1}(-\sqrt{2}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **is equal to:** [1 mark]

(A) $\frac{11\pi}{12}$

(B) $\frac{5\pi}{12}$

(C) $-\frac{5\pi}{12}$

(D) $\frac{7\pi}{12}$

Correct Answer: (D) $\frac{7\pi}{12}$

Solution:

Concept:

- The principal range of $\sec^{-1} x$ is $[0, \pi] \setminus \{\pi/2\}$.
- The principal range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$.

Step 1: Find the principal angles For $\sec\theta = -\sqrt{2}$, we have $\cos\theta = -1/\sqrt{2}$. The principal range chooses the second-quadrant angle $\theta = 3\pi/4$. Also, $\tan(\pi/6) = 1/\sqrt{3}$, and $\pi/6$ lies in the principal range of inverse tangent. Thus the two values are $3\pi/4$ and $\pi/6$.

Step 2: Subtract with a common denominator The required difference is

$$\frac{3\pi}{4} - \frac{\pi}{6} = \frac{9\pi - 2\pi}{12} = \frac{7\pi}{12}.$$

Adding instead gives the tempting $11\pi/12$, while $5\pi/12$ and its negative do not follow from the principal angles. In particular, a negative input to inverse secant does not imply a negative principal angle.

Step 3: Conclusion The value is $7\pi/12$. Hence, the correct option is (D).

Quick Tip to Solve: Convert inverse secant to cosine, select its principal quadrant, then subtract the inverse tangent angle.

8. If $\tan^{-1}(x^2 - y^2) = a$, where a is a constant, then $\frac{dy}{dx}$ is: [1 mark]

(A) $\frac{x}{y}$

(B) $-\frac{x}{y}$

(C) $\frac{a}{x}$

(D) $\frac{a}{y}$

Correct Answer: (A) $\frac{x}{y}$

Solution:

Concept:

- The derivative of $\tan^{-1} u$ is $u\prime/(1 + u^2)$.
- Implicit differentiation treats y as a function of x; the constant a has derivative zero.

Step 1: Differentiate the equation Using the chain rule on $\tan^{-1}(x^2 - y^2) = a$, we obtain

$$\frac{2x - 2y\, dy/dx}{1 + (x^2 - y^2)^2} = 0.$$

The denominator is strictly positive for real x, y, so the numerator must vanish.

Step 2: Isolate the derivative Thus $2x - 2y\, dy/dx = 0$, giving $y\, dy/dx = x$. At points where $y \neq 0$, this yields $dy/dx = x/y$. The negative sign in option (B) is incorrect because the differentiated $-y^2$ term moves to the other side. Options (C) and (D) insert a, although its derivative is zero. At $y = 0$, this quotient is not a finite derivative formula.

Step 3: Conclusion $\frac{dy}{dx} = \frac{x}{y}$, wherever the derivative is defined by this quotient. Hence, the correct option is (A).

Quick Tip to Solve: After differentiating, discard the positive inverse-tangent denominator and solve 2x-2y y-prime = 0.

9. Let $f(x) = x^2,\ x \in \mathbb{R}$. Then, which of the following statements is incorrect? [1 mark]

(A) Minimum value of f does not exist.

(B) There is no point of maximum value of f in $\mathbb{R}$.

(C) f is continuous at $x = 0$.

(D) f is differentiable at $x = 0$.

Correct Answer: (A) Minimum value of f does not exist.

Solution:

Concept:

- A minimum is attained at a point where the function has its least value.
- A polynomial is continuous and differentiable on all of $\mathbb{R}$.

Quick Tip to Solve: Underline "incorrect" first, then use $x^2 \geq 0$ with equality at zero.

Step 1: Test extrema For every real x, $x^2 \geq 0$, and $f(0) = 0$. Thus a minimum value does exist, namely zero at the origin. Conversely, choosing larger values of $|x|$ makes x^2 arbitrarily large, so there is no maximum on the real line. Statement (A) is false and statement (B) is true.

Step 2: Check the local statements At zero, $\lim_{x\to 0} x^2 = 0 = f(0)$, proving continuity. Also $f\prime(0) = \lim_{h\to 0}(h^2 - 0)/h = \lim_{h\to 0} h = 0$, proving differentiability. Hence statements (C) and (D) are true, so neither answers the request for an incorrect statement.

Step 3: Conclusion The incorrect statement is (A): the minimum value does exist.

Quick Tip to Solve: Underline "incorrect" first, then use $x^2 \geq 0$ with equality at zero.

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10. $\int \frac{x+5}{(x+6)^2} e^x \, dx$ **is equal to:** [1 mark]

(A) $\log(x+6) + C$

(B) $e^x + C$

(C) $\frac{e^x}{x+6} + C$

(D) $-\frac{1}{(x+6)^2} + C$

Correct Answer: (C) $\frac{e^x}{x+6} + C$

Solution:

Concept:

- An antiderivative is verified by differentiating it.
- The quotient rule is $(u/v)\prime = (u\prime v - uv\prime)/v^2$.

Step 1: Recognise the quotient derivative Put $u = e^x$ and $v = x + 6$. For $x \neq -6$, the quotient rule gives

$$\frac{d}{dx}\left(\frac{e^x}{x+6}\right) = \frac{e^x(x+6) - e^x}{(x+6)^2} = \frac{(x+5)e^x}{(x+6)^2}.$$

This is exactly the given integrand.

Step 2: Eliminate the other candidates The derivative of option (A) is $1/(x+6)$, that of (B) is e^x, and that of (D) is $2/(x+6)^3$. None equals the required expression on an interval. The integration constant disappears on differentiation, so it cannot repair these mismatches.

Step 3: Conclusion $\int \frac{x+5}{(x+6)^2} e^x \, dx = \frac{e^x}{x+6} + C$. The correct option is (C).

Quick Tip to Solve: Differentiate the quotient option: its numerator becomes $(x+6) - 1 = x + 5$.

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11. Let $f'(x) = 3(x^2 + 2x) - \frac{4}{x^3} + 5,\ f(1) = 0$. Then, $f(x)$ is: [1 mark]

(A) $x^3 + 3x^2 + \frac{2}{x^2} + 5x + 11$

(B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$

(C) $x^3 + 3x^2 - \frac{2}{x^2} + 5x - 11$

(D) $x^3 - 3x^2 - \frac{2}{x^2} + 5x - 11$

Correct Answer: (B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$

Solution:

Concept:

- Integrate each term of $f\prime(x)$ and add a constant.
- The condition $f(1) = 0$ determines the constant on the interval containing 1.

Step 1: Integrate the printed derivative Expand $f\prime(x) = 3x^2 + 6x - 4x^{-3} + 5$. Termwise integration gives

$$f(x) = x^3 + 3x^2 + 2x^{-2} + 5x + C, \qquad x \neq 0.$$

In particular, the inverse-square term is positive because $\int -4x^{-3}\,dx = 2x^{-2}$.

Step 2: Apply the initial value and compare options Substitute $x = 1$: $0 = 1 + 3 + 2 + 5 + C = 11 + C$, so $C = -11$. Option (A) has the wrong constant. Option (C) has a negative inverse-square term, whose derivative has the wrong sign. Option (D) also reverses the quadratic term. Only (B) has both the derivative and the stated initial value.

Step 3: Conclusion $f(x) = x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$. The correct option is (B).

Quick Tip to Solve: Check the sign of the inverse-square term before using $f(1) = 0$.

12. The order and degree of the differential equation $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right) = x\log\left(\frac{d^2y}{dx^2}\right)$ are respectively: [1 mark]

(A) $0, 3$

(B) $2, 1$

(C) 2, not defined

(D) 1, not defined

Correct Answer: (C) 2, not defined

Solution:

Concept:

- Order is the order of the highest derivative occurring in the equation.
- Degree is defined only when the equation is polynomial in its derivatives after valid algebraic simplification.

Quick Tip to Solve: Inspect functions of derivatives before counting powers for degree.

Step 1: Identify the highest derivative The given equation contains $y'' = d^2y/dx^2$, both as a standalone term and inside the logarithm. It contains no derivative of order greater than two. Therefore its order is two; a logarithm does not remove the derivative from the equation.

Step 2: Check whether degree exists The expression $\log(y'')$ makes the equation non-polynomial in its derivatives. Exponentiating would instead introduce an exponential containing derivatives, and does not convert it to a polynomial. Thus degree is not defined. Option (B) incorrectly reads degree one from the standalone second derivative; (A) and (D) give the wrong order.

Step 3: Conclusion Order $= 2$, degree is not defined. The correct option is (C).

Quick Tip to Solve: Inspect functions of derivatives before counting powers for degree.

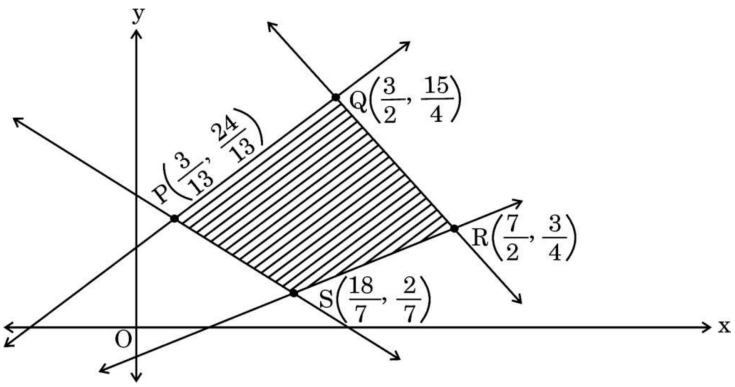
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13. For a Linear Programming Problem (LPP), the given objective function is $Z = x + 2y$. The feasible region $PQRS$ determined by the set of constraints is shown as a shaded region in the graph.

(Note: The figure is not to scale)

$P = \left(\frac{3}{13}, \frac{24}{13}\right)$, $Q = \left(\frac{3}{2}, \frac{15}{4}\right)$, $R = \left(\frac{7}{2}, \frac{3}{4}\right)$, $S = \left(\frac{18}{7}, \frac{2}{7}\right)$

Which of the following statements is correct? [1 mark]



(A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$

(B) Z is maximum at $R\left(\frac{7}{2}, \frac{3}{4}\right)$

(C) (Value of Z at P) $>$ (Value of Z at Q)

(D) (Value of Z at Q) $<$ (Value of Z at R)

Correct Answer: (A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$

Solution:

Concept:

- A linear objective on a bounded polygonal feasible region attains extrema at vertices.
- Evaluate $Z = x + 2y$ using the exact printed vertex coordinates.

Quick Tip to Solve: Use exact coordinates, since the printed graph explicitly is not to scale.

Step 1: Evaluate the objective at each vertex The four values are

$$Z(P) = \frac{3}{13} + 2\frac{24}{13} = \frac{51}{13},$$
$$Z(Q) = \frac{3}{2} + 2\frac{15}{4} = 9,$$
$$Z(R) = \frac{7}{2} + 2\frac{3}{4} = 5,$$
$$Z(S) = \frac{18}{7} + 2\frac{2}{7} = \frac{22}{7}.$$

The graph supplies a bounded feasible quadrilateral, so these vertex values suffice.

Step 2: Compare the statements The ordering is $22/7 < 51/13 < 5 < 9$. For the first comparison, $22 \times 13 = 286 < 357 = 51 \times 7$. Thus S gives the minimum and Q gives the maximum. Option (B) is false because R is not the maximum. Options (C) and (D) reverse the comparisons with Q.

Step 3: Conclusion Z is minimum at $S(18/7, 2/7)$. The correct option is (A).

Quick Tip to Solve: Use exact coordinates, since the printed graph explicitly is not to scale.

14. The area of the region bounded by the curve $y^2 = x$ between $x = 0$ and $x = 1$ is: [1 mark]

(A) $\frac{3}{2}$ sq units

(B) $\frac{2}{3}$ sq units

(C) 3 sq units

(D) $\frac{4}{3}$ sq units

Correct Answer: (D) $\frac{4}{3}$ sq units

Solution:

Concept:

- The parabola $y^2 = x$ has upper and lower branches $y = \sqrt{x}$ and $y = -\sqrt{x}$.
- Area between curves is the integral of upper ordinate minus lower ordinate.

Step 1: Choose the full vertical strip For $0 \le x \le 1$, the bounded region extends from $-\sqrt{x}$ to $\sqrt{x}$. Its strip height is $\sqrt{x} - (-\sqrt{x}) = 2\sqrt{x}$. Integrating only the positive branch would count just half of the enclosed region.

Step 2: Evaluate and test the options The area is

$$A = \int_0^1 2x^{1/2}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3}.$$

The value $2/3$ in option (B) is the upper-half area. The values $3/2$ and 3 in (A) and (C) do not equal the definite integral for both branches.

Step 3: Conclusion The area is $\frac{4}{3}$ square units. The correct option is (D).

Quick Tip to Solve: For $y^2 = x$, include both signs of y in the strip height.

15. Let $|\vec{a}| = 5$ and $-2 \le \lambda \le 1$. Then, the range of $|\lambda\vec{a}|$ is: [1 mark]

(A) $[5, 10]$

(B) $[-2, 5]$

(C) $[-2, 1]$

(D) $[-10, 5]$

Correct Answer: The correct range is $[0, 10]$; none of the printed options is correct.

Solution:

Concept:

- The magnitude of a scalar multiple is $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$.
- Magnitude is nonnegative even when the scalar is negative.

Step 1: Find the absolute-value range The scalar interval is $[-2, 1]$, which contains zero. Hence the least possible $|\lambda|$ is zero, attained at $\lambda = 0$, and the greatest is two, attained at $\lambda = -2$. Every intermediate absolute value is attained on $[-2, 0]$, so $|\lambda|$ has exactly the range $[0, 2]$.

Step 2: Scale and inspect every printed option With $|\vec{a}| = 5$, the range is $5[0, 2] = [0, 10]$. Option (A), $[5, 10]$, omits magnitudes between zero and five. Options (B), (C) and (D) include negative numbers, which cannot be vector magnitudes, and also fail to give the upper endpoint ten. Thus none of the four printed options states the correct range.

Step 3: Conclusion The range is $[0, 10]$. None of the printed options is correct.

Quick Tip to Solve: Apply the absolute value to the scalar before multiplying by five.

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16. The solution for the differential equation $\log\left(\dfrac{dy}{dx}\right) = 3x + 4y$ is: [1 mark]

(A) $3e^{4y} + 4e^{-3x} + C = 0$

(B) $e^{3x+4y} + C = 0$

(C) $3e^{-3y} + 4e^{4x} + 12C = 0$

(D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Correct Answer: (D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Solution:

Concept:

- Exponentiating $\log(y\prime) = 3x + 4y$ gives $y\prime = e^{3x+4y}$.
- For separable equations, collect all y terms with dy before integration.

Step 1: Separate and integrate The logarithm requires $y\prime > 0$, consistent with its exponential form. Separating gives $e^{-4y}\,dy = e^{3x}\,dx$. Hence

$$-\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + C.$$

Both antiderivatives retain the factors from differentiating their exponents.

Quick Tip to Solve: Retain the factors $-1/4$ and $1/3$ until clearing denominators.

Step 2: Compare the printed forms Multiply by twelve and move all terms to one side: $3e^{-4y} + 4e^{3x} + 12C = 0$. This matches (D). Differentiating (A) gives $y\prime = e^{-3x-4y}$, with both exponents reversed. Option (B) would make $3x + 4y$ constant and give $y\prime = -3/4$, outside the logarithm domain. Option (C) gives $y\prime = (16/9)e^{4x+3y}$, also incorrect.

Step 3: Conclusion $3e^{-4y} + 4e^{3x} + 12C = 0$. The correct option is (D).

Quick Tip to Solve: Retain the factors $-1/4$ and $1/3$ until clearing denominators.

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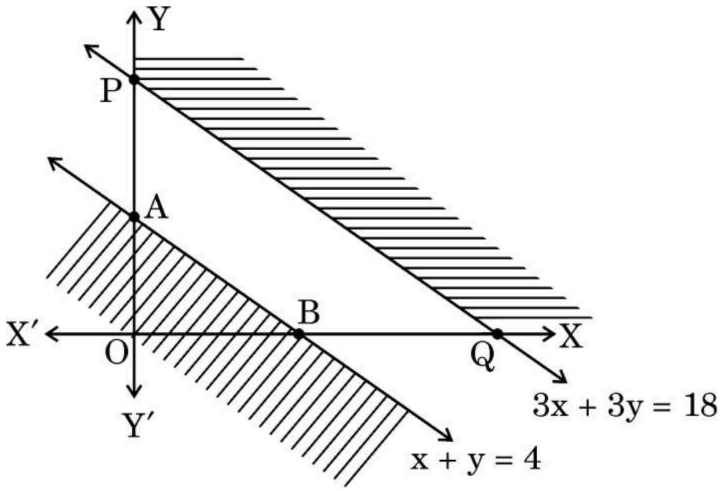
17. In a Linear Programming Problem (LPP), the objective function $Z = 2x + 5y$ is to be maximised under the following constraints:

$x + y \leq 4,\ 3x + 3y \geq 18,\ x, y \geq 0$

Study the graph and select the correct option.

(Note: The figure is not to scale)

The solution of the given LPP: [1 mark]



(A) lies in the shaded unbounded region.

(B) lies in $\triangle AOB$.

(C) does not exist.

(D) lies in the combined region of $\triangle AOB$ and unbounded shaded region.

Correct Answer: (C) does not exist.

Solution:

Concept:

- A feasible point must satisfy every constraint simultaneously; the feasible region is their common intersection.
- The objective function can be optimised only over the feasible region.

Step 1: Simplify the second inequality Dividing $3x + 3y \geq 18$ by the positive number 3 preserves the inequality and gives $x + y \geq 6$. The first constraint requires $x + y \leq 4$.

Quick Tip to Solve: Reduce proportional constraints first: incompatible bounds on the same expression immediately show infeasibility.

Step 2: Check the common region and the options A feasible point would therefore have to satisfy $6 \le x + y \le 4$, which is impossible. The triangle AOB satisfies the first inequality only, whereas the upper shaded region satisfies the second only. Neither shaded region, nor their union, satisfies both conditions together. Thus (A), (B) and (D) cannot describe a feasible solution; the coefficients in Z do not remove this contradiction.

Step 3: Conclusion The feasible region is empty, so the solution does not exist. Hence, the correct option is (C).

Quick Tip to Solve: Reduce proportional constraints first: incompatible bounds on the same expression immediately show infeasibility.

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18. Chances that three persons A, B, and C go to the market are 30%, 60% and 50% respectively. The probability that at least one will go to the market is: [1 mark]

(A) $\frac{14}{10}$

(B) $\frac{43}{50}$

(C) $\frac{9}{100}$

(D) $\frac{7}{50}$

Correct Answer: (B) $\frac{43}{50}$

Solution:

Concept:

- For independent events, the probability that none occurs is the product of their complementary probabilities.
- The event that at least one person goes is the complement of the event that all three stay away.

Step 1: Find the probability that nobody goes Let A, B, C denote the respective events of going to the market. Their complementary probabilities are $0.70, 0.40, 0.50$. Multiplying the probabilities of the independent decisions gives

$$P(A^c \cap B^c \cap C^c) = \frac{7}{10} \cdot \frac{4}{10} \cdot \frac{5}{10} = \frac{7}{50}.$$

Step 2: Use the complement and compare the options Therefore

$$P(A \cup B \cup C) = 1 - \frac{7}{50} = \frac{43}{50}.$$

Option (D) is the probability of nobody going, not of at least one going. Option (A) exceeds 1, so it cannot be a probability, and option (C) does not equal the computed complementary probability.

Step 3: Conclusion The required probability is $\frac{43}{50}$. Hence, the correct option is (B).

Quick Tip to Solve: For at least one independent occurrence, multiply the failure probabilities and subtract the product from 1.

19. Assertion (A): If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 256$ and $|\vec{b}| = 8$, then $|\vec{a}| = 2$.

Reason (R): $\sin^2\theta + \cos^2\theta = 1$ **and** $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ **and** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- If θ is the angle between two nonzero vectors, their dot and cross products involve $\cos\theta$ and $\sin\theta$, respectively.
- The identity $\sin^2\theta + \cos^2\theta = 1$ gives $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$.

Step 1: Judge the Assertion Using the identity, the given equality becomes $|\vec{a}|^2 \cdot 8^2 = 256$. Hence $|\vec{a}|^2 = 4$. A magnitude is nonnegative, so $|\vec{a}| = 2$; the Assertion is true.

Step 2: Judge the Reason The trigonometric identity and both stated product formulae are valid. Squaring the scalar dot product also removes any possible negative sign from $\cos\theta$. Thus the Reason is true.

Step 3: Check the explanatory link Squaring the two product formulae and adding produces exactly the identity used to determine $|\vec{a}|$. The Reason therefore explains the Assertion directly, excluding (B). Since neither statement is false, (C) and (D) are also excluded.

Step 4: Conclusion Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: When squared dot and cross products are added, factor out the squared magnitudes and use the trigonometric identity.

20. Assertion (A): Let $f(x) = e^x$ and $g(x) = \log x$. Then $(f+g)(x) = e^x + \log x$, where domain of $(f+g)$ is $\mathbb{R}$.

Reason (R): Dom$(f+g)$ = Dom$(f) \cap$ Dom(g). [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- The sum $(f+g)(x)$ is defined only where both $f(x)$ and $g(x)$ are defined.
- Over the real numbers, e^x is defined for all real x, while $\log x$ requires $x > 0$.

Step 1: Judge the Assertion The formula $(f+g)(x) = e^x + \log x$ is correct wherever both terms exist. However, at $x = 0$ or a negative real number the logarithm is undefined. Hence the asserted domain $\mathbb{R}$ is incorrect and the Assertion as a whole is false.

Step 2: Judge the Reason Both function values must exist before they can be added, so $\text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g)$. The Reason is true.

Step 3: Apply the domain rule The actual domain is

$$\mathbb{R} \cap (0, \infty) = (0, \infty).$$

Thus (A), (B) and (C), each of which declares the Assertion true, are excluded. The Reason gives the correct rule and reveals the error in the asserted domain.

Step 4: Conclusion The Assertion is false, but the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For a sum of functions, intersect their domains; a logarithm forces its argument to be strictly positive.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). Differentiate $\sqrt{e^{\sqrt{2x}}}$ with respect to $e^{\sqrt{2x}}$, for $x > 0$. [2 marks]

Solution:

Concept:

- To differentiate $u(x)$ with respect to $v(x)$, use $\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$ when $dv/dx \neq 0$.
- Equivalently, express u directly as a function of v and apply the power rule.

Quick Tip to Solve: Set the expression named after with respect to equal to a new variable; the outer square root then gives the derivative immediately.

Step 1: Express the dependent quantity using the new variable Let $v = e^{\sqrt{2x}}$ and $u = \sqrt{e^{\sqrt{2x}}}$. Since an exponential is positive, $u = v^{1/2}$. The required derivative is with respect to v, not with respect to x. By the power rule,

$$\frac{du}{dv} = \frac{1}{2}v^{-1/2}.$$

Step 2: Verify using differentiation with respect to x For $x > 0$, $\frac{d}{dx}\sqrt{2x} = \frac{1}{\sqrt{2x}}$. Thus

$$\frac{dv}{dx} = \frac{e^{\sqrt{2x}}}{\sqrt{2x}},$$
$$\frac{du}{dx} = \frac{e^{\sqrt{2x}/2}}{2\sqrt{2x}}.$$

The first derivative is nonzero, and their ratio is

$$\frac{du/dx}{dv/dx} = \frac{1}{2}e^{-\sqrt{2x}/2} = \frac{1}{2\sqrt{e^{\sqrt{2x}}}}.$$

This agrees with the direct power-rule calculation.

Step 3: Conclusion $\frac{d\sqrt{e^{\sqrt{2x}}}}{d(e^{\sqrt{2x}})} = \frac{1}{2\sqrt{e^{\sqrt{2x}}}}$.

Quick Tip to Solve: Set the expression named after with respect to equal to a new variable; the outer square root then gives the derivative immediately.

OR

21(b). If $(x)^y = (y)^x$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- Logarithmic differentiation transforms variable powers into products: $\log(x^y) = y\log x$, for positive real bases.
- When $y = y(x)$, use the product rule and $\frac{d}{dx}\log y = \frac{1}{y}\frac{dy}{dx}$.

Step 1: Take logarithms On a positive real differentiable branch of $x^y = y^x$, take logarithms of both sides to obtain

$$y\log x = x\log y.$$

This avoids applying the ordinary constant-exponent power rule incorrectly to a power in which both the base and exponent vary.

Step 2: Differentiate implicitly and collect the derivative Differentiating both products with respect

Quick Tip to Solve: Take logarithms before differentiating variable powers, then collect every term containing dy/dx on the same side.

to x, remembering that y depends on x, gives

$$\frac{dy}{dx}\log x + \frac{y}{x} = \log y + \frac{x}{y}\frac{dy}{dx}.$$

Move all derivative terms to the left and the remaining terms to the right:

$$\left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}.$$

Division by the coefficient yields the required derivative wherever that coefficient is nonzero. Multiplying numerator and denominator by xy gives an equivalent form without nested fractions.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{\log y - y/x}{\log x - x/y} = \frac{xy\log y - y^2}{xy\log x - x^2}}$$

The formula applies where its denominator is nonzero.

Quick Tip to Solve: Take logarithms before differentiating variable powers, then collect every term containing dy/dx on the same side.

22(a). If $\vec{a}$ and $\vec{b}$ are position vectors of point A and point B respectively, find the position vector of point C on BA produced such that $BC = 3BA$. [2 marks]

Solution:

Concept:

- The directed vector from a point with position vector $\vec{b}$ to one with position vector $\vec{a}$ is $\overrightarrow{BA} = \vec{a} - \vec{b}$.
- On a ray, distances multiply the corresponding directed displacement by a positive scalar. Position vectors are measured from the same origin O.

Step 1: Identify the direction of the produced line The line BA produced means that we start at B, pass through A, and continue beyond A to C. Thus B, A, C occur in this order. Since $BC = 3BA$, the vectors $\overrightarrow{BC}$ and $\overrightarrow{BA}$ have the same direction, and

$$\overrightarrow{BC} = 3\overrightarrow{BA} = 3(\vec{a} - \vec{b}).$$

The sign is positive because the extension is beyond A, not in the opposite direction from B.

Step 2: Add the displacement to the starting position Write the vector path from O to C through B:

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \vec{b} + 3(\vec{a} - \vec{b}).$$

Expanding gives $\overrightarrow{OC} = 3\vec{a} - 2\vec{b}$. As a check, $\overrightarrow{AC} = \overrightarrow{OC} - \vec{a} = 2(\vec{a} - \vec{b})$, so $AC = 2BA$, consistent with $BC = BA + AC = 3BA$.

Step 3: Conclusion The required position vector is $\boxed{\overrightarrow{OC} = 3\vec{a} - 2\vec{b}}$.

Quick Tip to Solve: For a line BA produced, begin at B and add the required positive multiple of the displacement A minus B.

OR

22(b). **Vector $\vec{r}$ is inclined at equal angles to the three axes x, y and z. If magnitude of $\vec{r}$ is $5\sqrt{3}$ units, then find $\vec{r}$.** [2 marks]

Solution:

Concept:

- If a vector makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$, and $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$.

Step 1: Use the equality of the three angles As the vector is inclined at equal angles to the three axes, $\alpha = \beta = \gamma$; hence $l = m = n$. Substitute this equality in the direction-cosine identity:

$$l^2 + l^2 + l^2 = 1 \quad \Longrightarrow \quad 3l^2 = 1.$$

Therefore $l = m = n = 1/\sqrt{3}$, or all three are $-1/\sqrt{3}$. Mixed signs would give unequal angles with the positive axes, so they are excluded.

Step 2: Recover the components from the magnitude The magnitude is $|\vec{r}| = 5\sqrt{3}$. For the positive common direction cosine,

$$\vec{r} = 5\sqrt{3}\left(\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right) = 5\hat{i} + 5\hat{j} + 5\hat{k}.$$

The negative common cosine gives the opposite vector. Both have magnitude $\sqrt{25 + 25 + 25} = 5\sqrt{3}$; each makes three equal angles. If the equal angles are taken to be acute, only the positive choice applies.

Step 3: Conclusion $\boxed{\vec{r} = \pm 5(\hat{i} + \hat{j} + \hat{k})}$, with a common sign. For equal acute angles, $\vec{r} = 5\hat{i} + 5\hat{j} + 5\hat{k}$.

Quick Tip to Solve: Equal angles force equal direction cosines; use their squares to find the common cosine before multiplying by the magnitude.

23. **Determine those values of x for which $f(x) = \dfrac{2}{x} - 5,\ x \neq 0$ is increasing or decreasing.** [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where $f'(x) > 0$ and strictly decreasing where $f'(x) < 0$.
- The function and derivative must be defined throughout the interval. A point excluded from the domain splits the intervals to be considered.

Step 1: Differentiate on the domain The given function is $f(x) = 2x^{-1} - 5$, with domain $\mathbb{R} \setminus \{0\}$. By

Quick Tip to Solve: For a reciprocal function, split at the excluded point before applying the derivative sign test; a negative derivative does not bridge a gap in the domain.

the power rule,

$$f'(x) = 2(-1)x^{-2} - 0 = -\frac{2}{x^2}.$$

The constant -5 shifts the function vertically and makes no contribution to the derivative. At $x = 0$, neither the function nor this derivative is defined.

Step 2: Determine the derivative sign on each interval For every allowed value $x \neq 0$, $x^2 > 0$. Consequently $-2/x^2 < 0$ on both $(-\infty, 0)$ and $(0, \infty)$. There is no value in the domain for which the derivative is positive, and it never vanishes. Apply the monotonicity test separately on the two intervals: the function decreases on each. We do not join them across zero because the function is discontinuous there, and comparisons across the gap need not satisfy decreasing behaviour.

Step 3: Conclusion $\boxed{f \text{ is strictly decreasing on } (-\infty, 0) \text{ and } (0, \infty).}$ It is increasing on no interval of its domain.

Quick Tip to Solve: For a reciprocal function, split at the excluded point before applying the derivative sign test; a negative derivative does not bridge a gap in the domain.

24. Find the domain of $f(x) = \sin^{-1}(-x^2)$. [2 marks]

Solution:

Concept:

- The real inverse sine function $\sin^{-1} t$ is defined precisely for $-1 \leq t \leq 1$.
- For a composite function, impose the domain condition of the outer function on the whole inner expression, here $t = -x^2$.

Step 1: Impose the inverse sine restriction For $f(x) = \sin^{-1}(-x^2)$ to have a real value, its input to inverse sine must lie between -1 and 1, inclusive. Thus

$$-1 \leq -x^2 \leq 1.$$

For real x, $x^2 \geq 0$, so $-x^2 \leq 0 \leq 1$. The upper bound is therefore automatically satisfied; only the lower bound can restrict x.

Step 2: Solve the remaining inequality and check endpoints The remaining restriction is $-x^2 \geq -1$. Multiplying both sides by -1 reverses the inequality and gives $x^2 \leq 1$. Equivalently, $|x| \leq 1$, so $-1 \leq x \leq 1$. At either endpoint $x = \pm 1$, the input is -1, for which inverse sine is defined. At $x = 0$, the input is zero. Outside this interval, $x^2 > 1$, so $-x^2 < -1$, and there is no real inverse sine value.

Step 3: Conclusion The domain is $\boxed{[-1, 1]}$.

Quick Tip to Solve: Apply the inverse sine restriction to its input, then reverse the inequality when multiplying by minus one; retain both endpoints.

25. Find the value of λ if the following lines are perpendicular to each other:

$l_1 : \dfrac{1-x}{-3} = \dfrac{3y-2}{2\lambda} = \dfrac{z-3}{3}$

$l_2 : \dfrac{x-1}{3\lambda} = \dfrac{1-y}{1} = \dfrac{2z-5}{3}$ [2 marks]

Solution:

Concept:

- A line with parameter t written as $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ has direction ratios (a, b, c).
- Perpendicular lines have orthogonal direction vectors, so their direction ratios satisfy $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Step 1: Extract the actual direction ratios Put each ratio of l_1 equal to t. Then

$$1 - x = -3t, \quad 3y - 2 = 2\lambda t, \quad z - 3 = 3t.$$

Thus $x = 1 + 3t$, $y = 2/3 + (2\lambda/3)t$, $z = 3 + 3t$, giving direction ratios $(3, 2\lambda/3, 3)$. For l_2, equate its ratios to s:

$$x = 1 + 3\lambda s, \quad y = 1 - s, \quad z = 5/2 + (3/2)s.$$

Its direction ratios are $(3\lambda, -1, 3/2)$. The factors 3 in $3y - 2$ and 2 in $2z - 5$ must be accounted for; the printed denominators alone are not the direction ratios.

Step 2: Apply the perpendicularity condition Take the scalar product of the two direction vectors and set it equal to zero:

$$3(3\lambda) + \frac{2\lambda}{3}(-1) + 3\left(\frac{3}{2}\right) = 0.$$

Collecting the terms gives

$$9\lambda - \frac{2\lambda}{3} + \frac{9}{2} = 0 \quad \Longrightarrow \quad \frac{25\lambda}{3} = -\frac{9}{2}.$$

Multiply by $3/25$ to obtain $\lambda = -27/50$. This is nonzero, so both lines have well-defined symmetric forms at the required value.

Step 3: Conclusion The required value is $\boxed{\lambda = -\dfrac{27}{50}}$.

Quick Tip to Solve: First solve the symmetric equations for x, y and z in terms of a parameter; use those coefficients, not the raw denominators, in the dot product.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. If $A = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 4 \\ 0 & 5 & 1 \end{pmatrix}$ **and** $C = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ **are three matrices, then find** ABC**.** [3 marks]

Solution:

Concept:

- Matrix multiplication is defined when the number of columns of the first matrix equals the number of rows of the second. Here the orders of A, B, C are $1 \times 3, 3 \times 3, 3 \times 1$, respectively.
- The entry of a product is the dot product of the corresponding row and column. Associativity gives $ABC = (AB)C = A(BC)$; it does not allow changing the order of the factors.

Step 1: Compute the row matrix AB Multiply the row of A by each column of B, retaining the negative sign in the second entry of A:

$$AB = \begin{pmatrix} 1(2) + (-1)(-1) + 0(0) & 1(0) + (-1)(3) + 0(5) & 1(1) + (-1)(4) + 0(1) \end{pmatrix} = \begin{pmatrix} 3 & -3 & -3 \end{pmatrix}.$$

The product has order 1×3, so it can now multiply the column matrix C.

Step 2: Multiply by the column matrix Using the entries of C in their printed order,

$$(AB)C = \begin{pmatrix} 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3(2) - 3(3) - 3(4) \end{pmatrix} = \begin{pmatrix} 6 - 9 - 12 \end{pmatrix}.$$

As a check, $BC = \begin{pmatrix} 8 \\ 23 \\ 19 \end{pmatrix}$, and $A(BC) = \begin{pmatrix} 8 - 23 \end{pmatrix}$, which agrees with the same computation.

Step 3: Conclusion Therefore, $\boxed{ABC = \begin{pmatrix} -15 \end{pmatrix}}$, a 1×1 matrix.

Quick Tip to Solve: For this row matrix, compute AB by subtracting the second row of B from its first row; the third row contributes zero.

27. Consider the Linear Programming Problem, where the objective function $Z = (x + 4y)$ needs to be minimized subject to constraints

$2x + y \geq 1000$

$x + 2y \geq 800$

$x, y \geq 0$.

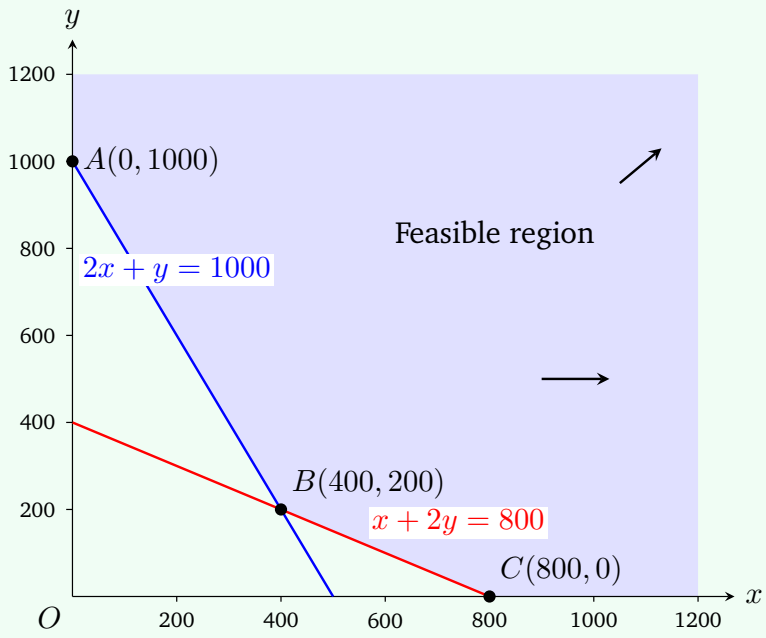
Draw a neat graph of the feasible region and find the minimum value of Z. [3 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes satisfying all constraints. For a linear objective, compare its values at feasible corner points; for an unbounded region, also establish that no feasible point gives a smaller value.
- For $Z = x + 4y$, evaluate the objective at each vertex. Non-negativity restricts the graph to the first quadrant.

Step 1: Draw and identify the feasible region The line $2x + y = 1000$ meets the axes at $(500, 0)$ and $(0, 1000)$. The line $x + 2y = 800$ meets them at $(800, 0)$ and $(0, 400)$. The origin does not satisfy either inequality, so take the side away from the origin for each line. Their common portion in the first quadrant is shaded in the graph. The lower boundary consists of two line segments followed by the positive-axis ray from $(800, 0)$; the region extends upwards and to the right.



The shaded feasible region extends without bound upwards and to the right.

Step 2: Find the corner points and objective values At the intersection, subtract $x + 2y = 800$ from twice $2x + y = 1000$ to get $3x = 1200$. Hence $x = 400$ and $y = 200$. The feasible vertices and their

Quick Tip to Solve: Use $Z = (x + 2y) + 2y$: the second constraint and $y \geq 0$ give an immediate lower bound of 800, which the graph confirms.

objective values are

Vertex	$Z = x + 4y$
$(0, 1000)$	4000
$(400, 200)$	1200
$(800, 0)$	800

The other intercepts $(500, 0)$ and $(0, 400)$ violate one of the inequalities and are therefore excluded.

Step 3: Confirm the minimum for the unbounded region Every feasible point satisfies $x + 2y \geq 800$ and $y \geq 0$, so

$$Z = x + 4y = (x + 2y) + 2y \geq 800.$$

Equality requires $y = 0$ and $x = 800$. This point also satisfies $2x + y = 1600 \geq 1000$, so the bound is attained within the shaded region. This verifies the minimum despite the unbounded feasible region.

Step 4: Conclusion The minimum value is $\boxed{Z_{\min} = 800}$, attained uniquely at $\boxed{(x, y) = (800, 0)}$.

Quick Tip to Solve: Use $Z = (x + 2y) + 2y$: the second constraint and $y \geq 0$ give an immediate lower bound of 800, which the graph confirms.

28(a). Find the distance of the point $P(2, 4, -1)$ from the line $\dfrac{x+5}{1} = \dfrac{y+3}{4} = \dfrac{z-6}{-9}$. [3 marks]

Solution:

Concept:

- A point on a line with point A and direction vector $\mathbf{d}$ is $\mathbf{r} = \mathbf{a} + t\mathbf{d}$. At the perpendicular foot H from P, $\overrightarrow{HP} \cdot \mathbf{d} = 0$.
- The shortest distance from a point to a line is the length of its perpendicular segment. For coordinates (u, v, w), the length of a vector is $\sqrt{u^2 + v^2 + w^2}$.

Step 1: Parametrize the printed line Set each ratio equal to t. Then

$$x = -5 + t, \qquad y = -3 + 4t, \qquad z = 6 - 9t.$$

Thus $A = (-5, -3, 6)$ is on the line and $\mathbf{d} = (1, 4, -9)$ is its direction vector. Since $P = (2, 4, -1)$, $\overrightarrow{AP} = (7, 7, -7)$. Note that the numerator $y + 3$ gives the base coordinate -3, not 3.

Step 2: Find the perpendicular foot Write $H = (-5 + t, -3 + 4t, 6 - 9t)$. Perpendicularity requires

$$\begin{aligned} \overrightarrow{HP} \cdot \mathbf{d} &= (7 - t) + 4(7 - 4t) - 9(-7 + 9t) \\ &= 98 - 98t = 0. \end{aligned}$$

Hence $t = 1$, and the perpendicular foot is $H = (-4, 1, -3)$. This point satisfies all three ratios in the line equation, each equal to 1.

Step 3: Calculate the segment length Now $\overrightarrow{HP} = (2 + 4, 4 - 1, -1 + 3) = (6, 3, 2)$. Therefore

$$PH = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{36 + 9 + 4} = \sqrt{49}.$$

Quick Tip to Solve: Set the three line ratios equal to t, then use the dot product with (1,4,-9) to find the perpendicular foot directly.

Also $(6,3,2)\cdot(1,4,-9) = 6+12-18 = 0$, confirming that the segment used is perpendicular to the line.

Step 4: Conclusion The required shortest distance is $\boxed{7 \text{ units}}$.

Quick Tip to Solve: Set the three line ratios equal to t, then use the dot product with (1,4,-9) to find the perpendicular foot directly.

OR

28(b). Let the position vectors of the points A, B and C be $3\hat{i}-\hat{j}-2\hat{k}$, $\hat{i}+2\hat{j}-\hat{k}$ and $\hat{i}+5\hat{j}+3\hat{k}$ respectively. Find the vector and cartesian equations of the line passing through A and parallel to line BC. [3 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{d}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- The direction vector of BC is $\overrightarrow{BC} = \vec{c} - \vec{b}$. Its components are direction ratios; a zero component means that the corresponding coordinate is constant.

Step 1: Find the direction of the given line From the given position vectors, $A = (3,-1,-2)$, $B = (1,2,-1)$ and $C = (1,5,3)$. Subtract the coordinates of the starting point B from those of C:

$$\overrightarrow{BC} = (1-1)\hat{i} + (5-2)\hat{j} + (3-(-1))\hat{k} = 3\hat{j} + 4\hat{k}.$$

Thus suitable direction ratios are $0, 3, 4$. The zero first component is essential: movement along this direction never changes the x-coordinate.

Step 2: Use the point and direction to obtain both forms Taking $\vec{a} = 3\hat{i}-\hat{j}-2\hat{k}$ and the direction vector above, the line is parametrised by

$$\vec{r} = (3\hat{i}-\hat{j}-2\hat{k}) + \lambda(3\hat{j}+4\hat{k}), \quad \lambda \in \mathbb{R}.$$

Equating components gives $x = 3$, $y = -1+3\lambda$ and $z = -2+4\lambda$. Eliminating the parameter from the last two gives $(y+1)/3 = (z+2)/4$. Do not divide $x-3$ by zero: retain $x = 3$ separately. At $\lambda = 0$ the point is A, and the coefficient of λ is exactly $\overrightarrow{BC}$, checking both required conditions.

Step 3: Conclusion The vector equation is $\boxed{\vec{r} = 3\hat{i}-\hat{j}-2\hat{k} + \lambda(3\hat{j}+4\hat{k}),\ \lambda \in \mathbb{R}}$. The Cartesian equations are $\boxed{x = 3, \quad \dfrac{y+1}{3} = \dfrac{z+2}{4}}$.

Quick Tip to Solve: If a direction ratio is zero, write its coordinate as a constant instead of placing zero in a symmetric denominator.

29(a). Differentiate $y = \sin^{-1}(3x - 4x^3)$ w.r.t. x, if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. [3 marks]

Solution:

Concept:

- The principal values of $\sin^{-1} u$ lie in $[-\pi/2, \pi/2]$. Therefore $\sin^{-1}(\sin v) = v$ only when v lies in that interval.
- The triple-angle identity is $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, and $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$.

Step 1: Check the principal-value branch Put $\theta = \sin^{-1} x$, so $x = \sin\theta$. Since $-1/2 \le x \le 1/2$, monotonicity of the principal inverse sine gives $-\pi/6 \le \theta \le \pi/6$. Consequently $-\pi/2 \le 3\theta \le \pi/2$. This range check is necessary before simplifying the composition: without it, the inverse sine of a sine need not return the original angle.

Step 2: Simplify first and then differentiate The identity gives

$$3x - 4x^3 = 3\sin\theta - 4\sin^3\theta = \sin 3\theta.$$

Since 3θ lies in the principal-value range established above,

$$y = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x.$$

Differentiating this equivalent expression at interior points of the specified interval yields

$$\frac{dy}{dx} = 3\frac{d}{dx}(\sin^{-1} x) = \frac{3}{\sqrt{1-x^2}}.$$

At the two endpoints this expression gives the corresponding one-sided derivatives, both equal to $2\sqrt{3}$. The ordinary two-sided derivative is stated on the open interval; the original function is prescribed on a closed interval, so only one-sided derivatives within that domain apply at its boundary.

Step 3: Conclusion For $-1/2 < x < 1/2$, $\boxed{y\prime = 3/\sqrt{1-x^2}}$. Each one-sided endpoint derivative is $2\sqrt{3}$.

Quick Tip to Solve: For inverse sine of a triple-angle expression, bound three times the substituted angle before cancelling inverse sine and sine.

OR

29(b). Differentiate $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to x, when $x \in (0, 1)$. [3 marks]

Solution:

Concept:

- The identity $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ converts the rational expression into a double angle.

Quick Tip to Solve: Use x = tan of an angle for this rational double-angle form, then check that twice the angle lies in the inverse-cosine principal range.

- The principal range of $\cos^{-1} u$ is $[0, \pi]$, and $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$.

Step 1: Choose an angle and verify its range Let $\theta = \tan^{-1} x$; then $x = \tan\theta$. The stated condition $0 < x < 1$ implies $0 < \theta < \pi/4$, hence $0 < 2\theta < \pi/2$. This is inside the principal range $[0, \pi]$ of inverse cosine. Thus the inverse cosine will return 2θ, without a change of sign or any additional branch.

Step 2: Apply the identity and differentiate By the double-angle identity,

$$\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta.$$

Therefore

$$y = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1} x.$$

Differentiate to obtain $dy/dx = 2/(1+x^2)$. As a check on the sign, the inner rational function decreases as positive x increases, while inverse cosine is decreasing; their composition must increase. Direct differentiation also agrees: the inner derivative is $-4x/(1+x^2)^2$, and the square root in the inverse-cosine formula is $2x/(1+x^2)$, because $x > 0$. Their quotient, with the extra minus sign of inverse cosine, is $2/(1+x^2)$.

Step 3: Conclusion $\boxed{\dfrac{dy}{dx} = \dfrac{2}{1+x^2}}$, for $0 < x < 1$.

Quick Tip to Solve: Use x = tan of an angle for this rational double-angle form, then check that twice the angle lies in the inverse-cosine principal range.

30(a). A student wants to pair up natural numbers in such a way that they satisfy the equation $2x + y = 41,\ x, y \in \mathbb{N}$. Find the domain and range of the relation. Check if the relation thus formed is reflexive, symmetric and transitive. Hence, state whether it is an equivalence relation or not.

[3 marks]

Solution:

Concept:

- The domain of a relation consists of its first coordinates, and the range consists of its second coordinates. Here $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- Reflexivity requires every (a, a); symmetry requires (b, a) whenever (a, b) occurs; transitivity requires (a, c) whenever (a, b) and (b, c) occur. An equivalence relation satisfies all three.

Step 1: Enumerate the permitted coordinate values Write $y = 41 - 2x$. Since both coordinates are positive integers, $41 - 2x \geq 1$, so $1 \leq x \leq 20$. Conversely each integer in this interval gives a positive integer y. Thus

$$R = \{(x, 41-2x) : x = 1, 2, \ldots, 20\}.$$

The first pair is $(1, 39)$, the last is $(20, 1)$, and the second coordinates decrease by two. Hence the domain is $\{1, 2, \ldots, 20\}$, and the range is $\{1, 3, 5, \ldots, 39\}$.

Step 2: Test reflexivity and symmetry using counterexamples This relation on $\mathbb{N}$ is not reflexive:

Quick Tip to Solve: To disprove transitivity, find two permitted pairs with a common middle coordinate, then check their first-to-last pair explicitly.

$(1,1) \notin R$, since $2(1)+1 = 3 \neq 41$. Indeed a diagonal pair would require $3a = 41$, which has no natural-number solution. It is not symmetric either: $(1,39) \in R$, whereas $(39,1) \notin R$, since $2(39)+1 = 79 \neq 41$. A single such counterexample suffices to disprove each universal property.

Step 3: Test transitivity Choose $(11,19)$ and $(19,3)$. Both belong to the relation because $2(11)+19 = 41$ and $2(19)+3 = 41$. Transitivity would require $(11,3) \in R$, but $2(11)+3 = 25 \neq 41$. Thus the relation is not transitive. Since the required three properties fail, it cannot be an equivalence relation.

Step 4: Conclusion $\boxed{\text{Dom}\, R = \{1,2,\ldots,20\}, \quad \text{Ran}\, R = \{1,3,\ldots,39\}}$. The relation is neither reflexive, nor symmetric, nor transitive; hence it is not an equivalence relation.

Quick Tip to Solve: To disprove transitivity, find two permitted pairs with a common middle coordinate, then check their first-to-last pair explicitly.

OR

30(b). Show that the function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is a set of natural numbers, given by $f(n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$ is a bijection. [3 marks]

Solution:

Concept:

- A function is bijective if it is injective (equal images imply equal inputs) and surjective (every element of the codomain has a preimage).
- Here $\mathbb{N} = \{1,2,3,\ldots\}$. The rule switches each consecutive pair of natural numbers and changes the parity of every input.

Step 1: Check the function is well defined For an even input n, we have $n \geq 2$, so $n-1$ is a positive odd integer. For an odd input n, the image $n+1$ is a positive even integer. Thus both branches give images in $\mathbb{N}$.

Step 2: Prove injectivity Let $f(a) = f(b)$. If both inputs are even, then $a-1 = b-1$, giving $a = b$. If both inputs are odd, then $a+1 = b+1$, again giving $a = b$. If their parities differ, their images have different parities: the even input gives an odd image and the odd input gives an even image. Such images cannot be equal. Hence equality of images always implies equality of inputs, proving injectivity.

Step 3: Prove surjectivity Take any $m \in \mathbb{N}$. If m is odd, choose the even input $n = m+1$; then $f(n) = n-1 = m$. If m is even, choose the odd input $n = m-1$, which is positive since $m \geq 2$; then $f(n) = n+1 = m$. These cases cover every element of the codomain, so f is surjective.

Step 4: Conclusion The function is both injective and surjective. Therefore $\boxed{f \text{ is a bijection}}$.

Quick Tip to Solve: Applying the rule twice gives $f(f(n)) = n$, so this function is its own inverse.

31. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed three times, find the probability distribution of number of tails. Hence, find the mean of the distribution. [3 marks]

Solution:

Concept:

- Let X denote the number of tails in three independent tosses. Then X takes the values $0, 1, 2, 3$.
- For independent trials with tail probability p, the binomial formula is $P(X = k) = \binom{3}{k} p^k (1-p)^{3-k}$. The mean is the weighted sum $E(X) = \sum k P(X = k)$.

Step 1: Find the probabilities of head and tail Write $P(T) = p$. The given bias means $P(H) = 3p$. Since head and tail are the only outcomes, $p + 3p = 1$. Thus $P(T) = \frac{1}{4}$ and $P(H) = \frac{3}{4}$. The more likely outcome is head, so the count being modelled uses success probability $1/4$, not $3/4$.

Step 2: Calculate the probability distribution Exactly k tails can occupy $\binom{3}{k}$ positions. Each such sequence has probability $(1/4)^k (3/4)^{3-k}$. Therefore

$$\begin{aligned} P(X = 0) &= (3/4)^3 = 27/64, \\ P(X = 1) &= 3(1/4)(3/4)^2 = 27/64, \\ P(X = 2) &= 3(1/4)^2(3/4) = 9/64, \\ P(X = 3) &= (1/4)^3 = 1/64. \end{aligned}$$

All these probabilities are nonnegative and their sum is $(27 + 27 + 9 + 1)/64 = 1$, as required for a distribution.

Step 3: Compute the mean Using the weighted average of the possible tail counts,

$$E(X) = 0 \cdot \frac{27}{64} + 1 \cdot \frac{27}{64} + 2 \cdot \frac{9}{64} + 3 \cdot \frac{1}{64} = \frac{48}{64} = \frac{3}{4}.$$

This agrees with the binomial mean $np = 3(1/4)$.

Step 4: Conclusion The distribution is $\boxed{P(X = 0, 1, 2, 3) = \frac{1}{64}(27, 27, 9, 1)}$, respectively, and its mean is $\boxed{\frac{3}{4}}$.

Quick Tip to Solve: For a count of tails, use the tail probability $1/4$; the mean is three times that probability.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Solve the differential equation: $x^2 y\,dx - (x^3 + y^3)\,dy = 0$. [5 marks]

Solution:

Concept:

- A homogeneous equation can be simplified by using the ratio of its variables. Here taking x as a function of y makes the substitution $x = vy$ convenient.

Quick Tip to Solve: Treat x as a function of y: the substitution $x = vy$ cancels the linear ratio term immediately.

- With $v = x/y$, the product rule gives $dx/dy = v + y\,dv/dy$. Separation then puts all factors of v on one side and those of y on the other.

Step 1: Express the equation in homogeneous form The given differential equation is

$$x^2 y\,dx - (x^3 + y^3)\,dy = 0.$$

On a portion of a solution where $x \neq 0$ and $y \neq 0$, divide by $x^2 y\,dy$ to obtain

$$\frac{dx}{dy} = \frac{x^3 + y^3}{x^2 y} = \frac{x}{y} + \frac{y^2}{x^2}.$$

The right side depends only on x/y, which confirms homogeneity and suggests treating y as the independent variable.

Step 2: Substitute and separate the variables Put $x = vy$, so $dx/dy = v + y\,dv/dy$. Substitution gives

$$v + y\frac{dv}{dy} = v + \frac{1}{v^2}.$$

Subtract the common term v and multiply by v^2/y:

$$v^2\frac{dv}{dy} = \frac{1}{y}, \qquad v^2\,dv = \frac{dy}{y}.$$

This cancellation is why the choice $x = vy$ is particularly efficient here.

Step 3: Integrate and restore the original variables Integrating both sides yields

$$\frac{v^3}{3} = \ln|y| + C_1.$$

Multiplication by three and absorption of $3C_1$ into an arbitrary constant C gives

$$v^3 = 3\ln|y| + C.$$

Since $v = x/y$, this becomes

$$\frac{x^3}{y^3} = 3\ln|y| + C, \qquad y \neq 0.$$

The absolute value allows either sign of y on an interval that does not cross zero.

Step 4: Verify the result and check a branch lost by division Differentiate $x^3/y^3 - 3\ln|y| = C$ in differential form:

$$\frac{3x^2}{y^3}\,dx - \left(\frac{3x^3}{y^4} + \frac{3}{y}\right)dy = 0.$$

Multiplying by $y^4/3$ recovers $x^2 y\,dx - (x^3 + y^3)\,dy = 0$. Also, the constant function $y = 0$ satisfies the original equation directly because $dy = 0$; it was excluded by division by y. The line $x = 0$ is not a further solution branch, since for varying y it would require $-y^3\,dy = 0$.

Step 5: Conclusion The general nonzero branch is $\boxed{x^3 = y^3\big(3\ln|y| + C\big),\ y \neq 0}$. The additional constant solution is $\boxed{y = 0}$.

Quick Tip to Solve: Treat x as a function of y: the substitution $x = vy$ cancels the linear ratio term immediately.

OR

32(b). Solve the differential equation $(1+x^2)\dfrac{dy}{dx}+2xy-4x^2=0$ subject to initial condition $y(0)=0$. [5 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\dfrac{dy}{dx}+P(x)y=Q(x)$. Its integrating factor is $I(x)=e^{\int P(x)\,dx}$, which makes the left side the derivative of $I(x)y$.
- The product rule gives $\dfrac{d}{dx}[(1+x^2)y]=(1+x^2)\dfrac{dy}{dx}+2xy$. An initial condition fixes the constant of integration and selects a particular solution.

Step 1: Put the equation in linear form Move the term independent of y to the right side of the given equation:

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2.$$

Since $1+x^2>0$ for every real x, division by this expression is valid throughout the real line. Thus

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}.$$

Comparing with the standard linear form, we have $P(x)=\dfrac{2x}{1+x^2}$ and $Q(x)=\dfrac{4x^2}{1+x^2}$.

Step 2: Calculate the integrating factor Use $u=1+x^2$, so that $du=2x\,dx$, to obtain

$$\int P(x)\,dx=\int\frac{2x}{1+x^2}\,dx=\log(1+x^2).$$

There is no absolute-value difficulty because $1+x^2$ is positive. Consequently, a suitable integrating factor is

$$I(x)=e^{\log(1+x^2)}=1+x^2.$$

A nonzero constant multiplier of the integrating factor would give the same family of solutions, so it is enough to use this simplest choice.

Step 3: Integrate the resulting product derivative Multiplying the linear equation by $1+x^2$ gives

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2.$$

The first two terms are exactly the derivative of $(1+x^2)y$, by the product rule. Therefore

$$\frac{d}{dx}\left[(1+x^2)y\right]=4x^2,$$
$$(1+x^2)y=\int 4x^2\,dx=\frac{4x^3}{3}+C.$$

The constant C must be retained until the given initial condition is applied.

Step 4: Apply the initial condition and check the expression Substituting $x=0$ and $y(0)=0$ into the

Quick Tip to Solve: Recognise $(1+x^2)y'+2xy$ as $\dfrac{d}{dx}[(1+x^2)y]$; integrate $4x^2$ directly and then use $y(0)=0$.

integrated equation gives $0 = C$. Thus the particular solution has the expression $y = \dfrac{4x^3}{3(1+x^2)}$. Its derivative is

$$y' = \frac{4x^2(3+x^2)}{3(1+x^2)^2}.$$

On substituting these expressions into the left side of the original equation, we obtain

$$\begin{aligned}(1+x^2)y' + 2xy - 4x^2 &= \frac{4x^2(3+x^2)+8x^4}{3(1+x^2)} - 4x^2 \\ &= 4x^2 - 4x^2 = 0.\end{aligned}$$

Also the expression takes the value zero at $x = 0$, so both the differential equation and its initial condition are satisfied.

Step 5: Conclusion The required particular solution is $\boxed{y = \dfrac{4x^3}{3(1+x^2)}}$, valid for all real x.

Quick Tip to Solve: Recognise $(1+x^2)y' + 2xy$ as $\dfrac{d}{dx}[(1+x^2)y]$; integrate $4x^2$ directly and then use $y(0) = 0$.

33. Use integration to find the area of the region enclosed by curve $y = -x^2$ and the straight lines $x = -3,\ x = 2$ and $y = 0$. Sketch a rough figure to illustrate the bounded region. [5 marks]

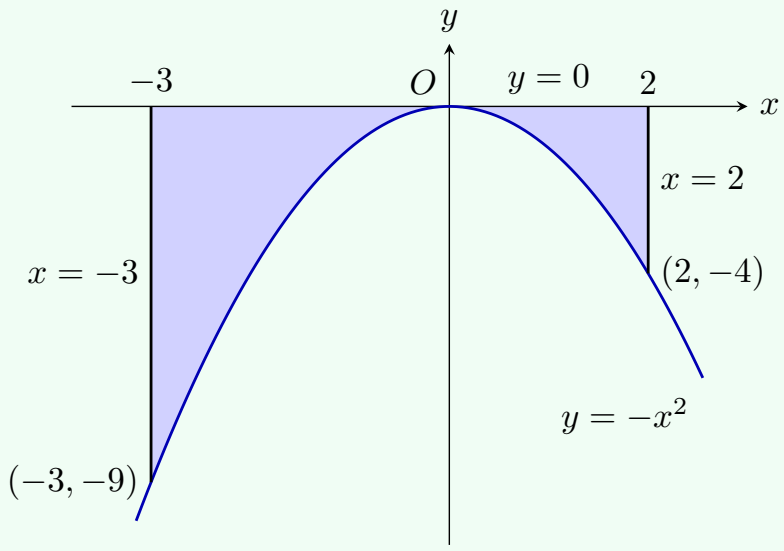
Solution:

Concept:

- For a region between two curves, its area is $A = \int_a^b [y_{\text{upper}} - y_{\text{lower}}]\, dx$, where a, b are the limiting abscissae. The integrand is the non-negative height of a vertical strip.
- Here the parabola $y = -x^2$ opens downwards and meets the x-axis $y = 0$ at the origin. Thus $y = 0$ is the upper boundary and $y = -x^2$ is the lower boundary throughout $[-3, 2]$.
- The power rule gives $\int x^2\, dx = x^3/3 + C$. A definite integral of a strip height gives a positive geometrical area, even when the region lies below the x-axis.

Step 1: Identify and sketch the bounded region At the two given vertical boundaries, the points on the parabola are $(-3, -9)$ and $(2, -4)$. The corresponding points on $y = 0$ are $(-3, 0)$ and $(2, 0)$. The parabola touches the horizontal axis at $(0, 0)$; the region consists of the two shaded portions between the parabola and the axis, joined at the origin. The vertical lines $x = -3$ and $x = 2$ close its outer ends. The following rough sketch shows all four prescribed boundaries.

Quick Tip to Solve: Use upper minus lower: below the axis, the strip height is $0 - (-x^2) = x^2$. Substitute the negative lower limit carefully, and do not double one side because the limits are unequal.



Shaded region between $y = 0$ and $y = -x^2$, from $x = -3$ to $x = 2$.

Step 2: Set up the area integral using vertical strips Take a vertical strip at any abscissa x between -3 and 2, with small width dx. Its upper end lies at $y = 0$, while its lower end lies at $y = -x^2$. Hence its height is $0 - (-x^2) = x^2$, so its area is $x^2\,dx$. Adding these strip areas across the full interval gives

$$A = \int_{-3}^{2} [0 - (-x^2)]\,dx = \int_{-3}^{2} x^2\,dx.$$

Although the parabola lies below the axis, the strip height is positive. Integrating $-x^2$ alone would give a signed integral rather than the required geometrical area.

Step 3: Evaluate and check the two portions Applying the power rule and substituting the upper and lower limits in order,

$$\begin{aligned} A &= \left[\frac{x^3}{3}\right]_{-3}^{2} \\ &= \frac{2^3}{3} - \frac{(-3)^3}{3} \\ &= \frac{8}{3} - \left(-\frac{27}{3}\right) = \frac{8+27}{3} = \frac{35}{3}. \end{aligned}$$

As a check, the portion from -3 to 0 has area $\int_{-3}^{0} x^2\,dx = 9$, and the portion from 0 to 2 has area $\int_{0}^{2} x^2\,dx = 8/3$. Both are positive and their sum agrees with the single integral. The unequal widths mean the two portions must not be treated as equal by symmetry.

Step 4: Conclusion The area enclosed by the given parabola and straight lines is $\boxed{\frac{35}{3} \text{ square units}}$.

Quick Tip to Solve: Use upper minus lower: below the axis, the strip height is $0 - (-x^2) = x^2$. Substitute the negative lower limit carefully, and do not double one side because the limits are unequal.

34(a). Find: $\displaystyle\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$ [5 marks]

Solution:

Concept:

- A proper rational function can be integrated by partial fractions. A repeated linear factor $(x-1)^2$ requires both $A/(x-1)$ and $B/(x-1)^2$, while the distinct factor $x+3$ requires $D/(x+3)$.
- The elementary antiderivatives are $\int (x-a)^{-1}\,dx = \ln|x-a|$ and $\int (x-a)^{-2}\,dx = -(x-a)^{-1}$. Absolute values make the logarithmic expressions valid on each interval where the original integrand is defined.

Step 1: Set up the partial fractions The numerator has degree two and the denominator has degree three, so polynomial division is unnecessary. Write

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{x+3}.$$

Multiplying by the common denominator gives the polynomial identity

$$x^2+1 = A(x-1)(x+3) + B(x+3) + D(x-1)^2.$$

Both terms corresponding to the repeated factor must be retained; omitting the squared-denominator term would prevent this identity from holding.

Step 2: Determine the three coefficients In the polynomial identity, substituting $x=1$ eliminates the terms containing A and D:

$$2 = 4B \quad\Longrightarrow\quad B = \frac{1}{2}.$$

Substituting $x=-3$ similarly isolates D:

$$10 = 16D \quad\Longrightarrow\quad D = \frac{5}{8}.$$

Comparing coefficients of x^2 gives $1 = A + D$, hence $A = 3/8$. These substitutions are made in the polynomial identity, not in the original rational function, which is undefined at those two points. The constant coefficient provides a check: $-3A + 3B + D = -9/8 + 12/8 + 5/8 = 1$.

Step 3: Integrate term by term and check Therefore the required integral becomes

$$\begin{aligned}&\frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{5}{8}\int \frac{dx}{x+3}\\ &\quad= \frac{3}{8}\ln|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\ln|x+3| + C.\end{aligned}$$

Differentiating the rational term gives $+1/[2(x-1)^2]$, so its negative sign is essential. Differentiating the entire expression reproduces the three partial fractions and hence the original integrand. The result applies on any interval avoiding $x=1$ and $x=-3$, with an arbitrary integration constant on that interval.

Quick Tip to Solve: After clearing denominators, substitute the two roots first: x=1 immediately gives the repeated-factor coefficient and x=-3 gives the other simple-factor coefficient.

Step 4: Conclusion

$$\boxed{\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\ln|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\ln|x+3| + C.}$$

Quick Tip to Solve: After clearing denominators, substitute the two roots first: x=1 immediately gives the repeated-factor coefficient and x=-3 gives the other simple-factor coefficient.

OR

34(b). Evaluate: $\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$ [5 marks]

Solution:

Concept:

- Reflection of a definite integral gives $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Here $a = \pi/2$, and the denominator is unchanged because $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$.
- The identity $\sin x + \cos x = \sqrt{2}\cos(x - \pi/4)$ reduces the remaining integral to a secant integral. Use $\int \sec t\,dt = \ln|\sec t + \tan t| + C$.

Step 1: Reflect the integral about its midpoint Denote the required integral by

$$I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx.$$

Set $x = \pi/2 - t$. Then $dx = -dt$, and the limits $x = 0, \pi/2$ become $t = \pi/2, 0$, respectively. Reversing these limits removes the negative sign, so

$$I = \int_0^{\pi/2} \frac{\pi/2 - t}{\cos t + \sin t}\,dt.$$

Renaming the integration variable to x makes the denominator identical to that in the original expression. This symmetry is the reason the numerator involving the variable can be eliminated.

Step 2: Add the two equal expressions Adding the original and reflected integrals yields

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{x + (\pi/2 - x)}{\sin x + \cos x}\,dx \\ &= \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}. \end{aligned}$$

Thus, if $J = \int_0^{\pi/2} (\sin x + \cos x)^{-1}\,dx$, then $I = (\pi/4)J$. On the whole integration interval the denominator is positive, so both integrals are finite and these operations introduce no singularity.

Step 3: Evaluate the symmetric secant integral Using $t = x - \pi/4$, the new limits are $-\pi/4$ and $\pi/4$.

Quick Tip to Solve: When the denominator is unchanged under x replaced by pi/2-x, add the reflected integral to the original before attempting a trigonometric substitution.

Therefore

$$J = \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec t \, dt$$
$$= \frac{1}{\sqrt{2}} \left[\ln |\sec t + \tan t| \right]_{-\pi/4}^{\pi/4}$$
$$= \frac{1}{\sqrt{2}} \left[\ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) \right].$$

Both endpoint arguments are positive. Since $(\sqrt{2}-1)(\sqrt{2}+1) = 1$, we have $\ln(\sqrt{2}-1) = -\ln(\sqrt{2}+1)$. Hence $J = \sqrt{2}\ln(1+\sqrt{2})$. Substitution into $I = (\pi/4)J$ gives the required exact value. In particular, the factor $\pi/4$ comes from dividing the sum by two; retaining $\pi/2$ here would double the answer incorrectly.

Step 4: Conclusion

$$\boxed{\int_0^{\pi/2} \frac{x}{\sin x + \cos x} \, dx = \frac{\pi}{2\sqrt{2}} \ln(1+\sqrt{2}).}$$

Quick Tip to Solve: When the denominator is unchanged under x replaced by pi/2-x, add the reflected integral to the original before attempting a trigonometric substitution.

35. Find the foot of the perpendicular drawn from point $(2, -1, 5)$ to the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. Also, find the length of the perpendicular. [5 marks]

Solution:

Concept:

- A line through $A(a, b, c)$ with direction vector $\mathbf{d} = (l, m, n)$ has points $H = A + t\mathbf{d}$. Here t is a real parameter.
- The foot of the perpendicular from P satisfies $(H - P) \cdot \mathbf{d} = 0$, since the perpendicular segment is orthogonal to the direction of the line.
- The distance between $P(x_1, y_1, z_1)$ and $H(x_2, y_2, z_2)$ is $PH = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

Step 1: Parameterise the given line Let the common ratio in the symmetric equation be t. Then every point on the line has coordinates

$$x = 11 + 10t, \qquad y = -2 - 4t, \qquad z = -8 - 11t.$$

Thus the line passes through $A = (11, -2, -8)$ and has direction vector $\mathbf{d} = (10, -4, -11)$. Write its required foot as $H = (11 + 10t, -2 - 4t, -8 - 11t)$, and retain all three signs from the printed line.

Step 2: Apply the perpendicularity condition For the given point $P = (2, -1, 5)$, the vector from P to H is

$$\overrightarrow{PH} = (9 + 10t, -1 - 4t, -13 - 11t).$$

Quick Tip to Solve: Set the common line ratio equal to one parameter, then use a dot product with $(10, -4, -11)$; verify the resulting point in all three ratios.

Since PH is perpendicular to the line, its dot product with $\mathbf{d}$ vanishes:

$$\begin{aligned} 10(9+10t) - 4(-1-4t) - 11(-13-11t) &= 0, \\ 90 + 100t + 4 + 16t + 143 + 121t &= 0, \\ 237 + 237t &= 0. \end{aligned}$$

Therefore $t = -1$. The coefficient $237 = 10^2 + (-4)^2 + (-11)^2$ is positive, so the parameter and hence the foot are unique.

Step 3: Find and verify the foot Substituting $t = -1$ gives

$$H = (11 - 10, -2 + 4, -8 + 11) = (1, 2, 3).$$

This point lies on the line because

$$\frac{1-11}{10} = \frac{2+2}{-4} = \frac{3+8}{-11} = -1.$$

Also, $\overrightarrow{PH} = (-1, 3, -2)$, and

$$(-1)(10) + 3(-4) + (-2)(-11) = -10 - 12 + 22 = 0.$$

These checks establish both required properties: membership of the line and perpendicularity to its direction.

Step 4: Calculate the length of the perpendicular Using the coordinates of the verified foot,

$$\begin{aligned} PH &= \sqrt{(1-2)^2 + (2-(-1))^2 + (3-5)^2} \\ &= \sqrt{(-1)^2 + 3^2 + (-2)^2} \\ &= \sqrt{1+9+4} = \sqrt{14}. \end{aligned}$$

The perpendicular is the shortest segment from the point to the line. Its length is positive, as expected since the given point does not coincide with the foot.

Step 5: Conclusion The foot of the perpendicular is $\boxed{(1, 2, 3)}$, and its length is $\boxed{\sqrt{14} \text{ units}}$.

Quick Tip to Solve: Set the common line ratio equal to one parameter, then use a dot product with $(10, -4, -11)$; verify the resulting point in all three ratios.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

A shop selling electronic items sells smartphones of only three reputed companies A, B and C because chances of their manufacturing a defective smartphone are only 5%, 4% and 2% respectively. In his inventory he has 25% smartphones from company A, 35% smartphones from company B and 40% smartphones from company C.

A person buys a smartphone from this shop. [4 marks]

36(i). Find the probability that it was defective. [2 marks]

Solution:

Concept:

- Let A, B, C denote the mutually exclusive manufacturer events, and let D denote that the purchased smartphone is defective.
- The law of total probability gives $P(D) = P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C)$, because the three companies exhaust the inventory.

Step 1: Record the given probabilities The inventory proportions give $P(A) = 0.25$, $P(B) = 0.35$ and $P(C) = 0.40$; they sum to 1. The defect rates are conditional on company: $P(D \mid A) = 0.05$, $P(D \mid B) = 0.04$, and $P(D \mid C) = 0.02$. A defect rate alone is not the probability that a randomly purchased phone is defective.

Step 2: Weight each defect rate by its inventory share The disjoint ways of buying a defective smartphone are buying a defective A phone, a defective B phone, or a defective C phone. Therefore

$$\begin{aligned} P(D) &= (0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02) \\ &= 0.0125 + 0.014 + 0.008 \\ &= 0.0345 = \frac{69}{2000}. \end{aligned}$$

Adding weighted contributions accounts for the different numbers of phones from each company; an unweighted average would incorrectly treat the inventory shares as equal.

Step 3: Conclusion The probability that the purchased smartphone is defective is $\boxed{\frac{69}{2000} = 0.0345}$, or 3.45%.

Quick Tip to Solve: Multiply each company share by its own defect rate before adding; convert both percentages to decimals first.

36(ii). What is the probability that this defective smartphone was manufactured by company B?

[2 marks]

Solution:

Concept:

- Let D be the defect event and B the event that the manufacturer is company B. The requested probability is $P(B \mid D)$.
- Bayes' theorem gives $P(B \mid D) = P(B)P(D \mid B)/P(D)$, where the denominator includes defective phones from all three manufacturers.

Step 1: Find the joint probability and total defect probability Company B accounts for 35% of the phones and has a 4% defect rate. Hence

$$P(B \cap D) = P(B)P(D \mid B) = 0.35 \times 0.04 = 0.014.$$

By total probability,

$$P(D) = 0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02 = 0.0345.$$

These are probabilities among all purchased phones before any defect information is supplied.

Step 2: Condition on the smartphone being defective Knowing the smartphone is defective restricts the sample space to D. The B defective phones form a subset of that space, so

$$\begin{aligned} P(B \mid D) &= \frac{P(B \cap D)}{P(D)} \\ &= \frac{0.014}{0.0345} = \frac{140}{345} = \frac{28}{69}. \end{aligned}$$

This differs from $P(D \mid B) = 0.04$: that probability conditions on a B phone, whereas the question conditions on a defective phone. The resulting value lies between 0 and 1, as a conditional probability must.

Step 3: Conclusion The probability that the defective smartphone was manufactured by B is $\boxed{\frac{28}{69}}$ (approximately 0.4058).

Quick Tip to Solve: Use B's weighted defective contribution as numerator and the sum of all weighted defective contributions as denominator.

37. Case Study - 2

Three students, Neha, Rani and Sam go to a market to purchase stationery items. Neha buys 4 pens, 3 notepads and 2 erasers and pays Rs. 60. Rani buys 2 pens, 4 notepads and 6 erasers for Rs. 90. Sam pays Rs. 70 for 6 pens, 2 notepads and 3 erasers.

Based upon the above information, answer the following questions: [4 marks]

37(i). Form the equations required to solve the problem of finding the price of each item, and express it in the matrix form $AX = B$. [1 mark]

Solution:

Concept:

- Let x, y, z be the prices in rupees of one pen, one notepad and one eraser, respectively.
- In $AX = B$, each row of A contains the item quantities for one purchase, X contains the unknown unit prices, and B contains the corresponding total bills.

Step 1: Translate the three purchases Adding the costs of the purchased items gives

$$\begin{aligned} 4x + 3y + 2z &= 60 \quad \text{(Neha)}, \\ 2x + 4y + 6z &= 90 \quad \text{(Rani)}, \\ 6x + 2y + 3z &= 70 \quad \text{(Sam)}. \end{aligned}$$

The price variables must occur in the same order in each equation.

Step 2: Arrange the coefficients in matrix form Keeping the equation order Neha, Rani, Sam and the variable order pen, notepad, eraser, the system is

$$\underbrace{\begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}}_{B}.$$

Each matrix row multiplied by X reproduces the matching purchase equation.

Step 3: Conclusion The required equations are $4x + 3y + 2z = 60$, $2x + 4y + 6z = 90$, and $6x + 2y + 3z = 70$, with $AX = B$ as displayed.

Quick Tip to Solve: Fix the variable order pen, notepad, eraser before entering the quantities into each row of the coefficient matrix.

37(ii). Find $|A|$ and confirm if it is possible to find A^{-1}. [1 mark]

Solution:

Concept:

- A square matrix A has an inverse if and only if $|A| \neq 0$. A zero determinant would make the matrix singular.
- Expand a 3×3 determinant along its first row using alternating cofactor signs $+, -, +$.

Quick Tip to Solve: Check the middle cofactor sign carefully: its minor is negative, so its contribution to this determinant is positive.

Step 1: Expand the determinant For the coefficient matrix of the purchases,

$$\begin{aligned}|A| &= \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} \\ &= 4(4 \times 3 - 6 \times 2) - 3(2 \times 3 - 6 \times 6) \\ &\quad + 2(2 \times 2 - 4 \times 6) \\ &= 4(12 - 12) - 3(6 - 36) + 2(4 - 24).\end{aligned}$$

The negative sign of the middle cofactor is essential.

Step 2: Evaluate and test invertibility The three contributions are 0, 90, and -40, respectively. Thus

$$|A| = 0 + 90 - 40 = 50.$$

Since this value is nonzero, the coefficient matrix is nonsingular; therefore the inverse exists. There is no need to calculate all cofactors just to confirm its existence.

Step 3: Conclusion $\boxed{|A| = 50}$. Since $|A| \neq 0$, $\boxed{A^{-1} \text{ exists}}$.

Quick Tip to Solve: Check the middle cofactor sign carefully: its minor is negative, so its contribution to this determinant is positive.

37(iii)(a). Find A^{-1}, if possible, and write the formula to find X. [2 marks]

Solution:

Concept:

- For a square matrix A, an inverse exists exactly when $|A| \neq 0$. Its inverse is $A^{-1} = \dfrac{\text{adj } A}{|A|}$.
- The adjugate is the transpose of the cofactor matrix. If $AX = B$, multiplication on the left by A^{-1} gives $X = A^{-1}B$.

Step 1: Set up the coefficient matrix and determinant Let the prices of a pen, notepad and eraser be p, n, e, respectively. The purchases give $AX = B$, where

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} p \\ n \\ e \end{pmatrix}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Expanding along the first row,

$$|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 50.$$

Thus the matrix is invertible.

Step 2: Calculate the cofactors and use the inverse formula The signed minors form the cofactor

Quick Tip to Solve: Compute the signed cofactors first, then transpose: omitting the transpose gives the wrong inverse.

matrix

$$C = \begin{pmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{pmatrix}.$$

For example, $C_{12} = -(2 \cdot 3 - 6 \cdot 6) = 30$, while $C_{21} = -(3 \cdot 3 - 2 \cdot 2) = -5$. Transposing this matrix gives

$$\operatorname{adj} A = C^T = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}.$$

Divide each entry by the determinant. To find the price vector, multiply the inverse on the left of B; the order of multiplication matters.

Step 3: Conclusion

$$\boxed{A^{-1} = \frac{1}{50}\begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}}, \qquad \boxed{X = A^{-1}B}.$$

Quick Tip to Solve: Compute the signed cofactors first, then transpose: omitting the transpose gives the wrong inverse.

OR

37(iii)(b). Find $A^2 - 8I$, where I is an identity matrix. [2 marks]

Solution:

Concept:

- The entry (i, j) of A^2 is the scalar product of row i of A and column j of A.
- The identity matrix I has ones on the diagonal and zeros elsewhere, so subtracting $8I$ changes only the diagonal entries.

Step 1: Multiply the coefficient matrix by itself The purchase equations have coefficient matrix

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}.$$

Use ordinary matrix multiplication, rather than squaring entries individually. For example, $(A^2)_{11} = 4 \cdot 4 + 3 \cdot 2 + 2 \cdot 6 = 34$, $(A^2)_{12} = 4 \cdot 3 + 3 \cdot 4 + 2 \cdot 2 = 28$, and $(A^2)_{13} = 4 \cdot 2 + 3 \cdot 6 + 2 \cdot 3 = 32$. For the second row, the products are $2 \cdot 4 + 4 \cdot 2 + 6 \cdot 6 = 52$, $2 \cdot 3 + 4 \cdot 4 + 6 \cdot 2 = 34$, and $2 \cdot 2 + 4 \cdot 6 + 6 \cdot 3 = 46$.

Quick Tip to Solve: For a scalar multiple of the identity, subtract the scalar only from diagonal entries, never from all nine entries.

Completing the third row similarly gives

$$A^2 = \begin{pmatrix} 34 & 28 & 32 \\ 52 & 34 & 46 \\ 46 & 32 & 33 \end{pmatrix}.$$

Notice that A is not symmetric, so the off-diagonal entries need not match.

Step 2: Subtract the scalar multiple of the identity Here

$$8I = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Subtract corresponding entries. The new diagonal entries are $34-8=26$, $34-8=26$, and $33-8=25$. Every off-diagonal entry stays as it was in A^2.

Step 3: Conclusion

$$\boxed{A^2 - 8I = \begin{pmatrix} 26 & 28 & 32 \\ 52 & 26 & 46 \\ 46 & 32 & 25 \end{pmatrix}}.$$

Quick Tip to Solve: For a scalar multiple of the identity, subtract the scalar only from diagonal entries, never from all nine entries.

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38. Case Study - 3

A ladder of fixed length $'h'$ is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions: [4 marks]

38(i). Express the distance (y) between the wall and foot of the ladder in terms of $'h'$ and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer. [1 mark]

Solution:

Concept:

- The wall and ground are perpendicular, so the ladder is the hypotenuse: $x^2 + y^2 = h^2$.
- The area of the right triangle is half the product of its perpendicular sides: $A = \frac{1}{2}xy$.

Step 1: Express the horizontal distance The ladder length h is fixed and positive. Rearranging the Pythagorean relation gives $y^2 = h^2 - x^2$. Since y is a distance, take the nonnegative root, not the negative root. Thus $y = \sqrt{h^2 - x^2}$, with $0 \le x \le h$.

Step 2: Express the area using only the height Substitute the horizontal distance into the area formula:

$$A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}.$$

Quick Tip to Solve: Eliminate the horizontal distance with Pythagoras before forming the area as a function of height.

For a nondegenerate triangle, $0 < x < h$. At either endpoint one perpendicular side is zero and the area is zero. The fixed ladder length is the hypotenuse, so it must not be used as the base in the area formula.

Step 3: Conclusion

$$\boxed{y = \sqrt{h^2 - x^2}}, \qquad \boxed{A = \frac{x}{2}\sqrt{h^2 - x^2}}.$$

Quick Tip to Solve: Eliminate the horizontal distance with Pythagoras before forming the area as a function of height.

38(ii). Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point. [1 mark]

Solution:

Concept:

- For a fixed ladder length h, the area is $A(x) = \frac{1}{2}x(h^2 - x^2)^{1/2}$.
- Apply the product and chain rules, treating h as constant; an interior critical point satisfies $A\prime(x) = 0$.

Step 1: Differentiate the area function For $0 < x < h$,

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{h^2 - x^2} + \frac{x}{2} \cdot \frac{-2x}{2\sqrt{h^2 - x^2}}$$
$$= \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}.$$

The negative sign in the chain-rule term comes from differentiating $-x^2$.

Step 2: Locate the interior critical point The denominator is positive throughout $0 < x < h$, so the derivative vanishes precisely when $h^2 - 2x^2 = 0$. This yields $x = \pm h/\sqrt{2}$, but a height is positive and only $x = h/\sqrt{2}$ lies in the interval. The endpoint $x = h$ makes the displayed derivative undefined, but is a boundary configuration, not the interior stationary point sought here.

Step 3: Conclusion

$$\boxed{\frac{dA}{dx} = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}}, \qquad \boxed{x = \frac{h}{\sqrt{2}}}.$$

Quick Tip to Solve: On the interior interval the denominator is positive, so solve only the numerator equation for the critical height.

38(iii)(a). Show that the area (A) of the right triangle is maximum at the critical point. [2 marks]

Solution:

Concept:

- The area is $A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}$, with $0 \le x \le h$ and fixed $h > 0$.
- If a differentiable function changes from increasing to decreasing at a stationary point, that point is a maximum.

Step 1: Use the derivative and its sign Differentiating the area gives

$$A\prime(x) = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}, \qquad 0 < x < h.$$

The critical height is $x = h/\sqrt{2}$. The denominator is positive on the entire open interval. If $0 < x < h/\sqrt{2}$, then $2x^2 < h^2$ and hence $A\prime(x) > 0$. If $h/\sqrt{2} < x < h$, then $2x^2 > h^2$ and hence $A\prime(x) < 0$. Thus the area increases up to the critical height and decreases after it.

Step 2: Check the whole interval and evaluate the area The sign change from positive to negative proves a maximum at the critical point. Since this increasing-then-decreasing behaviour covers the whole interior, and $A(0) = A(h) = 0$, it is also the absolute maximum on the closed physical interval. At $x = h/\sqrt{2}$, Pythagoras gives $y = \sqrt{h^2 - h^2/2} = h/\sqrt{2}$. Therefore

$$A\left(\frac{h}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{h}{\sqrt{2}} \cdot \frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

The two perpendicular sides are equal at this configuration.

Step 3: Conclusion The area is maximum at $\boxed{x = h/\sqrt{2}}$, with $\boxed{A_{\max} = h^2/4}$.

Quick Tip to Solve: Test the numerator sign on both sides of the stationary point; the denominator stays positive throughout the physical interior.

OR

38(iii)(b). If the foot of the ladder whose length is 5 m, is being pulled towards the wall such that the rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the ladder is 3 m away from the wall? [2 marks]

Solution:

Concept:

- The fixed ladder length imposes $x^2 + y^2 = 25$, with distances measured in metres.
- Differentiate with respect to time: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. A decreasing distance has a negative time derivative.

Step 1: Find the height and assign the signed rate At the instant in question, $y = 3\,\text{m}$ and the ladder length is $5\,\text{m}$. Pythagoras gives

$$x = \sqrt{25 - 9} = 4\,\text{m}.$$

Quick Tip to Solve: Write the decreasing horizontal rate as negative before substituting into the differentiated Pythagorean relation.

Use the positive root because x is the height above the ground. The foot moves towards the wall, so the horizontal distance decreases at $2\,\text{m/s}$. Therefore $dy/dt = -2\,\text{m/s}$, not $+2\,\text{m/s}$.

Step 2: Differentiate the length constraint and substitute The distances x and y both depend on time, while the ladder length stays constant. Differentiating $x^2 + y^2 = 25$ gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Hence

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}\,\text{m/s}.$$

The positive sign confirms that the top of the ladder moves upwards as its foot approaches the wall. Substitution is made after differentiation because the given distances hold only at one instant.

Step 3: Conclusion The height on the wall increases at $\boxed{\frac{3}{2}\,\text{m/s}}$.

Quick Tip to Solve: Write the decreasing horizontal rate as negative before substituting into the differentiated Pythagorean relation.