CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-7-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

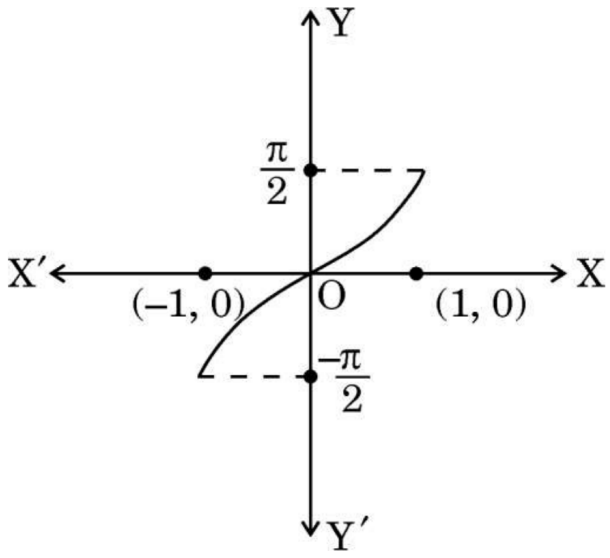
(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. Study the given graph. It illustrates: [1 mark]



(A) $y = \tan^{-1} x$

(B) $y = \cos^{-1} x$

(C) $y = \sec^{-1} x$

(D) $y = \sin^{-1} x$

Correct Answer: (D) $y = \sin^{-1} x$

Solution:

Concept:

- The principal inverse sine function has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
- Its graph is increasing and passes through $(0, 0)$, with endpoints $(-1, -\pi/2)$ and $(1, \pi/2)$.

Step 1: Read the domain and endpoints The given curve runs from $x = -1$ to $x = 1$, passes through the origin and rises from $-\pi/2$ to $\pi/2$. These values agree with $\sin^{-1}(-1) = -\pi/2$, $\sin^{-1}(0) = 0$ and $\sin^{-1}(1) = \pi/2$.

Step 2: Compare the other inverse functions The graph of $\tan^{-1} x$ has domain $\mathbb{R}$ and never reaches $\pm\pi/2$. The graph of $\cos^{-1} x$ is decreasing and passes through $(0, \pi/2)$. The function $\sec^{-1} x$ is undefined for $-1 < x < 1$, so it cannot pass through the origin. These exclude (A), (B) and (C).

Step 3: Conclusion The graph illustrates $y = \sin^{-1} x$. Hence, the correct option is (D).

Quick Tip to Solve: Distinguish inverse sine from inverse tangent using the finite endpoints at $x = \pm 1$; both graphs pass through the origin.

2. If $\begin{bmatrix} 2x-1 & 3x \\ 0 & y^2-1 \end{bmatrix} = \begin{bmatrix} x+3 & 12 \\ 0 & 35 \end{bmatrix}$**, then the value of** $(x-y)$ **is:** [1 mark]

(A) 2 or 10

(B) -2 or 10

(C) 2 or -10

(D) -2 or -10

Correct Answer: (B) -2 or 10

Solution:

Concept:

- Two matrices of the same order are equal exactly when their corresponding entries are equal.
- If $y^2 = a^2$ for a nonzero real number a, both roots $y = a$ and $y = -a$ must be considered.

Step 1: Equate corresponding entries Comparing the top-left entries gives $2x - 1 = x + 3$, hence $x = 4$. The top-right entries give $3x = 12$, confirming the same value. The lower-left entries already agree. Comparing the lower-right entries gives

$$y^2 - 1 = 35 \quad \Rightarrow \quad y^2 = 36 \quad \Rightarrow \quad y = \pm 6.$$

Step 2: Calculate both differences For $y = 6$, the difference is $x - y = 4 - 6 = -2$. For $y = -6$, it is $x - y = 4 - (-6) = 10$. Option (A) replaces -2 by 2, option (C) reverses both signs, and option (D) replaces 10 by -10; none gives the required pair.

Step 3: Conclusion The possible values of $x - y$ are -2 or 10. Hence, the correct option is (B).

Quick Tip to Solve: For equal matrices, compare entries directly and retain both square roots when solving $y^2 = 36$.

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3. Let A **be a square matrix of order** 3**. If** $|A| = 5$**, then** $|\operatorname{adj} A|$ **is:** [1 mark]

(A) 5

(B) 125

(C) 25

(D) -5

Correct Answer: (C) 25

Solution:

Concept:

- For a square matrix A of order n, $A \operatorname{adj} A = |A| I_n$, where I_n is the identity matrix.
- Determinants satisfy $|AB| = |A||B|$ and $|kI_n| = k^n$; consequently $|\operatorname{adj} A| = |A|^{n-1}$.

Step 1: Apply the adjoint identity Here the order is $n = 3$ and $|A| = 5 \neq 0$. Taking determinants in the identity for the adjoint gives

$$|A|\,|\operatorname{adj} A| = \big||A| I_3\big| = |A|^3.$$

Quick Tip to Solve: For an order-three matrix, square its determinant to find the determinant of its adjoint.

Division by the nonzero determinant $|A|$ leaves $|\operatorname{adj} A| = |A|^2$.

Step 2: Substitute and compare the choices Thus $|\operatorname{adj} A| = 5^2 = 25$. The value 5 in (A) omits the required power, 125 in (B) uses the order itself instead of one less than the order, and -5 in (D) introduces an unsupported negative sign.

Step 3: Conclusion The determinant of the adjoint is 25. Hence, the correct option is (C).

Quick Tip to Solve: For an order-three matrix, square its determinant to find the determinant of its adjoint.

4. If A and B are two square matrices each of order 3 with $|A| = 3$ and $|B| = 5$, then $|2AB|$ is: [1 mark]

(A) 30

(B) 120

(C) 15

(D) 225

Correct Answer: (B) 120

Solution:

Concept:

- For square matrices of the same order, the determinant of a product is the product of their determinants: $|AB| = |A||B|$.
- If M has order n, multiplying every entry by a scalar k gives $|kM| = k^n|M|$, since a factor k comes from each row.

Step 1: Find the determinant of the product Both matrices have order three, so their product is also a square matrix of order three. Apply the product rule to the given determinant values:

$$|AB| = |A||B| = 3 \times 5 = 15.$$

Step 2: Account for the scalar multiplier Multiplication of AB by 2 multiplies each of its three rows by 2. Therefore

$$|2AB| = 2^3|AB| = 8 \times 15 = 120.$$

Option (A), 30, counts the factor 2 only once. Option (C), 15, omits it altogether. Option (D), 225, squares $|AB|$ instead of applying the scalar rule.

Step 3: Conclusion The required determinant is $|2AB| = 120$. Hence, the correct option is (B).

Quick Tip to Solve: In a determinant of order three, a scalar multiplying the whole matrix contributes its cube.

5. The matrix $A = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$ is a/an: [1 mark]

(A) scalar matrix

(B) identity matrix

(C) null matrix

(D) symmetric matrix

Correct Answer: (D) symmetric matrix

Solution:

Concept:

- A real square matrix A is symmetric when $A^T = A$, where A^T denotes its transpose.
- A diagonal matrix has zero off-diagonal entries. It is scalar only if all its diagonal entries are equal, and it is an identity matrix only if all diagonal entries are 1.

Step 1: Compare the matrix with its transpose Transposition interchanges entries on opposite sides of the main diagonal. Here every off-diagonal entry is zero, so each such pair remains unchanged. The diagonal entries $\sqrt{3}, \sqrt{2}, \sqrt{5}$ remain in their original positions. Hence $A^T = A$.

Step 2: Check the remaining matrix types The three diagonal entries are unequal, so (A), scalar matrix, does not apply. They are not all 1, excluding the identity matrix in (B). The diagonal entries are nonzero, so this is not the null matrix in (C). Being diagonal is fully consistent with being symmetric.

Step 3: Conclusion The given matrix is a symmetric matrix. Hence, the correct option is (D).

Quick Tip to Solve: Every real diagonal matrix is symmetric; equal diagonal entries are required only for the stronger scalar classification.

6. What is the total number of possible matrices of order 3×3 with each entry as $\sqrt{2}$ or $\sqrt{3}$? [1 mark]

(A) 9

(B) 512

(C) 615

(D) 64

Correct Answer: (B) 512

Solution:

Concept:

- A matrix of order $m \times n$ contains mn entry positions, and matrices differ when even one corresponding entry differs.
- By the multiplication principle, if each of r independent positions has k possible entries, the number of arrangements is k^r.

Step 1: Count the independent entry positions A 3×3 matrix has $3 \cdot 3 = 9$ positions. Every position can contain either $\sqrt{2}$ or $\sqrt{3}$. These are two distinct real numbers, and the question imposes no

Quick Tip to Solve: Count all nine entries separately; two permitted values at each entry give 2^9, regardless of the values themselves.

symmetry condition or restriction on repetitions.

Step 2: Multiply the choices for all positions There are two choices for each of the nine entries, so the multiplication principle gives

$$N = \underbrace{2 \times 2 \times \cdots \times 2}_{9 \text{ factors}} = 2^9 = 512.$$

Option (A), 9, counts positions rather than matrices. Option (D), $64 = 2^6$, allows only six independent positions. Option (C), 615, is not the product of the specified choices.

Step 3: Conclusion The total number of possible matrices is 512. Hence, the correct option is (B).

Quick Tip to Solve: Count all nine entries separately; two permitted values at each entry give 2^9, regardless of the values themselves.

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7. Domain of $f(x) = \cos^{-1} x + \sin x$ is: [1 mark]

(A) $\mathbb{R}$

(B) $(-1, 1)$

(C) $[-1, 1]$

(D) φ

Correct Answer: (C) $[-1, 1]$

Solution:

Concept:

- The real inverse cosine function $\cos^{-1} x$ is defined exactly when $-1 \leq x \leq 1$.
- The domain of a sum of real functions is the intersection of the domains of its terms; $\sin x$ is defined for every real x.

Step 1: Determine the simultaneous restrictions The first term, $\cos^{-1} x$, requires $x \in [-1, 1]$. The second term, $\sin x$, places no additional restriction. Both terms therefore exist precisely on

$$[-1, 1] \cap \mathbb{R} = [-1, 1].$$

Step 2: Check whether the endpoints are included At $x = 1$, $\cos^{-1}(1) = 0$, and at $x = -1$, $\cos^{-1}(-1) = \pi$; the sine term is also defined at both endpoints. Thus the open interval in (B) excludes permitted values. Option (A) incorrectly includes values outside the domain of inverse cosine, while (D), the empty set, excludes all permitted inputs.

Step 3: Conclusion The domain of f is $[-1, 1]$. Hence, the correct option is (C).

Quick Tip to Solve: For $\cos^{-1} x + \sin x$, use the inverse cosine restriction and keep both endpoints; sine adds no restriction.

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8. If $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **is continuous at** $x = 0$**, then the value of** a **is:** [1 mark]

(A) ± 1

(B) -1

(C) 0

(D) 1

Correct Answer: (A) ± 1

Solution:

Concept:

- Continuity at the origin requires $\lim_{x \to 0} f(x) = f(0)$.
- The standard trigonometric limit is $\lim_{t \to 0} \sin t/t = 1$, with angles measured in radians.

Step 1: Evaluate the limiting value For a fixed nonzero value of a, introduce ax in the denominator to use the standard limit:

$$\lim_{x \to 0} \frac{\sin^2(ax)}{x^2} = a^2 \lim_{x \to 0} \left(\frac{\sin(ax)}{ax} \right)^2 = a^2.$$

If $a = 0$, the expression is zero for every nonzero x, so its limit is again $a^2 = 0$.

Step 2: Apply the continuity condition The defined value at the origin is $f(0) = 1$; hence continuity requires $a^2 = 1$. Both positive and negative roots satisfy this equation, since the sine is squared. Options (B) and (D) each omit one admissible value, while option (C) would give a limiting value of zero instead of one.

Step 3: Conclusion The complete set of possible values is $a = \pm 1$. Hence, the correct option is (A).

Quick Tip to Solve: Factor out the square of the coefficient inside sine before equating the limit to the value at the origin.

9. If $f : \mathbb{R} \to \mathbb{R}$ **is defined as** $f(x) = 2x - \sin x$**, then** f **is:** [1 mark]

(A) a decreasing function

(B) an increasing function

(C) maximum at $x = \dfrac{\pi}{2}$

(D) maximum at $x = 0$

Correct Answer: (B) an increasing function

Solution:

Concept:

- A differentiable function whose derivative is positive throughout an interval is strictly increasing there.
- For every real x, $-1 \leq \cos x \leq 1$; this bounds a derivative involving cosine.

Quick Tip to Solve: Bound $\cos x$ by one to determine the derivative's sign without solving any trigonometric equation.

Step 1: Differentiate the function Differentiating $f(x) = 2x - \sin x$ term by term gives

$$f'(x) = 2 - \cos x.$$

The derivative exists at every real number because both terms are differentiable on $\mathbb{R}$.

Step 2: Determine the sign on the whole domain Since the greatest possible value of $\cos x$ is one, $f'(x) \geq 2 - 1 = 1 > 0$ for all real x. Thus a decrease is impossible, ruling out (A). In particular, $f'(\pi/2) = 2$ and $f'(0) = 1$, so neither point is a stationary maximum. Also strict increase on all of $\mathbb{R}$ means that a larger argument always gives a larger value, ruling out (C) and (D).

Step 3: Conclusion The function is strictly increasing on $\mathbb{R}$. Hence, the correct option is (B).

Quick Tip to Solve: Bound $\cos x$ by one to determine the derivative's sign without solving any trigonometric equation.

10. If R be a relation defined as aRb iff $|a - b| > 0$, $a, b \in \mathbb{R}$, then R is: [1 mark]

(A) reflexive

(B) symmetric

(C) transitive

(D) symmetric and transitive

Correct Answer: (B) symmetric

Solution:

Concept:

- Reflexivity requires aRa for every element, and symmetry requires $aRb \Rightarrow bRa$.
- Transitivity requires aRb and bRc together to imply aRc; one counterexample disproves it.

Step 1: Test reflexivity and symmetry The condition $|a - b| > 0$ is equivalent to $a \neq b$. For every real a, $|a - a| = 0$, so the relation is not reflexive and (A) fails. On the other hand, $|b - a| = |a - b|$. Therefore, whenever aRb, the same positive distance gives bRa; the relation is symmetric.

Step 2: Check transitivity with a counterexample Take $a = 0$, $b = 1$ and $c = 0$. Then $|0 - 1| = 1 > 0$ and $|1 - 0| = 1 > 0$, so $0R1$ and $1R0$. But $|0 - 0| = 0$, so $0R0$ is false. Transitivity therefore fails. Both (C) and (D) incorrectly assert that the relation is transitive.

Step 3: Conclusion The relation is symmetric, but neither reflexive nor transitive. Hence, the correct option is (B).

Quick Tip to Solve: For a relation expressing unequal elements, set the first and third elements equal to test transitivity quickly.

11. If $f(x) = -2x^8$, then the correct statement is: [1 mark]

(A) $f'\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$

(B) $f'\left(\frac{1}{2}\right) = -f'\left(-\frac{1}{2}\right)$

(C) $-f'\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right)$

(D) $f\left(\frac{1}{2}\right) = -f\left(-\frac{1}{2}\right)$

Correct Answer: (B) $f'\left(\frac{1}{2}\right) = -f'\left(-\frac{1}{2}\right)$

Solution:

Concept:

- The power rule gives $\frac{d}{dx}x^n = nx^{n-1}$.
- An odd function takes opposite values at opposite arguments: $g(-x) = -g(x)$.

Step 1: Calculate the derivative at both arguments Differentiating the given function yields $f'(x) = -16x^7$, an odd function. Direct substitution gives

$$f'\left(\frac{1}{2}\right) = -\frac{1}{8}, \qquad f'\left(-\frac{1}{2}\right) = \frac{1}{8}.$$

These values are opposite rather than equal, so option (A) fails and the equality in (B) is satisfied.

Step 2: Check the statements involving the original function The original function has the even power eight, so

$$f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = -\frac{1}{128}.$$

In (C), the left side is $1/8$ while the right side is $-1/128$, so they differ. In (D), the proposed right side is $1/128$, which also differs from the left side $-1/128$.

Step 3: Conclusion We have $f'(1/2) = -f'(-1/2)$. Hence, the correct option is (B).

Quick Tip to Solve: An even power gives an even function, but differentiation reduces the power by one and makes its derivative odd here.

12. For a function $f(x)$, which of the following holds true? [1 mark]

(A) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(B) $\int_{-a}^{a} f(x)\,dx = 0$, if f is an even function

(C) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an odd function

(D) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(2a+x)\,dx$

Correct Answer: (A) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Solution:

Concept:

- A change of variable in a definite integral must change both its differential and its limits.
- On symmetric limits an odd integrable function has zero integral, whereas an even one gives twice its integral over the positive half.

Step 1: Verify the reflection identity Set $t = a+b-x$. Then $dx = -dt$, and the limits $x = a, b$ become $t = b, a$, respectively. Thus, whenever the integrals exist,

$$\int_a^b f(a+b-x)\,dx = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt.$$

Since the integration variable is a dummy variable, this is the equality in (A).

Step 2: Reject the remaining general claims Take $a = 1$. For (B), the even function $f(x) = 1$ gives $\int_{-1}^{1} f(x)\,dx = 2$, not zero. For (C), the odd function $f(x) = x$ gives zero on the left but $2\int_0^1 x\,dx = 1$ on the right. For (D), $f(x) = 1$ gives 2 on the left and $1 - 1 = 0$ on the right. Each is therefore false as a general identity.

Step 3: Conclusion Reflection about the midpoint preserves the definite integral: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Hence, the correct option is (A).

Quick Tip to Solve: Under $x \mapsto a+b-x$, reverse the bounds and include the minus sign; the two reversals cancel.

13. $\int \frac{e^{9\log x} - e^{8\log x}}{e^{6\log x} - e^{5\log x}}\,dx$ **is equal to:** [1 mark]

(A) $x + C$

(B) $\frac{x^2}{2} + C$

(C) $\frac{x^4}{4} + C$

(D) $\frac{x^3}{3} + C$

Correct Answer: (C) $\frac{x^4}{4} + C$

Solution:

Concept:

- For the natural logarithm, $e^{k\log x} = x^k$ when $x > 0$.
- The power rule for integration is $\int x^n\,dx = x^{n+1}/(n+1) + C$ for $n \neq -1$, where C is a constant.

Step 1: Simplify the exponential expressions Rewrite each exponential as a power of x, and factor numerator and denominator:

$$\frac{e^{9\log x} - e^{8\log x}}{e^{6\log x} - e^{5\log x}} = \frac{x^9 - x^8}{x^6 - x^5} = \frac{x^8(x-1)}{x^5(x-1)} = x^3.$$

Cancellation is valid on the original domain $x > 0$, $x \neq 1$, since the logarithms require positivity and the denominator must be nonzero.

Step 2: Apply and check the integration rule Integrating the reduced integrand gives $\int x^3\,dx = x^4/4 + C$, on each interval of its domain. To check the other choices, differentiating (A), (B) and (D) gives 1, x and x^2, respectively; none equals x^3 throughout such an interval.

Step 3: Conclusion The integral is $\frac{x^4}{4} + C$. Hence, the correct option is (C).

Quick Tip to Solve: Convert exponentials of logarithms to powers first; factor out the lowest power to expose the common factor.

14. $\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$ **is equal to:** [1 mark]

(A) $\frac{\sin^{-1}(a^x)}{\log_e a} + C$

(B) $\log_e(1-a^{2x}) + C$

(C) $\frac{\cos^{-1}(a^x)}{\log_e a} + C$

(D) $\frac{\sin^{-1}(a^x)}{a^x} + C$

Correct Answer: (A) $\frac{\sin^{-1}(a^x)}{\log_e a} + C$

Solution:

Concept:

- For $a > 0$, $\frac{d}{dx}a^x = a^x \log_e a$; take $a \neq 1$ for the stated integrand.
- The standard inverse-sine integral is $\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + C$, on $|t| < 1$.

Step 1: Substitute the exponential expression Let $t = a^x$. Then $dt = a^x \log_e a\,dx$. On the real domain where $a^{2x} < 1$, the integral becomes

$$\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx = \frac{1}{\log_e a}\int \frac{dt}{\sqrt{1-t^2}} = \frac{\sin^{-1} t}{\log_e a} + C.$$

Step 2: Verify the factor and signs Substituting back $t = a^x$, differentiation of (A) cancels its factor $\log_e a$ and recovers the integrand. Option (C) has the opposite derivative because the derivative of inverse cosine is negative. Option (B) differentiates to $-2a^{2x}\log_e a/(1-a^{2x})$, which has the wrong numerator and denominator. Option (D) introduces an extra term through differentiation of its denominator a^x, so it also fails.

Step 3: Conclusion The integral is $\frac{\sin^{-1}(a^x)}{\log_e a} + C$. Hence, the correct option is (A).

Quick Tip to Solve: When substituting a^x, divide by $\log_e a$; choose inverse sine for a positive derivative with a square-root denominator.

15. A coin is tossed and a card is selected at random from a well shuffled pack of 52 playing cards. The probability of getting head on the coin and a face card from the pack is: [1 mark]

(A) $\frac{2}{13}$

(B) $\frac{3}{26}$

(C) $\frac{19}{26}$

(D) $\frac{3}{13}$

Correct Answer: (B) $\frac{3}{26}$

Solution:

Concept:

- For independent events H and F, $P(H \cap F) = P(H)P(F)$. Here H denotes a head and F denotes a face card.
- A standard deck has three face cards (jack, queen and king) in each of four suits, so there are 12 face cards among 52 cards.

Step 1: Find the individual probabilities The coin has two equally likely outcomes, giving $P(H) = 1/2$. The favourable cards are the twelve face cards, hence $P(F) = 12/52 = 3/13$. An ace is not a face card.

Step 2: Use independence and compare the options The card selection does not depend on the coin outcome. Thus

$$P(H \cap F) = \frac{1}{2} \times \frac{3}{13} = \frac{3}{26}.$$

Option (D) counts only the card event and ignores the head requirement. Options (A) and (C) do not equal the product of the two probabilities.

Step 3: Conclusion The required probability is $\boxed{3/26}$. Hence, the correct option is (B).

Quick Tip to Solve: Count only J, Q and K as face cards, then multiply their probability by $1/2$ for the head.

16. A student tries to tie ropes, parallel to each other, from one end of the wall to the other. If one rope is along the vector $3\hat{i} + 15\hat{j} + 6\hat{k}$ and the other is along the vector $2\hat{i} + 10\hat{j} + \lambda\hat{k}$, then the value of λ is: [1 mark]

(A) 6

(B) 1

(C) $\frac{1}{4}$

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- Two nonzero vectors are parallel if one is a scalar multiple of the other: $\vec{b} = k\vec{a}$.
- For parallel vectors, corresponding components must have the same ratio; all three components must satisfy the same scalar multiplier.

Step 1: Find the common scalar multiplier Let $\vec{a} = 3\hat{i} + 15\hat{j} + 6\hat{k}$ and $\vec{b} = 2\hat{i} + 10\hat{j} + \lambda\hat{k}$. Comparing the first two components gives

$$k = \frac{2}{3} = \frac{10}{15}.$$

Thus the second rope must have every component equal to $2/3$ times the corresponding first-rope component.

Step 2: Check the third component and the options The third component must obey

$$\lambda = \frac{2}{3} \times 6 = 4.$$

Indeed, $(2, 10, 4) = \frac{2}{3}(3, 15, 6)$. The other proposed values $6, 1, 1/4$ give third-component ratios $1, 1/6, 1/24$, respectively, none equal to $2/3$.

Step 3: Conclusion $\boxed{\lambda = 4}$. Hence, the correct option is (D).

Quick Tip to Solve: Use the first two components to identify the multiplier, then apply it directly to the third component.

17. If $P(A) = \frac{1}{5}$, $P(B) = \frac{3}{5}$ and $P\left(\frac{A}{B}\right) = \frac{2}{5}$, then $P(A' \cap B')$ is: [1 mark]

(A) $\frac{11}{25}$

(B) $\frac{19}{25}$

(C) $\frac{8}{25}$

(D) $\frac{6}{25}$

Correct Answer: No consistent probability model exists for the printed data, because $P(A \cap B) = 6/25 > P(A) = 5/25$. Formal substitution gives $11/25$, option (A), as the intended result.

Solution:

Concept:

- Conditional probability gives $P(A \cap B) = P(A \mid B)P(B)$, provided $P(B) > 0$.
- De Morgan's law and inclusion-exclusion give $P(A' \cap B') = 1 - P(A) - P(B) + P(A \cap B)$.

Step 1: Apply the printed conditional probability Interpreting the printed $P(A/B)$ as $P(A \mid B)$, the stated values formally give

$$P(A \cap B) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}.$$

This is the intersection term, not the requested probability; it is option (D).

Step 2: Find the complement of the union Using the printed values,

$$P(A' \cap B') = 1 - \frac{1}{5} - \frac{3}{5} + \frac{6}{25} = \frac{11}{25}.$$

Option (B), $19/25$, is the formal complement of the intersection, whereas option (C), $8/25$, is not the result of the required formula.

Step 3: Check consistency of the supplied data The printed numbers have a mathematical defect: $P(A \cap B) = 6/25$ exceeds $P(A) = 5/25$, which cannot occur for actual events because $A \cap B \subseteq A$. Thus no probability model satisfies all three given values; $11/25$ is the intended formal answer obtained by substitution.

Step 4: Conclusion Using the intended substitution, $\boxed{11/25}$, option (A). The printed probability data are internally inconsistent.

Quick Tip to Solve: For neither event, use $1 - P(A) - P(B) + P(A \mid B)P(B)$, and also check that the intersection cannot exceed either event.

18. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ for any two vectors, then vectors $\vec{a}$ and $\vec{b}$ are: [1 mark]

(A) orthogonal vectors

(B) parallel to each other

(C) unit vectors

(D) collinear vectors

Correct Answer: (A) orthogonal vectors

Solution:

Concept:

- The squared magnitude satisfies $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Two vectors are orthogonal precisely when their scalar product is zero; for nonzero vectors this means their angle is 90°.

Step 1: Square and expand the equal magnitudes Both magnitudes are nonnegative, so squaring preserves the equality. Expanding with the dot product gives

$$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}.$$

Step 2: Cancel the common terms and test the choices The squared-magnitude terms cancel, leaving $4\vec{a} \cdot \vec{b} = 0$, hence $\vec{a} \cdot \vec{b} = 0$. No condition fixes either magnitude to one, so unit vectors are not implied. Nonzero parallel or collinear vectors have a nonzero dot product and therefore do not satisfy this condition.

Step 3: Conclusion The vectors are $\boxed{\text{orthogonal}}$. Hence, the correct option is (A).

Quick Tip to Solve: Equal lengths of a vector sum and difference are tested by squaring; the only surviving term is the dot product.

19. Assertion (A): $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **is continuous at** $x = 0$.

Reason (R): When $x \to 0$, $\sin \frac{1}{x}$ **is a finite value between** -1 **and** 1. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- Continuity at zero requires $\lim_{x \to 0} f(x) = f(0)$.
- The squeeze theorem applies when $|f(x)| \leq g(x)$ and $g(x) \to 0$; bounded oscillations times a factor tending to zero tend to zero.

Step 1: Judge the Assertion For $x \neq 0$,

$$\left| x \sin \frac{1}{x} \right| \leq |x|.$$

Since $|x| \to 0$, the squeeze theorem gives $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$. Hence the Assertion is true.

Step 2: Judge the Reason For every nonzero real x, $-1 \leq \sin(1/x) \leq 1$. The Reason states this boundedness as x approaches zero, and is true. It does not assert that $\sin(1/x)$ has a limit; the sine factor continues to oscillate.

Step 3: Decide whether the Reason explains the Assertion The boundedness supplied by the Reason yields the estimate $|x \sin(1/x)| \leq |x|$, which proves the required limit. Thus it directly explains continuity. Option (B) incorrectly denies that link, while (C) and (D) incorrectly reject a true statement.

Step 4: Conclusion Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Bound the absolute value by $|x|$; never claim that the oscillating factor $\sin(1/x)$ itself has a limit.

20. Assertion (A): Set of values of $\sec^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is a null set.

Reason (R): $\sec^{-1} x$ is defined for $x \in \mathbb{R} - (-1, 1)$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The real inverse secant is defined only for $x \in (-\infty, -1] \cup [1, \infty) = \mathbb{R} \setminus (-1, 1)$.
- For real angles where secant is defined, $\sec\theta = 1/\cos\theta$ has absolute value at least one.

Step 1: Judge the Assertion The official assertion uses $\sqrt{3}/2$. Since $0 < \sqrt{3}/2 < 1$, this number is outside the real domain of inverse secant. Equivalently, $\sec\theta = \sqrt{3}/2$ would require $\cos\theta = 2/\sqrt{3} > 1$, impossible for a real angle. The Assertion is true.

Step 2: Judge the Reason The stated domain $\mathbb{R} - (-1, 1)$ is exactly the set of real numbers of absolute value at least one. The endpoints -1 and 1 are included, but numbers strictly between them are excluded. Thus the Reason is true.

Step 3: Decide the explanatory connection The domain rule excludes the particular argument $\sqrt{3}/2$, so it directly explains why the asserted set of real values is empty. Consequently option (B) wrongly denies the explanation, and options (C) and (D) wrongly classify one statement as false.

Step 4: Conclusion Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Before evaluating an inverse secant, test whether its argument has absolute value at least one.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). 10 **identical blocks are marked with '0' on two of them, '1' on three of them, '2' on four of them and '3' on one of them and put in a box. If X denotes the number written on the block, then write the probability distribution of X and calculate its mean.** [2 marks]

Solution:

Concept:

- For an equally likely selection, $P(X = x)$ equals the number of blocks labelled x divided by the total number of blocks.
- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$; a valid distribution has nonnegative probabilities summing to one.

Step 1: List the possible values and their probabilities The ten blocks are equally likely to be selected. There are two blocks marked 0, three marked 1, four marked 2, and one marked 3. Therefore the probability distribution is

x	0	1	2	3
$P(X = x)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{10}$

The possible values are the labels, not the numbers of blocks carrying those labels.

Step 2: Check the distribution and calculate the mean The probabilities are all nonnegative and their sum is $(2 + 3 + 4 + 1)/10 = 1$. The weighted average of the labels is

$$E(X) = 0\left(\frac{2}{10}\right) + 1\left(\frac{3}{10}\right) + 2\left(\frac{4}{10}\right) + 3\left(\frac{1}{10}\right)$$
$$= \frac{0 + 3 + 8 + 3}{10} = \frac{14}{10} = \frac{7}{5}.$$

The blocks labelled zero contribute probability but contribute zero to the numerator of the mean.

Step 3: Conclusion The probabilities for $X = 0, 1, 2, 3$ are respectively $\boxed{1/5,\ 3/10,\ 2/5,\ 1/10}$, and the mean is $\boxed{E(X) = 7/5 = 1.4}$.

Quick Tip to Solve: Compute the mean as the sum of all ten printed labels divided by ten; do not average the four distinct labels equally.

OR

21(b). In a village of 8000 people, 3000 go out of the village to work and 4000 are women. It is noted that 30% of women go out of the village to work. What is the probability that a randomly chosen individual is either a woman or a person working outside the village? [2 marks]

Solution:

Concept:

- For equally likely individuals, $P(E) = n(E)/n(S)$, where $n(E)$ counts individuals satisfying event E.
- For two events, $P(W \cup O) = P(W) + P(O) - P(W \cap O)$; subtract the intersection because it is counted twice.

Step 1: Identify the overlap of the two groups Let W be the event that the selected individual is a woman, and O the event that the individual works outside the village. Here $n(S) = 8000$, $n(W) = 4000$ and $n(O) = 3000$. Since the stated percentage refers to women, the number of women working outside is

$$n(W \cap O) = \frac{30}{100} \times 4000 = 1200.$$

Step 2: Apply the addition rule The word "either" here asks for the union: people satisfying at least one of the two descriptions, including those satisfying both. Thus

$$n(W \cup O) = 4000 + 3000 - 1200 = 5800,$$
$$P(W \cup O) = \frac{5800}{8000} = \frac{29}{40}.$$

Subtracting the overlap once prevents the 1200 women who work outside from being counted twice.

Step 3: Conclusion The required probability is $\boxed{\frac{29}{40}}$, or 0.725.

Quick Tip to Solve: Compute the overlap as 30% of 4000 women, not of the total village population, before using the union rule.

22. If $\begin{pmatrix} 3 & -1 \\ 0 & 1 \\ 2 & -3 \end{pmatrix} A = \begin{pmatrix} 2 \\ -5 \\ -17 \end{pmatrix}$, then find matrix A. [2 marks]

Solution:

Concept:

- A 3×2 matrix multiplied by A gives a 3×1 column only when A has order 2×1.
- Equality of matrices requires every pair of corresponding entries to be equal; all the resulting scalar equations must hold simultaneously.

Quick Tip to Solve: In a rectangular matrix equation, substitute the candidate into every row; two equations may give entries that fail the remaining equation.

Step 1: Express the unknown matrix as a column Write $A = \begin{pmatrix} a \\ b \end{pmatrix}$. Multiplying the matrices and comparing corresponding rows gives

$$\begin{pmatrix} 3a - b \\ b \\ 2a - 3b \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -17 \end{pmatrix},$$

so the required equations are $3a - b = 2$, $b = -5$ and $2a - 3b = -17$. A candidate must satisfy all three, even though there are only two unknown entries.

Step 2: Check the consistency of all three rows The second row gives $b = -5$. Substituting this into the first row yields

$$3a + 5 = 2 \quad \Longrightarrow \quad a = -1.$$

However, these forced values give the third entry as

$$2a - 3b = 2(-1) - 3(-5) = -2 + 15 = 13 \neq -17.$$

Therefore the unique candidate from the first two rows fails the third row. The three equations are inconsistent as printed.

Step 3: Conclusion No matrix A satisfies the given equation.

Quick Tip to Solve: In a rectangular matrix equation, substitute the candidate into every row; two equations may give entries that fail the remaining equation.

---

23. Let $f : A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}$, where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. Discuss the bijectivity of the function. [2 marks]

Solution:

Concept:

- A function is injective if $f(u) = f(v)$ implies $u = v$ for all u, v in its domain.
- A function is surjective onto B if every $y \in B$ has a preimage in A. A bijection has both properties.

Step 1: Verify the codomain and prove injectivity For $x \neq 3$, the denominator is nonzero. Also $(x-2)/(x-3) \neq 1$, since equality would imply $-2 = -3$. Thus the formula takes values in B. Now suppose $f(u) = f(v)$, with $u, v \neq 3$. Cross multiplication is valid and gives

$$(u-2)(v-3) = (v-2)(u-3),$$
$$uv - 3u - 2v + 6 = uv - 3v - 2u + 6,$$
$$u = v.$$

Hence f is one-one.

Step 2: Produce a preimage for every element of the codomain Let $y \in B$, so $y \neq 1$. Solving

Quick Tip to Solve: When finding a preimage, also check that it avoids the excluded domain value; here $x - 3 = 1/(y-1)$ does this immediately.

$y = (x-2)/(x-3)$ gives

$$(y-1)x = 3y-2 \quad \Longrightarrow \quad x = \frac{3y-2}{y-1}.$$

This is a real number, and

$$x - 3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Thus $x \in A$. Further, $x - 2 = y/(y-1)$, so $f(x) = y$. Every element of B is attained, proving that f is onto.

Step 3: Conclusion The function is both one-one and onto; hence $\boxed{f : A \to B \text{ is bijective}}$.

Quick Tip to Solve: When finding a preimage, also check that it avoids the excluded domain value; here $x - 3 = 1/(y-1)$ does this immediately.

24. In a Linear Programming Program (LPP) for objective function $Z = 14x - 10y$ subject to constraints $x + y \leq 8$, $3x - 2y \geq -6$, $x, y \geq 0$, shade the feasible region and mark the corner points in a neatly drawn graph. [2 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes representing all constraints. Non-strict inequalities include the boundary lines.
- The conditions $x, y \geq 0$ restrict the graph to the first quadrant; corners are intersections of boundary lines or axes.

Step 1: Draw the boundaries and select the half-planes The line $x + y = 8$ passes through $(8, 0)$ and $(0, 8)$. The inequality selects the side containing the origin. Rearrange the other constraint as

$$3x - 2y \geq -6 \quad \Longleftrightarrow \quad y \leq \frac{3}{2}x + 3.$$

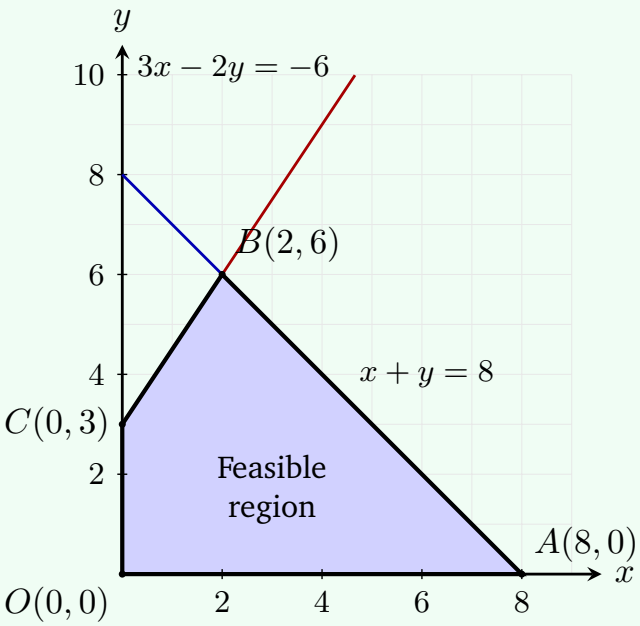
Its boundary passes through $(0, 3)$ and $(2, 6)$. Shade below both lines in the first quadrant.

Step 2: Calculate and mark the vertices At the intersection of the two oblique boundaries,

$$x + y = 8, \qquad 3x - 2y = -6,$$
$$3x - 2(8 - x) = -6, \qquad 5x = 10.$$

Thus the lines meet at $(2, 6)$. The feasible axis intervals are $0 \leq x \leq 8$ on the x-axis and $0 \leq y \leq 3$ on the y-axis. These give the remaining vertices.

Quick Tip to Solve: When dividing $-2y \geq -6 - 3x$ by -2, reverse the inequality; this places the feasible region below the rising line.



Feasible region $OABC$ and all four corner points.

Step 3: Conclusion The shaded feasible region is the quadrilateral $OABC$, including its boundary, with corner points $\boxed{(0,0),\ (8,0),\ (2,6),\ (0,3)}$.

Quick Tip to Solve: When dividing $-2y \geq -6 - 3x$ by -2, reverse the inequality; this places the feasible region below the rising line.

25(a). Differentiate $\left(\frac{5^x}{x^5}\right)$ with respect to x. [2 marks]

Solution:

Concept:

- For a positive constant a, $\frac{d}{dx}(a^x) = a^x \ln a$. The logarithm factor is essential when the base is not e.
- The quotient rule is $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$, where $v \neq 0$. Also $\frac{d}{dx}(x^5) = 5x^4$.

Step 1: Identify the numerator and denominator derivatives Let $y = 5^x/x^5$, which is defined for $x \neq 0$. Take $u = 5^x$ and $v = x^5$. The numerator is an exponential function, whereas the denominator is a power function, so their differentiation rules differ:

$$u' = 5^x \ln 5, \qquad v' = 5x^4.$$

Quick Tip to Solve: Distinguish 5^x from x^5: their derivatives are $5^x \ln 5$ and $5x^4$, respectively, before applying the quotient rule.

Step 2: Apply and simplify the quotient rule Keeping the order $vu' - uv'$ in the numerator, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^5(5^x \ln 5) - 5^x(5x^4)}{x^{10}} \\ &= \frac{5^x x^4(x \ln 5 - 5)}{x^{10}} \\ &= \frac{5^x(x \ln 5 - 5)}{x^6}.\end{aligned}$$

Cancelling x^4 is valid throughout the original domain $x \neq 0$. The same expression follows by writing $y = 5^x x^{-5}$ and applying the product rule.

Step 3: Conclusion $\boxed{\frac{d}{dx}\left(\frac{5^x}{x^5}\right) = \frac{5^x(x \ln 5 - 5)}{x^6}}, \quad x \neq 0.$

Quick Tip to Solve: Distinguish 5^x from x^5: their derivatives are $5^x \ln 5$ and $5x^4$, respectively, before applying the quotient rule.

OR

25(b). If $-2x^2 - 5xy + y^3 = 76$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x, even when the equation does not isolate y.
- The product rule gives $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$, while the chain rule gives $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$.

Step 1: Differentiate both sides with respect to x In the given equation, the derivative of the constant 76 is zero. Apply the power rule to $-2x^2$, the product rule to $-5xy$, and the chain rule to y^3:

$$-4x - 5\left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0.$$

In particular, y cannot be held constant when differentiating the product xy.

Step 2: Collect the derivative terms Expand the bracket and move the terms without the derivative to the other side:

$$\begin{aligned}-4x - 5y - 5x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 0, \\ (3y^2 - 5x)\frac{dy}{dx} &= 4x + 5y.\end{aligned}$$

Division by the coefficient $3y^2 - 5x$ gives the derivative wherever that coefficient is nonzero.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{4x + 5y}{3y^2 - 5x}} \qquad (3y^2 - 5x \neq 0).$$

Quick Tip to Solve: In this equation, attach dy/dx to the derivative of y^3, and use both terms of the product rule for xy before collecting derivatives.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26(a). Let $2x+5y-1=0$ and $3x+2y-7=0$ represent the equations of two lines on which the ants are moving on the ground. Using matrix method, find a point common to the paths of the ants. [3 marks]

Solution:

Concept:

- A point common to two lines has coordinates (x, y) that satisfy both line equations simultaneously.
- Write a linear system as $AX = B$, where A is the coefficient matrix, X is the column of unknowns, and B is the constants column. If $|A| \neq 0$, its unique solution is $X = A^{-1}B$.

Step 1: Express the two line equations as a matrix equation Move the constant terms to the right-hand sides, preserving the order of the unknowns:

$$2x + 5y = 1, \qquad 3x + 2y = 7.$$

Thus the required coordinates satisfy

$$\underbrace{\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}_{B}.$$

Step 2: Find the inverse of the coefficient matrix The determinant is

$$|A| = 2 \cdot 2 - 5 \cdot 3 = 4 - 15 = -11 \neq 0.$$

Hence the lines meet at exactly one point. Interchange the diagonal entries, change the signs of the off-diagonal entries, and divide by the determinant:

$$A^{-1} = -\frac{1}{11} \begin{pmatrix} 2 & -5 \\ -3 & 2 \end{pmatrix}.$$

Step 3: Multiply by the constants column and verify Premultiplying $AX = B$ by A^{-1} yields

$$\begin{aligned} X &= A^{-1}B \\ &= -\frac{1}{11} \begin{pmatrix} 2 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} 2-35 \\ -3+14 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}. \end{aligned}$$

Checking in the original equations gives $2(3) + 5(-1) - 1 = 0$ and $3(3) + 2(-1) - 7 = 0$, so the coordinates lie on both paths.

Step 4: Conclusion The point common to the paths of the ants is $\boxed{(3, -1)}$.

Quick Tip to Solve: Keep the variable order (x, y) fixed in both rows; then check the resulting point in both original line equations to catch sign errors.

OR

26(b). A shopkeeper sells 50 Chemistry, 60 Physics and 35 Maths books on day I and sells 40 Chemistry, 45 Physics and 50 Maths books on day II. If the selling price for each such subject book is ₹150 (Chemistry), ₹175 (Physics) and ₹180 (Maths), then find his total sale in two days, using matrix method. If cost price of all the books together is ₹35,000, what profit did he earn after the sale of two days? [3 marks]

Solution:

Concept:

- In a quantity matrix, each row can represent a day and each column a subject. Multiplying by the corresponding column of prices gives the revenue for each day.
- Total sale is the sum of the daily revenues, and profit equals total sale minus total cost price. All monetary amounts below are in rupees.

Step 1: Arrange the quantities and prices in matching order Use the subject order Chemistry, Physics, Mathematics in every row. The first row records day I and the second row records day II:

$$Q = \begin{pmatrix} 50 & 60 & 35 \\ 40 & 45 & 50 \end{pmatrix}, \qquad P = \begin{pmatrix} 150 \\ 175 \\ 180 \end{pmatrix}.$$

Here Q has order 2×3 and P has order 3×1, so their product is a 2×1 column. Its entries give the sales on the two days in their original order.

Step 2: Compute the daily sales by matrix multiplication Multiply each quantity row by the price column:

$$QP = \begin{pmatrix} 50(150) + 60(175) + 35(180) \\ 40(150) + 45(175) + 50(180) \end{pmatrix}$$
$$= \begin{pmatrix} 7500 + 10500 + 6300 \\ 6000 + 7875 + 9000 \end{pmatrix} = \begin{pmatrix} 24300 \\ 22875 \end{pmatrix}.$$

Thus the sale on day I is 24,300 rupees and the sale on day II is 22,875 rupees. The multiplication accounts for all three subjects on each day.

Step 3: Add the revenues and subtract the cost A row of ones adds the daily entries:

$$\text{Total sale} = \begin{pmatrix} 1 & 1 \end{pmatrix} QP = 24300 + 22875 = 47175.$$

The given cost price of all books sold over both days is 35,000 rupees, so it is subtracted once from this combined revenue:

$$\text{Profit} = 47175 - 35000 = 12175.$$

Step 4: Conclusion The total sale in two days is $\boxed{47{,}175 \text{ rupees}}$, and the shopkeeper earns a profit of $\boxed{12{,}175 \text{ rupees}}$.

Quick Tip to Solve: Match the price column to Chemistry, Physics, Mathematics before multiplying; subtract the combined cost only after adding the two daily revenues.

27(a). Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x^3 - 5$, $\forall x \in \mathbb{R}$ is one-one and onto.

[3 marks]

Solution:

Concept:

- A function is one-one (injective) if equality of images forces equality of their inputs: $f(a) = f(b) \Rightarrow a = b$.
- A function is onto (surjective) if every element of its codomain is the image of at least one element of its domain. A real number always has a unique real cube root.

Step 1: Prove equality of images implies equality of inputs Take arbitrary real numbers a, b and suppose that $f(a) = f(b)$. By the given formula,

$$\begin{aligned} 4a^3 - 5 &= 4b^3 - 5, \\ a^3 &= b^3, \\ (a-b)(a^2 + ab + b^2) &= 0. \end{aligned}$$

To justify that the inputs must agree over the real numbers, observe that

$$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} \geq 0.$$

If this second factor is zero, both squares vanish, giving $b = 0$ and then $a = 0$. Otherwise the first factor must vanish. In either case $a = b$. Therefore no two distinct real inputs have the same image, and f is one-one.

Step 2: Construct a preimage for every real output Let y be any element of the codomain $\mathbb{R}$. Solving $f(x) = y$ gives

$$4x^3 - 5 = y \quad \Longrightarrow \quad x^3 = \frac{y+5}{4} \quad \Longrightarrow \quad x = \sqrt[3]{\frac{y+5}{4}}.$$

This cube root is a real number for every real y, including when $y + 5$ is negative. Thus it belongs to the specified domain. Substitution confirms

$$f\left(\sqrt[3]{\frac{y+5}{4}}\right) = 4\left(\frac{y+5}{4}\right) - 5 = y.$$

Every element of $\mathbb{R}$ is therefore attained by the function, proving that it is onto.

Step 3: Conclusion The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x^3 - 5$, is both one-one and onto, and hence is bijective.

Quick Tip to Solve: For a nonzero multiple of x^3 plus a constant, prove injectivity by equating images and prove surjectivity by solving for a real cube-root preimage.

OR

27(b). Let R be a relation defined on a set $\mathbb{N}$ of natural numbers such that $R = \{(x, y) : xy$ is a square of a natural number, $x, y \in \mathbb{N}\}$. Determine if the relation R is an equivalence relation.

[3 marks]

Solution:

Concept:

- A relation is an equivalence relation precisely when it is reflexive, symmetric and transitive on its underlying set.
- A natural number is a perfect square if and only if every exponent in its prime factorisation is even. This criterion lets us test products without taking uncertain square roots.

Step 1: Verify reflexivity Take any $x \in \mathbb{N}$. The product of x with itself is $x \cdot x = x^2$, which is the square of the natural number x. Thus $(x, x) \in R$ for every $x \in \mathbb{N}$, so R is reflexive.

Step 2: Verify symmetry Suppose $(x, y) \in R$. Then $xy = m^2$ for some $m \in \mathbb{N}$. Since multiplication is commutative, $yx = xy = m^2$, so $(y, x) \in R$. This implication holds for every related pair; hence R is symmetric.

Step 3: Verify transitivity using prime exponents Suppose $(x, y) \in R$ and $(y, z) \in R$. Thus both xy and yz are perfect squares. For any prime p, let its exponents in x, y, z be a, b, c, respectively, allowing exponent zero when that prime is absent. The square condition says that $a + b$ and $b + c$ are both even. Therefore

$$a + c = (a + b) + (b + c) - 2b$$

is even. This holds for every prime, so every prime exponent in xz is even. Hence xz is a square of a natural number and $(x, z) \in R$. Therefore R is transitive.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive. Therefore, R is an equivalence relation on $\mathbb{N}$.

Quick Tip to Solve: For a relation defined by a product being a square, track the parity of each prime exponent; adding the two even exponent sums proves transitivity directly.

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28. Show that the derivative of $\tan^{-1}(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, with respect to x is equal to $\frac{1}{2}$.

[3 marks]

Solution:

Concept:

- The chain rule for the principal inverse tangent is $\frac{d}{dx}\tan^{-1} u = \frac{u'}{1 + u^2}$.
- Use $\frac{d}{dx}\sec x = \sec x \tan x$, $\frac{d}{dx}\tan x = \sec^2 x$, and the identity $1 + \tan^2 x = \sec^2 x$.

Step 1: Differentiate the composite function Put $y = \tan^{-1}(\sec x + \tan x)$ and $u = \sec x + \tan x$. On

Quick Tip to Solve: In this derivative, expand $1 + (\sec x + \tan x)^2$ and use $1 + \tan^2 x = \sec^2 x$; it becomes twice the chain-rule numerator.

the given interval $-\pi/2 < x < \pi/2$, we have $\cos x > 0$, so both trigonometric terms are defined and differentiable. Applying the chain rule gives

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{1 + (\sec x + \tan x)^2}.$$

Step 2: Factor the numerator and denominator The numerator is $\sec x(\sec x + \tan x)$. Expand the denominator and replace $1 + \tan^2 x$ by $\sec^2 x$:

$$\begin{aligned} 1 + (\sec x + \tan x)^2 &= 1 + \sec^2 x + 2\sec x \tan x + \tan^2 x \\ &= 2\sec^2 x + 2\sec x \tan x \\ &= 2\sec x(\sec x + \tan x). \end{aligned}$$

Thus the denominator is exactly twice the numerator.

Step 3: Check that cancellation is valid Since $\cos x > 0$ and $-1 < \sin x < 1$ on the stated open interval,

$$\sec x > 0, \qquad \sec x + \tan x = \frac{1 + \sin x}{\cos x} > 0.$$

Consequently the common factor $\sec x(\sec x + \tan x)$ never vanishes anywhere in the domain and can safely be cancelled in the derivative expression.

Step 4: Conclusion Therefore,

$$\boxed{\frac{d}{dx}\tan^{-1}(\sec x + \tan x) = \frac{1}{2}}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Quick Tip to Solve: In this derivative, expand $1 + (\sec x + \tan x)^2$ and use $1 + \tan^2 x = \sec^2 x$; it becomes twice the chain-rule numerator.

29. Find dimensions of a rectangle of perimeter 12 cm which will generate maximum volume when swept along a circular rotation keeping the shorter side fixed as the axis. [3 marks]

Solution:

Concept:

- Rotating a rectangle about one of its sides generates a cylinder. Its height is the length of the fixed side, while its radius is the length of the perpendicular side.
- A cylinder of radius r and height h has volume $V = \pi r^2 h$. Use the perimeter constraint to express the volume in one variable and examine the sign of its derivative.

Step 1: Express the cylinder dimensions using the perimeter Let the shorter side, which is the fixed axis, be h cm, and let the longer side be r cm. The rotation gives a cylinder of height h and radius r. The perimeter condition is

$$2(h + r) = 12 \quad \Longrightarrow \quad r = 6 - h.$$

Quick Tip to Solve: The side used as the rotation axis becomes the cylinder's height. Here set the shorter side equal to h, so that the radius is $6 - h$, before differentiating.

Positivity and $h < r$ give the admissible interval $0 < h < 3$.

Step 2: Differentiate the volume function Substituting the constraint into the volume formula gives

$$V(h) = \pi h(6-h)^2.$$

Applying the product rule and factoring,

$$\begin{aligned} V'(h) &= \pi\big[(6-h)^2 - 2h(6-h)\big] \\ &= \pi(6-h)(6-3h) \\ &= 3\pi(6-h)(2-h). \end{aligned}$$

The stationary values are $h = 2$ and $h = 6$; only $h = 2$ belongs to $0 < h < 3$.

Step 3: Establish the maximum over the admissible interval For $0 < h < 2$, both factors $6-h$ and $2-h$ are positive, hence $V'(h) > 0$. For $2 < h < 3$, the first remains positive while the second is negative, hence $V'(h) < 0$. The volume therefore increases up to $h = 2$ and decreases afterwards throughout the remaining domain, proving that this stationary point is the absolute maximum. The corresponding other side is $r = 6 - 2 = 4\,\text{cm}$.

Step 4: Conclusion The required rectangle has dimensions $\boxed{4\,\text{cm} \times 2\,\text{cm}}$, with the 2 cm side as the fixed axis. The maximum volume is $\pi(4)^2(2) = \boxed{32\pi\ \text{cm}^3}$.

Quick Tip to Solve: The side used as the rotation axis becomes the cylinder's height. Here set the shorter side equal to h, so that the radius is $6-h$, before differentiating.

30(a). The scalar product of the vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \lambda\hat{i} - 2\hat{j} - 3\hat{k}$ is equal to 1. Find the value of λ. [3 marks]

Solution:

Concept:

- A unit vector in the direction of a nonzero vector $\vec{v}$ is $\hat{v} = \vec{v}/|\vec{v}|$; its direction is the same as that of $\vec{v}$.
- The scalar product is obtained by multiplying corresponding components and adding: $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.

Step 1: Form the sum and its unit vector Add corresponding components of the two given vectors:

$$\vec{b} + \vec{c} = (\lambda + 2)\hat{i} - 6\hat{j} + 2\hat{k}.$$

This vector is nonzero for every real λ, since its second component is -6. Its magnitude and the required unit vector are

$$|\vec{b} + \vec{c}| = \sqrt{(\lambda+2)^2 + 40}, \qquad \hat{u} = \frac{(\lambda+2)\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{(\lambda+2)^2 + 40}}.$$

Quick Tip to Solve: Normalize the sum of the vectors, not the vectors separately, and check the sign of the numerator after squaring.

Step 2: Apply the scalar-product condition Using $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, the given condition becomes

$$\vec{a} \cdot \hat{u} = \frac{(\lambda + 2) + 6 + 4}{\sqrt{(\lambda + 2)^2 + 40}} = 1.$$

Hence $\lambda + 12 = \sqrt{(\lambda + 2)^2 + 40}$, whose right side is positive. Squaring and simplifying gives

$$\begin{aligned} (\lambda + 12)^2 &= (\lambda + 2)^2 + 40, \\ \lambda^2 + 24\lambda + 144 &= \lambda^2 + 4\lambda + 44, \\ 20\lambda &= -100. \end{aligned}$$

Step 3: Check the value before accepting it The candidate is $\lambda = -5$. In the original unsquared expression, the numerator is 7 and the denominator is $\sqrt{9 + 40} = 7$. Thus their ratio is indeed 1, so squaring has introduced no extraneous answer.

Step 4: Conclusion The required value is $\boxed{\lambda = -5}$.

Quick Tip to Solve: Normalize the sum of the vectors, not the vectors separately, and check the sign of the numerator after squaring.

OR

30(b). Find the shortest distance between the lines:

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$

$\vec{r} = (\hat{i} + 4\hat{k}) + \mu(3\hat{i} - 6\hat{j} + 9\hat{k})$ [3 marks]

Solution:

Concept:

- Two lines are parallel when their direction vectors are nonzero scalar multiples of each other.
- For parallel lines through points A, B with common direction vector $\vec{d}$, their shortest distance is $D = |\overrightarrow{AB} \times \vec{d}|/|\vec{d}|$. This is the perpendicular component of the displacement between the lines.

Step 1: Identify the points and directions The first line passes through $A = (2, -1, 3)$ and has direction vector $\vec{d} = (1, -2, 3)$. The second passes through $B = (1, 0, 4)$ and has direction vector

$$(3, -6, 9) = 3(1, -2, 3) = 3\vec{d}.$$

Therefore the lines are parallel. The displacement from the first chosen point to the second is

$$\overrightarrow{AB} = B - A = (-1, 1, 1).$$

This is not a multiple of $\vec{d}$, so the two parallel lines are distinct.

Step 2: Calculate the perpendicular separation The magnitude of the cross product is the area of the parallelogram formed by $\overrightarrow{AB}$ and $\vec{d}$. Dividing this area by the base length $|\vec{d}|$ gives the perpendicular

Quick Tip to Solve: Check the direction vectors first: proportional directions require the parallel-line distance formula, since the skew-line formula would have a zero denominator.

distance:

$$\overrightarrow{AB} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$
$$= (3+2)\hat{i} - (-3-1)\hat{j} + (2-1)\hat{k}$$
$$= 5\hat{i} + 4\hat{j} + \hat{k}.$$

Consequently,

$$|\overrightarrow{AB} \times \vec{d}| = \sqrt{25+16+1} = \sqrt{42}, \qquad |\vec{d}| = \sqrt{1+4+9} = \sqrt{14}.$$

The required ratio is $\sqrt{42}/\sqrt{14}$. Using the longer direction vector $3\vec{d}$ would multiply both numerator and denominator by 3, leaving this ratio unchanged.

Step 3: Conclusion The shortest distance between the lines is $\boxed{\sqrt{3} \text{ units}}$.

Quick Tip to Solve: Check the direction vectors first: proportional directions require the parallel-line distance formula, since the skew-line formula would have a zero denominator.

31. In the Linear Programming Problem, find the minimum value of the objective function $Z = 18x + 10y$, subject to the following constraints:

$4x + y \geq 20$

$2x + 3y \geq 30$

$x, y \geq 0$ [3 marks]

Solution:

Concept:

- The feasible region consists of points satisfying all the constraints simultaneously. A finite optimum of a linear objective on a polygonal feasible region can be checked at its corner points.
- For an unbounded feasible region, a candidate minimum must be justified. A nonnegative weighted sum of the constraints can give a lower bound for the objective throughout the region.

Step 1: Determine the feasible corner points The constraints require first-quadrant points on or above both boundary lines. Their intersection satisfies

$$4x + y = 20, \qquad 2x + 3y = 30.$$

Substitute $y = 20 - 4x$ into the second equation:

$$2x + 3(20 - 4x) = 30 \quad \Rightarrow \quad -10x = -30.$$

Thus the intersection is $(3, 8)$. On the y-axis, both inequalities hold only when $y \geq 20$, giving the corner $(0, 20)$. On the x-axis, both hold only when $x \geq 15$, giving $(15, 0)$. The other individual line intercepts, $(5, 0)$ and $(0, 10)$, each violate one constraint and are not feasible corners.

Quick Tip to Solve: For this unbounded region, combine the constraints with positive weights $17/5$ and $11/5$ to prove the candidate minimum directly.

Step 2: Evaluate the objective at each corner Substitute the three feasible vertices into $Z = 18x + 10y$:

$$\begin{aligned} Z(0, 20) &= 200, \\ Z(3, 8) &= 54 + 80 = 134, \\ Z(15, 0) &= 270. \end{aligned}$$

The smallest of these values occurs at the intersection of the two boundary lines. Since the feasible region extends indefinitely upwards and to the right, we also verify a global lower bound.

Step 3: Establish the minimum on the unbounded region Write the objective as a positive weighted sum of the two constrained expressions:

$$\begin{aligned} Z &= \frac{17}{5}(4x + y) + \frac{11}{5}(2x + 3y) \\ &\geq \frac{17}{5}(20) + \frac{11}{5}(30) \\ &= 68 + 66 = 134. \end{aligned}$$

Equality is achieved when both constraints are equalities, which holds at the feasible point $(3, 8)$. Thus no other point in the unbounded region can give a smaller value.

Step 4: Conclusion The minimum value is $\boxed{Z_{\min} = 134}$, attained at $\boxed{(x, y) = (3, 8)}$.

Quick Tip to Solve: For this unbounded region, combine the constraints with positive weights $17/5$ and $11/5$ to prove the candidate minimum directly.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Find: $\int \frac{x}{(x-1)(x^2+4)}\,dx$. [5 marks]

Solution:

Concept:

- A proper rational function with a linear factor and an irreducible quadratic factor is decomposed using a constant numerator over the linear factor and a linear numerator over the quadratic factor.
- The standard antiderivatives are $\int du/u = \ln|u| + C$ and $\int dx/(x^2 + a^2) = a^{-1}\tan^{-1}(x/a) + C$, where $a > 0$.

Step 1: Set up the partial fractions Let I denote the required indefinite integral. Since the rational function is proper and $x^2 + 4$ has no real linear factor, write

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 4}.$$

Quick Tip to Solve: Set $x = 1$ in the cleared-denominator identity to get $A = 1/5$ immediately; retain a linear numerator over $x^2 + 4$.

Multiplication by $(x-1)(x^2+4)$, for $x \neq 1$, gives the polynomial identity

$$\begin{aligned} x &= A(x^2+4) + (Bx+C)(x-1) \\ &= (A+B)x^2 + (C-B)x + (4A-C). \end{aligned}$$

Step 2: Determine the three coefficients Equating the coefficients of x^2, x and the constant terms on the two sides gives

$$A+B=0, \qquad C-B=1, \qquad 4A-C=0.$$

Thus $B=-A$ and $C=4A$. Substituting these into $C-B=1$ yields $5A=1$, so $A=1/5$, $B=-1/5$, and $C=4/5$. Splitting the quadratic numerator into its two terms now gives three standard integrals.

Step 3: Integrate the logarithmic and inverse tangent terms Integrating the three terms separately,

$$I = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x\,dx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}.$$

Use $u=x^2+4$, $du=2x\,dx$, for the middle integral, and the inverse tangent formula with $a=2$ for the last. The respective antiderivatives are

$$\frac{1}{5}\ln|x-1|, \qquad -\frac{1}{10}\ln(x^2+4), \qquad \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right).$$

Since $x^2+4>0$, only the linear factor needs absolute values. Combine the integration constants into C on any interval excluding $x=1$.

Step 4: Conclusion

$$\boxed{\int \frac{x\,dx}{(x-1)(x^2+4)} = \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln(x^2+4) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C.}$$

Quick Tip to Solve: Set $x=1$ in the cleared-denominator identity to get $A=1/5$ immediately; retain a linear numerator over x^2+4.

OR

32(b). Evaluate: $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. [5 marks]

Solution:

Concept:

- Reflection of a definite integral gives $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.
- Here $\sin(\pi-x)=\sin x$ and $\cos^2(\pi-x)=\cos^2 x$.
- The substitution $u=\cos x$ gives $du=-\sin x\,dx$, reducing the remaining integral to $\int du/(1+u^2) = \tan^{-1}u + C$.

Quick Tip to Solve: When the factor multiplying x is unchanged by $x \mapsto \pi - x$, add the reflected integral first; then replace x by the average $\pi/2$.

Step 1: Reflect the integral about the midpoint of its interval Put

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$$

The integrand is continuous on $[0, \pi]$. Substitute $x = \pi - t$, so $dx = -dt$ and the endpoints change from $0, \pi$ to $\pi, 0$. Reverse the limits and rename t as x to obtain

$$\begin{aligned} I &= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx \\ &= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx. \end{aligned}$$

Although the cosine changes sign under reflection, its square remains unchanged.

Step 2: Add the two equal representations Add the original and reflected expressions; the factors x and $\pi - x$ sum to π:

$$\begin{aligned} 2I &= \int_0^{\pi} \frac{\{x + (\pi - x)\}\sin x}{1 + \cos^2 x}\, dx \\ &= \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx. \end{aligned}$$

Thus it remains to evaluate $J = \int_0^{\pi} \sin x/(1 + \cos^2 x)\, dx$, and then use $I = (\pi/2)J$.

Step 3: Evaluate the remaining definite integral with transformed limits Take $u = \cos x$, giving $du = -\sin x\, dx$. When $x = 0$, $u = 1$; when $x = \pi$, $u = -1$. Therefore

$$\begin{aligned} J &= -\int_1^{-1} \frac{du}{1 + u^2} = \int_{-1}^{1} \frac{du}{1 + u^2} \\ &= \left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

Substitute this value into $2I = \pi J$.

Step 4: Conclusion

$$\boxed{\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.}$$

Quick Tip to Solve: When the factor multiplying x is unchanged by $x \mapsto \pi - x$, add the reflected integral first; then replace x by the average $\pi/2$.

33(a). Find the point Q on the line $\dfrac{2x+4}{6} = \dfrac{y+1}{2} = \dfrac{-2z+6}{-4}$ at a distance of $3\sqrt{2}$ from the point $P(1,2,3)$. [5 marks]

Solution:

Concept:

- A common parameter converts a symmetric line equation into coordinates of a general point on the line. Each coordinate must be obtained by solving its own numerator equation carefully.
- For points $Q(x,y,z)$ and $P(x_0,y_0,z_0)$, the distance formula is $PQ^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$. A prescribed distance can give two points on a line.

Step 1: Parameterise the given line Let the common value of the three ratios be t. Then

$$2x+4 = 6t, \qquad y+1 = 2t, \qquad -2z+6 = -4t.$$

Solving these equations gives $x = 3t-2$, $y = 2t-1$ and $z = 2t+3$. Thus a general point on the line is

$$Q(t) = (3t-2,\ 2t-1,\ 2t+3).$$

Step 2: Apply the prescribed distance from the fixed point The coordinate differences from $P(1,2,3)$ are $3t-3$, $2t-3$ and $2t$. Since $PQ = 3\sqrt{2}$, squaring this nonnegative distance gives

$$\begin{aligned}(3t-3)^2 + (2t-3)^2 + (2t)^2 &= 18,\\ (9t^2-18t+9) + (4t^2-12t+9) + 4t^2 &= 18,\\ 17t^2 - 30t &= 0,\\ t(17t-30) &= 0.\end{aligned}$$

Therefore the admissible parameter values are $t = 0$ and $t = 30/17$. Neither root is to be discarded: both represent points on the given line at the required distance.

Step 3: Substitute and check both parameter values At $t = 0$, the coordinates are $(-2,-1,3)$, whose squared distance from P is $(-3)^2+(-3)^2+0^2 = 18$. At $t = 30/17$, substitution gives

$$x = \frac{90-34}{17} = \frac{56}{17}, \quad y = \frac{60-17}{17} = \frac{43}{17}, \quad z = \frac{60+51}{17} = \frac{111}{17}.$$

For this point, the squared distance is

$$PQ^2 = \frac{39^2+9^2+60^2}{17^2} = \frac{5202}{289} = 18.$$

Step 4: Conclusion The two possible positions are

$$\boxed{Q = (-2,-1,3) \quad \text{or} \quad Q = \left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right).}$$

Quick Tip to Solve: Keep the common parameter until after applying the distance formula; factor the resulting quadratic and retain every root that satisfies the distance.

OR

33(b). Find the image of the point $(-1, 5, 2)$ in the line $\frac{2x-4}{2} = \frac{y}{2} = \frac{2-z}{3}$. Find the length of the line segment joining the given point and the image point. [5 marks]

Solution:

Concept:

- The image P' of a point P in a line is obtained using the perpendicular foot H from P to that line. The line bisects PP' at right angles, so $P' = 2H - P$.
- If the line has direction vector $\mathbf{d}$, the perpendicular foot satisfies $\overrightarrow{PH} \cdot \mathbf{d} = 0$. The length joining a point and its image is $PP' = 2PH$.

Step 1: Write a general point on the reflecting line Denote the given point by $P = (-1, 5, 2)$. Put the common value of the ratios equal to t:

$$\frac{2x-4}{2} = \frac{y}{2} = \frac{2-z}{3} = t.$$

Thus $x = t + 2$, $y = 2t$ and $z = 2 - 3t$. The line has direction vector $\mathbf{d} = (1, 2, -3)$. Write the perpendicular foot as

$$H = (t + 2,\ 2t,\ 2 - 3t).$$

The numerator $2x - 4$ must first be divided by its coefficient of x; the direction coefficient for x is 1, not 2.

Step 2: Locate the perpendicular foot The vector from P to H is

$$\overrightarrow{PH} = (t + 3,\ 2t - 5,\ -3t).$$

Since this vector is perpendicular to the given line, its scalar product with the direction vector vanishes:

$$\begin{aligned}(t+3) + 2(2t-5) + (-3)(-3t) &= 0,\\ t + 3 + 4t - 10 + 9t &= 0,\\ 14t - 7 &= 0.\end{aligned}$$

Hence $t = 1/2$, giving $H = (5/2, 1, 1/2)$. Its three line ratios are all $1/2$, confirming that it lies on the line.

Step 3: Reflect the point through the perpendicular foot If the image is $P' = (x', y', z')$, then H is the midpoint of PP'. Therefore

$$\begin{aligned}x' &= 2\left(\frac{5}{2}\right) - (-1) = 6,\\ y' &= 2(1) - 5 = -3,\\ z' &= 2\left(\frac{1}{2}\right) - 2 = -1.\end{aligned}$$

The candidate image is consequently $(6, -3, -1)$. The joining vector is $(7, -8, -3)$, and its scalar product with the line direction is $7 - 16 + 9 = 0$. Together with the midpoint check, this verifies the reflection.

Quick Tip to Solve: Find the perpendicular foot using one scalar-product equation, then double its coordinates and subtract the original point to obtain the image.

Step 4: Calculate the requested segment length Apply the three-dimensional distance formula directly to P and P':

$$\begin{aligned} PP' &= \sqrt{(6+1)^2 + (-3-5)^2 + (-1-2)^2} \\ &= \sqrt{7^2 + (-8)^2 + (-3)^2} \\ &= \sqrt{49 + 64 + 9} = \sqrt{122}. \end{aligned}$$

As a second check, $PH^2 = (7/2)^2 + (-4)^2 + (-3/2)^2 = 122/4$, so $2PH = \sqrt{122}$ agrees with the direct calculation.

Step 5: Conclusion The image of the given point in the line is $\boxed{(6, -3, -1)}$, and the length of the segment joining the point and its image is $\boxed{\sqrt{122} \text{ units}}$.

Quick Tip to Solve: Find the perpendicular foot using one scalar-product equation, then double its coordinates and subtract the original point to obtain the image.

34. Solve the differential equation $(x^2 + y^2)\,dx + xy\,dy = 0$, $y(1) = 1$. [5 marks]

Solution:

Concept:

- A differential equation is homogeneous when its slope can be expressed as a function of y/x. The substitution $y = vx$ then gives $dy/dx = v + x\,dv/dx$.
- For a separable equation, collect terms involving v with dv and those involving x with dx, then integrate both sides. The initial condition determines the constant of integration.
- When taking a square root to express y, its sign must agree with the prescribed initial value and remain consistent on the solution interval.

Step 1: Express the equation in homogeneous form Dividing the given equation by dx gives

$$x^2 + y^2 + xy\frac{dy}{dx} = 0.$$

In a neighbourhood of the initial point $(1, 1)$, both x and y are nonzero. Therefore,

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{xy} = -\frac{1 + (y/x)^2}{y/x}.$$

The right-hand side depends only on the ratio y/x, so the homogeneous substitution applies.

Step 2: Substitute and separate the variables Put $y = vx$, where v is a function of x. Using the product rule,

$$v + x\frac{dv}{dx} = -\frac{1 + v^2}{v}, \qquad x\frac{dv}{dx} = -\frac{1 + 2v^2}{v}.$$

Thus the separated equation is

$$\frac{v}{1 + 2v^2}\,dv = -\frac{dx}{x}.$$

The denominator $1 + 2v^2$ is strictly positive for real v, so this separation does not discard any real

Quick Tip to Solve: Alternatively, set $z = y^2$: the equation becomes $z' + 2z/x = -2x$, whose integrating factor is x^2. This gives the same first integral quickly.

constant-ratio solution.

Step 3: Integrate and return to the original variables Since the derivative of $1+2v^2$ is $4v$, integration yields

$$\frac{1}{4}\ln(1+2v^2) = -\ln|x| + C_1.$$

Multiplying by 4 and exponentiating gives

$$x^4(1+2v^2) = C, \qquad x^4 + 2x^2y^2 = C.$$

Here the factor x^4 comes from $e^{4\ln|x|}$; hence it has the same value on either side of zero, although the solution through the given point lies on the positive side.

Step 4: Apply the initial condition and check the family Substituting $x = 1$ and $y = 1$ gives $C = 1 + 2 = 3$. Solving the resulting relation for y^2 gives

$$y^2 = \frac{3 - x^4}{2x^2}.$$

The positive root is selected by $y(1) = 1$. The interval containing 1 on which this root is positive and differentiable is $0 < x < \sqrt[4]{3}$. As a direct check, differentiation of the implicit relation gives

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} = 0.$$

Dividing by $4x$ recovers exactly the given differential equation in derivative form, verifying the integration and the constant.

Step 5: Conclusion The required solution is $\boxed{x^4 + 2x^2y^2 = 3}$, with the branch through $(1, 1)$ given by

$$\boxed{y = \sqrt{\frac{3 - x^4}{2x^2}}}, \qquad 0 < x < \sqrt[4]{3}.$$

Quick Tip to Solve: Alternatively, set $z = y^2$: the equation becomes $z' + 2z/x = -2x$, whose integrating factor is x^2. This gives the same first integral quickly.

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35. A woman discovered a scratch along a straight line on a circular table top of radius 8 cm. She divided the table top into 4 equal quadrants and discovered the scratch passing through the origin inclined at an angle $\frac{\pi}{4}$ anticlockwise along the positive direction of x-axis. Using the method of integration, find the area of the region enclosed by the x-axis, the scratch, and the circular table top in the first quadrant. [5 marks]

Solution:

Concept:

- A circle of radius r centred at the origin has equation $x^2 + y^2 = r^2$. Its upper semicircle is $y = \sqrt{r^2 - x^2}$.

Quick Tip to Solve: Find the scratch-circle intersection before choosing limits: the top boundary changes there from the line to the circular arc, so two vertical-strip integrals are needed.

- A straight line through the origin making angle θ with the positive x-axis has equation $y = x\tan\theta$.
- The area between a nonnegative upper boundary $y = f(x)$ and the x-axis is $\int f(x)\,dx$. If the upper boundary changes at an intersection, split the integral there.

Step 1: Write the equations of the boundaries Take the centre of the circular tabletop as the origin, so that its four quadrants are the coordinate quadrants. The radius is 8 cm; hence the circle and its first-quadrant arc are

$$x^2 + y^2 = 64, \qquad y = \sqrt{64 - x^2}.$$

The scratch passes through the origin and has inclination $\pi/4$. Its slope is $\tan(\pi/4) = 1$, so its equation is $y = x$. The remaining boundary is $y = 0$, the positive x-axis.

Step 2: Find where the scratch meets the circle Substituting $y = x$ into the circle gives $2x^2 = 64$. Since the required intersection is in the first quadrant,

$$x = 4\sqrt{2}, \qquad y = 4\sqrt{2}.$$

For $0 \le x \le 4\sqrt{2}$, the upper edge of the required region is the scratch $y = x$. For $4\sqrt{2} \le x \le 8$, the upper edge is the circular arc. In both intervals the lower edge is the x-axis. Therefore the area is

$$A = \int_0^{4\sqrt{2}} x\,dx + \int_{4\sqrt{2}}^{8} \sqrt{64 - x^2}\,dx.$$

Integrating the circle over the entire interval from 0 to 8 would include the unwanted region above the scratch, so the split is essential.

Step 3: Evaluate the straight-line contribution The first integral gives

$$A_1 = \left[\frac{x^2}{2}\right]_0^{4\sqrt{2}} = \frac{32}{2} = 16\,\text{cm}^2.$$

This contribution corresponds to vertical strips below the scratch up to its intersection with the circle.

Step 4: Evaluate the circular-arc contribution and combine Use the standard antiderivative

$$\int \sqrt{64 - x^2}\,dx = \frac{x}{2}\sqrt{64 - x^2} + 32\sin^{-1}\left(\frac{x}{8}\right).$$

At $x = 8$, this expression is 16π. At $x = 4\sqrt{2}$, the square root is $4\sqrt{2}$ and $\sin^{-1}(\sqrt{2}/2) = \pi/4$, so it is $16 + 8\pi$. Thus

$$A_2 = 16\pi - (16 + 8\pi) = 8\pi - 16, \qquad A = A_1 + A_2 = 16 + 8\pi - 16.$$

All inverse trigonometric angles here are measured in radians. As a check, the desired region subtends $\pi/4$ at the centre, exactly half the first quadrant; the integrated area must therefore be half the quadrant's area.

Step 5: Conclusion The area enclosed by the x-axis, the scratch and the circular boundary in the first quadrant is $\boxed{8\pi\ \text{cm}^2}$.

Quick Tip to Solve: Find the scratch-circle intersection before choosing limits: the top boundary changes there from the line to the circular arc, so two vertical-strip integrals are needed.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1: Based upon the results of regular medical check-ups in a hospital, it was found that out of 1000 people, 700 were very healthy, 200 maintained average health and 100 had a poor health record.

Let A_1: People with good health,

A_2: People with average health,

and A_3: People with poor health.

During a pandemic, the data expressed that the chances of people contracting the disease from category A_1, A_2 and A_3 are 25%, 35% and 50%, respectively.

Based upon the above information, answer the following questions: [4 marks]

36(i). A person was tested randomly. What is the probability that he/she has contracted the disease? [2 marks]

Solution:

Concept:

- The three health categories form a partition: every person belongs to exactly one of A_1, A_2, A_3.
- For the event D of contracting the disease, the theorem of total probability gives $P(D) = \sum_{j=1}^{3} P(A_j)P(D \mid A_j)$.

Step 1: Write the category and conditional probabilities A person is selected randomly from the whole population, so the category probabilities are the proportions in the group of 1000 people:

$$P(A_1) = \frac{700}{1000} = 0.7, \quad P(A_2) = \frac{200}{1000} = 0.2, \quad P(A_3) = \frac{100}{1000} = 0.1.$$

The disease probabilities within these categories are $0.25, 0.35, 0.50$, respectively. These are conditional probabilities, not the proportions of the whole population who contract the disease.

Step 2: Weight each disease probability by its category size The mutually exclusive ways to contract the disease are to belong to one of the three categories and contract it there. Thus,

$$\begin{aligned} P(D) &= (0.7)(0.25) + (0.2)(0.35) + (0.1)(0.50) \\ &= 0.175 + 0.070 + 0.050 = 0.295. \end{aligned}$$

The probabilities must be weighted because the categories do not contain equal numbers of people.

Step 3: Conclusion The probability that the randomly tested person has contracted the disease is $\boxed{\frac{59}{200} = 0.295}$, or 29.5%.

Quick Tip to Solve: Multiply each category's fraction of the population by its disease rate, then add the three products; do not average the three rates directly.

36(ii). Given that the person has not contracted the disease, what is the probability that the person is from category A_2? [2 marks]

Solution:

Concept:

- If D denotes contracting the disease, then D' denotes not contracting it, and $P(D' \mid A_j) = 1 - P(D \mid A_j)$.
- Bayes' theorem gives $P(A_2 \mid D') = \dfrac{P(A_2)P(D' \mid A_2)}{P(D')}$, where the denominator includes disease-free people from every health category.

Step 1: Find the probability of not contracting the disease The category probabilities are $P(A_1) = 0.7$, $P(A_2) = 0.2$ and $P(A_3) = 0.1$. Taking the complements of the disease rates gives

$$P(D' \mid A_1) = 0.75, \quad P(D' \mid A_2) = 0.65, \quad P(D' \mid A_3) = 0.50.$$

Apply total probability to the three mutually exclusive categories:

$$P(D') = (0.7)(0.75) + (0.2)(0.65) + (0.1)(0.50) = 0.705.$$

Step 2: Condition on the person being disease-free The probability that a randomly selected person both belongs to the average-health category and has not contracted the disease is

$$P(A_2 \cap D') = P(A_2)P(D' \mid A_2) = (0.2)(0.65) = 0.13.$$

Restricting the sample space to disease-free people, Bayes' theorem gives

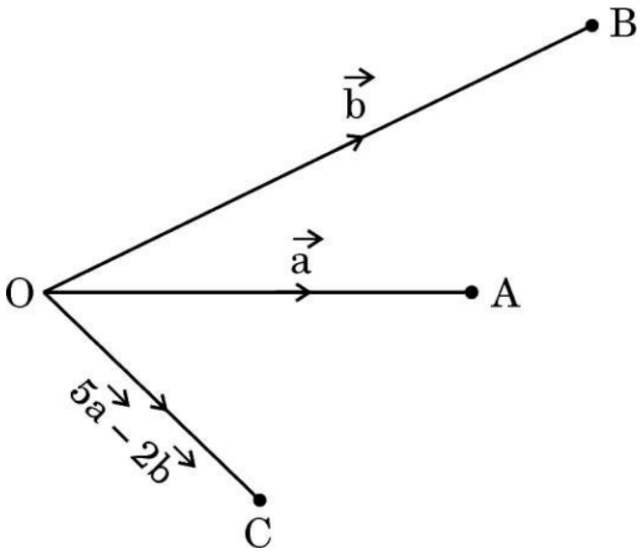
$$P(A_2 \mid D') = \frac{0.13}{0.705} = \frac{130}{705} = \frac{26}{141}.$$

Step 3: Conclusion Given that the person has not contracted the disease, the probability of belonging to category A_2 is $\boxed{\dfrac{26}{141}}$.

Quick Tip to Solve: First complement every disease rate, then divide the disease-free contribution from category A_2 by the total disease-free probability.

37. Case Study - 2: Three friends A, B and C move out from the same location O at the same time in three different directions to reach their destinations. They move out on straight paths and decide that A and B after reaching their destinations will meet up with C at his predecided destination, following straight paths from A to C and B to C in such a way that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = 5\vec{a} - 2\vec{b}$ respectively.

Based upon the above information, answer the following questions: [4 marks]



37(i). Complete the given figure to explain their entire movement plan along the respective vectors. [1 mark]

Solution:

Concept:

- A displacement vector has its tail at the starting point and its arrowhead at the destination.
- The displacement from P to Q is $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$, using the same origin O.

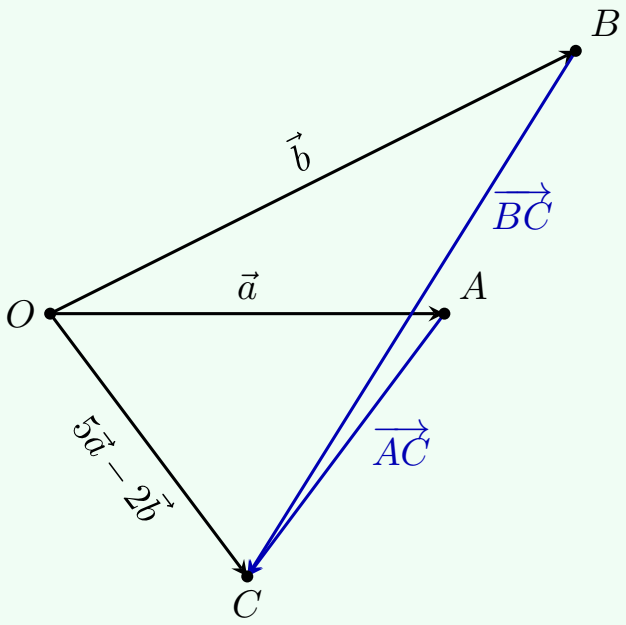
Step 1: Retain the three initial journeys Keep the given arrows from O to A, B and C, labelled $\vec{a}$, $\vec{b}$ and $5\vec{a} - 2\vec{b}$, respectively. These represent the simultaneous departures from the common location.

Step 2: Complete the journeys to the meeting point Add straight directed segments from A to C and from B to C, with both arrowheads at C. Their labels follow from subtracting the starting position vector:

$$\overrightarrow{AC} = 4\vec{a} - 2\vec{b}, \qquad \overrightarrow{BC} = 5\vec{a} - 3\vec{b}.$$

Thus the two added paths show the friends at A and B travelling to the friend waiting at C.

Quick Tip to Solve: Put both added arrowheads at the meeting point C; reversing either arrow would represent a journey away from the meeting.



Completed movement plan (schematic)

Step 3: Conclusion The completed movement plan contains the original three arrows and the added arrows $\boxed{A \longrightarrow C \text{ and } B \longrightarrow C}$, as shown.

Quick Tip to Solve: Put both added arrowheads at the meeting point C; reversing either arrow would represent a journey away from the meeting.

37(ii). Find vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$. [1 mark]

Solution:

Concept:

- With a common origin O, a displacement from one endpoint to another is the destination position vector minus the starting position vector.
- The triangle law gives $\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ and $\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$.

Step 1: Subtract the position vector of A For the journey from A to C, use $\overrightarrow{OC}=5\vec{a}-2\vec{b}$ and $\overrightarrow{OA}=\vec{a}$:

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(5\vec{a}-2\vec{b})-\vec{a}=4\vec{a}-2\vec{b}.$$

Step 2: Subtract the position vector of B For the journey from B to C, the starting position vector is $\overrightarrow{OB}=\vec{b}$. Therefore,

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=(5\vec{a}-2\vec{b})-\vec{b}=5\vec{a}-3\vec{b}.$$

In both calculations, subtraction in the reverse order would give the vector pointing away from C.

Step 3: Conclusion The required vectors are $\boxed{\overrightarrow{AC}=4\vec{a}-2\vec{b}}$ and $\boxed{\overrightarrow{BC}=5\vec{a}-3\vec{b}}$.

Quick Tip to Solve: Use destination minus starting point: subtract $\vec{a}$ for AC, but subtract $\vec{b}$ for BC.

37(iii)(a). If $\vec{a} \cdot \vec{b} = 1$, distance of O to A is 1 km and that from O to B is 2 km, then find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Also, find $|\vec{a} \times \vec{b}|$. [2 marks]

Solution:

Concept:

- The scalar product satisfies $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where $0 \le \theta \le \pi$ is the angle between the vectors.
- The cross-product magnitude is $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$; it equals the area of the parallelogram on the two vectors.

Step 1: Use the scalar product to determine the angle Since $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, the stated distances give $|\vec{a}| = 1$ and $|\vec{b}| = 2$ when distances are measured in kilometres. Substituting the given scalar product,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{1}{1 \times 2} = \frac{1}{2}.$$

The angle between two vectors is taken in $[0, \pi]$, so this determines $\theta = \pi/3$. The positive scalar product is consistent with an acute angle.

Step 2: Use the angle to calculate the cross-product magnitude Substitute the vector lengths and the angle into the cross-product formula:

$$|\vec{a} \times \vec{b}| = (1)(2)\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}.$$

As a check, $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = 4 - 1 = 3$, agreeing with the positive square root above.

Step 3: Conclusion The angle is $\boxed{60^\circ}$, and $\boxed{|\vec{a} \times \vec{b}| = \sqrt{3}}$, with area units km^2 when the vectors are measured in kilometres.

Quick Tip to Solve: Divide the dot product by the product of lengths to get the cosine; then use the same lengths with the sine for the cross-product magnitude.

OR

37(iii)(b). If $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, then find a unit vector perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$. [2 marks]

Solution:

Concept:

- For nonparallel vectors $\vec{u}$ and $\vec{v}$, the nonzero cross product $\vec{u} \times \vec{v}$ is perpendicular to both.
- A perpendicular unit vector is obtained by normalising: $\hat{n} = \dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$. Its negative gives the other permissible direction.

Step 1: Form the sum and difference of the given vectors Using $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, add

Quick Tip to Solve: The identity $(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = -2(\vec{a} \times \vec{b})$ gives a shortcut; always divide by the magnitude at the end.

and subtract corresponding components:

$$\vec{u} = \vec{a} + \vec{b} = 2\hat{i} + 3\hat{k}, \qquad \vec{v} = \vec{a} - \vec{b} = 2\hat{i} - 2\hat{j} + 5\hat{k}.$$

The cross product of these two vectors gives a common perpendicular direction.

Step 2: Calculate and normalise the cross product Expanding the determinant with the alternating signs in its first row,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 3 \\ 2 & -2 & 5 \end{vmatrix}$$
$$= 6\hat{i} - 4\hat{j} - 4\hat{k} = 2(3\hat{i} - 2\hat{j} - 2\hat{k}).$$

Its magnitude is $\sqrt{36 + 16 + 16} = 2\sqrt{17}$. Dividing gives

$$\hat{n} = \frac{3\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{17}}.$$

The numerator has dot products $6 - 6 = 0$ with $\vec{u}$ and $6 + 4 - 10 = 0$ with $\vec{v}$; its squared length is 17, confirming both perpendicularity and unit normalisation.

Step 3: Conclusion One required unit vector is $\boxed{\frac{3\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{17}}}$. Its negative is also a valid answer.

Quick Tip to Solve: The identity $(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = -2(\vec{a} \times \vec{b})$ gives a shortcut; always divide by the magnitude at the end.

38. Case Study - 3: Camphor is a waxy, colourless solid with strong aroma that evaporates through the process of sublimation, if left in the open at room temperature.

A cylindrical camphor tablet whose height is equal to its radius (r) evaporates when exposed to air such that the rate of reduction of its volume is proportional to its total surface area. Thus, $\frac{dV}{dt} = kS$ is the differential equation, where V is the volume, S is the surface area and t is the time in hours.

Based upon the above information, answer the following questions: [4 marks]

38(i). Write the order and degree of the given differential equation. [1 mark]

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.
- Its degree is the exponent of the highest-order derivative when the equation is polynomial in derivatives.

Step 1: Identify the highest derivative Write the given equation as $\frac{dV}{dt} - kS = 0$. Its only derivative is the first derivative of volume V with respect to time t; no second or higher derivative occurs.

Step 2: Inspect the derivative's power The equation is already polynomial in its derivative, and dV/dt

Quick Tip to Solve: Inspect the derivative, not powers of the radius inside the volume or surface-area formula, when finding order and degree.

occurs to the first power. The factor kS contains no derivative, so it does not alter either the order or the degree.

Step 3: Conclusion The differential equation has $\boxed{\text{order} = 1, \quad \text{degree} = 1}$.

Quick Tip to Solve: Inspect the derivative, not powers of the radius inside the volume or surface-area formula, when finding order and degree.

38(ii). Substituting $V = \pi r^3$ and $S = 2\pi r^2$, we get the differential equation $\frac{dr}{dt} = \frac{2}{3}k$. Solve it, given that $r(0) = 5$ mm. [1 mark]

Solution:

Concept:

- For a constant rate $dr/dt = a$, integration gives $r = at + C$, where C is a constant of integration.
- An initial condition fixes C and selects the required particular solution.

Step 1: Integrate the supplied differential equation Using the differential equation stated in the question, separate the differentials and integrate with k constant:

$$dr = \frac{2k}{3}\,dt, \qquad r = \frac{2k}{3}t + C.$$

Step 2: Apply the initial radius At $t = 0$, the radius is 5 mm, so $5 = 0 + C$ and $C = 5$. Differentiating the resulting expression returns $dr/dt = 2k/3$, and substituting zero returns the given initial radius.

Step 3: Conclusion The required solution is $\boxed{r(t) = 5 + \frac{2k}{3}t}$, with r in millimetres and t in hours.

Quick Tip to Solve: A constant derivative gives a straight-line expression in time; substitute $t = 0$ immediately to determine its constant term.

38(iii)(a). If it is given that $r = 3$ mm when $t = 1$ hour, find the value of k. Hence, find t for $r = 0$ mm. [2 marks]

Solution:

Concept:

- The supplied equation $dr/dt = 2k/3$, with $r(0) = 5$, gives $r(t) = 5 + (2k/3)t$, where radius is in millimetres and time is in hours.
- A further radius-time measurement determines k. Complete evaporation corresponds to $r = 0$, since $V = \pi r^3$.

Step 1: Determine the constant from the later radius Substitute $r = 3$ and $t = 1$ into the particular

Quick Tip to Solve: The radius drops by 2 mm in one hour, so its constant slope is -2; equate that slope to $2k/3$ before finding the zero-radius time.

solution from part (ii):

$$3 = 5 + \frac{2k}{3}(1),$$
$$-2 = \frac{2k}{3},$$
$$k = -3\,\text{mm h}^{-1}.$$

The negative sign agrees with a decreasing radius. In the printed convention $dV/dt = kS$, reduction is represented by a negative value of k.

Step 2: Find when the radius becomes zero With this value of k, the radius equation is $r = 5 - 2t$. Set the radius equal to zero:

$$0 = 5 - 2t \quad \Longrightarrow \quad t = \frac{5}{2}\,\text{h}.$$

This is the elapsed time measured from $t = 0$. The physical model is used only until this time; a negative radius afterwards has no meaning.

Step 3: Conclusion $\boxed{k = -3\,\text{mm h}^{-1}}$, and the tablet reaches zero radius at $\boxed{t = 2.5\,\text{h}}$, or 2 hours 30 minutes after the initial instant.

Quick Tip to Solve: The radius drops by 2 mm in one hour, so its constant slope is -2; equate that slope to $2k/3$ before finding the zero-radius time.

OR

38(iii)(b). If it is given that $r = 1$ mm when $t = 1$ hour, find the value of k. Hence, find t for $r = 0$ mm. [2 marks]

Solution:

Concept:

- Integration of the supplied equation $dr/dt = 2k/3$, using $r(0) = 5$, gives $r(t) = 5 + (2k/3)t$, where radius is in millimetres and time is in hours.
- This alternative supplies a new radius-time measurement to determine k. Vanishing volume $V = \pi r^3$ occurs at zero radius.

Step 1: Use the measurement for this alternative Here the radius is 1 mm at $t = 1$ h. Insert these values into the solution from part (ii):

$$1 = 5 + \frac{2k}{3}(1),$$
$$-4 = \frac{2k}{3},$$
$$k = -6\,\text{mm h}^{-1}.$$

The negative constant is consistent with evaporation: both the radius and volume decrease as time increases under the printed equation $dV/dt = kS$.

Step 2: Calculate the zero-radius time Substitution of this constant gives $r = 5 - 4t$. At complete

Quick Tip to Solve: Recompute the slope for this alternative: $(1 - 5)/(1 - 0) = -4\,\text{mm h}^{-1}$. Do not carry over the constant from the other alternative.

evaporation,

$$0 = 5 - 4t \quad \Longrightarrow \quad t = \frac{5}{4}\,\text{h}.$$

This is the total time from the initial instant, rather than the additional time after the measurement. The radius remains nonnegative for $0 \le t \le 5/4$, the physical interval of this solution.

Step 3: Conclusion $\boxed{k = -6\,\text{mm h}^{-1}}$, and the radius becomes zero at $\boxed{t = 1.25\,\text{h}}$, or 1 hour 15 minutes after the initial instant.

Quick Tip to Solve: Recompute the slope for this alternative: $(1-5)/(1-0) = -4\,\text{mm h}^{-1}$. Do not carry over the constant from the other alternative.