CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-6-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All questions are compulsory.**

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $\tan^{-1}(x^2 - y^2) = a$, where a is a constant, then $\frac{dy}{dx}$ is: [1 mark]

(A) $\frac{x}{y}$

(B) $-\frac{x}{y}$

(C) $\frac{a}{x}$

(D) $\frac{a}{y}$

Correct Answer: (A) $\frac{x}{y}$

Solution:

Concept:

- For a constant a, $\frac{da}{dx} = 0$. Treat y as a function of x when differentiating.
- The chain rule gives $\frac{d}{dx}\tan^{-1} u = \frac{u'}{1+u^2}$.

Step 1: Differentiate implicitly Apply the chain rule to the given identity, remembering that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$:

$$\frac{2x - 2y\frac{dy}{dx}}{1 + (x^2 - y^2)^2} = 0.$$

Step 2: Isolate the derivative The denominator is strictly positive, so the numerator must vanish:

$$2x - 2y\frac{dy}{dx} = 0 \quad \Longrightarrow \quad y\frac{dy}{dx} = x.$$

At a point where $y \neq 0$, divide by y. A negative sign would come from mishandling the derivative of $-y^2$. Options (C) and (D) incorrectly use the constant a in place of the variable dependence.

Step 3: Conclusion $\frac{dy}{dx} = \frac{x}{y}$, wherever the derivative is defined. Hence, the correct option is (A).

Quick Tip to Solve: Take tangent first to obtain $x^2 - y^2 = \tan a$; its right side is constant, so implicit differentiation is immediate.

2. If $A = \begin{pmatrix} 0 & 0 & -5 \\ 0 & 3 & 0 \\ 4.3 & 0 & 0 \end{pmatrix}$, then A is a: [1 mark]

(A) skew-symmetric matrix

(B) scalar matrix

(C) diagonal matrix

(D) square matrix

Correct Answer: (D) square matrix

Solution:

Concept:

- A square matrix has the same number of rows and columns.
- A diagonal matrix has zero off-diagonal entries; a scalar matrix is diagonal with equal diagonal entries. A skew-symmetric matrix satisfies $A^T = -A$.

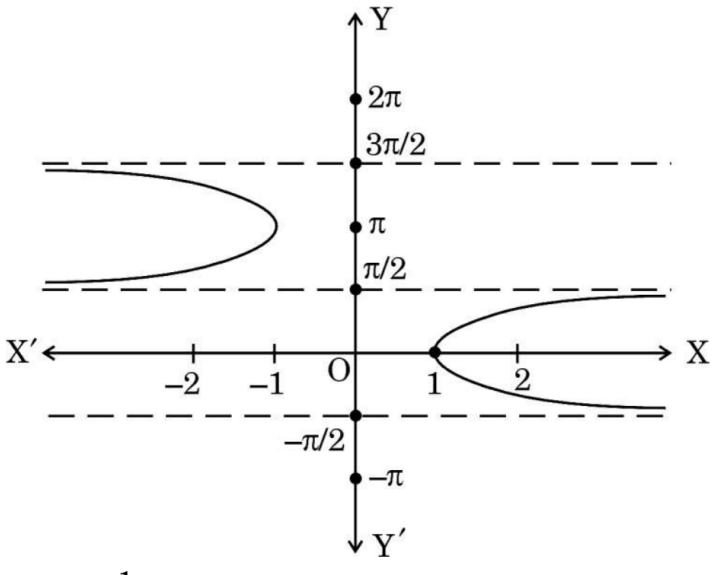
Step 1: Count the rows and columns The displayed matrix has three rows and three columns. Its order is therefore 3×3, regardless of the numerical values of its entries.

Step 2: Check the narrower classifications The entries $a_{13} = -5$ and $a_{31} = 4.3$ are off the main diagonal and nonzero, so the matrix is neither diagonal nor scalar. It is not skew-symmetric: that condition would require every diagonal entry to be zero, whereas $a_{22} = 3$. Moreover, $a_{31} \neq -a_{13}$, because $4.3 \neq 5$. Thus options (A), (B) and (C) each fail their defining condition.

Step 3: Conclusion A is a square matrix of order 3×3. Hence, the correct option is (D).

Quick Tip to Solve: First count rows and columns; then rule out diagonal and skew-symmetric types by inspecting one offending entry.

3. The graph shown below depicts : [1 mark]



(A) $y = \sec^{-1} x$

(B) $y = \sec x$

(C) $y = \csc^{-1} x$

(D) $y = \csc x$

Correct Answer: (A) $y = \sec^{-1} x$

Solution:

Concept:

- For the inverse secant relation, $x = \sec y$; its principal function has domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi] \setminus \{\pi/2\}$.
- The endpoint values distinguish inverse secant from inverse cosecant: $\sec^{-1}(1) = 0$ and $\sec^{-1}(-1) = \pi$.

Step 1: Read the turning points and asymptotes The right-hand portion passes through $(1, 0)$, and the left-hand portion turns at $(-1, \pi)$. The curves extend horizontally towards lines such as $y = \pi/2$. These features agree with $x = \sec y$, because secant has magnitude at least one and becomes unbounded near odd multiples of $\pi/2$.

Step 2: Identify the inverse relation Solving $x = \sec y$ for the angle gives the inverse secant relation. The printed drawing includes reflected, nonprincipal branches; the principal function uses the upper right branch and the lower left branch, with $0 \le y \le \pi$, $y \ne \pi/2$. For inverse cosecant, the endpoint at $x = 1$ would instead be $y = \pi/2$. The direct functions $y = \sec x$ and $y = \csc x$ have vertical asymptotes and repeated branches along the horizontal axis, unlike this drawing.

Step 3: Conclusion The indicated inverse-trigonometric relation is $y = \sec^{-1} x$, with the principal branches selected as above. Hence, the correct option is (A).

Quick Tip to Solve: Check the endpoint at $x = 1$: angle zero identifies inverse secant, whereas angle $\pi/2$ identifies inverse cosecant.

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4. $\sec^{-1}(-\sqrt{2}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **is equal to:** [1 mark]

(A) $\frac{11\pi}{12}$

(B) $\frac{5\pi}{12}$

(C) $-\frac{5\pi}{12}$

(D) $\frac{7\pi}{12}$

Correct Answer: (D) $\frac{7\pi}{12}$

Solution:

Concept:

- The principal range of $\sec^{-1} x$ is $[0, \pi] \setminus \{\pi/2\}$; a negative argument gives a second-quadrant angle.
- The principal range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$.

Step 1: Evaluate each principal value Since $\cos(3\pi/4) = -1/\sqrt{2}$, we have $\sec(3\pi/4) = -\sqrt{2}$, with

Quick Tip to Solve: For a negative inverse secant argument, select the second-quadrant principal angle before doing the angle arithmetic.

$3\pi/4$ in the required range. Also, $\tan(\pi/6) = 1/\sqrt{3}$. Therefore,

$$\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}, \qquad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Step 2: Subtract using a common denominator Keep the printed subtraction sign:

$$\frac{3\pi}{4} - \frac{\pi}{6} = \frac{9\pi - 2\pi}{12}.$$

Option (A) would result from adding these two angles. Option (B), $5\pi/12$, does not equal the difference of the principal values. Option (C) is negative, whereas the second-quadrant angle $3\pi/4$ is larger than $\pi/6$, so their difference is positive.

Step 3: Conclusion The value is $\frac{7\pi}{12}$. Hence, the correct option is (D).

Quick Tip to Solve: For a negative inverse secant argument, select the second-quadrant principal angle before doing the angle arithmetic.

5. Let both AB' and $B'A$ be defined for matrices A and B. If order of A is $n \times m$, then the order of B is: [1 mark]

(A) $n \times n$

(B) $n \times m$

(C) $m \times m$

(D) $m \times n$

Correct Answer: (B) $n \times m$

Solution:

Concept:

- The product of a $p \times q$ matrix and an $r \times s$ matrix is defined precisely when $q = r$.
- Transposition interchanges rows and columns: if B has order $r \times s$, then $B\prime$ has order $s \times r$.

Step 1: Use the condition for the first product Let the order of B be $r \times s$, so $B\prime$ is $s \times r$. Since A is $n \times m$, the product $AB\prime$ is defined only if its inner dimensions agree:

$$m = s.$$

Step 2: Use the condition for the second product For $B\prime A$, the factors have orders $s \times r$ and $n \times m$. Thus the second compatibility condition is

$$r = n.$$

Together these conditions specify both dimensions of B. The square orders in (A) and (C), and the reversed order in (D), cannot satisfy both conditions for arbitrary n, m; they work only in special cases such as $n = m$.

Quick Tip to Solve: Write a temporary order $r \times s$ for B, transpose it, and compare inner dimensions separately in both products.

Step 3: Conclusion The order of B is $r \times s = n \times m$. Hence, the correct option is (B).

Quick Tip to Solve: Write a temporary order $r \times s$ for B, transpose it, and compare inner dimensions separately in both products.

6. Sum of two skew-symmetric matrices of same order is always a/an: [1 mark]

(A) skew-symmetric matrix

(B) symmetric matrix

(C) null matrix

(D) identity matrix

Correct Answer: (A) skew-symmetric matrix

Solution:

Concept:

- A real matrix A is skew-symmetric when $A^T = -A$.
- Transpose distributes over addition: $(A + B)^T = A^T + B^T$, for matrices of the same order.

Step 1: Apply the defining property Let A and B be the two given skew-symmetric matrices. Their equal orders make the sum $S = A + B$ defined. By hypothesis, $A^T = -A$ and $B^T = -B$.

Step 2: Transpose their sum Using linearity of transpose gives

$$S^T = (A + B)^T = A^T + B^T = -A - B = -(A + B) = -S.$$

This is exactly the skew-symmetric condition. The sum can be zero for a special pair $B = -A$, but it need not be zero. For example, adding a nonzero skew-symmetric matrix to itself gives twice that matrix. Such a nonzero sum is not symmetric. An identity matrix is not skew-symmetric because its diagonal entries are one, not zero.

Step 3: Conclusion The sum is always a skew-symmetric matrix. Hence, the correct option is (A).

Quick Tip to Solve: For a sum of skew-symmetric matrices, transpose the whole sum and factor out the common minus sign.

7. If $y = a\cos(\log x) + b\sin(\log x)$, then $x^2y_2 + xy_1$ is: [1 mark]

(A) $\cot(\log x)$

(B) y

(C) $-y$

(D) $\tan(\log x)$

Correct Answer: (C) $-y$

Solution:

Concept:

- For $x > 0$, $\dfrac{d}{dx}\log x = 1/x$. Here $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$.
- Use both the chain rule for the trigonometric functions and the product rule when differentiating xy_1.

Step 1: Differentiate once and multiply by x Applying the chain rule to both terms gives

$$y_1 = \frac{-a\sin(\log x) + b\cos(\log x)}{x}.$$

Consequently, $xy_1 = -a\sin(\log x) + b\cos(\log x)$.

Step 2: Differentiate the resulting identity The product rule on the left and the chain rule on the right give

$$y_1 + xy_2 = \frac{-a\cos(\log x) - b\sin(\log x)}{x}.$$

Multiplying by x expresses the required combination in terms of the original function. The minus signs arise from differentiating sine and cosine in succession, so option (B) has the wrong sign. Options (A) and (D) are cotangent and tangent expressions, which do not represent this linear combination for arbitrary constants a, b.

Step 3: Conclusion $x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x) = -y$. Hence, the correct option is (C).

Quick Tip to Solve: Multiply the first derivative by x before differentiating again; this avoids expanding a quotient for the second derivative.

8. If $f(x) = \begin{cases} \dfrac{\log(1+ax)+\log(1-bx)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ **is continuous at** $x = 0$**, then the value of** k **is:** [1 mark]

(A) a

(B) $a + b$

(C) $a - b$

(D) b

Correct Answer: (C) $a - b$

Solution:

Concept:

- Continuity at zero requires $\lim_{x\to 0} f(x) = f(0) = k$.
- The standard logarithmic limit is $\lim_{t\to 0} \log(1+t)/t = 1$.

Step 1: Separate the logarithmic limits For sufficiently small x, both logarithm arguments are positive. Hence

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+ax)}{x} + \lim_{x\to 0} \frac{\log(1-bx)}{x}.$$

The first limit is a, and the second is $-b$, using the standard limit. If either coefficient is zero, its logarithmic term vanishes identically.

Step 2: Apply continuity and compare the options The limiting value is $a - b$, so continuity forces $k = a - b$. Option (B) misses the negative sign in $1 - bx$; options (A) and (D) retain only one contribution. Thus the expression in option (C) is the required general value.

Step 3: Conclusion $\boxed{k = a - b}$. Hence, the correct option is (C).

Quick Tip to Solve: In $\log(1+cx)/x$, read off the coefficient c, keeping its sign.

9. $f(x) = x^x$ **has a critical point at:** [1 mark]

(A) $x = e$

(B) $x = e^{-1}$

(C) $x = 0$

(D) $x = 1$

Correct Answer: (B) $x = e^{-1}$

Solution:

Concept:

- For the real logarithmic treatment of x^x, use $x > 0$.
- An interior critical point occurs where the derivative vanishes or fails to exist.

Step 1: Differentiate logarithmically Put $y = x^x$. Since $x > 0$, $\log y = x \log x$. Differentiating gives $y^{-1}dy/dx = \log x + 1$, hence

$$f\prime(x) = x^x(\log x + 1).$$

This derivative exists throughout the positive domain.

Quick Tip to Solve: For x^x, differentiate $x \log x$ first; solve $1 + \log x = 0$.

Step 2: Find the zero and reject the other choices Because $x^x > 0$, a zero derivative requires $\log x + 1 = 0$, giving $x = e^{-1}$. At $x = e$ the logarithmic factor is 2, and at $x = 1$ it is 1, so neither is critical. The point $x = 0$ is outside the positive domain used for the given real differentiable function.

Step 3: Conclusion The critical point is at $\boxed{x = e^{-1}}$. Hence, the correct option is (B).

Quick Tip to Solve: For x^x, differentiate $x \log x$ first; solve $1 + \log x = 0$.

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10. The solution for the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ is: [1 mark]

(A) $3e^{4y} + 4e^{-3x} + C = 0$

(B) $e^{3x+4y} + C = 0$

(C) $3e^{-3y} + 4e^{4x} + 12C = 0$

(D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Correct Answer: (D) $3e^{-4y} + 4e^{3x} + 12C = 0$

Solution:

Concept:

- Exponentiating $\log(dy/dx) = 3x + 4y$ gives $dy/dx = e^{3x+4y} > 0$.
- A separable differential equation is integrated after putting all y-terms with dy and all x-terms with dx.

Step 1: Separate variables and integrate The equation becomes $e^{-4y}\,dy = e^{3x}\,dx$. Therefore

$$-\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + C.$$

Multiplying by 12 and bringing every term to one side gives $3e^{-4y} + 4e^{3x} + 12C = 0$.

Step 2: Check the signs and coefficients against all options Differentiating the obtained relation gives $-12e^{-4y}y\prime + 12e^{3x} = 0$, so $y\prime = e^{3x+4y}$, as required. Printed option (A) instead gives $y\prime = e^{-3x-4y}$. Option (C) gives $y\prime = (16/9)e^{4x+3y}$. Option (B), on implicit differentiation, gives $y\prime = -3/4$, which cannot have the stated real logarithm. Only (D) fits.

Step 3: Conclusion $\boxed{3e^{-4y} + 4e^{3x} + 12C = 0}$. Hence, the correct option is (D).

Quick Tip to Solve: After exponentiating, check the solution by implicit differentiation; exponential signs matter.

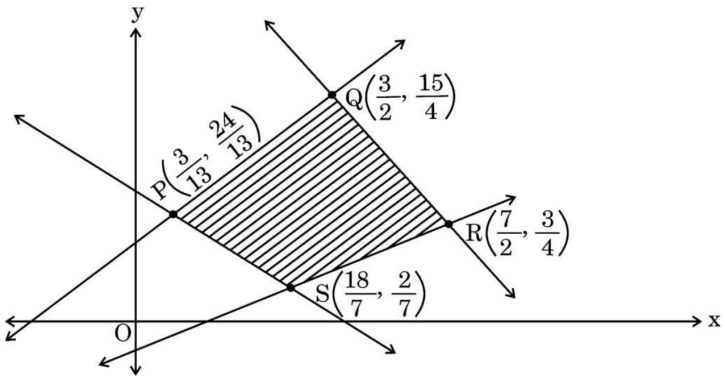
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11. For a Linear Programming Problem (LPP), the given objective function is $Z = x + 2y$. The feasible region $PQRS$ determined by the set of constraints is shown as a shaded region in the graph.

(Note: The figure is not to scale)

$P = \left(\frac{3}{13}, \frac{24}{13}\right)$, $Q = \left(\frac{3}{2}, \frac{15}{4}\right)$, $R = \left(\frac{7}{2}, \frac{3}{4}\right)$, $S = \left(\frac{18}{7}, \frac{2}{7}\right)$

Which of the following statements is correct? [1 mark]



(A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$

(B) Z is maximum at $R\left(\frac{7}{2}, \frac{3}{4}\right)$

(C) (Value of Z at P) $>$ (Value of Z at Q)

(D) (Value of Z at Q) $<$ (Value of Z at R)

Correct Answer: (A) Z is minimum at $S\left(\frac{18}{7}, \frac{2}{7}\right)$

Solution:

Concept:

- A linear objective function on a bounded feasible polygon has its extrema at corner points.
- Here $Z = x + 2y$, so substitute both coordinates of each vertex.

Step 1: Evaluate the objective at each vertex At the four vertices, substitution in $Z = x + 2y$ gives

$$Z(P) = \frac{3}{13} + \frac{48}{13} = \frac{51}{13}, \quad Z(Q) = \frac{3}{2} + \frac{15}{2} = 9,$$
$$Z(R) = \frac{7}{2} + \frac{3}{2} = 5, \qquad Z(S) = \frac{18}{7} + \frac{4}{7} = \frac{22}{7}.$$

Step 2: Compare the four objective values The values satisfy $22/7 < 51/13 < 5 < 9$. Thus the minimum occurs at S, and the maximum occurs at Q. Statement (A) is true. Statement (B) wrongly puts the maximum at R; statements (C) and (D) reverse the actual comparisons with the value at Q. The scale of the sketch does not affect these exact substitutions.

Step 3: Conclusion Z is minimum at $\boxed{S(18/7, 2/7)}$, with value $22/7$. Hence, the correct option is (A).

Quick Tip to Solve: For a polygonal feasible region, make a corner-value list instead of judging the picture visually.

12. The order and degree of the given differential equation $\left[\frac{d^2y}{dx^2} - 1\right]^2 = \frac{dy}{dx}$ are respectively: [1 mark]

(A) $2, 2$

(B) 2, not defined

(C) $1, 2$

(D) 1, not defined

Correct Answer: (A) $2, 2$

Solution:

Concept:

- The order is the order of the highest derivative present in the equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The equation contains $y\prime = dy/dx$ and $y\prime\prime = d^2y/dx^2$. No derivative of order greater than two appears. Therefore the order is 2, irrespective of the square outside the bracket.

Step 2: Write the polynomial form and find its degree Expanding and bringing the right-hand side to the left gives

$$(y\prime\prime)^2 - 2y\prime\prime + 1 - y\prime = 0.$$

This is a polynomial in the derivatives. The highest-order derivative $y\prime\prime$ has highest exponent 2, so the degree is defined and equals 2. Options with order one confuse the first derivative on the right with the highest derivative; the undefined-degree options do not apply to this polynomial equation.

Step 3: Conclusion The order and degree are $\boxed{2, 2}$, respectively. Hence, the correct option is (A).

Quick Tip to Solve: Expand the bracket: order tracks which derivative appears, while degree tracks its highest power.

13. Let $f'(x) = 3(x^2 + 2x) - \frac{4}{x^3} + 5,\ f(1) = 0$. Then, $f(x)$ is: [1 mark]

(A) $x^3 + 3x^2 + \frac{2}{x^2} + 5x + 11$

(B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$

(C) $x^3 + 3x^2 - \frac{2}{x^2} + 5x - 11$

(D) $x^3 - 3x^2 - \frac{2}{x^2} + 5x - 11$

Correct Answer: (B) $x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11$

Solution:

Concept:

- An antiderivative of x^n is $x^{n+1}/(n+1)$ for $n \neq -1$.
- The initial value $f(1) = 0$ fixes the additive integration constant.

Step 1: Integrate the printed derivative term by term The derivative is $3x^2 + 6x - 4x^{-3} + 5$. Hence

$$f(x) = x^3 + 3x^2 + 2x^{-2} + 5x + C.$$

In particular, integrating $-4x^{-3}$ gives $2x^{-2}$, because division by -2 changes the sign.

Step 2: Use the initial value and check the options Substitution of $x = 1$ gives $0 = 1+3+2+5+C = 11 + C$, so $C = -11$. Option (A) has the wrong constant. Option (C) has the wrong sign of $2/x^2$, whose derivative would then be $+4/x^3$. Option (D) also differentiates its quadratic term to $-6x$ instead of $+6x$. Option (B) satisfies both the derivative and initial condition.

Step 3: Conclusion $\boxed{f(x) = x^3 + 3x^2 + \frac{2}{x^2} + 5x - 11}$. Hence, the correct option is (B).

Quick Tip to Solve: Integrate the negative power before using $f(1)$; differentiate the reciprocal term to check its sign.

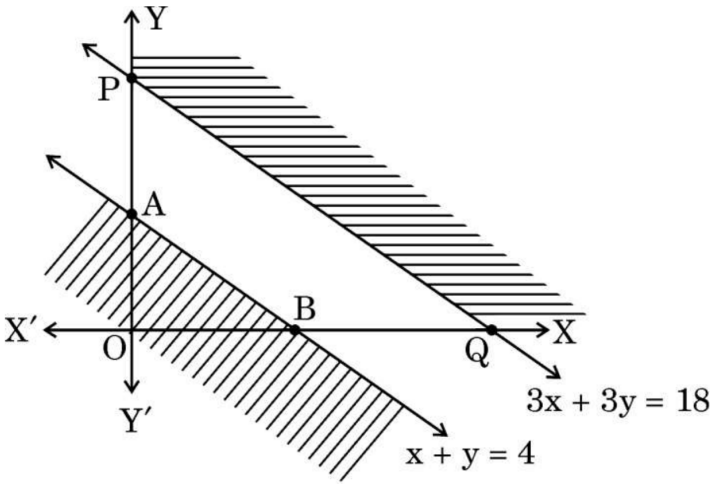
14. In a Linear Programming Problem (LPP), the objective function $Z = 2x + 5y$ is to be maximised under the following constraints:

$x + y \leq 4,\ 3x + 3y \geq 18,\ x, y \geq 0$

Study the graph and select the correct option.

(Note: The figure is not to scale)

The solution of the given LPP: [1 mark]



(A) lies in the shaded unbounded region.

(B) lies in $\triangle AOB$.

(C) does not exist.

(D) lies in the combined region of $\triangle AOB$ and unbounded shaded region.

Correct Answer: (C) does not exist.

Solution:

Concept:

- The feasible region is the intersection of all constraint regions.
- An objective function has a feasible optimum only if at least one point satisfies every constraint.

Step 1: Reduce the second constraint The inequality $3x + 3y \geq 18$, divided by the positive number 3, is $x + y \geq 6$. The first constraint simultaneously requires $x + y \leq 4$. These two conditions cannot hold for the same point.

Step 2: Interpret the graph and inspect the choices The triangle AOB corresponds to the first constraint in the non-negative quadrant, whereas the shaded unbounded region beyond PQ corresponds to the second. The two feasible sides have no common point. Taking their union, as suggested in (D), would satisfy only one constraint at a time; (A) and (B) likewise use individual regions. The required intersection is empty, so the value of $2x + 5y$ cannot be maximised over feasible points.

Step 3: Conclusion The solution of the given LPP does not exist. Hence, the correct option is (C).

Quick Tip to Solve: Before optimising, simplify parallel constraints: $x + y \leq 4$ and $x + y \geq 6$ already prove infeasibility.

15. The area of the region bounded by the curve $y^2 = x$ between $x = 0$ and $x = 1$ is: [1 mark]

(A) $\frac{3}{2}$ sq units

(B) $\frac{2}{3}$ sq units

(C) 3 sq units

(D) $\frac{4}{3}$ sq units

Correct Answer: (D) $\frac{4}{3}$ sq units

Solution:

Concept:

- For a region between an upper curve y_u and a lower curve y_l, its area is $\int_a^b (y_u - y_l)\,dx$.
- The parabola $y^2 = x$ has two branches $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Step 1: Identify both boundaries For every x between 0 and 1, the upper boundary is $\sqrt{x}$ and the lower boundary is $-\sqrt{x}$. Thus the vertical width is $2\sqrt{x}$. The entire bounded region includes both halves on opposite sides of the x-axis.

Step 2: Evaluate the area

$$A = \int_0^1 2\sqrt{x}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3}.$$

Option (B), $2/3$, counts only the upper half. Options (A), $3/2$, and (C), 3, do not equal the integral over the full vertical width.

Step 3: Conclusion The required area is $\boxed{\frac{4}{3}}$ square units. Hence, the correct option is (D).

Quick Tip to Solve: When the equation is $y^2 = x$, include the negative branch as well as the positive branch when finding the whole bounded area.

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16. $\int \frac{x+5}{(x+6)^2} e^x\,dx$ is equal to: [1 mark]

(A) $\log(x+6) + C$

(B) $e^x + C$

(C) $\frac{e^x}{x+6} + C$

(D) $\frac{-1}{(x+6)^2} + C$

Correct Answer: (C) $\frac{e^x}{x+6} + C$

Solution:

Concept:

- An antiderivative can be identified by differentiating the proposed options.

Quick Tip to Solve: Look for a numerator one less than the linear denominator factor: differentiating $e^x/(x+6)$ produces exactly that difference.

- The quotient rule gives $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, where $v \neq 0$.

Step 1: Differentiate the matching expression The numerator $x + 5$ is one less than the denominator factor $x + 6$. Therefore test $e^x/(x + 6)$:

$$\frac{d}{dx}\left(\frac{e^x}{x+6}\right) = \frac{e^x(x+6) - e^x}{(x+6)^2} = \frac{(x+5)e^x}{(x+6)^2}.$$

This agrees with the integrand for $x \neq -6$.

Step 2: Check the other options The derivatives of options (A), (B) and (D) are respectively $1/(x+6)$, e^x, and $2/(x+6)^3$. None reproduces both the exponential factor and the stated rational factor. Adding a constant C leaves the derivative unchanged.

Step 3: Conclusion

$$\boxed{\int \frac{x+5}{(x+6)^2} e^x \, dx = \frac{e^x}{x+6} + C}.$$

Hence, the correct option is (C).

Quick Tip to Solve: Look for a numerator one less than the linear denominator factor: differentiating $e^x/(x+6)$ produces exactly that difference.

17. Let $|\vec{a}| = 5$ and $-2 \leq \lambda \leq 1$. Then, the range of $|\lambda\vec{a}|$ is: [1 mark]

(A) $[5, 10]$

(B) $[-2, 5]$

(C) $[-2, 1]$

(D) $[-10, 5]$

Correct Answer: No printed option is defensible: the correct range is $[0, 10]$, because $|\lambda\vec{a}| = 5|\lambda|$.

Solution:

Concept:

- The magnitude of a scalar multiple is $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$. A magnitude is always nonnegative.
- To determine a range, check interior values as well as the endpoints of the scalar interval.

Step 1: Find the range of the absolute scalar The interval $[-2, 1]$ contains zero, so the minimum of $|\lambda|$ is 0, attained at $\lambda = 0$. Its maximum is 2, attained at $\lambda = -2$. Every number between 0 and 2 occurs as $|\lambda|$, by taking λ in $[-2, 0]$.

Step 2: Scale the interval and check the choices Using $|\vec{a}| = 5$, we obtain $0 \leq |\lambda\vec{a}| = 5|\lambda| \leq 10$. Option (A) omits values below 5, including the attainable value zero. Options (B), (C) and (D) include negative magnitudes and also do not give the required interval. The interval $[-10, 5]$ describes signed scalar values 5λ, not magnitudes.

Step 3: Conclusion The range is $\boxed{[0, 10]}$. None of the printed options is correct.

Quick Tip to Solve: If the scalar interval crosses zero, the smallest vector magnitude is zero; multiply the largest absolute endpoint by the vector magnitude.

18. A meeting will be held only if all three members A, B and C are present. The probability that member A does not turn up is 0.10, member B does not turn up is 0.20 and member C does not turn up is 0.05. The probability of the meeting being cancelled is: [1 mark]

(A) 0.35

(B) 0.316

(C) 0.001

(D) 0.65

Correct Answer: (B) 0.316

Solution:

Concept:

- The meeting takes place precisely when all three members attend; cancellation is the complement of that event.
- For independent attendance events, multiply the probabilities of attendance: $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Step 1: Calculate the probability of all attending The attendance probabilities are $P(A) = 1 - 0.10 = 0.90$, $P(B) = 1 - 0.20 = 0.80$, and $P(C) = 1 - 0.05 = 0.95$. Using the independent-attendance interpretation of the question,

$$P(\text{meeting held}) = 0.90 \times 0.80 \times 0.95 = 0.684.$$

Step 2: Take the complement and check the options

$$P(\text{cancelled}) = 1 - 0.684 = 0.316.$$

Option (A), 0.35, simply adds the absence probabilities, double-counting overlapping absences. Option (C), 0.001, is the probability that all three are absent, whereas just one absence cancels the meeting. Option (D), 0.65, is the complement of the incorrect simple sum. Independence is implicit in this intended calculation; marginal probabilities alone do not determine their joint probability.

Step 3: Conclusion The intended probability of cancellation is $\boxed{0.316}$. Hence, the correct option is (B).

Quick Tip to Solve: For an event cancelled by any absence, calculate the probability of no absences first, then subtract that product from one.

19. Assertion (A): If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 256$ and $|\vec{b}| = 8$, then $|\vec{a}| = 2$.

Reason (R): $\sin^2\theta + \cos^2\theta = 1$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- If θ is the angle between vectors, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.
- The identity $\sin^2\theta + \cos^2\theta = 1$ combines the squared cross and dot products.

Step 1: Judge the assertion Using the vector identity,

$$256 = |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 = 64|\vec{a}|^2.$$

Hence $|\vec{a}|^2 = 4$. A magnitude is nonnegative, so $|\vec{a}| = 2$, and Assertion (A) is true.

Step 2: Judge the reason All three statements in Reason (R) are standard identities. The angle between vectors lies in $[0, \pi]$, where $\sin\theta \geq 0$; squaring the dot product also removes its possible sign. Thus Reason (R) is true.

Step 3: Check the explanatory connection Squaring the cross-product and dot-product formulae and adding gives $|\vec{a}|^2|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2$. This is exactly the equation used to establish the assertion. Therefore (B) is excluded because the explanation is direct, while (C) and (D) give incorrect truth values.

Step 4: Conclusion Both Assertion (A) and Reason (R) are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Combine squared dot and cross products immediately into the product of squared magnitudes, then take the nonnegative square root.

20. Assertion (A): Let $f(x) = e^x$ and $g(x) = \log x$. Then $(f+g)(x) = e^x + \log x$, where domain of $(f+g)$ is $\mathbb{R}$.

Reason (R): Dom$(f+g)$ = Dom$(f) \cap$ Dom(g). [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A real-valued sum of functions is defined only where both summands are defined.
- The exponential e^x is defined on $\mathbb{R}$, whereas the real logarithm $\log x$ requires $x > 0$.

Step 1: Judge the assertion The formula $(f+g)(x) = e^x + \log x$ is valid whenever both functions exist. But $\log x$ is undefined as a real number for $x \leq 0$, so the claimed domain $\mathbb{R}$ is incorrect. Therefore Assertion (A), taken as a whole, is false.

Step 2: Judge the reason To add $f(x)$ and $g(x)$, each value must exist. Thus $\text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g)$, and Reason (R) is true.

Step 3: Apply the domain rule Here

$$\text{Dom}(f+g) = \mathbb{R} \cap (0, \infty) = (0, \infty).$$

The reason directly exposes the error in the assertion. Options (A) and (B) require a true assertion, and option (C) requires a false reason; each conflicts with these domain checks.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For a sum involving $\log x$, intersect the domains before accepting any claim that the sum is defined for all real numbers.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). If $\vec{a}$ and $\vec{b}$ are position vectors of point A and point B respectively, find the position vector of point C on BA produced such that $BC = 3BA$. [2 marks]

Solution:

Concept:

- If $\vec{a}$ and $\vec{b}$ are position vectors of A and B, then $\overrightarrow{BA} = \vec{a} - \vec{b}$.
- The position vector of a new point is the starting position vector plus the displacement to that point.

Quick Tip to Solve: For BA produced, start at B: write $\vec{c} = \vec{b} + k(\vec{a} - \vec{b})$, where $k = BC/BA$.

Step 1: Translate the directed geometry into vectors Let O be the origin and write $\overrightarrow{OC} = \vec{c}$. The phrase BA produced places C beyond A in the direction from B to A, giving the order B, A, C. Since $BC = 3BA$, the vectors $\overrightarrow{BC}$ and $\overrightarrow{BA}$ have the same direction and their lengths are in the ratio $3 : 1$. Consequently,

$$\overrightarrow{BC} = 3\overrightarrow{BA} = 3(\vec{a} - \vec{b}).$$

Step 2: Add the displacement to the starting point By vector addition along the path O to B to C,

$$\begin{aligned}\vec{c} &= \overrightarrow{OB} + \overrightarrow{BC} \\ &= \vec{b} + 3(\vec{a} - \vec{b}) \\ &= 3\vec{a} - 2\vec{b}.\end{aligned}$$

This also gives $\overrightarrow{AC} = \vec{c} - \vec{a} = 2(\vec{a} - \vec{b})$, consistent with C lying beyond A and $AC = 2BA$. The direction is essential: adding the displacement to A instead of B would place C too far along the line.

Step 3: Conclusion The position vector of C is $\boxed{3\vec{a} - 2\vec{b}}$.

Quick Tip to Solve: For BA produced, start at B: write $\vec{c} = \vec{b} + k(\vec{a} - \vec{b})$, where $k = BC/BA$.

OR

21(b). Vector $\vec{r}$ is inclined at equal angles to the three axes x, y and z. If magnitude of $\vec{r}$ is $5\sqrt{3}$ units, then find $\vec{r}$. [2 marks]

Solution:

Concept:

- The direction cosines of a vector are $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$, where the angles are measured with the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$. A vector of magnitude R is $\vec{r} = R(l\hat{i} + m\hat{j} + n\hat{k})$.

Step 1: Use the equality of the angles Equal angles give equal direction cosines, so let $l = m = n = t$. Substitution in the direction-cosine identity gives

$$3t^2 = 1, \qquad t = \pm\frac{1}{\sqrt{3}}.$$

All three signs must be the same: choosing different signs would give different angles with the positive axes.

Step 2: Multiply by the given magnitude With $R = 5\sqrt{3}$, each component is $Rt = \pm 5$, hence

$$\vec{r} = \pm 5(\hat{i} + \hat{j} + \hat{k}).$$

The positive choice makes equal acute angles with the axes; the negative choice makes equal obtuse angles. In either case the magnitude is $\sqrt{25 + 25 + 25} = 5\sqrt{3}$, as required.

Quick Tip to Solve: Equal angles mean equal signed components. Divide the magnitude by $\sqrt{3}$, and keep the same sign for all three components.

Step 3: Conclusion $\boxed{\vec{r} = \pm 5(\hat{i} + \hat{j} + \hat{k})}$ units. If the equal angles are taken to be acute, $\vec{r} = 5\hat{i} + 5\hat{j} + 5\hat{k}$ units.

Quick Tip to Solve: Equal angles mean equal signed components. Divide the magnitude by $\sqrt{3}$, and keep the same sign for all three components.

22. Find the domain of $f(x) = \sin^{-1}(-x^2)$. [2 marks]

Solution:

Concept:

- The real inverse sine function $\sin^{-1} u$ is defined exactly when $-1 \leq u \leq 1$.
- For every real number x, $x^2 \geq 0$; therefore $-x^2 \leq 0$. The restriction must be imposed on the entire argument $-x^2$.

Step 1: Apply the inverse sine condition For $f(x) = \sin^{-1}(-x^2)$ to be real, its argument must satisfy

$$-1 \leq -x^2 \leq 1.$$

The upper inequality holds automatically because $-x^2 \leq 0 \leq 1$. Thus only the lower bound restricts the input x.

Step 2: Solve the remaining inequality From $-1 \leq -x^2$, multiplication by -1 reverses the inequality and gives $x^2 \leq 1$. Consequently,

$$-1 \leq x \leq 1.$$

Both endpoints are allowed: when $x = \pm 1$, the argument is -1, and $\sin^{-1}(-1) = -\pi/2$ exists. If $|x| > 1$, the argument is below -1, so there is no real function value.

Step 3: Conclusion The domain is $\boxed{[-1, 1]}$.

Quick Tip to Solve: Check the argument of the inverse sine, not its output. For a negative square, the lower bound -1 gives the only restriction.

23. Find the interval in which $f(x) = x + \dfrac{1}{x}$, $x \neq 0$ is always increasing. [2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where its first derivative is positive. First identify the domain and then test the sign of the derivative.
- The function $f(x) = x + 1/x$ is defined for $x \neq 0$, and $\dfrac{d}{dx}(x^{-1}) = -x^{-2}$.

Quick Tip to Solve: For $1 - 1/x^2$, multiply by the positive quantity x^2 to test its sign, and keep the two intervals separate across $x = 0$.

Step 1: Differentiate the given function Using the power rule on each term gives

$$f\prime(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}.$$

For every permitted input, $x^2 > 0$, so the denominator is positive. The sign of the derivative is therefore determined entirely by the numerator $x^2 - 1$.

Step 2: Find where the derivative is positive The condition for a positive derivative is

$$x^2 - 1 > 0 \quad \Longleftrightarrow \quad |x| > 1.$$

Thus $x < -1$ or $x > 1$. On $(-1, 0)$ and $(0, 1)$, the derivative is negative, so those intervals are decreasing. At $x = -1$ and $x = 1$, the derivative is zero; at $x = 0$, the original function is undefined, so no interval may cross that point.

Step 3: Conclusion f is increasing separately on $\boxed{(-\infty, -1) \text{ and } (1, \infty)}$.

Quick Tip to Solve: For $1 - 1/x^2$, multiply by the positive quantity x^2 to test its sign, and keep the two intervals separate across $x = 0$.

24(a). Differentiate $\sqrt{e^{\sqrt{2x}}}$ with respect to $e^{\sqrt{2x}}$, for $x > 0$. [2 marks]

Solution:

Concept:

- To differentiate one function $y(x)$ with respect to another $u(x)$, use $\frac{dy}{du} = \frac{dy/dx}{du/dx}$, provided $du/dx \neq 0$.
- Here $u = e^{\sqrt{2x}} > 0$ and $y = \sqrt{u}$. The power rule gives $\frac{d}{du}u^{1/2} = \frac{1}{2}u^{-1/2}$.

Step 1: Identify the variable of differentiation Put $u = e^{\sqrt{2x}}$. The expression to be differentiated becomes $y = \sqrt{u} = u^{1/2}$, so

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}.$$

This substitution matters because the question asks for differentiation with respect to $e^{\sqrt{2x}}$, rather than with respect to x.

Step 2: Verify the derivative ratio For $x > 0$, the chain rule gives

$$\frac{du}{dx} = \frac{e^{\sqrt{2x}}}{\sqrt{2x}}, \qquad \frac{dy}{dx} = \frac{e^{\sqrt{2x}}/2}{2\sqrt{2x}}.$$

The derivative du/dx is positive and nonzero. Dividing cancels the common factor $1/\sqrt{2x}$, leaving

$$\frac{dy}{du} = \frac{1}{2}e^{-\sqrt{2x}/2}.$$

Quick Tip to Solve: When both expressions contain the same inner exponential, name it u first: the problem then becomes differentiating $\sqrt{u}$.

Step 3: Conclusion The required derivative is $\boxed{\frac{1}{2\sqrt{e^{\sqrt{2x}}}}}$, for $x > 0$.

Quick Tip to Solve: When both expressions contain the same inner exponential, name it u first: the problem then becomes differentiating $\sqrt{u}$.

OR

24(b). If $(x)^y = (y)^x$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- For positive bases x, y, logarithms turn variable powers into products: $\ln(x^y) = y \ln x$ and $\ln(y^x) = x \ln y$.
- Treat y as a function of x. Apply the product rule and use $\frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}$.

Step 1: Take logarithms and differentiate implicitly From $x^y = y^x$, taking natural logarithms gives

$$y \ln x = x \ln y.$$

Differentiating both sides with respect to x yields

$$\frac{dy}{dx} \ln x + \frac{y}{x} = \ln y + \frac{x}{y} \frac{dy}{dx}.$$

The term $(1/y)(dy/dx)$ on the right is essential because y varies with x.

Step 2: Collect the derivative terms Bring all terms containing dy/dx to the left and the remaining terms to the right:

$$\frac{dy}{dx} \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x}.$$

At a regular point where $\ln x - x/y \neq 0$, division isolates the derivative. Multiplying numerator and denominator by xy also gives the equivalent form

$$\frac{dy}{dx} = \frac{y(x \ln y - y)}{x(y \ln x - x)}.$$

Step 3: Conclusion $\boxed{\frac{dy}{dx} = \frac{\ln y - y/x}{\ln x - x/y}}$, for positive x, y where the denominator is nonzero.

Quick Tip to Solve: For an equality with variables in both bases and exponents, take logarithms before differentiating and collect every dy/dx term together.

25. Find the angle at which the given lines are inclined to each other:

$l_1 : \dfrac{x-5}{2} = \dfrac{y+3}{1} = \dfrac{z-1}{-3}$

$l_2 : \dfrac{x}{3} = \dfrac{y-1}{2} = \dfrac{z+5}{-1}$ [2 marks]

Solution:

Concept:

- The denominators in a line's symmetric equation give its direction ratios; the coordinates in the numerators locate a point on it.
- For direction vectors $\vec{a}$ and $\vec{b}$, the acute angle θ between the lines satisfies $\cos\theta = \dfrac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|}$.

Step 1: Read the direction vectors From the denominators of the two symmetric equations, choose

$$\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}, \qquad \vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}.$$

The shifts $x-5, y+3, z-1$ and $x, y-1, z+5$ determine positions, so they do not enter the angle formula.

Step 2: Apply the scalar-product formula Compute the scalar product and magnitudes:

$$\vec{a}\cdot\vec{b} = 2(3) + 1(2) + (-3)(-1) = 11,$$
$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14},$$
$$|\vec{b}| = \sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}.$$

Consequently, $\cos\theta = |11|/(\sqrt{14}\sqrt{14}) = 11/14$. The absolute value selects the conventional acute angle between the unoriented lines.

Step 3: Conclusion The angle at which the lines are inclined is $\boxed{\theta = \cos^{-1}\left(\dfrac{11}{14}\right)}$, approximately 38.21°.

Quick Tip to Solve: For the angle between symmetric-form lines, use only the three denominators of each line as direction ratios.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. Find the value of x, if $\begin{pmatrix}1 & x & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{pmatrix}\begin{pmatrix}1\\ 2\\ x\end{pmatrix} = O$. [3 marks]

Solution:

Concept:

- Matrix multiplication is performed by taking the scalar product of each row of the first matrix with each column of the second, keeping the given order.
- A 1×3 row multiplied by a 3×3 matrix and a 3×1 column produces a 1×1 matrix. Equating this to O means its single entry is zero.
- Matrix multiplication is associative, so the row and middle matrix may be multiplied first; it is not necessary to change the order of the factors.

Step 1: Multiply the row by the middle matrix Let the three given factors be R, M, C, respectively. The entries of RM are obtained from the three columns of M:

$$\begin{aligned}(RM)_1 &= 1(1) + x(2) + 1(15) = 16 + 2x,\\ (RM)_2 &= 1(3) + x(5) + 1(3) = 6 + 5x,\\ (RM)_3 &= 1(2) + x(1) + 1(2) = 4 + x.\end{aligned}$$

Thus $RM = \begin{pmatrix}16 + 2x & 6 + 5x & 4 + x\end{pmatrix}$. In particular, the first column contains 15, while the bottom entry of the second column is 3.

Step 2: Multiply by the column and solve Take the scalar product with $C = \begin{pmatrix}1 & 2 & x\end{pmatrix}^T$. The stated equation becomes

$$\begin{aligned}(16 + 2x)(1) + (6 + 5x)(2) + (4 + x)x &= 0,\\ 16 + 2x + 12 + 10x + 4x + x^2 &= 0,\\ x^2 + 16x + 28 &= 0,\\ (x + 2)(x + 14) &= 0.\end{aligned}$$

The constant product is $2 \times 14 = 28$, and the sum of the factors' constants is 16. Both roots are allowed because there is no denominator or additional restriction on x. Substitution gives $4 - 32 + 28 = 0$ at $x = -2$, and $196 - 224 + 28 = 0$ at $x = -14$.

Step 3: Conclusion The required values are $\boxed{x = -2 \text{ or } x = -14}$.

Quick Tip to Solve: Multiply the row by the middle matrix first, then check both roots in the resulting scalar quadratic.

27(a). Find the distance of the point $P(2,4,-1)$ from the line $\dfrac{x+5}{1}=\dfrac{y+3}{4}=\dfrac{z-6}{-9}$. [3 marks]

Solution:

Concept:

- A line $\vec{r}=\vec{a}+t\vec{b}$ passes through the point with position vector $\vec{a}$ and has direction vector $\vec{b}$.
- The shortest distance from a point to a line is measured along its perpendicular to that line. If H is the perpendicular foot from P, then $\overrightarrow{HP}\cdot\vec{b}=0$.
- For a point A on the line, the parameter of the foot is $t=\dfrac{\overrightarrow{AP}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$; the distance is then $|\overrightarrow{HP}|$.

Step 1: Parameterise the line and identify the displacement Set the common value of the three ratios equal to t. Every point on the line has coordinates

$$x=-5+t,\qquad y=-3+4t,\qquad z=6-9t.$$

Hence take $A=(-5,-3,6)$ and $\vec{b}=(1,4,-9)$. For $P=(2,4,-1)$,

$$\overrightarrow{AP}=(7,7,-7),\qquad \vec{b}\cdot\vec{b}=1+16+81=98.$$

The signs in A follow by setting the numerators of the line equation equal to zero.

Step 2: Locate the perpendicular foot Write $H=A+t\vec{b}$. Then $\overrightarrow{HP}=\overrightarrow{AP}-t\vec{b}$. Enforce perpendicularity:

$$\begin{aligned}(\overrightarrow{AP}-t\vec{b})\cdot\vec{b}&=0,\\ 7(1)+7(4)+(-7)(-9)-98t&=0,\\ 98-98t&=0.\end{aligned}$$

This gives $t=1$, so $H=(-4,1,-3)$. As a check, H lies on the line because all three original ratios equal 1.

Step 3: Compute the perpendicular length Now $\overrightarrow{HP}=(6,3,2)$, whose scalar product with the line direction is $6+12-18=0$, confirming that it is perpendicular. Therefore

$$PH=\sqrt{6^2+3^2+2^2}=\sqrt{36+9+4}=\sqrt{49}.$$

Since this segment is perpendicular to the line, its length is the shortest distance required.

Step 4: Conclusion The distance of P from the given line is $\boxed{7\text{ units}}$.

Quick Tip to Solve: First set the ratios to a parameter; solving one perpendicularity equation then gives the foot and its distance.

OR

27(b). Let the position vectors of the points A, B and C be $3\hat{i}-\hat{j}-2\hat{k}$, $\hat{i}+2\hat{j}-\hat{k}$ and $\hat{i}+5\hat{j}+3\hat{k}$ respectively. Find the vector and cartesian equations of the line passing through A and parallel to line BC. [3 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a non-zero direction vector $\vec{d}$, has vector equation $\vec{r}=\vec{a}+\lambda\vec{d}$, with $\lambda\in\mathbb{R}$.
- The direction vector of BC is obtained by subtracting position vectors: $\overrightarrow{BC}=\vec{c}-\vec{b}$.
- If a direction component is zero, that coordinate is constant on the line. State this separately in the Cartesian equation instead of dividing by zero.

Step 1: Determine the direction of the line BC Read the given points as

$$A=(3,-1,-2),\qquad B=(1,2,-1),\qquad C=(1,5,3).$$

Subtract the coordinates of B from those of C:

$$\overrightarrow{BC}=(1-1,5-2,3-(-1))=(0,3,4).$$

Thus BC has direction vector $3\hat{j}+4\hat{k}$. A line parallel to BC can use this same non-zero vector; the zero first component means that motion along the line does not change the x-coordinate.

Step 2: Use point A to form the vector equation Since the required line passes through A, take its position vector as the fixed vector:

$$\vec{r}=(3\hat{i}-\hat{j}-2\hat{k})+\lambda(3\hat{j}+4\hat{k}),\qquad \lambda\in\mathbb{R}.$$

At $\lambda=0$, this gives precisely A, and its parameter coefficient is the direction of BC. These two checks establish the required point and parallelism.

Step 3: Eliminate the parameter for the Cartesian form Equating coordinates in the vector equation gives

$$x=3,\qquad y=-1+3\lambda,\qquad z=-2+4\lambda.$$

From the last two equations, $\lambda=(y+1)/3=(z+2)/4$. The first coordinate must remain as the separate condition $x=3$; a symmetric fraction with a zero denominator would not be defined. Together these conditions describe the same line for every real value of λ.

Step 4: Conclusion Vector equation: $\boxed{\vec{r}=3\hat{i}-\hat{j}-2\hat{k}+\lambda(3\hat{j}+4\hat{k}),\ \lambda\in\mathbb{R}}$. Cartesian equation: $\boxed{x=3,\quad \dfrac{y+1}{3}=\dfrac{z+2}{4}}$.

Quick Tip to Solve: Subtract C minus B for the parallel direction; a zero direction ratio becomes a fixed-coordinate equation.

28. Consider the Linear Programming Problem, where the objective function $Z = (x + 4y)$ needs to be minimized subject to constraints

$2x + y \geq 1000$

$x + 2y \geq 800$

$x, y \geq 0$.

Draw a neat graph of the feasible region and find the minimum value of Z. [3 marks]

Solution:

Concept:

- Each linear inequality represents a half-plane. The feasible region is their intersection with the first quadrant, since $x, y \geq 0$.
- For a linear objective function $Z = x + 4y$, compare its values at the corner points of the feasible region. If the region is unbounded, also verify that no feasible point gives a smaller value.

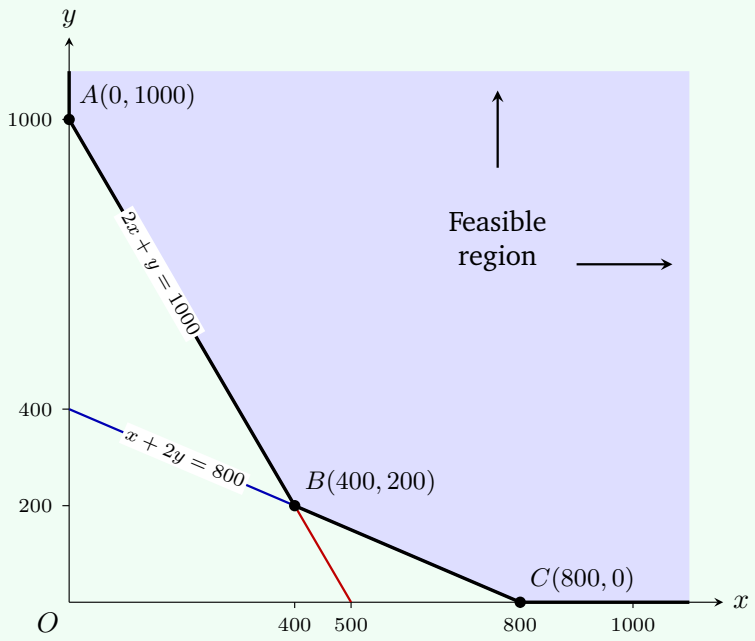
Step 1: Draw the constraint lines and select the feasible sides The line $2x + y = 1000$ passes through $(500, 0)$ and $(0, 1000)$. The line $x + 2y = 800$ passes through $(800, 0)$ and $(0, 400)$. Draw these lines using their intercepts. Since the origin does not satisfy either inequality, choose the side away from the origin for each line. Restrict to the first quadrant and shade their common region. It extends indefinitely upwards and to the right; the shading shown in the graph is its portion inside the plotting window.

Step 2: Determine the corner points At the intersection of the two lines,

$$2x + y = 1000,$$
$$x + 2y = 800.$$

Doubling the second equation and subtracting the first gives $3y = 600$, so $y = 200$ and $x = 400$. On the y-axis both inequalities require $y \geq 1000$, while on the x-axis they require $x \geq 800$. Thus the corner points are $A = (0, 1000)$, $B = (400, 200)$ and $C = (800, 0)$.

Quick Tip to Solve: For an unbounded region, certify the minimum directly: write $x+4y = (x+2y)+2y \geq 800$.



Feasible region in the first quadrant; the shaded region continues upwards and rightwards.

Step 3: Evaluate the objective and check the unbounded region The objective values at the corners are

$$Z(A) = 0 + 4(1000) = 4000,$$
$$Z(B) = 400 + 4(200) = 1200,$$
$$Z(C) = 800 + 4(0) = 800.$$

The feasible region is unbounded, so verify this candidate globally. At every feasible point,

$$Z = x + 4y = (x + 2y) + 2y \geq 800 + 0 = 800.$$

Equality requires $y = 0$ and $x + 2y = 800$, and hence $x = 800$. This point also satisfies $2x + y = 1600 \geq 1000$, so the lower bound is attained.

Step 4: Conclusion The minimum value is $\boxed{Z_{\min} = 800}$, attained uniquely at $\boxed{(x, y) = (800, 0)}$. The feasible region is the shaded unbounded region with corners $(0, 1000)$, $(400, 200)$ and $(800, 0)$.

Quick Tip to Solve: For an unbounded region, certify the minimum directly: write $x+4y = (x+2y)+2y \geq 800$.

29(a). **A student wants to pair up natural numbers in such a way that they satisfy the equation $2x + y = 41,\ x, y \in \mathbb{N}$. Find the domain and range of the relation. Check if the relation thus formed is reflexive, symmetric and transitive. Hence, state whether it is an equivalence relation or not.**

[3 marks]

Solution:

Concept:

- The domain of a relation consists of its first coordinates, and the range consists of its second coordinates. Here natural numbers start at 1.
- A relation on $\mathbb{N}$ is reflexive when every (a, a) belongs to it, symmetric when (a, b) implies (b, a), and transitive when (a, b) and (b, c) imply (a, c).
- An equivalence relation must have all three properties. A single counterexample disproves any one of these universal properties.

Step 1: Find the relation, domain and range Write $y = 41 - 2x$. Since $x, y \in \mathbb{N}$, we need $x \geq 1$ and $41 - 2x \geq 1$. Thus $1 \leq x \leq 20$, and the relation is

$$R = \{(x, 41 - 2x) : x = 1, 2, \ldots, 20\}.$$

Its pairs begin $(1, 39), (2, 37)$ and end $(19, 3), (20, 1)$. Therefore,

$$\text{Dom}(R) = \{1, 2, \ldots, 20\}, \qquad \text{Ran}(R) = \{1, 3, 5, \ldots, 39\}.$$

The second coordinate decreases by 2 as the first coordinate increases by 1, explaining why the range contains precisely these odd numbers.

Step 2: Test reflexivity For (a, a) to belong to R, it would be necessary that $2a + a = 41$, or $3a = 41$. This has no natural-number solution. In particular, $(1, 1) \notin R$, although $1 \in \mathbb{N}$. Hence R is not reflexive.

Step 3: Test symmetry The pair $(1, 39)$ belongs to R, because $2(1) + 39 = 41$. Its reverse $(39, 1)$ does not belong, since $2(39) + 1 = 79 \neq 41$. Thus a pair in the relation need not have its reverse in the relation, and R is not symmetric.

Step 4: Test transitivity Choose pairs whose middle coordinates agree: $(11, 19) \in R$, since $2(11) + 19 = 41$, and $(19, 3) \in R$, since $2(19) + 3 = 41$. Transitivity would require $(11, 3) \in R$. However, $2(11) + 3 = 25 \neq 41$, so this pair is absent. Therefore R is not transitive.

Step 5: Conclusion The domain is $\boxed{\{1, 2, \ldots, 20\}}$, and the range is $\boxed{\{1, 3, 5, \ldots, 39\}}$. The relation is neither reflexive, nor symmetric, nor transitive; hence it is $\boxed{\text{not an equivalence relation}}$.

Quick Tip to Solve: To disprove transitivity, find two valid pairs with a shared middle entry, then test the required first-to-last pair.

OR

29(b). **Show that the function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is a set of natural numbers, given by $f(n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases}$ is a bijection.** [3 marks]

Solution:

Concept:

- A function is injective if equal outputs force equal inputs; it is surjective if each element of the codomain has a preimage. A function possessing both properties is bijective.
- Here f subtracts 1 from an even natural number and adds 1 to an odd natural number. Thus it exchanges each adjacent pair $2k-1, 2k$, where $k \in \mathbb{N}$.

Step 1: Check that the function maps natural numbers into natural numbers If n is even, then $n \geq 2$, so $f(n) = n - 1 \geq 1$ is an odd natural number. If n is odd, then $f(n) = n + 1 \geq 2$ is an even natural number. Thus both rules yield values in the stated codomain $\mathbb{N}$. In particular, the initial pairs are exchanged as $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and so on.

Step 2: Prove injectivity by considering the parities of the inputs Suppose $f(a) = f(b)$, where $a, b \in \mathbb{N}$. If both are even, then $a - 1 = b - 1$, giving $a = b$. If both are odd, then $a + 1 = b + 1$, again giving $a = b$. If one is even and the other is odd, their outputs have opposite parities: an even input gives an odd output, while an odd input gives an even output. Those outputs cannot be equal. Hence equal outputs always imply equal inputs, proving that f is injective.

Step 3: Construct a preimage for each element of the codomain Let $m \in \mathbb{N}$ be arbitrary. If m is odd, choose $n = m + 1$. Then n is an even natural number and

$$f(n) = n - 1 = (m + 1) - 1 = m.$$

If m is even, choose $n = m - 1$. Since $m \geq 2$, this n is an odd natural number and

$$f(n) = n + 1 = (m - 1) + 1 = m.$$

The two cases cover every natural number m, including $m = 1$, whose preimage is 2. Therefore every codomain element is attained, proving that f is surjective.

Step 4: Conclusion The function $f : \mathbb{N} \to \mathbb{N}$ is both injective and surjective, so $\boxed{f \text{ is a bijection}}$. Its inverse is itself, since applying the same adjacent-pair exchange twice returns the original number.

Quick Tip to Solve: Notice the swaps $2k - 1 \leftrightarrow 2k$: every natural number has exactly one partner and returns after two applications.

30(a). Differentiate $y = \sin^{-1}(3x - 4x^3)$ w.r.t. x, if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. [3 marks]

Solution:

Concept:

- The triple-angle identity is $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. It converts the cubic inside the inverse sine into a trigonometric expression.
- The principal range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$. Therefore, $\sin^{-1}(\sin t) = t$ only after checking that t lies in this range.
- For $-1 < x < 1$, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

Step 1: Choose a substitution and check its range Put $x = \sin\theta$, taking $\theta = \sin^{-1} x$ in the principal range. Since the given interval is $-1/2 \le x \le 1/2$, the monotonicity of inverse sine gives

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}, \qquad -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}.$$

This range check is essential: it ensures that inverse sine returns 3θ, with no change of branch.

Step 2: Simplify the function before differentiating Using the triple-angle identity,

$$\begin{aligned} y &= \sin^{-1}(3\sin\theta - 4\sin^3\theta) \\ &= \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1} x. \end{aligned}$$

Thus the given expression agrees throughout the specified closed interval with a much simpler inverse trigonometric function.

Step 3: Differentiate the simplified expression At each interior point of the given interval,

$$\frac{dy}{dx} = 3\frac{d}{dx}(\sin^{-1} x) = \frac{3}{\sqrt{1-x^2}}.$$

Differentiating the simplified form also avoids an apparent $0/0$ in the direct chain-rule expression at $x = \pm 1/2$. For the function restricted to the stated interval, the corresponding inward one-sided derivatives at both endpoints are $3/\sqrt{3/4} = 2\sqrt{3}$.

Step 4: Conclusion Hence, $\boxed{\frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}}$ for $-1/2 < x < 1/2$. At the endpoints of the stated interval, the inward one-sided derivative has value $2\sqrt{3}$.

Quick Tip to Solve: Before replacing inverse sine of a sine by its angle, use the given interval to check the principal range.

OR

30(b). Differentiate $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to x, when $x \in (0,1)$. [3 marks]

Solution:

Concept:

- The double-angle identity $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ suggests using $x = \tan\theta$.
- The principal range of $\cos^{-1}$ is $[0, \pi]$. Hence $\cos^{-1}(\cos t) = t$ when t lies in this interval.
- The derivative of inverse tangent is $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Step 1: Set up the substitution with the correct branch Let $x = \tan\theta$, where $\theta = \tan^{-1} x$. Because $0 < x < 1$, we have

$$0 < \theta < \frac{\pi}{4}, \qquad 0 < 2\theta < \frac{\pi}{2}.$$

In particular, 2θ lies inside the principal range of inverse cosine. This verifies that the double-angle substitution does not introduce an extra angle or a sign change.

Step 2: Reduce the inverse cosine expression Substituting $x = \tan\theta$ and applying the double-angle formula gives

$$\begin{aligned} y &= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \\ &= \cos^{-1}(\cos 2\theta) = 2\theta \\ &= 2\tan^{-1} x. \end{aligned}$$

The equality holds for every x in the specified open interval, so the simplified expression may be differentiated there.

Step 3: Differentiate and check the sign We now obtain

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1} x) = \frac{2}{1+x^2}.$$

This derivative is positive. Indeed, as a positive x increases, $(1-x^2)/(1+x^2)$ decreases, while inverse cosine is decreasing; their composition therefore increases, consistent with this result. The condition $x > 0$ is essential when simplifying the square root in a direct chain-rule calculation.

Step 4: Conclusion Therefore, $\boxed{\frac{dy}{dx} = \frac{2}{1+x^2}}$, for $0 < x < 1$.

Quick Tip to Solve: Recognise the ratio as cosine of twice an inverse-tangent angle, and check that the angle lies in the inverse-cosine range.

31. Bag I contains 4 white and 5 black balls. Bag II contains 6 white and 7 black balls. A ball drawn randomly from bag I is transferred to bag II and then a ball is drawn randomly from bag II. Find the probability that the ball drawn is white. [3 marks]

Solution:

Concept:

- A random draw from a bag of equally likely balls gives probability equal to the number of favourable balls divided by the total number of balls.
- For mutually exclusive and exhaustive events E_1, E_2, the law of total probability gives $P(W) = P(E_1)P(W \mid E_1) + P(E_2)P(W \mid E_2)$.

Step 1: Identify the two possible transfers Let E_1 denote transfer of a white ball from bag I, and E_2 transfer of a black ball from bag I. Let W denote drawing a white ball from bag II after the transfer. Bag I initially has $4 + 5 = 9$ balls, so

$$P(E_1) = \frac{4}{9}, \qquad P(E_2) = \frac{5}{9}.$$

Exactly one of these transfer events occurs, and their probabilities add to one. They therefore form the complete set of cases needed for total probability.

Step 2: Find the conditional probabilities for bag II Bag II initially has 6 white and 7 black balls, a total of 13. After either transfer it contains 14 balls. If a white ball is transferred, its composition becomes 7 white and 7 black, giving

$$P(W \mid E_1) = \frac{7}{14}.$$

If a black ball is transferred, its composition becomes 6 white and 8 black, giving

$$P(W \mid E_2) = \frac{6}{14}.$$

These conditional probabilities differ because the colour transferred changes the number of favourable balls in the second bag.

Step 3: Combine the two mutually exclusive routes Weight each conditional probability by the probability of its corresponding transfer:

$$\begin{aligned} P(W) &= P(E_1)P(W \mid E_1) + P(E_2)P(W \mid E_2) \\ &= \frac{4}{9} \cdot \frac{7}{14} + \frac{5}{9} \cdot \frac{6}{14} \\ &= \frac{28 + 30}{126} = \frac{58}{126} = \frac{29}{63}. \end{aligned}$$

As a check, this weighted average lies between $6/14$ and $7/14$, the two possible conditional probabilities, as required.

Step 4: Conclusion The probability that the ball drawn from bag II is white is $\boxed{\frac{29}{63}}$.

Quick Tip to Solve: Increase the total of bag II by one in both transfer cases, then weight its white-ball probabilities by the transfer probabilities.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Solve the differential equation: $x^2y\,dx - (x^3+y^3)\,dy = 0$**.** [5 marks]

Solution:

Concept:

- A differential equation is homogeneous when its differential coefficients are homogeneous functions of the same degree. Here both x^2y and x^3+y^3 have degree three.
- Taking x as a function of y, use $x = vy$, where $v = x/y$. The product rule gives $dx/dy = v + y\,dv/dy$.
- After substitution, separate the variables and integrate. The formula $\int dy/y = \ln|y| + C$ applies on intervals with $y \neq 0$; any solution excluded by division must be checked in the original equation.

Step 1: Express the equation with y as the independent variable The given equation is

$$x^2y\,dx - (x^3+y^3)\,dy = 0.$$

On a region where $x \neq 0$ and $y \neq 0$, move the second term to the other side and divide by $x^2y\,dy$:

$$\frac{dx}{dy} = \frac{x^3+y^3}{x^2y} = \frac{x}{y} + \frac{y^2}{x^2}.$$

The right-hand side depends only on the ratio x/y. Choosing x as the dependent variable makes the homogeneous substitution particularly simple.

Step 2: Substitute x = vy and separate the variables Let $x = vy$, so that $x/y = v$, $y^2/x^2 = 1/v^2$, and

$$\frac{dx}{dy} = v + y\frac{dv}{dy}.$$

Substitute these expressions into the equation:

$$\begin{aligned} v + y\frac{dv}{dy} &= v + \frac{1}{v^2}, \\ y\frac{dv}{dy} &= \frac{1}{v^2}, \\ v^2\,dv &= \frac{dy}{y}. \end{aligned}$$

The v-dependent factors and the y-dependent factors are now on opposite sides, so the equation is separable.

Step 3: Integrate and restore the original variables Integrating both sides gives

$$\frac{v^3}{3} = \ln|y| + C_1.$$

Quick Tip to Solve: Here choose $x = vy$, rather than $y = vx$: the v terms cancel immediately, leaving $v^2\,dv = dy/y$. Check $y = 0$ separately after division.

Restore $v = x/y$, multiply by three, and absorb $3C_1$ into an arbitrary constant C:

$$\frac{x^3}{y^3} - 3\ln|y| = C, \qquad y \neq 0.$$

The absolute value keeps the logarithm real for either sign of y. The constant is arbitrary on each interval of the solution.

Step 4: Verify the implicit relation and check the excluded solution Differentiate the implicit expression directly:

$$d\left(\frac{x^3}{y^3} - 3\ln|y|\right) = \frac{3x^2}{y^3}\,dx - \left(\frac{3x^3}{y^4} + \frac{3}{y}\right)dy.$$

Multiplying the equation $d(\cdots) = 0$ by $y^4/3$ recovers exactly

$$x^2y\,dx - (x^3 + y^3)\,dy = 0.$$

Also, $y = 0$ gives $dy = 0$, and both terms in the original equation vanish. Thus this constant solution, lost when dividing by y, is valid separately.

Step 5: Conclusion The general implicit solution for $y \neq 0$ is

$$\boxed{\frac{x^3}{y^3} - 3\ln|y| = C}.$$

Equivalently, $x^3 = y^3(3\ln|y| + C)$. The original equation also admits the constant solution $\boxed{y = 0}$.

Quick Tip to Solve: Here choose $x = vy$, rather than $y = vx$: the v terms cancel immediately, leaving $v^2\,dv = dy/y$. Check $y = 0$ separately after division.

OR

32(b). Solve the differential equation $(1 + x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$ subject to initial condition $y(0) = 0$. [5 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $dy/dx + P(x)y = Q(x)$, where P and Q are functions of the independent variable x.
- Its integrating factor is $\text{IF} = e^{\int P(x)\,dx}$. Multiplication by this factor makes the left-hand side the derivative of $y\,\text{IF}$.
- Integrate $d(y\,\text{IF})/dx = Q(x)\,\text{IF}$, then use the prescribed initial value to determine the arbitrary constant.

Step 1: Write the equation in linear standard form Begin with the printed differential equation:

$$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0.$$

Quick Tip to Solve: Notice directly that $(1 + x^2)y' + 2xy$ is the product derivative $\left[(1 + x^2)y\right]'$. Integrate $4x^2$, then set $x = 0$ to find the constant.

Move the last term to the right and divide by $1 + x^2$. This denominator is positive for every real x, so the division excludes no real value:

$$\frac{dy}{dx} + \frac{2x}{1 + x^2} y = \frac{4x^2}{1 + x^2}.$$

Comparing with $dy/dx + P(x)y = Q(x)$, we identify

$$P(x) = \frac{2x}{1 + x^2}, \qquad Q(x) = \frac{4x^2}{1 + x^2}.$$

The equation is linear because y and its first derivative occur only to the first power.

Step 2: Find the integrating factor Calculate the integral of $P(x)$. The numerator $2x$ is precisely the derivative of the denominator $1 + x^2$, hence

$$\int P(x)\, dx = \int \frac{2x}{1 + x^2}\, dx = \ln(1 + x^2).$$

A nonzero constant multiple of an integrating factor is also an integrating factor, so we take the simplest choice:

$$\text{IF} = e^{\ln(1+x^2)} = 1 + x^2.$$

There is no need for an absolute value inside this logarithm, since $1 + x^2 > 0$ on the real line.

Step 3: Multiply through and integrate the product derivative Multiplication of the standard-form equation by $1 + x^2$ gives

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2.$$

By the product rule, the entire left-hand side equals $d\big[(1 + x^2)y\big]/dx$. Thus

$$\frac{d}{dx}\big[(1 + x^2)y\big] = 4x^2.$$

Integrate with respect to x:

$$(1 + x^2)y = \int 4x^2\, dx = \frac{4x^3}{3} + C.$$

This relation is the general solution before imposing the initial condition.

Step 4: Apply the initial condition and verify the result Substitute $x = 0$ and $y(0) = 0$ in the integrated relation:

$$(1 + 0^2) \cdot 0 = \frac{4 \cdot 0^3}{3} + C \quad \Longrightarrow \quad C = 0.$$

Hence the required function satisfies $(1 + x^2)y = 4x^3/3$. As a check, differentiation of this identity gives

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2,$$

which is the original equation with its last term moved to the right. At $x = 0$, the function is zero, so it also satisfies the initial condition. The expression is defined for every real x.

Quick Tip to Solve: Notice directly that $(1 + x^2)y' + 2xy$ is the product derivative $\big[(1 + x^2)y\big]'$. Integrate $4x^2$, then set $x = 0$ to find the constant.

Step 5: Conclusion The particular solution satisfying $y(0) = 0$ is

$$\boxed{y(x) = \frac{4x^3}{3(1+x^2)}}, \qquad x \in \mathbb{R}.$$

Quick Tip to Solve: Notice directly that $(1+x^2)y' + 2xy$ is the product derivative $\left[(1+x^2)y\right]'$. Integrate $4x^2$, then set $x = 0$ to find the constant.

---

33. Using integration, find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ bounded between the lines $x = -\frac{a}{2}$ to $x = \frac{a}{2}$. [5 marks]

Solution:

Concept:

- The area between two curves is the integral of upper ordinate minus lower ordinate over the stated interval of x.
- For an ellipse with positive semi-axes a, b, the upper and lower halves are $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$.
- Symmetry about both axes allows the required central area to be calculated as four times its first-quadrant portion; use $x = a\sin\theta$ to integrate the square root.

Step 1: Express the vertical strip height Rearranging the given ellipse gives

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right), \qquad y_{\text{upper}} = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y_{\text{lower}} = -\frac{b}{a}\sqrt{a^2 - x^2}.$$

Each vertical strip extends from the lower to the upper ellipse, so its height is twice the positive ordinate. The required area S is

$$S = \frac{2b}{a}\int_{-a/2}^{a/2}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_{0}^{a/2}\sqrt{a^2 - x^2}\,dx.$$

Evenness of the integrand gives the last equality.

Step 2: Transform the definite integral Set $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$. The limits $x = 0$ and $x = a/2$ correspond to $\theta = 0$ and $\theta = \pi/6$. On this interval the cosine is positive, giving $\sqrt{a^2 - x^2} = a\cos\theta$. Hence

$$S = 4ab\int_{0}^{\pi/6}\cos^2\theta\,d\theta = 2ab\int_{0}^{\pi/6}(1 + \cos 2\theta)\,d\theta.$$

The radical has become a standard trigonometric integral.

Quick Tip to Solve: Use four times the first-quadrant integral from 0 to $a/2$; integrating only the upper half misses a factor of two.

Step 3: Evaluate at the transformed limits Integrating the constant and cosine terms separately yields

$$\begin{aligned} S &= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/6} \\ &= 2ab\left(\frac{\pi}{6} + \frac{\sin(\pi/3)}{2}\right) \\ &= 2ab\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right). \end{aligned}$$

Both terms at the lower limit vanish.

Step 4: Conclusion The area of the ellipse between the given vertical lines is

$$\boxed{ab\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \text{ square units}}.$$

Quick Tip to Solve: Use four times the first-quadrant integral from 0 to $a/2$; integrating only the upper half misses a factor of two.

34(a). Find: $\displaystyle\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$ [5 marks]

Solution:

Concept:

- A proper rational function can be resolved into partial fractions. The repeated factor $(x-1)^2$ requires terms with both denominators $x-1$ and $(x-1)^2$.
- Use $\int \frac{dx}{x-c} = \ln|x-c| + C$ and the power rule $\int (x-c)^{-2}dx = -(x-c)^{-1} + C$, on an interval excluding the poles.

Step 1: Set up the partial fractions The numerator has degree two and the denominator degree three, so no polynomial division is necessary. Introduce constants A, B, D and write

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{D}{x+3}.$$

Multiplying through by the common denominator gives the polynomial identity

$$x^2 + 1 = A(x-1)(x+3) + B(x+3) + D(x-1)^2.$$

Although the original rational expression is undefined at its poles, this cleared polynomial identity can be evaluated there to determine coefficients efficiently.

Step 2: Determine the three coefficients Substitution of $x = 1$ eliminates the terms containing A and

Quick Tip to Solve: Include both powers of the repeated factor before substituting $x = 1$ and $x = -3$; check the negative sign when integrating $(x-1)^{-2}$.

D, giving

$$2 = 4B \quad \Longrightarrow \quad B = \frac{1}{2}.$$

Substitution of $x = -3$ eliminates the first two terms, giving

$$10 = 16D \quad \Longrightarrow \quad D = \frac{5}{8}.$$

Comparing coefficients of x^2 in the identity yields

$$A + D = 1 \quad \Longrightarrow \quad A = 1 - \frac{5}{8} = \frac{3}{8}.$$

To check all coefficients, expand the right side: its linear coefficient is $2A + B - 2D = 0$ and its constant coefficient is $-3A + 3B + D = 1$, agreeing with the numerator $x^2 + 1$.

Step 3: Integrate each elementary term Inserting the coefficients and using linearity, the integral becomes

$$I = \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int (x-1)^{-2}\,dx + \frac{5}{8}\int \frac{dx}{x+3}.$$

The first and third terms give logarithms with absolute values, while the middle term integrates by the power rule. Thus

$$I = \frac{3}{8}\ln|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\ln|x+3| + C.$$

In particular, the inverse-power term carries a negative sign. Differentiating it gives $1/[2(x-1)^2]$, while the logarithms give $3/[8(x-1)]$ and $5/[8(x+3)]$; together they reconstruct the original integrand. The constant C is arbitrary on each interval avoiding $x = 1, -3$.

Step 4: Conclusion

$$\boxed{\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\ln|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\ln|x+3| + C}.$$

Quick Tip to Solve: Include both powers of the repeated factor before substituting $x = 1$ and $x = -3$; check the negative sign when integrating $(x-1)^{-2}$.

OR

34(b). Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$ [5 marks]

Solution:

Concept:

- For an integrable function on $[0, a]$, reflection of the variable gives $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. This is the definite-integral symmetry property often called King's rule.
- Here $\sin(\pi/2 - x) + \cos(\pi/2 - x) = \sin x + \cos x$, so reflection preserves the denominator and

Quick Tip to Solve: Whenever the denominator is unchanged by $x \mapsto a - x$, add the reflected integral to replace a numerator x by the constant a.

changes the numerator from x to $\pi/2 - x$.

- The identity $\sin x + \cos x = \sqrt{2}\cos(x - \pi/4)$ reduces the remaining integral to a secant integral, with $\int \sec t\, dt = \ln|\sec t + \tan t| + C$.

Step 1: Reflect the variable in the given integral Denote the required value by

$$I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx.$$

On this interval the denominator is positive, so the integrand is continuous and the reflection property applies. Put $x = \pi/2 - t$. Then $dx = -dt$, and the limits $x = 0, \pi/2$ become $t = \pi/2, 0$. Reversing these limits cancels the negative sign and gives

$$I = \int_0^{\pi/2} \frac{\pi/2 - t}{\cos t + \sin t}\, dt.$$

Renaming the dummy variable t as x allows the two representations of I to be added directly.

Step 2: Add the two representations The numerators add to the constant $\pi/2$, while the common denominator stays the same. Thus

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{x + (\pi/2 - x)}{\sin x + \cos x}\, dx \\ &= \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}. \end{aligned}$$

If $J = \int_0^{\pi/2} (\sin x + \cos x)^{-1}\, dx$, then $I = \pi J/4$. The symmetry has eliminated the factor x without integration by parts.

Step 3: Evaluate the remaining secant integral Using the angle-shift identity and then $t = x - \pi/4$, we obtain

$$\begin{aligned} J &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec t\, dt \\ &= \frac{1}{\sqrt{2}} \left[\ln|\sec t + \tan t|\right]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right). \end{aligned}$$

Both logarithm arguments are positive. Since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, the ratio equals $(\sqrt{2} + 1)^2$. Therefore $J = \sqrt{2}\ln(1 + \sqrt{2})$. Substitution in $I = \pi J/4$ determines the required value.

Step 4: Conclusion Hence,

$$\boxed{\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx = \frac{\pi}{2\sqrt{2}} \ln(1 + \sqrt{2}).}$$

Quick Tip to Solve: Whenever the denominator is unchanged by $x \mapsto a - x$, add the reflected integral to replace a numerator x by the constant a.

35. Show that the line passing through the points $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line joining points $C(3, 9, 4)$ and $D(-4, 4, 4)$. [5 marks]

Solution:

Concept:

- The line through points with position vectors $\mathbf{a}$ and $\mathbf{b}$ has parametric equation $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, where t is any real number.
- Two lines intersect when their parametric equations describe the same point for some pair of real parameter values. All three coordinates must agree, so agreement of only two coordinates is insufficient in three dimensions.
- A zero component of a direction vector means that coordinate is constant along the line. This is most naturally handled in parametric form, without dividing by the zero direction component.

Step 1: Write a parametric equation for the line through A and B For the given points $A(0, -1, -1)$ and $B(4, 5, 1)$, a direction vector is

$$\overrightarrow{AB} = (4 - 0,\ 5 - (-1),\ 1 - (-1)) = (4, 6, 2).$$

Consequently every point on the full line AB has coordinates

$$(x, y, z) = (0, -1, -1) + t(4, 6, 2) = (4t, -1 + 6t, -1 + 2t), \quad t \in \mathbb{R}.$$

Taking $t = 0$ recovers A, and $t = 1$ recovers B, checking both the base point and the direction. The question concerns lines, whose parameters need not lie between zero and one.

Step 2: Write the second line and locate the only possible common point The direction vector of the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$ is

$$\overrightarrow{CD} = (-4 - 3,\ 4 - 9,\ 4 - 4) = (-7, -5, 0).$$

Its parametric coordinates are therefore

$$(x, y, z) = (3 - 7u, 9 - 5u, 4), \quad u \in \mathbb{R}.$$

In particular, every point of this line has $z = 4$. A common point must thus satisfy $-1 + 2t = 4$ on the first line, giving $t = 5/2$. Substituting this value in its other two coordinates gives

$$x = 4\left(\frac{5}{2}\right) = 10, \qquad y = -1 + 6\left(\frac{5}{2}\right) = 14.$$

Hence the only possible intersection point is $P(10, 14, 4)$; it remains to verify that the second line actually contains it.

Step 3: Verify the candidate point on the second line Equating its x-coordinate to 10 yields

$$3 - 7u = 10 \quad \Longrightarrow \quad u = -1.$$

Quick Tip to Solve: Since C and D both have $z = 4$, first impose $z = 4$ on line AB, then verify its resulting x and y on line CD.

At this same parameter value its remaining coordinates are

$$y = 9 - 5(-1) = 14, \qquad z = 4.$$

Thus the second line gives $(10, 14, 4)$, exactly the point found on the first line. Equivalently, the vector expressions are

$$A + \frac{5}{2}(B - A) = (10, 14, 4) = C - (D - C).$$

The values $t = 5/2$ and $u = -1$ are admissible real parameters. They place the intersection on the extensions of the given segments, which are included in the lines specified by the question.

Step 4: Conclusion The two lines have the common point $\boxed{P(10, 14, 4)}$. Hence the line through A and B intersects the line through C and D, as required.

Quick Tip to Solve: Since C and D both have $z = 4$, first impose $z = 4$ on line AB, then verify its resulting x and y on line CD.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

A ladder of fixed length $'h'$ is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions: [4 marks]

36(i). Express the distance (y) between the wall and foot of the ladder in terms of $'h'$ and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer. [1 mark]

Solution:

Concept:

- For a right triangle, the square of its hypotenuse equals the sum of the squares of the perpendicular sides.
- The area is $A = \frac{1}{2}xy$, where x is the vertical height and y is the horizontal distance from the wall.

Step 1: Relate the ladder to the two distances The wall and the ground are perpendicular, so the ladder of fixed length h forms the hypotenuse. Pythagoras gives

$$x^2 + y^2 = h^2.$$

Hence $y^2 = h^2 - x^2$. As y is a distance, take the non-negative square root, not the negative root.

Step 2: Express the triangular area in one variable Substitute the horizontal distance into the base-times-height formula:

$$A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}.$$

Quick Tip to Solve: Use the ladder as the hypotenuse and choose the positive square root because the foot-to-wall separation is a distance.

This expression uses the fixed parameter h and the variable height x. The geometric domain is $0 \le x \le h$; for a non-degenerate triangle, $0 < x < h$.

Step 3: Conclusion

$$\boxed{y = \sqrt{h^2 - x^2}, \qquad A = \frac{x}{2}\sqrt{h^2 - x^2}}.$$

Quick Tip to Solve: Use the ladder as the hypotenuse and choose the positive square root because the foot-to-wall separation is a distance.

36(ii). Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point. [1 mark]

Solution:

Concept:

- The area is $A(x) = \frac{1}{2}x(h^2 - x^2)^{1/2}$, with fixed ladder length $h > 0$.
- Use the product rule and chain rule. An interior critical point occurs where A is differentiable and $A\prime(x) = 0$.

Step 1: Differentiate the area For $0 < x < h$, applying the product and chain rules gives

$$\begin{aligned} A\prime(x) &= \frac{1}{2}\sqrt{h^2 - x^2} + \frac{x}{2}\frac{-2x}{2\sqrt{h^2 - x^2}} \\ &= \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}. \end{aligned}$$

The factor h is constant, since the ladder length does not change.

Step 2: Locate the interior critical point The denominator is positive on $0 < x < h$, so the derivative vanishes precisely when $h^2 - 2x^2 = 0$. Thus $x^2 = h^2/2$. Height is non-negative, so only $x = h/\sqrt{2}$ is admissible; the negative root is outside the geometric domain. The endpoint $x = h$ is checked separately for extrema, not treated as an interior stationary point.

Step 3: Conclusion

$$\boxed{\frac{dA}{dx} = \frac{h^2 - 2x^2}{2\sqrt{h^2 - x^2}}, \qquad x_{\text{critical}} = \frac{h}{\sqrt{2}}}.$$

Quick Tip to Solve: Combine the two derivative terms over one denominator; the positive denominator means the numerator alone locates the stationary height.

36(iii)(a). Show that the area (A) of the right triangle is maximum at the critical point. [2 marks]

Solution:

Concept:

- The first-derivative test proves a maximum when the derivative changes from positive to negative at the critical point.
- Here $A(x) = \frac{1}{2}x\sqrt{h^2 - x^2}$ on $[0, h]$, and $A\prime(x) = \frac{h^2-2x^2}{2\sqrt{h^2-x^2}}$ on $(0, h)$.

Step 1: Examine the sign before the critical height The stationary height is $c = h/\sqrt{2}$. For every x in $0 < x < c$, we have $2x^2 < h^2$. The numerator $h^2 - 2x^2$ is therefore positive. The denominator $2\sqrt{h^2 - x^2}$ is also positive, so $A\prime(x) > 0$. Thus the area increases continuously as the height approaches c from below.

Step 2: Examine the sign after the critical height For $c < x < h$, we have $2x^2 > h^2$, making the numerator negative while the denominator remains positive. Therefore $A\prime(x) < 0$, and the area decreases after c. This change from increasing to decreasing establishes a strict maximum at the critical point.

Step 3: Check the whole allowed interval At both endpoints the triangle is degenerate: $A(0) = A(h) = 0$. At $x = c$, the horizontal distance is also $h/\sqrt{2}$, and

$$A(c) = \frac{1}{2}\frac{h}{\sqrt{2}}\frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

Since the function increases up to c and decreases thereafter, this is the global maximum on $[0, h]$.

Step 4: Conclusion The area is maximum at $\boxed{x = h/\sqrt{2}}$, with $\boxed{A_{\max} = h^2/4}$.

Quick Tip to Solve: The derivative denominator stays positive inside the domain, so compare the sign of h squared minus twice x squared on either side of the critical height.

OR

36(iii)(b). If the foot of the ladder whose length is 5 m, is being pulled towards the wall such that the rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the ladder is 3 m away from the wall? [2 marks]

Solution:

Concept:

- The fixed ladder length gives $x^2 + y^2 = 25$, where x and y are distances in metres.
- Differentiate with respect to time t: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. A decreasing distance has a negative time derivative.

Step 1: Record the rate with its correct sign The foot moves towards the wall, so its horizontal distance y decreases at 2 m/s. Thus

$$\frac{dy}{dt} = -2 \text{ m/s}.$$

Quick Tip to Solve: Assign a negative sign to the decreasing horizontal distance first; use the 3-4-5 triangle to obtain the instantaneous height.

The ladder remains of length $5\,\text{m}$, so its right-triangle relation is $x^2 + y^2 = 5^2$. At the specified instant, $y = 3\,\text{m}$, giving

$$x = \sqrt{25 - 9} = 4\,\text{m}.$$

The positive root is required because x is a height.

Step 2: Differentiate before substituting the instant The distances change with time, so differentiate the fixed-length relation with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$
$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}.$$

Substitute $x = 4$, $y = 3$, and $dy/dt = -2$:

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}\,\text{m/s}.$$

The positive result agrees with the ladder top rising as the foot approaches the wall.

Step 3: Conclusion The height on the wall is increasing at $\boxed{\frac{3}{2}\,\text{m/s}}$.

Quick Tip to Solve: Assign a negative sign to the decreasing horizontal distance first; use the 3-4-5 triangle to obtain the instantaneous height.

37. Case Study - 2

A shop selling electronic items sells smartphones of only three reputed companies A, B and C because chances of their manufacturing a defective smartphone are only 5%, 4% and 2% respectively. In his inventory he has 25% smartphones from company A, 35% smartphones from company B and 40% smartphones from company C.

A person buys a smartphone from this shop. [4 marks]

37(i). Find the probability that it was defective. [2 marks]

Solution:

Concept:

- Let E_A, E_B, E_C denote the events that the selected smartphone is made by companies A, B and C. These events form a mutually exclusive and exhaustive partition.
- For the defective event D, the total-probability formula is $P(D) = \sum_{j=A,B,C} P(E_j)P(D \mid E_j)$.

Step 1: Translate the inventory and defect rates into probabilities The inventory shares give

$$P(E_A) = 0.25, \qquad P(E_B) = 0.35, \qquad P(E_C) = 0.40.$$

Quick Tip to Solve: Multiply each company inventory percentage by its own defect percentage, then add the three products.

Their sum is 1, as required for all possible manufacturers. The conditional defect probabilities are

$$P(D \mid E_A) = 0.05, \quad P(D \mid E_B) = 0.04, \quad P(D \mid E_C) = 0.02.$$

Each defect rate applies within that manufacturer's share, rather than to the entire inventory.

Step 2: Add the three mutually exclusive defect contributions A defective purchase can come from exactly one of the three companies. Weight each company's defect probability by its inventory share:

$$\begin{aligned} P(D) &= (0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02) \\ &= 0.0125 + 0.014 + 0.008 \\ &= 0.0345. \end{aligned}$$

Simply averaging the three defect percentages would be incorrect because the inventory shares differ.

Step 3: Conclusion The probability that the purchased smartphone is defective is $\boxed{0.0345 = \frac{69}{2000}}$, or 3.45%.

Quick Tip to Solve: Multiply each company inventory percentage by its own defect percentage, then add the three products.

37(ii). What is the probability that this defective smartphone was manufactured by company B?

[2 marks]

Solution:

Concept:

- Let E_B mean manufacture by company B, and D mean that the smartphone is defective. The required probability is $P(E_B \mid D)$.
- Bayes' theorem gives $P(E_B \mid D) = \frac{P(E_B)P(D|E_B)}{P(D)}$, where the denominator includes defective phones from every company.

Step 1: Find the probability of a defective phone from B Company B supplies 35% of the inventory and has a 4% conditional defect rate. Therefore the joint probability of selecting a phone that is both from B and defective is

$$P(E_B \cap D) = P(E_B)P(D \mid E_B) = (0.35)(0.04) = 0.014.$$

This numerator counts only the defective phones made by B.

Step 2: Condition on the complete defective population The total probability of a defective purchase is

$$\begin{aligned} P(D) &= (0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02) \\ &= 0.0345. \end{aligned}$$

Quick Tip to Solve: For Bayes' theorem, put the defective contribution from B in the numerator and the defective contributions from all three companies in the denominator.

As the phone is already known to be defective, divide the joint probability by this total:

$$P(E_B \mid D) = \frac{0.014}{0.0345} = \frac{140}{345} = \frac{28}{69}.$$

The 4% defect rate alone is $P(D \mid E_B)$; it does not answer the reversed conditional question.

Step 3: Conclusion The probability that the defective smartphone was manufactured by company B is $\boxed{\frac{28}{69}}$.

Quick Tip to Solve: For Bayes' theorem, put the defective contribution from B in the numerator and the defective contributions from all three companies in the denominator.

38. Case Study - 3

Three students, Neha, Rani and Sam go to a market to purchase stationery items. Neha buys 4 pens, 3 notepads and 2 erasers and pays Rs. 60. Rani buys 2 pens, 4 notepads and 6 erasers for Rs. 90. Sam pays Rs. 70 for 6 pens, 2 notepads and 3 erasers.

Based upon the above information, answer the following questions: [4 marks]

38(i). Form the equations required to solve the problem of finding the price of each item, and express it in the matrix form $AX = B$. [1 mark]

Solution:

Concept:

- The total purchase cost is the sum of quantity times unit price for each stationery item.
- In $AX = B$, the columns of A must follow the same item order as the entries of the price vector X; each row corresponds to one student's purchase.

Step 1: Assign unit prices and form the equations Let x, y, z be the prices in rupees of one pen, one notepad and one eraser, respectively. Neha's purchase gives $4x + 3y + 2z = 60$, Rani's gives $2x + 4y + 6z = 90$, and Sam's gives $6x + 2y + 3z = 70$.

Step 2: Arrange the coefficients and totals Keeping the order pens, notepads, erasers in all three rows, define

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Multiplying each row of A by X reproduces the corresponding total cost equation. The totals belong in B, not in the coefficient matrix.

Step 3: Conclusion The required matrix equation is

$$\boxed{\begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}}.$$

Quick Tip to Solve: Fix the item order first, then place one student's quantities in each row and that student's payment in the matching row of B.

38(ii). Find $|A|$ and confirm if it is possible to find A^{-1}. [1 mark]

Solution:

Concept:

- A square matrix A has an inverse if and only if its determinant is nonzero.
- Expanding a 3×3 determinant along its first row uses alternating signs $+, -, +$ on the corresponding 2×2 minors.

Step 1: Identify the coefficient matrix With the columns in the order pens, notepads, erasers and the rows in the order Neha, Rani, Sam, the purchase equations give

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}.$$

Its determinant, rather than its individual entries, decides whether the inverse exists.

Step 2: Expand the determinant Expanding along the first row gives

$$\begin{aligned} |A| &= 4(4 \cdot 3 - 6 \cdot 2) - 3(2 \cdot 3 - 6 \cdot 6) + 2(2 \cdot 2 - 4 \cdot 6) \\ &= 4(0) - 3(-30) + 2(-20) \\ &= 90 - 40 = 50. \end{aligned}$$

The zero first minor does not make the entire determinant zero: both remaining signed terms must be included.

Step 3: Conclusion $\boxed{|A| = 50 \neq 0}$. Thus A is nonsingular and it is possible to find $\boxed{A^{-1}}$.

Quick Tip to Solve: Expand along the first row here: its first minor vanishes, leaving only the two signed contributions 90 and -40.

38(iii)(a). Find A^{-1}, if possible, and write the formula to find X. [2 marks]

Solution:

Concept:

- If $|A| \neq 0$, the inverse is $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j. Left multiplication of $AX = B$ by A^{-1} gives the price vector.

Step 1: Verify that the inverse exists The coefficient matrix and its determinant are

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \qquad |A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 50.$$

Quick Tip to Solve: After finding all nine cofactors, transpose the array before dividing by 50; this avoids confusing the cofactor matrix with the inverse.

Since the determinant is nonzero, division by it in the inverse formula is valid.

Step 2: Calculate the cofactors and transpose them Using the alternating cofactor signs, the cofactor matrix is

$$C = \begin{pmatrix} 12-12 & -(6-36) & 4-24 \\ -(9-4) & 12-12 & -(8-18) \\ 18-8 & -(24-4) & 16-6 \end{pmatrix} = \begin{pmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{pmatrix}.$$

Therefore

$$\operatorname{adj} A = C^T = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}.$$

Transposing is essential: the cofactor matrix itself is not the adjugate.

Step 3: Apply the inverse to the matrix equation From $AX = B$, multiplying on the left gives

$$A^{-1}AX = A^{-1}B \quad \Longrightarrow \quad IX = A^{-1}B.$$

Here $X = (x, y, z)^T$ contains the prices and $B = (60, 90, 70)^T$ contains the students' payments. The multiplication order must be $A^{-1}B$ because B is a column vector.

Step 4: Conclusion

$$\boxed{A^{-1} = \frac{1}{50}\begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}}, \qquad \boxed{X = A^{-1}B}.$$

Quick Tip to Solve: After finding all nine cofactors, transpose the array before dividing by 50; this avoids confusing the cofactor matrix with the inverse.

OR

38(iii)(b). Find $A^2 - 8I$, where I is an identity matrix. [2 marks]

Solution:

Concept:

- The square of a matrix means $A^2 = A \cdot A$, using row-by-column multiplication; it does not mean squaring each entry separately.
- The identity matrix of order three has ones on the main diagonal and zeros elsewhere. Consequently subtracting $8I$ subtracts eight only from the three diagonal entries.

Step 1: Square the coefficient matrix For the purchase equations,

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}.$$

Quick Tip to Solve: Compute A times A by row-column products first; then change only the diagonal entries when subtracting 8I.

Each entry of A^2 is a row-column dot product. For example, its first row is obtained from

$$(A^2)_{11} = 4 \cdot 4 + 3 \cdot 2 + 2 \cdot 6 = 34,$$
$$(A^2)_{12} = 4 \cdot 3 + 3 \cdot 4 + 2 \cdot 2 = 28,$$
$$(A^2)_{13} = 4 \cdot 2 + 3 \cdot 6 + 2 \cdot 3 = 32.$$

The same multiplication for the second and third rows gives

$$A^2 = \begin{pmatrix} 34 & 28 & 32 \\ 52 & 34 & 46 \\ 46 & 32 & 33 \end{pmatrix}.$$

Step 2: Subtract the scalar multiple of the identity Since A has order three, use the identity matrix of that same order:

$$8I = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Matrix subtraction acts entry by entry. The diagonal entries become $34 - 8$, $34 - 8$ and $33 - 8$, while every off-diagonal entry remains the same.

Step 3: Conclusion

$$\boxed{A^2 - 8I = \begin{pmatrix} 26 & 28 & 32 \\ 52 & 26 & 46 \\ 46 & 32 & 25 \end{pmatrix}}.$$

Quick Tip to Solve: Compute A times A by row-column products first; then change only the diagonal entries when subtracting 8I.