

CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-7-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

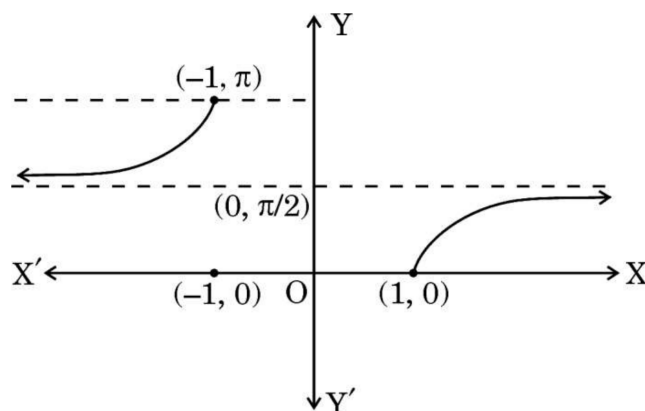
- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The given graph illustrates:

[1 mark]



- (A) $y = \sec^{-1} x$
- (B) $y = \cot^{-1} x$
- (C) $y = \tan^{-1} x$
- (D) $y = \operatorname{cosec}^{-1} x$

Correct Answer: (A) $y = \sec^{-1} x$

Solution:

Concept:

- The principal inverse secant has domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi] \setminus \{\pi/2\}$.
- Identify an inverse trigonometric graph by its domain, endpoint values and horizontal asymptote.

Step 1: Read the endpoints and branches The graph has no points for $-1 < x < 1$. Its right branch begins at $(1, 0)$, and its left branch ends at $(-1, \pi)$. Both branches approach the horizontal line $y = \pi/2$. These agree with $\sec^{-1}(1) = 0$ and $\sec^{-1}(-1) = \pi$.

Step 2: Check the other inverse functions Both $\cot^{-1} x$ and $\tan^{-1} x$ are defined for every real x , so neither can have the displayed gap. The inverse cosecant has endpoint values $\operatorname{cosec}^{-1}(1) = \pi/2$ and $\operatorname{cosec}^{-1}(-1) = -\pi/2$, which disagree with the graph.

Step 3: Conclusion The graph represents $y = \sec^{-1} x$. Hence, the correct option is (A).

Quick Tip to Solve: Check the points at $x = 1$ and $x = -1$: inverse secant gives $0, \pi$, while inverse cosecant gives $\pi/2, -\pi/2$.

2. Let A be a square matrix of order 3. If $|A| = 5$, then $|\text{adj } A|$ is:

[1 mark]

- (A) 5
- (B) 125
- (C) 25
- (D) -5

Correct Answer: (C) 25

Solution:

Concept:

- For a square matrix A of order n , the adjoint identity is $A \text{adj } A = |A|I_n$, where I_n is the identity matrix.
- Taking determinants gives $|A| |\text{adj } A| = |A|^n$; when $|A| \neq 0$, this implies $|\text{adj } A| = |A|^{n-1}$.

Step 1: Substitute the order and determinant The matrix has order $n = 3$ and its determinant is 5, which is nonzero. Therefore the exponent in the adjoint formula is $3 - 1 = 2$:

$$|\text{adj } A| = 5^{3-1} = 5^2 = 25.$$

Step 2: Check the printed options Option (A), 5, merely repeats the determinant of A . Option (B), 125, uses the exponent 3 instead of 2. Option (D), -5, introduces an unjustified negative sign. Only option (C) equals the computed value.

Step 3: Conclusion $|\text{adj } A| = 25$. Hence, the correct option is (C).

Quick Tip to Solve: For the adjoint of an order-three matrix, square the given determinant; do not cube it.

3. If A and B are two square matrices each of order 3 with $|A| = 3$ and $|B| = 5$, then $|2AB|$ is: [1 mark]

- (A) 30
- (B) 120
- (C) 15
- (D) 225

Correct Answer: (B) 120

Solution:

Concept:

- The determinant of a product of square matrices of the same order is $|AB| = |A| |B|$.
- For a matrix M of order n and a scalar k , $|kM| = k^n |M|$, since the scalar multiplies all n rows.

Step 1: Evaluate the determinant of the product Both matrices have order 3, so their product also has order 3. Using the given determinants,

$$|AB| = |A| |B| = 3 \times 5 = 15.$$

Quick Tip to Solve: Read the matrix order before moving a scalar outside a determinant: order three makes 2 become 2^3 .

Step 2: Apply the scalar factor to all rows Multiplication of AB by 2 multiplies each of its three rows by 2. Hence,

$$|2AB| = 2^3|AB| = 8 \times 15 = 120.$$

Option (A), 30, uses only one factor of 2; option (C), 15, ignores the scalar entirely. Option (D), 225, squares $|AB|$, which is not the required operation. Option (B) alone agrees with the calculation.

Step 3: Conclusion $|2AB| = 120$. Hence, the correct option is (B).

Quick Tip to Solve: Read the matrix order before moving a scalar outside a determinant: order three makes 2 become 2^3 .

4. What is the total number of possible matrices of order 3×3 with each entry as $\sqrt{2}$ or $\sqrt{3}$? [1 mark]

- (A) 9
- (B) 512
- (C) 615
- (D) 64

Correct Answer: (B) 512

Solution:

Concept:

- A matrix of order $m \times n$ has mn distinct entry positions, with a fixed row and column for each entry.
- By the multiplication principle, if each of r positions has k independent choices, the number of arrangements is k^r .

Step 1: Count the positions and choices A 3×3 matrix has $3 \times 3 = 9$ positions. At each position we may choose either $\sqrt{2}$ or $\sqrt{3}$, giving two distinct choices. Repetition is allowed, and choosing an entry does not restrict any other position.

Step 2: Multiply the independent choices The nine positions therefore give

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9 = 512.$$

Option (A), 9, counts positions instead of matrices. Option (D), $64 = 2^6$, accounts for only six independent positions. Option (C), 615, is not the product of the nine binary choices. Option (B) matches the count.

Step 3: Conclusion The total number of possible matrices is 512. Hence, the correct option is (B).

Quick Tip to Solve: For two allowed values in a 3×3 matrix, first count all nine positions, then use 2^9 .

5. Domain of $f(x) = \cos^{-1} x + \sin x$ is:

[1 mark]

- (A) $\mathbb{R}$
- (B) $(-1, 1)$
- (C) $[-1, 1]$
- (D) φ

Correct Answer: (C) $[-1, 1]$

Solution:

Concept:

- For a real-valued inverse cosine, its argument must satisfy $-1 \leq x \leq 1$.
- A sum of two functions is defined on the intersection of their domains; $\sin x$ is defined for all real x .

Step 1: Apply both domain conditions The term $\cos^{-1} x$ requires $x \in [-1, 1]$, whereas $\sin x$ imposes no further restriction. Both terms exist precisely when

$$x \in [-1, 1] \cap \mathbb{R} = [-1, 1].$$

Step 2: Check the endpoints and alternatives At $x = 1$, $\cos^{-1}(1) = 0$, and at $x = -1$, $\cos^{-1}(-1) = \pi$; the sine term is also defined at both endpoints. Thus the interval is closed, ruling out (B). Values outside this interval are excluded, ruling out (A), while $x = 0$ is allowed, ruling out the empty set in (D).

Step 3: Conclusion The domain is $[-1, 1]$. Hence, the correct option is (C).

Quick Tip to Solve: For a sum involving an inverse trigonometric term, first restrict its argument and then intersect with the other term's domain.

6. The matrix $A = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$ is a/an:

[1 mark]

- (A) scalar matrix
- (B) identity matrix
- (C) null matrix
- (D) symmetric matrix

Correct Answer: (D) symmetric matrix

Solution:

Concept:

- A real square matrix is symmetric if $A^T = A$, where A^T is its transpose.
- A scalar matrix is diagonal with all diagonal entries equal; the identity has all diagonal entries equal to 1, and the null matrix has every entry zero.

Step 1: Compare the matrix with its transpose All off-diagonal entries are zero. Transposition leaves

Quick Tip to Solve: Any real diagonal matrix equals its transpose; check equality of its diagonal entries separately before calling it scalar.

each diagonal entry in its original position and interchanges only equal zero entries. Therefore

$$A^T = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix} = A.$$

Step 2: Eliminate the remaining classifications The three diagonal entries $\sqrt{3}, \sqrt{2}, \sqrt{5}$ are distinct, so this is not a scalar matrix. They are not all 1, so it is not the identity matrix, and their nonzero values rule out the null matrix. A diagonal matrix need not be scalar, but it is always symmetric.

Step 3: Conclusion The given matrix is symmetric. Hence, the correct option is (D).

Quick Tip to Solve: Any real diagonal matrix equals its transpose; check equality of its diagonal entries separately before calling it scalar.

7. If $f(x) = -2x^8$, then the correct statement is:

[1 mark]

- (A) $f'\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$
(B) $f'\left(\frac{1}{2}\right) = -f'\left(-\frac{1}{2}\right)$
(C) $-f'\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$
(D) $f\left(\frac{1}{2}\right) = -f\left(-\frac{1}{2}\right)$

Correct Answer: (B) $f'\left(\frac{1}{2}\right) = -f'\left(-\frac{1}{2}\right)$

Solution:

Concept:

- The power rule gives $\frac{d}{dx}(x^n) = nx^{n-1}$. Thus differentiation reduces the exponent by one.
- An even power gives $f(-x) = f(x)$; an odd power changes sign under $x \mapsto -x$.

Step 1: Differentiate and substitute For $f(x) = -2x^8$, the derivative is $f'(x) = -16x^7$. Therefore

$$f'(1/2) = -\frac{1}{8}, \quad f'(-1/2) = \frac{1}{8}.$$

These derivatives are opposite and nonzero, so option (A), which equates them, is false.

Step 2: Compare the function values too Since the original exponent is even,

$$f(1/2) = f(-1/2) = -\frac{1}{128}.$$

Option (C) would equate $1/8$ to $-1/128$, which is false. Option (D) would equate $-1/128$ to $1/128$, also false. The opposite signs of the derivative values give precisely the relation in (B).

Quick Tip to Solve: Recognize that $-2x^8$ is even and its derivative $-16x^7$ is odd; use this parity to compare values at opposite inputs.

Step 3: Conclusion $f'(1/2) = -f'(-1/2)$. Hence, the correct option is (B).

Quick Tip to Solve: Recognize that $-2x^8$ is even and its derivative $-16x^7$ is odd; use this parity to compare values at opposite inputs.

8. If $f(x) = \begin{cases} 3ax - b, & x > 1 \\ 11, & x = 1 \\ -5ax - 2b, & x < 1 \end{cases}$ is continuous at $x = 1$, then the values of a and b are: [1 mark]

(A) $a = 3, b = 5$

(B) $a = 8, b = -1$

(C) $a = 1, b = -8$

(D) $a = -3, b = 5$

Correct Answer: (C) $a = 1, b = -8$

Solution:

Concept:

- Continuity at $x = 1$ requires $\lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$.
- For each linear branch, its one-sided limit is obtained by substituting $x = 1$ in that branch.

Step 1: Equate both limits to the given value The right-hand branch gives $3a - b$, while the left-hand branch gives $-5a - 2b$. Since the value at the junction is 11,

$$3a - b = 11, \quad -5a - 2b = 11.$$

Merely equating the two limits is insufficient: both must also equal $f(1)$.

Step 2: Solve and check the parameters From the first equation, $b = 3a - 11$. Substitution in the second gives

$$-5a - 2(3a - 11) = 11 \implies -11a = -11.$$

This gives $a = 1$ and $b = -8$; indeed, both limits are 11. Options (A), (B) and (D) give right-hand limits 4, 25 and -14 , respectively, and therefore fail continuity.

Step 3: Conclusion The required values are $a = 1, b = -8$. Hence, the correct option is (C).

Quick Tip to Solve: At the junction, equate each branch separately to 11; equal one-sided limits alone do not guarantee continuity.

9. If $\begin{bmatrix} 2x-1 & 3x \\ 0 & y^2-1 \end{bmatrix} = \begin{bmatrix} x+3 & 12 \\ 0 & 35 \end{bmatrix}$, then the value of $(x-y)$ is:

[1 mark]

- (A) 2 or 10
- (B) -2 or 10
- (C) 2 or -10
- (D) -2 or -10

Correct Answer: (B) -2 or 10

Solution:

Concept:

- Two matrices of the same order are equal if and only if every pair of corresponding entries is equal.
- An equation $y^2 = k$, with $k > 0$, has both real roots $y = \sqrt{k}$ and $y = -\sqrt{k}$.

Step 1: Compare corresponding matrix entries The top-left and top-right entries give

$$2x - 1 = x + 3 \implies x = 4, \quad 3x = 12 \implies x = 4.$$

These conditions agree. The bottom-left entries are both zero, so they impose no additional condition. The brackets denote matrices, so we compare entries rather than only their determinants.

Step 2: Retain both roots for the lower entry Equality of the bottom-right entries requires

$$y^2 - 1 = 35 \implies y^2 = 36 \implies y = 6 \text{ or } -6.$$

For the positive root, $x - y = 4 - 6 = -2$; for the negative root, $x - y = 4 - (-6) = 10$. Options (A) and (C) incorrectly include 2, while (D) incorrectly gives -10 for the second value.

Step 3: Conclusion Thus, $x - y = -2 \text{ or } 10$. Hence, the correct option is (B).

Quick Tip to Solve: Compare the easy top-right entries first to get x, and keep both signs when solving the squared y-entry.

10. Edge of a variable cube increases at the rate of 5 cm/s. The rate at which the surface area of the cube increases when the edge is 2 cm long is:

[1 mark]

- (A) $24 \text{ cm}^2/\text{s}$
- (B) $120 \text{ cm}^2/\text{s}$
- (C) $12 \text{ cm}^2/\text{s}$
- (D) $5 \text{ cm}^2/\text{s}$

Correct Answer: (B) $120 \text{ cm}^2/\text{s}$

Solution:

Concept:

- A cube with edge a has total surface area $S = 6a^2$, because its six faces are squares.

Quick Tip to Solve: Differentiate the total area $6a^2$ before putting $a=2$; then multiply by the given edge rate 5.

- When a varies with time t , the chain rule gives $\frac{dS}{dt} = \frac{dS}{da} \frac{da}{dt}$.

Step 1: Differentiate the surface-area formula Let a be measured in centimetres and t in seconds. Differentiating with respect to time, we obtain

$$\frac{dS}{dt} = 12a \frac{da}{dt}.$$

The factor da/dt is essential because the question asks for the rate of area change with time, not with edge length.

Step 2: Substitute the instantaneous edge and its rate At the stated instant, $a = 2$ cm and $da/dt = 5$ cm/s. Hence,

$$\frac{dS}{dt} = 12 \times 2 \times 5 = 120 \text{ cm}^2/\text{s}.$$

Option (A), 24, omits the time-rate factor; (C), 12, is not the surface-area rate; (D), 5, is the edge rate and has the wrong physical meaning.

Step 3: Conclusion The surface area increases at $\boxed{120 \text{ cm}^2/\text{s}}$. Hence, the correct option is (B).

Quick Tip to Solve: Differentiate the total area $6a$ squared before putting $a=2$; then multiply by the given edge rate 5.

11. $\int \frac{e^{9 \log x} - e^{8 \log x}}{e^{6 \log x} - e^{5 \log x}} dx$ is equal to:

[1 mark]

- (A) $x + C$
 (B) $\frac{x^2}{2} + C$
 (C) $\frac{x^4}{4} + C$
 (D) $\frac{x^3}{3} + C$

Correct Answer: (C) $\frac{x^4}{4} + C$

Solution:

Concept:

- For $x > 0$, the exponential-logarithm identity is $e^{n \log x} = x^n$, where $\log$ is the natural logarithm.
- The power rule gives $\int x^n dx = x^{n+1}/(n+1) + C$ for $n \neq -1$; C is an arbitrary constant.

Step 1: Simplify the integrand by factoring Replace each exponential by the corresponding power of x , then factor:

$$\frac{e^{9 \log x} - e^{8 \log x}}{e^{6 \log x} - e^{5 \log x}} = \frac{x^9 - x^8}{x^6 - x^5} = \frac{x^8(x-1)}{x^5(x-1)} = x^3.$$

The original expression requires $x > 0$ and $x \neq 1$. Cancellation is valid on each interval of this domain.

Quick Tip to Solve: Convert every exponential into a power of x first, cancel the common factor $x-1$, and only then integrate.

Step 2: Integrate and test the options Applying the power rule on either domain interval,

$$\int x^3 dx = \frac{x^4}{4} + C.$$

Differentiating options (A), (B), (C) and (D) gives 1, x , x^3 and x^2 , respectively. Thus only (C) differentiates to the simplified integrand throughout an interval.

Step 3: Conclusion The integral is $\frac{x^4}{4} + C$, on the intervals where the original integrand is defined. Hence, the correct option is (C).

Quick Tip to Solve: Convert every exponential into a power of x first, cancel the common factor $x-1$, and only then integrate.

12. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = 2x - \sin x$, then f is:

[1 mark]

- (A) a decreasing function
- (B) an increasing function
- (C) maximum at $x = \frac{\pi}{2}$
- (D) maximum at $x = 0$

Correct Answer: (B) an increasing function

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- For every real x , $-1 \leq \cos x \leq 1$; this bounds expressions involving the cosine function.

Step 1: Differentiate the given function Since $f(x) = 2x - \sin x$, differentiation gives

$$f'(x) = 2 - \cos x.$$

The function is differentiable for every real number, so the derivative test applies on the entire real line.

Step 2: Determine the sign everywhere Using the full range of cosine,

$$1 \leq 2 - \cos x \leq 3.$$

Thus $f'(x)$ is strictly positive for every real x , which excludes a decreasing function. At $x = \pi/2$, the derivative is 2, and at $x = 0$, it is 1; neither point can be a local maximum. Since the function keeps increasing, neither is a global maximum either. Hence (A), (C) and (D) fail.

Step 3: Conclusion The function is strictly increasing on $\mathbb{R}$. Hence, the correct option is (B).

Quick Tip to Solve: Bound 2 minus cosine between 1 and 3; this settles its sign everywhere without solving any trigonometric equation.

13. A student tries to tie ropes, parallel to each other, from one end of the wall to the other. If one rope is along the vector $3\hat{i} + 15\hat{j} + 6\hat{k}$ and the other is along the vector $2\hat{i} + 10\hat{j} + \lambda\hat{k}$, then the value of λ is: [1 mark]

- (A) 6
- (B) 1
- (C) $\frac{1}{4}$
- (D) 4

Correct Answer: (D) 4

Solution:

Concept:

- Two non-zero vectors are parallel when one is a scalar multiple of the other.
- Their corresponding components therefore have the same ratio; the scalar must work simultaneously for all three components.

Step 1: Find the common multiplier Write the second rope's vector as k times the first. Comparing the $\hat{i}$ and $\hat{j}$ components gives

$$2 = 3k \implies k = \frac{2}{3}, \quad 10 = 15k.$$

The second equation agrees with $k = 2/3$, confirming that the first two components are consistent with parallel directions.

Step 2: Match the remaining component The same multiplier must also relate the $\hat{k}$ components:

$$\lambda = 6k = 6 \times \frac{2}{3} = 4.$$

Substitution gives $(2, 10, 4) = \frac{2}{3}(3, 15, 6)$. In contrast, options (A), (B) and (C) would give third-component ratios 1, $1/6$ and $1/24$, respectively, none equal to $2/3$.

Step 3: Conclusion The required value is $\lambda = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Use the known component ratio $2/3$ once, then multiply the first vector's third component 6 by that same ratio.

14. $\int \frac{e^{-x}}{16 + 9e^{-2x}} dx$ is equal to:

[1 mark]

(A) $\frac{16}{9} \tan^{-1}(e^{-x}) + C$

(B) $-\frac{1}{12} \tan^{-1}\left(\frac{3e^{-x}}{4}\right) + C$

(C) $\tan^{-1}\left(\frac{e^{-x}}{4}\right) + C$

(D) $-\frac{1}{3} \tan^{-1}\left(\frac{e^{-x}}{4}\right) + C$

Correct Answer: (B) $-\frac{1}{12} \tan^{-1}\left(\frac{3e^{-x}}{4}\right) + C$

Solution:

Concept:

- Substitution $u = e^{-x}$ gives $du = -e^{-x} dx$, so the negative sign must be retained.
- The standard formula is $\int \frac{dv}{1+v^2} = \tan^{-1} v + C$, where C is an arbitrary constant.

Step 1: Substitute the exponential expression Put $u = e^{-x}$. Then the integral becomes

$$-\int \frac{du}{16 + 9u^2} = -\frac{1}{16} \int \frac{du}{1 + (3u/4)^2}.$$

Now take $v = 3u/4$, so $du = (4/3) dv$. This rescales the quadratic denominator to the standard inverse-tangent form.

Step 2: Integrate and check the sign and scale Thus,

$$-\frac{1}{12} \int \frac{dv}{1 + v^2} = -\frac{1}{12} \tan^{-1}\left(\frac{3e^{-x}}{4}\right) + C.$$

Differentiating this expression gives

$$-\frac{1}{12} \frac{-(3/4)e^{-x}}{1 + (9/16)e^{-2x}} = \frac{e^{-x}}{16 + 9e^{-2x}}.$$

Options (A) and (C) have negative derivatives, whereas the integrand is positive. Option (D) differentiates to $\frac{4e^{-x}}{3(16 + e^{-2x})}$, whose scale and denominator are incorrect.

Step 3: Conclusion The integral is $\boxed{-\frac{1}{12} \tan^{-1}\left(\frac{3e^{-x}}{4}\right) + C}$. Hence, the correct option is (B).

Quick Tip to Solve: With u equal to e to the power minus x , keep the minus from du and scale u by $3/4$ before applying the inverse-tangent formula.

15. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ for any two vectors, then vectors $\vec{a}$ and $\vec{b}$ are:

[1 mark]

- (A) orthogonal vectors
- (B) parallel to each other
- (C) unit vectors
- (D) collinear vectors

Correct Answer: (A) orthogonal vectors

Solution:

Concept:

- The square of a vector's magnitude is its scalar product with itself: $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Orthogonal vectors have zero scalar product; for nonzero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$, where θ is their angle.

Step 1: Square and expand both magnitudes Squaring the given equality gives

$$|\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2.$$

The squared magnitudes cancel, leaving $4\vec{a} \cdot \vec{b} = 0$.

Step 2: Interpret the zero scalar product Thus $\vec{a} \cdot \vec{b} = 0$, which is the condition for orthogonality. For nonzero vectors their angle is 90° . Parallel or collinear nonzero vectors have a nonzero scalar product, so (B) and (D) do not follow. No condition fixes either magnitude at one, so (C) does not follow either.

Step 3: Conclusion The vectors are orthogonal. Hence, the correct option is (A).

Quick Tip to Solve: When the magnitudes of a vector sum and difference are equal, square them and cancel the two squared magnitudes to isolate the scalar product.

16. A coin is tossed and a card is selected at random from a well shuffled pack of 52 playing cards. The probability of getting head on the coin and a face card from the pack is:

[1 mark]

- (A) $\frac{2}{13}$
- (B) $\frac{3}{26}$
- (C) $\frac{19}{26}$
- (D) $\frac{3}{13}$

Correct Answer: (B) $\frac{3}{26}$

Solution:

Concept:

- For independent events H and F , $P(H \cap F) = P(H)P(F)$.
- A standard deck contains three face cards (jack, queen and king) in each of four suits, giving 12

Quick Tip to Solve: Count face cards as $3 \times 4 = 12$, then halve their deck probability because a head is also required.

face cards among 52 cards.

Step 1: Find the two individual probabilities Let H denote a head and F a face card. The coin has two equally likely outcomes, so $P(H) = 1/2$. The card draw gives $P(F) = 12/52 = 3/13$; aces are not face cards.

Step 2: Multiply for the joint event The coin result does not affect the card selection, so the events are independent. Therefore

$$P(H \cap F) = \frac{1}{2} \times \frac{12}{52} = \frac{3}{26}.$$

Option (D) counts only the card condition and omits the head. Option (A) would use an incorrect face-card count. Option (C) is too large for an event that requires a head, whose probability is only $1/2$.

Step 3: Conclusion The required probability is $\frac{3}{26}$. Hence, the correct option is (B).

Quick Tip to Solve: Count face cards as $3 \times 4 = 12$, then halve their deck probability because a head is also required.

17. If A and B are two events such that $P(B) = \frac{1}{5}$, $P(A | B) = \frac{2}{3}$ and $P(A \cup B) = \frac{3}{5}$, then $P(A)$ is: [1 mark]

- (A) $\frac{10}{15}$
- (B) $\frac{2}{15}$
- (C) $\frac{1}{5}$
- (D) $\frac{8}{15}$

Correct Answer: (D) $\frac{8}{15}$

Solution:

Concept:

- Conditional probability satisfies $P(A | B) = P(A \cap B) / P(B)$ when $P(B) > 0$.
- The addition theorem is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; the overlap is subtracted to avoid counting it twice.

Step 1: Recover the probability of the intersection Multiply the conditional probability by $P(B)$:

$$P(A \cap B) = P(A | B)P(B) = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}.$$

This is the overlap of the events, not the probability of A alone.

Step 2: Rearrange the addition theorem Substituting the known union and intersection gives

$$P(A) = \frac{3}{5} - \frac{1}{5} + \frac{2}{15} = \frac{9 - 3 + 2}{15} = \frac{8}{15}.$$

Quick Tip to Solve: First turn a conditional probability into an intersection by multiplying by the probability of the conditioning event; then use the union formula.

Option (B) is only the intersection, option (C) repeats $P(B)$, and option (A) repeats the conditional probability $2/3$. None of those quantities equals $P(A)$ here.

Step 3: Conclusion The required probability is $\frac{8}{15}$. Hence, the correct option is (D).

Quick Tip to Solve: First turn a conditional probability into an intersection by multiplying by the probability of the conditioning event; then use the union formula.

18. For a function $f(x)$, which of the following holds true?

[1 mark]

- (A) $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$
 (B) $\int_{-a}^a f(x) dx = 0$, if f is an even function
 (C) $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$, if f is an odd function
 (D) $\int_0^{2a} f(x) dx = \int_0^a f(x) dx - \int_0^a f(2a+x) dx$

Correct Answer: (A) $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

Solution:

Concept:

- A substitution changes both the integrand and the limits; reversing integration limits changes the sign of the integral.
- On symmetric limits, an integrable odd function has zero integral, while an even function satisfies $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

Step 1: Verify the reflection identity In the right-hand integral of (A), put $t = a + b - x$. Then $dx = -dt$, and the limits $x = a, b$ become $t = b, a$, respectively. Hence

$$\int_a^b f(a+b-x) dx = - \int_b^a f(t) dt = \int_a^b f(t) dt.$$

The dummy integration variable can be renamed x , proving (A).

Step 2: Check the remaining identities Option (B) incorrectly assigns the odd-function result to an even function. For example, $f(x) = 1$ gives $2a$, not zero. Option (C) incorrectly assigns the even-function rule to an odd function: $f(x) = x$ gives zero on the left but a^2 on the right. In (D), taking $f(x) = 1$ gives $2a$ on the left and zero on the right for $a > 0$, so it is not a general identity either.

Step 3: Conclusion The valid general property is $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$. Hence, the correct option is (A).

Quick Tip to Solve: To verify the reflection property, use $t = a + b - x$; the minus sign from dx cancels the reversal of the limits.

19. Assertion (A): $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$.

Reason (R): When $x \rightarrow 0$, $\sin \frac{1}{x}$ is a finite value between -1 and 1 .

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- A function is continuous at zero if $\lim_{x \rightarrow 0} f(x) = f(0)$.
- Since $|\sin t| \leq 1$ for real t , multiplying a bounded sine factor by a factor tending to zero permits the squeeze theorem.

Step 1: Evaluate the Assertion For $x \neq 0$,

$$0 \leq \left| x \sin \frac{1}{x} \right| \leq |x|.$$

As $x \rightarrow 0$ from either side, $|x| \rightarrow 0$. The squeeze theorem gives $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$, so the Assertion is true.

Step 2: Evaluate the Reason For every nonzero real x , $-1 \leq \sin(1/x) \leq 1$. Thus the Reason correctly states that the sine factor stays finite and bounded near zero. It does not say that this oscillating factor has a unique limit; in fact that limit does not exist.

Step 3: Check the explanatory link The boundedness stated in the Reason gives the inequality used above. Multiplication by x , whose magnitude tends to zero, forces the product to tend to zero. Hence the Reason supplies the relevant explanation of continuity, ruling out (B); neither statement is false, ruling out (C) and (D).

Step 4: Conclusion Both statements are true and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: For $x \sin(1/x)$, bound the absolute value by $|x|$; do not attempt to assign a limit to $\sin(1/x)$ itself.

20. Assertion (A): Set of values of $\sec^{-1} \left(\frac{\sqrt{3}}{2} \right)$ is a null set.

Reason (R): $\sec^{-1} x$ is defined for $x \in \mathbb{R} - (-1, 1)$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The real inverse secant $\sec^{-1} t$ is defined only for $t \in (-\infty, -1] \cup [1, \infty)$.
- A real angle has $-1 \leq \cos \theta \leq 1$, so any defined value of $\sec \theta = 1/\cos \theta$ has magnitude at least one.

Step 1: Evaluate the Assertion The printed argument satisfies $0 < \sqrt{3}/2 < 1$. If a real angle θ satisfied $\sec \theta = \sqrt{3}/2$, it would have $\cos \theta = 2/\sqrt{3} > 1$, which is impossible. Thus no real value exists, and the stated set is empty. The Assertion is true.

Step 2: Evaluate the Reason The domain of the real inverse secant is exactly $\mathbb{R} - (-1, 1)$, with the endpoints -1 and 1 included. Therefore the Reason is true.

Step 3: Check the explanatory link The domain restriction in the Reason excludes $\sqrt{3}/2$ directly, because this number lies strictly between zero and one. It therefore explains why the set in the Assertion is empty. Option (B) denies this direct link, while (C) and (D) incorrectly declare one of the true statements false.

Step 4: Conclusion Both statements are true and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Before evaluating a real inverse secant, check that the argument has absolute value at least one; $\sqrt{3}/2$ fails this test.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). Differentiate $\left(\frac{5^x}{x^5}\right)$ with respect to x .

[2 marks]

Solution:

Concept:

- For differentiable functions u and v , with $v \neq 0$, the quotient rule gives $\frac{d}{dx}(u/v) = (vu' - uv')/v^2$.
- The exponential and power rules are $\frac{d}{dx}(a^x) = a^x \ln a$ for $a > 0$, and $\frac{d}{dx}(x^n) = nx^{n-1}$. Here $\ln$ is the natural logarithm.

Step 1: Differentiate the numerator and denominator Put $y = 5^x/x^5$. Its real domain excludes $x = 0$, since the denominator must be nonzero. Take $u = 5^x$ and $v = x^5$. Applying the exponential rule to u and the power rule to v , we obtain

$$u' = 5^x \ln 5, \quad v' = 5x^4.$$

The factor $\ln 5$ is necessary because the base of the exponential is 5, rather than e .

Step 2: Apply and simplify the quotient rule Substitute these derivatives while keeping the order $vu' - uv'$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^5(5^x \ln 5) - 5^x(5x^4)}{x^{10}} \\ &= \frac{5^x x^4(x \ln 5 - 5)}{x^{10}} \\ &= \frac{5^x(x \ln 5 - 5)}{x^6}, \quad x \neq 0. \end{aligned}$$

Factoring $5^x x^4$ from the numerator makes cancellation with the denominator immediate and leaves the domain restriction unchanged.

Step 3: Conclusion

$$\boxed{\frac{d}{dx} \left(\frac{5^x}{x^5} \right) = \frac{5^x(x \ln 5 - 5)}{x^6}, \quad x \neq 0.}$$

Quick Tip to Solve: Rewrite the expression as $5^x x^{-5}$ to check the quotient-rule result quickly with the product rule; retain $\ln 5$ when differentiating 5^x .

21(b). If $-2x^2 - 5xy + y^3 = 76$, then find $\frac{dy}{dx}$.

[2 marks]

Solution:

Concept:

- In an implicit relation, regard y as a differentiable function of x and differentiate both sides with respect to x .
- Apply the product rule to xy : $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$, and the chain rule to y^3 : $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$.

Step 1: Differentiate the given relation The right-hand side is a constant, so its derivative is zero. Differentiating each term of $-2x^2 - 5xy + y^3 = 76$ gives

$$-4x - 5\left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0.$$

The derivative of the product xy contains both terms; y cannot be treated as a constant.

Step 2: Collect the derivative terms Expand the product and move the terms without $\frac{dy}{dx}$ to the other side:

$$\begin{aligned} -5x\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 4x + 5y, \\ (3y^2 - 5x)\frac{dy}{dx} &= 4x + 5y. \end{aligned}$$

Now divide by $3y^2 - 5x$, wherever this denominator is nonzero.

Step 3: Conclusion $\frac{dy}{dx} = \frac{4x + 5y}{3y^2 - 5x}$, provided $3y^2 - 5x \neq 0$.

Quick Tip to Solve: Differentiate xy as $x \frac{dy}{dx} + y$; then group all $\frac{dy}{dx}$ terms before dividing.

22. If $A = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$, then find the value of K if $A^2 = 6A + KI_2$, where I_2 is an identity matrix.

[2 marks]

Solution:

Concept:

- The square of a matrix means the matrix product $A^2 = A \cdot A$, computed by multiplying rows by columns.
- Since $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix KI_2 has K on both diagonal entries and zero elsewhere. Equal matrices have equal corresponding entries.

Step 1: Calculate the square of the matrix Multiply the given matrix by itself, retaining the order of

Quick Tip to Solve: Compute $A^2 - 6A$; a correct result must have equal diagonal entries and zero off-diagonal entries.

its rows and columns:

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + 0(-1) & 1 \cdot 0 + 0 \cdot 5 \\ (-1) \cdot 1 + 5(-1) & (-1) \cdot 0 + 5 \cdot 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -6 & 25 \end{pmatrix}. \end{aligned}$$

Step 2: Compare with the required identity Subtract $6A$ from both sides of the given equation:

$$\begin{aligned} KI_2 &= A^2 - 6A \\ &= \begin{pmatrix} 1 & 0 \\ -6 & 25 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ -6 & 30 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}. \end{aligned}$$

Both diagonal entries agree, and both off-diagonal entries vanish, as required for a scalar multiple of the identity.

Step 3: Conclusion The required value is $K = -5$, so $A^2 = 6A - 5I_2$.

Quick Tip to Solve: Compute $A^2 - 6A$; a correct result must have equal diagonal entries and zero off-diagonal entries.

23(a). 10 identical blocks are marked with '0' on two of them, '1' on three of them, '2' on four of them and '3' on one of them and put in a box. If X denotes the number written on the block, then write the probability distribution of X and calculate its mean. [2 marks]

Solution:

Concept:

- For a randomly selected block, all ten blocks are equally likely. Thus $P(X = x)$ equals the number of blocks carrying x , divided by 10.
- The mean of a discrete random variable is the weighted sum $\mu = E(X) = \sum_x xP(X = x)$; the probabilities must add to 1.

Step 1: Write the probability distribution The possible values of X are 0, 1, 2, 3, occurring on 2, 3, 4, 1 blocks respectively. Hence the distribution is

x	0	1	2	3
$P(X = x)$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{10}$

These are nonnegative, and their sum is $(2 + 3 + 4 + 1)/10 = 1$, confirming that every block has been included.

Quick Tip to Solve: Check the mean by adding the numbers on all ten blocks and dividing by ten: $(0 + 3 + 8 + 3)/10$.

Step 2: Calculate the mean Multiply each possible number by its own probability and add:

$$\begin{aligned} E(X) &= 0 \left(\frac{2}{10} \right) + 1 \left(\frac{3}{10} \right) + 2 \left(\frac{4}{10} \right) + 3 \left(\frac{1}{10} \right) \\ &= \frac{0 + 3 + 8 + 3}{10} = \frac{14}{10}. \end{aligned}$$

Equal likelihood applies to the blocks, not to the four distinct numerical values, so their unweighted average would be inappropriate.

Step 3: Conclusion The probabilities at $X = 0, 1, 2, 3$ are respectively $\frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{10}$, and the mean is

$$\mu = \frac{7}{5} = 1.4.$$

Quick Tip to Solve: Check the mean by adding the numbers on all ten blocks and dividing by ten: $(0 + 3 + 8 + 3)/10$.

OR

23(b). In a village of 8000 people, 3000 go out of the village to work and 4000 are women. It is noted that 30% of women go out of the village to work. What is the probability that a randomly chosen individual is either a woman or a person working outside the village? [2 marks]

Solution:

Concept:

- For events W and O , the inclusive event either W or O is $W \cup O$, which includes people belonging to both.
- Inclusion-exclusion gives $n(W \cup O) = n(W) + n(O) - n(W \cap O)$. For uniform random selection, probability is the favourable count divided by the total count.

Step 1: Find the number in both groups Let W be the event that the chosen individual is a woman and O the event that the individual works outside the village. The given counts are $n(W) = 4000$ and $n(O) = 3000$. Since 30% of the women work outside,

$$n(W \cap O) = \frac{30}{100} \times 4000 = 1200.$$

The percentage is taken of the women, not of all 8000 villagers.

Step 2: Count the union and divide by the population Adding the two group sizes counts the 1200 women working outside twice. Subtract their count once:

$$\begin{aligned} n(W \cup O) &= 4000 + 3000 - 1200 = 5800, \\ P(W \cup O) &= \frac{5800}{8000}. \end{aligned}$$

Equivalently, the favourable group consists of all 4000 women together with the $3000 - 1200 = 1800$ men working outside.

Quick Tip to Solve: First find the overlap as 30% of 4000, then subtract it once when adding women and outside workers.

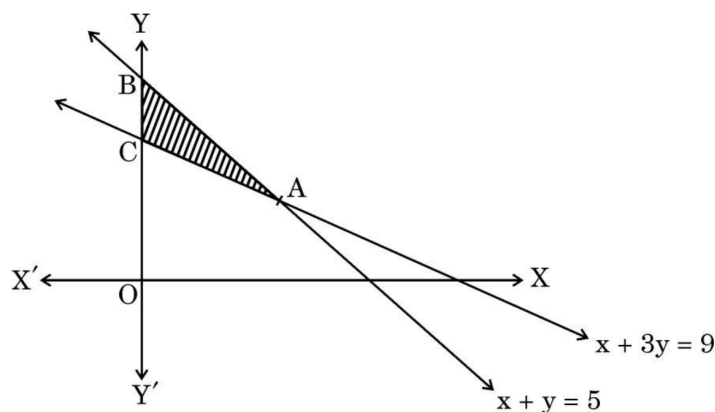
Step 3: Conclusion The required probability is $\frac{29}{40} = 0.725$.

Quick Tip to Solve: First find the overlap as 30% of 4000, then subtract it once when adding women and outside workers.

24. For a Linear Programming Problem, find $\min Z = 5x + 3y$ (where Z is the objective function) for the feasible region shaded in the given figure.

(Note: The figure is not to scale.)

[2 marks]



Solution:

Concept:

- A linear objective function attains its minimum on a nonempty bounded polygonal feasible region at a corner point.
- Find each vertex from the equations of its intersecting boundary lines, then evaluate the objective $Z = 5x + 3y$ at every vertex. The drawing need not be to scale.

Step 1: Determine the vertices of the shaded triangle The feasible region is the shaded triangle ABC . On the y -axis, $x = 0$; the line $x + y = 5$ therefore gives $B = (0, 5)$, while $x + 3y = 9$ gives $C = (0, 3)$. At A , solve both line equations:

$$\begin{aligned} x + y &= 5, & x + 3y &= 9, \\ 2y &= 4, & y &= 2, & x &= 3. \end{aligned}$$

Thus $A = (3, 2)$. The origin is outside the shaded region and is not a feasible candidate.

Step 2: Evaluate the objective at each vertex Substitute the coordinates of all three corner points:

Vertex	$Z = 5x + 3y$
$A(3, 2)$	$5(3) + 3(2) = 21$
$B(0, 5)$	$5(0) + 3(5) = 15$
$C(0, 3)$	$5(0) + 3(3) = 9$

Comparing these values identifies the smallest feasible objective value. Since C has a strictly smaller

Quick Tip to Solve: Read coordinates from the printed boundary equations, not the sketch's apparent scale; test only vertices of the shaded triangle.

value than the other vertices, the minimum occurs uniquely there.

Step 3: Conclusion The minimum is $Z_{\min} = 9$, attained at $(x, y) = (0, 3)$.

Quick Tip to Solve: Read coordinates from the printed boundary equations, not the sketch's apparent scale; test only vertices of the shaded triangle.

25. Let $f : A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3}$, where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. Discuss the bijectivity of the function. [2 marks]

Solution:

Concept:

- A function is one-one (injective) if $f(u) = f(v)$ implies $u = v$ for all inputs in its domain.
- A function is onto (surjective) if every element of its codomain has a preimage in the domain. A bijection is both one-one and onto.

Step 1: Check the mapping and prove it is one-one Since $x \neq 3$, the denominator is nonzero. Also,

$$f(x) = 1 + \frac{1}{x-3} \neq 1,$$

so the output lies in B . Now let $u, v \in A$ and suppose $f(u) = f(v)$. Cross-multiplying the nonzero denominators gives

$$\begin{aligned}(u-2)(v-3) &= (v-2)(u-3), \\ uv - 3u - 2v + 6 &= uv - 3v - 2u + 6, \\ u &= v.\end{aligned}$$

Thus distinct inputs cannot have the same image, so f is one-one.

Step 2: Prove every codomain element has a preimage Take any $y \in B$, so $y \neq 1$. Solving $y = (x-2)/(x-3)$ for x gives

$$(y-1)x = 3y-2, \quad x = \frac{3y-2}{y-1}.$$

This value is defined, and

$$x-3 = \frac{1}{y-1} \neq 0.$$

Hence $x \in A$. Substitution yields

$$f(x) = \frac{y/(y-1)}{1/(y-1)} = y.$$

Every permitted y therefore has a preimage, proving that f is onto.

Step 3: Conclusion The function $f : A \rightarrow B$ is both one-one and onto; hence it is bijective.

Quick Tip to Solve: Write $f(x) = 1 + 1/(x-3)$ to see the excluded output immediately; when proving onto, check that the solved preimage is not 3.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. In the Linear Programming Problem for objective function $Z = 18x + 10y$ subject to constraints $4x + y \geq 20$, $2x + 3y \geq 30$, $x, y \geq 0$, find the minimum value of Z . [3 marks]

Solution:

Concept:

- A feasible point satisfies all the inequalities simultaneously. A finite optimum of a linear objective on a polygonal feasible region occurs at a corner point.
- For an unbounded feasible region, compare the corner values and also establish that no feasible point gives a smaller objective value. A positive weighted sum of the constraints can provide this lower bound.

Step 1: Determine the feasible corner points The boundary lines are $4x + y = 20$ and $2x + 3y = 30$. Their intersection is obtained by substituting $y = 20 - 4x$ into the second equation:

$$2x + 3(20 - 4x) = 30 \implies x = 3, \quad y = 8.$$

On the y -axis, the constraints require $y \geq 20$ and $y \geq 10$, so the boundary corner is $(0, 20)$. On the x -axis they require $x \geq 5$ and $x \geq 15$, giving the corner $(15, 0)$. Thus the feasible region lies in the first quadrant above both boundary lines and is unbounded, with corners $(0, 20)$, $(3, 8)$, $(15, 0)$.

Step 2: Evaluate the objective and verify its lower bound Evaluate $Z = 18x + 10y$ at each feasible corner:

(x, y)	Z
$(0, 20)$	200
$(3, 8)$	$54 + 80 = 134$
$(15, 0)$	270

To confirm that the unbounded portion cannot give a lower value, use the original inequalities:

$$\begin{aligned} Z &= \frac{17}{5}(4x + y) + \frac{11}{5}(2x + 3y) \\ &\geq \frac{17}{5}(20) + \frac{11}{5}(30) = 134. \end{aligned}$$

Both coefficients are positive, so the inequality holds for every feasible point. Equality requires both constraints to be equalities, which occurs at their feasible intersection $(3, 8)$.

Step 3: Conclusion The minimum value is $Z_{\min} = 134$, attained at $x = 3, y = 8$.

Quick Tip to Solve: With greater-than constraints the region is unbounded; after comparing corners, express the objective as a positive sum of the constraints to certify the minimum.

27(a). The scalar product of the vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ with a unit vector along sum of vectors $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \lambda\hat{i} - 2\hat{j} - 3\hat{k}$ is equal to 1. Find the value of λ . [3 marks]

Solution:

Concept:

- The unit vector along a nonzero vector $\vec{v}$ is $\hat{v} = \vec{v}/|\vec{v}|$. Its direction is the same as that of $\vec{v}$.
- The scalar product is the sum of products of corresponding components: $\vec{a} \cdot \vec{v} = a_x v_x + a_y v_y + a_z v_z$. When squaring an equation containing a square root, check the final value in the unsquared equation.

Step 1: Form the required unit vector Add the given vectors component by component:

$$\begin{aligned}\vec{b} + \vec{c} &= (\lambda + 2)\hat{i} - 6\hat{j} + 2\hat{k}, \\ |\vec{b} + \vec{c}| &= \sqrt{(\lambda + 2)^2 + (-6)^2 + 2^2} \\ &= \sqrt{(\lambda + 2)^2 + 40}.\end{aligned}$$

The squared magnitude is at least 40, so the sum is never the zero vector. Therefore its unit vector is well-defined for every real λ :

$$\hat{u} = \frac{(\lambda + 2)\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{(\lambda + 2)^2 + 40}}.$$

Step 2: Apply the scalar-product condition and check the value Using $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{a} \cdot \hat{u} = 1$, we obtain

$$\frac{(\lambda + 2) + 6 + 4}{\sqrt{(\lambda + 2)^2 + 40}} = 1, \quad \lambda + 12 = \sqrt{(\lambda + 2)^2 + 40}.$$

The right side is positive, so any valid answer must have $\lambda + 12 > 0$. Squaring and simplifying gives

$$\begin{aligned}\lambda^2 + 24\lambda + 144 &= \lambda^2 + 4\lambda + 44, \\ 20\lambda &= -100, \\ \lambda &= -5.\end{aligned}$$

To check the sign and rule out a spurious root, at this value the unit vector is $\hat{u} = (-3\hat{i} - 6\hat{j} + 2\hat{k})/7$, and

$$\vec{a} \cdot \hat{u} = \frac{-3 + 6 + 4}{7} = 1.$$

Thus the candidate satisfies the original, unsquared condition.

Step 3: Conclusion The required value is $\lambda = -5$.

Quick Tip to Solve: Normalise the sum $\vec{b} + \vec{c}$, not each vector separately; after squaring, substitute the result in the original scalar-product equation.

27(b). Find the shortest distance between the lines:

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\vec{r} = (\hat{i} + 4\hat{k}) + \mu(3\hat{i} - 6\hat{j} + 9\hat{k})$$

[3 marks]

Solution:

Concept:

- Lines whose direction vectors are nonzero scalar multiples are parallel. Their shortest distance is the perpendicular separation between them.
- For parallel lines through position vectors $\vec{p}$ and $\vec{q}$, with common direction $\vec{d}$, the distance is $D = |(\vec{q} - \vec{p}) \times \vec{d}|/|\vec{d}|$. This follows from $|\vec{u} \times \vec{d}| = |\vec{u}||\vec{d}|\sin\theta$, where $\vec{u} = \vec{q} - \vec{p}$.

Step 1: Identify the parallel directions and a joining vector The lines pass through $P(2, -1, 3)$ and $Q(1, 0, 4)$, respectively. Their direction vectors are

$$\vec{d}_1 = \hat{i} - 2\hat{j} + 3\hat{k}, \quad \vec{d}_2 = 3\hat{i} - 6\hat{j} + 9\hat{k} = 3\vec{d}_1.$$

Thus the lines are parallel. Use the common direction $\vec{d} = \vec{d}_1$ and the joining vector

$$\vec{PQ} = (1 - 2)\hat{i} + (0 + 1)\hat{j} + (4 - 3)\hat{k} = -\hat{i} + \hat{j} + \hat{k}.$$

This joining vector is not parallel to $\vec{d}$, so the lines are distinct. Its component perpendicular to $\vec{d}$ measures their separation.

Step 2: Compute the perpendicular distance Calculate the cross product:

$$\begin{aligned} \vec{PQ} \times \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\ &= (3 + 2)\hat{i} - (-3 - 1)\hat{j} + (2 - 1)\hat{k} \\ &= 5\hat{i} + 4\hat{j} + \hat{k}. \end{aligned}$$

Hence the perpendicular separation is

$$D = \frac{\sqrt{5^2 + 4^2 + 1^2}}{\sqrt{1^2 + (-2)^2 + 3^2}} = \frac{\sqrt{42}}{\sqrt{14}} = \sqrt{3}.$$

As a check, $\vec{PQ} \cdot \vec{d} = -1 - 2 + 3 = 0$. Therefore PQ itself is perpendicular to both lines, and its length $\sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$ agrees with the calculation.

Step 3: Conclusion The shortest distance between the given parallel lines is $\boxed{\sqrt{3} \text{ units}}$.

Quick Tip to Solve: First compare the direction ratios. Here the joining vector $(-1, 1, 1)$ is already perpendicular to $(1, -2, 3)$, so its length directly gives the distance.

28. Differentiate $\log(x^x + \operatorname{cosec}^2 x)$ with respect to x .

[3 marks]

Solution:

Concept:

- For a positive differentiable function $u(x)$, the chain rule gives $\frac{d}{dx} \log u = \frac{u'}{u}$, where $\log$ denotes the natural logarithm.
- When both the base and exponent vary, logarithmic differentiation gives $\frac{d}{dx} x^x = x^x(1 + \log x)$, for $x > 0$.
- The trigonometric derivative is $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$; its square is differentiated using the chain rule.

Step 1: Differentiate the variable power Put $v = x^x$. On the real differentiable domain $x > 0$, take logarithms and differentiate:

$$\begin{aligned}\log v &= x \log x, \\ \frac{v'}{v} &= \log x + x \frac{1}{x} = \log x + 1, \\ v' &= x^x(1 + \log x).\end{aligned}$$

The ordinary power rule with a constant exponent cannot be used directly because the exponent also depends on x .

Step 2: Differentiate the trigonometric term Applying the chain rule to the square of $\operatorname{cosec} x$, for $\sin x \neq 0$, gives

$$\frac{d}{dx}(\operatorname{cosec}^2 x) = 2 \operatorname{cosec} x(-\operatorname{cosec} x \cot x) = -2 \operatorname{cosec}^2 x \cot x.$$

Step 3: Apply the outer logarithm rule Let $u = x^x + \operatorname{cosec}^2 x$, so the requested function is $\log u$. Add the derivatives of the two terms:

$$u' = x^x(1 + \log x) - 2 \operatorname{cosec}^2 x \cot x.$$

The outer logarithm requires this entire derivative to be divided by the entire argument u . Since $x^x > 0$ for $x > 0$, the argument is positive wherever the cosecant is defined.

Step 4: Conclusion Hence,

$$\boxed{\frac{d}{dx} \log(x^x + \operatorname{cosec}^2 x) = \frac{x^x(1 + \log x) - 2 \operatorname{cosec}^2 x \cot x}{x^x + \operatorname{cosec}^2 x}}.$$

This holds for $x > 0$ with $\sin x \neq 0$.

Quick Tip to Solve: Differentiate the sum inside the logarithm first, then divide by the unchanged sum. Use logarithmic differentiation only for the variable power x^x .

29. Show that of all the rectangles with a fixed perimeter, the square has the greatest area. [3 marks]

Solution:

Concept:

- A rectangle of length l and breadth b has perimeter $2(l + b)$ and area lb . Fixing the perimeter lets us express its area as a function of one side.
- At an interior maximum of a differentiable function, its first derivative is zero. A negative second derivative throughout the domain makes the function strictly concave and its stationary point the unique maximum.

Step 1: Express the area in one variable Let the fixed perimeter be $P > 0$. If the length is x , the breadth is determined by the perimeter condition:

$$2(x + b) = P \implies b = \frac{P}{2} - x.$$

Both sides of a rectangle must be positive, so $0 < x < P/2$. Its area is therefore

$$A(x) = x \left(\frac{P}{2} - x \right) = \frac{Px}{2} - x^2.$$

Here P is a constant; only x varies among rectangles with this perimeter.

Step 2: Find and test the stationary point Differentiate the area with respect to x :

$$A'(x) = \frac{P}{2} - 2x, \quad A''(x) = -2.$$

The stationary point satisfies

$$\frac{P}{2} - 2x = 0 \implies x = \frac{P}{4}.$$

This lies in $(0, P/2)$. Moreover, $A'(x) > 0$ when $x < P/4$, and $A'(x) < 0$ when $x > P/4$. Thus the area increases up to this point and decreases thereafter, establishing the greatest area over the whole allowable interval.

Step 3: Determine the corresponding breadth Substituting the maximizing length into the perimeter condition gives

$$b = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}.$$

The breadth equals the length. As a check, writing the area as

$$A(x) = \frac{P^2}{16} - \left(x - \frac{P}{4} \right)^2$$

shows directly that no other permitted value of x gives a larger area, since a real square is nonnegative.

Step 4: Conclusion For fixed perimeter P , the greatest area is $\boxed{P^2/16}$, attained only when both sides are $\boxed{P/4}$. Hence, of all rectangles with a fixed perimeter, the square has the greatest area.

Quick Tip to Solve: Eliminate one side using the fixed perimeter before differentiating. The maximum occurs when the two sides become equal; the negative second derivative confirms it.

30(a). Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 4x^3 - 5, \forall x \in \mathbb{R}$ is one-one and onto.

[3 marks]

Solution:

Concept:

- A function is one-one (injective) if equal outputs force equal inputs: $f(a) = f(b) \Rightarrow a = b$ for all inputs in its domain.
- A function is onto (surjective) if every element of its codomain has a preimage in its domain. To prove this, take an arbitrary codomain element y , solve $f(x) = y$, and check that the resulting x is allowed.

Step 1: Prove that equal outputs have equal inputs Take any $a, b \in \mathbb{R}$ and suppose $f(a) = f(b)$. Using the given rule,

$$\begin{aligned}4a^3 - 5 &= 4b^3 - 5, \\4a^3 &= 4b^3, \\a^3 &= b^3, \\a &= b.\end{aligned}$$

The final implication holds because a real number has a unique real cube root. Thus distinct real inputs cannot produce the same output, which verifies the definition of a one-one function.

Step 2: Find a preimage for an arbitrary real number Let y be any element of the codomain $\mathbb{R}$. Solving for an input that maps to it gives

$$\begin{aligned}4x^3 - 5 &= y, \\x^3 &= \frac{y+5}{4}, \\x &= \sqrt[3]{\frac{y+5}{4}}.\end{aligned}$$

A real cube root exists for every real radicand, including negative numbers and zero. Hence this candidate x belongs to the specified domain $\mathbb{R}$ for every real y .

Step 3: Verify the preimage in the function Substitution of the chosen input gives

$$f\left(\sqrt[3]{\frac{y+5}{4}}\right) = 4\left(\sqrt[3]{\frac{y+5}{4}}\right)^3 - 5 = y + 5 - 5 = y.$$

Since y was arbitrary, every real codomain element is attained. Therefore the range is all of $\mathbb{R}$, establishing that the function is onto.

Step 4: Conclusion The function $f(x) = 4x^3 - 5$, from $\mathbb{R}$ to $\mathbb{R}$, is both one-one and onto, and hence is bijective.

Quick Tip to Solve: For onto, exhibit $x = \sqrt[3]{(y+5)/4}$ for an arbitrary real y . Unlike a square root, a real cube root imposes no sign restriction on $y + 5$.

30(b). Let R be a relation defined on a set $\mathbb{N}$ of natural numbers such that $R = \{(x, y) : xy \text{ is a square of a natural number, } x, y \in \mathbb{N}\}$. Determine if the relation R is an equivalence relation.

[3 marks]

Solution:**Concept:**

- An equivalence relation on $\mathbb{N}$ must be reflexive, symmetric and transitive. We use $\mathbb{N} = \{1, 2, 3, \dots\}$.
- A positive integer is a perfect square exactly when the exponent of every prime in its prime factorisation is even.

Step 1: Check reflexivity Take any $x \in \mathbb{N}$. Since $x \cdot x = x^2$ is the square of the natural number x , the pair (x, x) belongs to R . This holds for every element of the set, so R is reflexive.

Step 2: Check symmetry Suppose $(x, y) \in R$. By the definition of the relation, $xy = m^2$ for some $m \in \mathbb{N}$. Multiplication is commutative, so $yx = xy = m^2$. Thus $(y, x) \in R$ whenever $(x, y) \in R$, and R is symmetric.

Step 3: Check transitivity using prime exponents Suppose $(x, y) \in R$ and $(y, z) \in R$. Then both xy and yz are perfect squares. For any prime p , let its exponents in x, y, z be a, b, c , respectively, allowing an exponent to be zero when the prime is absent. The square conditions say that $a + b$ and $b + c$ are even. Hence

$$a + c = (a + b) + (b + c) - 2b$$

is even. But $a + c$ is precisely the exponent of p in xz . Since this argument applies to every prime, all prime exponents in xz are even. Therefore xz is the square of a natural number, giving $(x, z) \in R$. This proves transitivity.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive. Hence R is an equivalence relation on $\mathbb{N}$.

Quick Tip to Solve: For transitivity of a perfect-square product relation, track the parity of each prime exponent; the shared middle exponent cancels twice.

31(a). Let $2x + 5y - 1 = 0$ and $3x + 2y - 7 = 0$ represent the equations of two lines on which the ants are moving on the ground. Using matrix method, find a point common to the paths of the ants. [3 marks]

Solution:**Concept:**

- A point common to two lines satisfies their equations simultaneously. A linear system can be written as $AX = B$, where A is the coefficient matrix and X is the column of unknowns.
- If $|A| \neq 0$, the unique solution is $X = A^{-1}B$. For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse is

Quick Tip to Solve: Write both constants on the right before forming the matrices, and verify the final ordered pair in both original line equations.

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Step 1: Express the simultaneous equations in matrix form Move the constant terms to the right-hand sides of the two line equations:

$$2x + 5y = 1, \quad 3x + 2y = 7.$$

In the order x, y , define

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Then the required common point is obtained by solving $AX = B$.

Step 2: Compute the inverse and multiply The determinant is $|A| = 2(2) - 5(3) = 4 - 15 = -11 \neq 0$, so the system has exactly one solution. The adjugate and inverse are

$$\text{adj } A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}, \quad A^{-1} = -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}.$$

Therefore

$$\begin{aligned} X = A^{-1}B &= -\frac{1}{11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} \\ &= -\frac{1}{11} \begin{bmatrix} 2 - 35 \\ -3 + 14 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \end{aligned}$$

Step 3: Verify the coordinates in both paths Substituting $x = 3$ and $y = -1$ into the original left-hand sides gives

$$2(3) + 5(-1) - 1 = 0, \quad 3(3) + 2(-1) - 7 = 0.$$

Thus the point found by the matrix method lies on each line, as required for the common point of the ants' paths.

Step 4: Conclusion The common point of the two paths is $\boxed{(3, -1)}$.

Quick Tip to Solve: Write both constants on the right before forming the matrices, and verify the final ordered pair in both original line equations.

31(b). A shopkeeper sells 50 Chemistry, 60 Physics and 35 Maths books on day I and sells 40 Chemistry, 45 Physics and 50 Maths books on day II. If the selling price for each such subject book is ₹150 (Chemistry), ₹175 (Physics) and ₹180 (Maths), then find his total sale in two days, using matrix method. If cost price of all the books together is ₹35,000, what profit did he earn after the sale of two days? [3 marks]

Solution:

Concept:

- A row of quantities multiplied by a column of corresponding unit prices gives the total selling value. The order of subjects must be the same in both matrices.
- Profit equals total selling price minus total cost price. The daily sales must first be added because the stated cost price covers all the books together.

Step 1: Form the quantity matrix and price column Use the subject order Chemistry, Physics, Mathematics. Put the quantities sold on day I in the first row and those sold on day II in the second row:

$$Q = \begin{bmatrix} 50 & 60 & 35 \\ 40 & 45 & 50 \end{bmatrix}, \quad P = \begin{bmatrix} 150 \\ 175 \\ 180 \end{bmatrix}.$$

Entries of P are selling prices in rupees per book. Since Q is of order 2×3 and P is of order 3×1 , the product QP is a 2×1 column giving the sales in rupees for the two days.

Step 2: Calculate sales by matrix multiplication Multiply each day's quantity row by the price column:

$$\begin{aligned} QP &= \begin{bmatrix} 50(150) + 60(175) + 35(180) \\ 40(150) + 45(175) + 50(180) \end{bmatrix} \\ &= \begin{bmatrix} 7,500 + 10,500 + 6,300 \\ 6,000 + 7,875 + 9,000 \end{bmatrix} = \begin{bmatrix} 24,300 \\ 22,875 \end{bmatrix}. \end{aligned}$$

Thus the sales on days I and II are respectively ₹24,300 and ₹22,875. Adding them, equivalently multiplying by the row matrix $\begin{bmatrix} 1 & 1 \end{bmatrix}$, gives

$$S = \begin{bmatrix} 1 & 1 \end{bmatrix} QP = 24,300 + 22,875 = 47,175.$$

Step 3: Deduct the total cost price The combined cost price of all books is ₹35,000. Therefore the excess of the total sales over this cost is

$$\text{Profit} = 47,175 - 35,000 = 12,175 \text{ rupees.}$$

The positive difference confirms that the shopkeeper earned a profit on the books sold over the two days.

Step 4: Conclusion Total sale in two days is ₹47,175, and the profit earned is ₹12,175.

Quick Tip to Solve: Keep Chemistry, Physics and Mathematics in the same order in each quantity row and the price column; subtract the combined cost only once.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Find: $\int \frac{3x+1}{(x-2)^2(x+2)} dx.$

[5 marks]

Solution:

Concept:

- For a proper rational function, resolve the integrand into partial fractions. A repeated linear factor $(x-a)^2$ requires separate terms with denominators $x-a$ and $(x-a)^2$.
- Use $\int (x-a)^{-1} dx = \ln|x-a| + C$ and $\int (x-a)^{-2} dx = -(x-a)^{-1} + C$, where C is an arbitrary constant of integration.

Step 1: Set up the partial fractions Introduce constants A, B, D and write

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{D}{x+2}.$$

Multiplying throughout by the common denominator gives the polynomial identity

$$3x+1 = A(x-2)(x+2) + B(x+2) + D(x-2)^2.$$

Although the original fractions require $x \neq \pm 2$, this polynomial identity can be evaluated at any value of x to determine the constants.

Step 2: Determine all three constants Substitute $x = 2$ to eliminate the terms containing A and D , and then substitute $x = -2$ to eliminate the first two terms:

$$\begin{aligned} 7 &= 4B \quad \Rightarrow B = \frac{7}{4}, \\ -5 &= 16D \quad \Rightarrow D = -\frac{5}{16}. \end{aligned}$$

The coefficient of x^2 on the left is zero. Thus $A + D = 0$, giving $A = 5/16$. As a check, the coefficient of x on the right is $B - 4D = 7/4 + 5/4 = 3$, and its constant term is $-4A + 2B + 4D = 1$.

Step 3: Integrate each elementary term Substitute the constants into the decomposition and use linearity of integration:

$$\begin{aligned} I &= \frac{5}{16} \int \frac{dx}{x-2} + \frac{7}{4} \int \frac{dx}{(x-2)^2} - \frac{5}{16} \int \frac{dx}{x+2} \\ &= \frac{5}{16} \ln|x-2| - \frac{7}{4(x-2)} - \frac{5}{16} \ln|x+2| + C. \end{aligned}$$

The squared denominator gives a negative reciprocal term, not a logarithm. Combining the logarithms is valid on every interval avoiding -2 and 2 . Differentiation of these three terms reproduces precisely the partial fractions above, verifying the antiderivative.

Quick Tip to Solve: Include both powers of the repeated factor $x-2$. Substituting $x = 2$ and $x = -2$ in the cleared identity finds two constants immediately; the x^2 coefficient gives the third.

Step 4: Conclusion Therefore,

$$\int \frac{3x+1}{(x-2)^2(x+2)} dx = \frac{5}{16} \ln \left| \frac{x-2}{x+2} \right| - \frac{7}{4(x-2)} + C.$$

Quick Tip to Solve: Include both powers of the repeated factor $x - 2$. Substituting $x = 2$ and $x = -2$ in the cleared identity finds two constants immediately; the x^2 coefficient gives the third.

OR

32(b). Evaluate: $\int_0^{\pi/2} \frac{x}{\cos x + \sin x} dx.$

[5 marks]

Solution:

Concept:

- The reflection property is $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$. Reflection about $\pi/4$ interchanges sine and cosine, leaving their sum unchanged.
- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ and $\int \sec t dt = \ln |\sec t + \tan t| + C$.

Step 1: Apply reflection to remove the numerator Denote the required integral by I . Its denominator is positive on $[0, \pi/2]$, so the integrand is continuous. Substituting $x = \pi/2 - u$ interchanges $\cos x$ and $\sin x$. After reversing the limits and renaming the variable,

$$I = \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x} dx.$$

Add this to the original integral. The two numerators sum to $\pi/2$, giving

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\cos x + \sin x}.$$

Thus $I = (\pi/4)J$, where J denotes the last integral without its prefactor.

Step 2: Transform the remaining integral Write $\cos x + \sin x = \sqrt{2} \cos(x - \pi/4)$ and put $t = x - \pi/4$. Then $dt = dx$, and the lower and upper limits become $-\pi/4$ and $\pi/4$:

$$J = \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec t dt = \frac{1}{\sqrt{2}} [\ln |\sec t + \tan t|]_{-\pi/4}^{\pi/4}.$$

The standard secant antiderivative is valid throughout these limits because $\cos t > 0$.

Step 3: Evaluate the endpoints At the upper and lower limits, $\sec t + \tan t$ equals $\sqrt{2} + 1$ and $\sqrt{2} - 1$,

Quick Tip to Solve: Reflect x to $\pi/2 - x$ before integrating: the denominator stays fixed, and adding the two forms replaces the numerator by $\pi/2$.

respectively. Both are positive. Therefore,

$$J = \frac{1}{\sqrt{2}} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \\ = \frac{1}{\sqrt{2}} \ln(\sqrt{2} + 1)^2 = \sqrt{2} \ln(\sqrt{2} + 1).$$

Here $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$ simplifies the ratio. Substitute this value of J into $I = (\pi/4)J$.

Step 4: Conclusion Therefore,

$$\int_0^{\pi/2} \frac{x}{\cos x + \sin x} dx = \frac{\pi}{2\sqrt{2}} \ln(1 + \sqrt{2}).$$

Quick Tip to Solve: Reflect x to $\pi/2 - x$ before integrating: the denominator stays fixed, and adding the two forms replaces the numerator by $\pi/2$.

33(a). Find the point Q on the line $\frac{2x+4}{6} = \frac{y+1}{2} = \frac{-2z+6}{-4}$ at a distance of $3\sqrt{2}$ from the point $P(1, 2, 3)$. [5 marks]

Solution:

Concept:

- Equating all three ratios of a line to a real parameter gives the coordinates of every point on the line.
- The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- A prescribed distance from a fixed point may give two points on a line.

Step 1: Parameterise the given line Put the common ratio equal to t :

$$\frac{2x+4}{6} = \frac{y+1}{2} = \frac{-2z+6}{-4} = t.$$

Solving these three linear equations separately gives

$$x = 3t - 2, \quad y = 2t - 1, \quad z = 2t + 3.$$

Thus an arbitrary point on the line is $Q(3t - 2, 2t - 1, 2t + 3)$.

Step 2: Apply the prescribed distance Since $P = (1, 2, 3)$, the coordinate differences from P to Q are

Quick Tip to Solve: Set the printed ratios equal to one parameter before reading direction ratios. Here the resulting quadratic factors with a zero root, so retain both points.

$3t - 3$, $2t - 3$, and $2t$. Using $PQ = 3\sqrt{2}$, we obtain

$$\begin{aligned}(3t - 3)^2 + (2t - 3)^2 + (2t)^2 &= 18, \\ 9t^2 - 18t + 9 + 4t^2 - 12t + 9 + 4t^2 &= 18, \\ 17t^2 - 30t &= 0, \\ t(17t - 30) &= 0.\end{aligned}$$

Hence the admissible parameter values are $t = 0$ and $t = 30/17$.

Step 3: Obtain and verify both points At $t = 0$, substitution gives $Q_1 = (-2, -1, 3)$. Its squared distance from P is $9 + 9 + 0 = 18$. At $t = 30/17$, the coordinates are

$$Q_2 = \left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17} \right).$$

Its differences from P are $39/17, 9/17, 60/17$, so

$$PQ_2^2 = \frac{39^2 + 9^2 + 60^2}{17^2} = \frac{5202}{289} = 18.$$

Both points lie on the line by their construction, and both meet the required distance. The two distinct real roots therefore give two valid points, and neither can be omitted.

Step 4: Conclusion The required points are $\boxed{(-2, -1, 3)}$ and $\boxed{\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17} \right)}$. Each is at a distance $3\sqrt{2}$ from $P(1, 2, 3)$.

Quick Tip to Solve: Set the printed ratios equal to one parameter before reading direction ratios. Here the resulting quadratic factors with a zero root, so retain both points.

OR

33(b). Find the image of the point $(-1, 5, 2)$ in the line $\frac{2x - 4}{2} = \frac{y}{2} = \frac{2 - z}{3}$. Find the length of the line segment joining the points (given point and the image point). [5 marks]

Solution:

Concept:

- The image A' of a point A in a line is obtained by taking the foot H of the perpendicular from A to the line and making H the midpoint of AA' . Thus $A' = 2H - A$.
- If $\vec{d}$ is a direction vector of the line, the perpendicularity condition is $\overrightarrow{AH} \cdot \vec{d} = 0$.
- The length joining a point to its image is twice the perpendicular distance from that point to the reflecting line: $AA' = 2AH$.

Step 1: Write the line in parametric form Denote the given point by $A = (-1, 5, 2)$. Equate the printed ratios to t :

$$\frac{2x - 4}{2} = \frac{y}{2} = \frac{2 - z}{3} = t.$$

Quick Tip to Solve: First solve for the perpendicular foot with a dot product, then double its coordinates and subtract the original point. Double the perpendicular distance to obtain the image distance.

It follows that

$$x = 2 + t, \quad y = 2t, \quad z = 2 - 3t.$$

Therefore a general point of the line is $(2 + t, 2t, 2 - 3t)$, and a direction vector is $\vec{d} = (1, 2, -3)$.

Step 2: Find the perpendicular foot Let $H = (2 + t, 2t, 2 - 3t)$ be the foot of the perpendicular. Then

$$\overrightarrow{AH} = (3 + t, 2t - 5, -3t).$$

Since AH is perpendicular to the line, take its dot product with $(1, 2, -3)$:

$$\begin{aligned}(3 + t) + 2(2t - 5) + (-3)(-3t) &= 0, \\ 14t - 7 &= 0, \\ t &= \frac{1}{2}.\end{aligned}$$

Substituting this parameter gives the unique perpendicular foot

$$H = \left(\frac{5}{2}, 1, \frac{1}{2}\right).$$

Step 3: Use the midpoint condition to obtain the image Write the image as $A' = (x', y', z')$. Since H bisects the segment joining the point and its image, the coordinate equations give

$$x' = 5 + 1 = 6, \quad y' = 2 - 5 = -3, \quad z' = 1 - 2 = -1.$$

As a check, the image displacement $(7, -8, -3)$ has dot product $7 - 16 + 9 = 0$ with the line direction.

Step 4: Calculate the length of the joining segment Apply the three-dimensional distance formula to $A = (-1, 5, 2)$ and $A' = (6, -3, -1)$:

$$\begin{aligned}AA' &= \sqrt{(6 + 1)^2 + (-3 - 5)^2 + (-1 - 2)^2} \\ &= \sqrt{49 + 64 + 9} = \sqrt{122}.\end{aligned}$$

Step 5: Conclusion The image of the given point in the line is $(6, -3, -1)$, and the length of the segment joining the point and its image is $\sqrt{122}$ units.

Quick Tip to Solve: First solve for the perpendicular foot with a dot product, then double its coordinates and subtract the original point. Double the perpendicular distance to obtain the image distance.

34. Solve the differential equation $(x - \sin y) dy + (\tan y) dx = 0$, **given** $y(0) = 0$.

[5 marks]

Solution:

Concept:

- A linear differential equation in x as a function of y has the form $\frac{dx}{dy} + P(y)x = Q(y)$, with integrating factor $e^{\int P(y) dy}$.
- After multiplying by the integrating factor, the left side becomes a product derivative. The initial condition $y(0) = 0$ specifies the point $(x, y) = (0, 0)$.
- Division by a trigonometric factor can lose a constant solution. Such a solution must be checked separately in the original equation.

Step 1: Write the equation in linear form Treat x as the dependent variable. Away from $\sin y = 0$, divide the given equation by $\tan y dy$:

$$\frac{dx}{dy} + x \cot y = \sin y \cot y = \cos y.$$

Hence $P(y) = \cot y$ and $Q(y) = \cos y$. The equation is linear in x , even though it does not initially appear linear in y .

Step 2: Find the integrating factor and integrate On any interval where $\sin y \neq 0$, an integrating factor is

$$\text{I.F.} = e^{\int \cot y dy} = e^{\log |\sin y|}.$$

A constant nonzero multiple of an integrating factor is also valid, so we may use $\sin y$. Multiplying through gives

$$\begin{aligned}\sin y \frac{dx}{dy} + x \cos y &= \sin y \cos y, \\ \frac{d}{dy}(x \sin y) &= \sin y \cos y, \\ x \sin y &= \frac{1}{2} \sin^2 y + C.\end{aligned}$$

The integrated relation extends continuously to the specified initial point.

Step 3: Apply the initial condition and check the branches Substituting $x = 0$, $y = 0$ into the integrated relation gives $C = 0$. Thus

$$x \sin y - \frac{1}{2} \sin^2 y = 0, \quad \sin y(2x - \sin y) = 0.$$

On the nonconstant branch through the origin, $\sin y = 2x$, so locally $y = \sin^{-1}(2x)$, with $-\frac{1}{2} < x < \frac{1}{2}$. Here $\sin^{-1}$ denotes the principal inverse sine. Its derivative satisfies $\cos y y' = 2$, and substitution in $(x - \sin y)y' + \tan y = 0$ gives zero. The division in the first step excluded $y \equiv 0$. Direct substitution shows that this constant function also satisfies both the original equation and the initial condition. Thus the given condition does not select a unique branch.

Step 4: Conclusion The implicit answer is $x \sin y = \frac{1}{2} \sin^2 y$. Through the origin, the solutions are

$$y = \sin^{-1}(2x), |x| < \frac{1}{2}, \text{ and } y \equiv 0.$$

Quick Tip to Solve: Make x the dependent variable and use the integrating factor $\sin y$. Before cancelling $\sin y$, check the lost constant branch $y = 0$ in the original equation.

35. A woman discovered a scratch along a straight line on a circular table top of radius 8 cm. She divided the table top into 4 equal quadrants and discovered the scratch passing through the origin inclined at an angle $\frac{\pi}{4}$ anticlockwise along the positive direction of x -axis. Find the area of the region enclosed by the x -axis, the scratch and the circular table top in the first quadrant, using integration. [5 marks]

Solution:

Concept:

- A circle of radius r centred at the origin has equation $x^2 + y^2 = r^2$. Its upper semicircle is $y = \sqrt{r^2 - x^2}$.
- A line through the origin making angle θ with the positive x -axis has equation $y = x \tan \theta$.
- Area between an upper curve $y = f(x)$ and the x -axis is $\int f(x) dx$. Split the integral where the upper boundary changes.

Step 1: Write the equations of the boundaries Place the centre of the table top at the origin and take the two perpendicular diameters as coordinate axes. The radius is 8 cm, so the circle and its first-quadrant arc are

$$x^2 + y^2 = 64, \quad y = \sqrt{64 - x^2}.$$

The scratch passes through the origin at angle $\pi/4$, giving

$$y = x \tan \frac{\pi}{4} = x.$$

The required region lies above the positive x -axis and below the scratch and the circular arc.

Step 2: Locate the intersection and split the area At the intersection of the scratch and the circular edge, substitute $y = x$:

$$2x^2 = 64 \Rightarrow x = y = 4\sqrt{2}.$$

Only the positive coordinates belong to the first quadrant. From $x = 0$ to $x = 4\sqrt{2}$, the upper boundary is the scratch $y = x$. From $x = 4\sqrt{2}$ to $x = 8$, it is the circular arc. Therefore

$$A = \int_0^{4\sqrt{2}} x dx + \int_{4\sqrt{2}}^8 \sqrt{64 - x^2} dx.$$

Step 3: Evaluate the two definite integrals The first integral is

$$\int_0^{4\sqrt{2}} x dx = \left[\frac{x^2}{2} \right]_0^{4\sqrt{2}} = 16.$$

For the circular part, use the standard antiderivative

$$\int \sqrt{64 - x^2} dx = \frac{x}{2} \sqrt{64 - x^2} + 32 \sin^{-1} \left(\frac{x}{8} \right).$$

At the upper limit $x = 8$, this expression equals 16π . At the lower limit $x = 4\sqrt{2}$, the square root is

Quick Tip to Solve: Split the vertical-strip integral at $x = 4\sqrt{2}$, where the line meets the circle. As a check, the region subtends $\pi/4$, so it is one-eighth of the full circle.

$4\sqrt{2}$, and $\sin^{-1}(1/\sqrt{2}) = \pi/4$. Consequently,

$$\begin{aligned}\int_{4\sqrt{2}}^8 \sqrt{64 - x^2} dx &= 16\pi - (16 + 8\pi) \\ &= 8\pi - 16.\end{aligned}$$

Adding the triangular part gives $A = 16 + (8\pi - 16)$. Both contributions are areas in square centimetres because the coordinates measure lengths in centimetres.

Step 4: Conclusion The area enclosed by the x -axis, the scratch and the circular table top in the first quadrant is $8\pi \text{ cm}^2$.

Quick Tip to Solve: Split the vertical-strip integral at $x = 4\sqrt{2}$, where the line meets the circle. As a check, the region subtends $\pi/4$, so it is one-eighth of the full circle.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1: Camphor is a waxy, colourless solid with strong aroma that evaporates through the process of sublimation, if left in the open at room temperature.

A cylindrical camphor tablet whose height is equal to its radius (r) evaporates when exposed to air such that the rate of reduction of its volume is proportional to its total surface area. Thus, $\frac{dV}{dt} = kS$ is the differential equation, where V is the volume, S is the surface area and t is the time in hours.

Based upon the above information, answer the following questions:

[4 marks]

36(i). Write the order and degree of the given differential equation.

[1 mark]

Solution:

Concept:

- The order is the order of the highest derivative appearing in a differential equation.
- The degree is the power of the highest-order derivative when the equation is a polynomial in its derivatives.

Step 1: Identify the highest derivative The given equation is $\frac{dV}{dt} = kS$, where V denotes volume and t denotes time. Its only derivative is the first derivative dV/dt ; there is no second or higher derivative.

Step 2: Inspect its power Write the equation as $\frac{dV}{dt} - kS = 0$. It is polynomial in the derivative, and dV/dt appears to the first power. The surface area S and proportionality constant k do not represent additional derivatives.

Step 3: Conclusion The differential equation has $\boxed{\text{order} = 1, \text{ degree} = 1}$.

Quick Tip to Solve: Inspect the derivative itself: its differentiation order gives the order, while its exponent gives the degree.

36(ii). Substituting $V = \pi r^3$ and $S = 2\pi r^2$, we get the differential equation $\frac{dr}{dt} = \frac{2}{3}k$. Solve it, given that $r(0) = 5$ mm. [1 mark]

Solution:

Concept:

- For a constant rate $dr/dt = a$, integration with respect to time gives $r = at + C$.
- An initial condition determines the integration constant; here r is in millimetres and t in hours.

Step 1: Integrate the supplied differential equation Using the differential equation given in the question, separate the variables:

$$dr = \frac{2k}{3} dt.$$

Since k is constant, integration yields

$$r = \frac{2k}{3}t + C.$$

The single constant C accounts for the initial radius.

Step 2: Use the initial radius Substitute $t = 0$ and $r(0) = 5$:

$$5 = \frac{2k}{3}(0) + C, \quad C = 5.$$

Differentiating the resulting expression returns $dr/dt = 2k/3$, so it satisfies both the given rate equation and the initial condition.

Step 3: Conclusion $r(t) = 5 + \frac{2k}{3}t$, with r in mm and t in hours, while the tablet exists.

Quick Tip to Solve: Integrate the constant rate first, then set time to zero to read off the initial radius directly.

36(iii)(a). If it is given that $r = 3$ mm when $t = 1$ hour, find the value of k . Hence, find t for $r = 0$ mm. [2 marks]

Solution:

Concept:

- Integrating the supplied rate equation $dr/dt = 2k/3$ with $r(0) = 5$ gives $r(t) = 5 + (2k/3)t$.
- A later radius determines k , and complete evaporation occurs at the first time when the radius becomes zero.

Step 1: Determine the proportionality constant Here r is measured in millimetres and t in hours. Insert the given observation $r = 3$ at $t = 1$ into the solution from part (ii):

$$3 = 5 + \frac{2k}{3}(1), \quad \frac{2k}{3} = -2, \quad k = -3.$$

The negative sign is consistent with the reduction in radius and volume. In the printed convention $dV/dt = kS$, evaporation therefore requires a negative k .

Quick Tip to Solve: The radius drops from 5 mm to 3 mm in one hour, so its constant rate is -2 mm per hour; use $2k/3 = -2$.

Step 2: Find the time at which the radius vanishes Substituting this constant into the radius equation gives $r(t) = 5 - 2t$. Thus the radius decreases by 2 mm each hour. Put $r = 0$:

$$0 = 5 - 2t, \quad 2t = 5, \quad t = \frac{5}{2}.$$

This is the total elapsed time measured from $t = 0$, when the radius was 5 mm. The model is physically used only until this time, since a negative radius would have no meaning.

Step 3: Conclusion $k = -3 \text{ mm h}^{-1}$, and the tablet disappears after $t = \frac{5}{2}$ hours, or 2 hours 30 minutes.

Quick Tip to Solve: The radius drops from 5 mm to 3 mm in one hour, so its constant rate is -2 mm per hour; use $2k/3 = -2$.

OR

36(iii)(b). If it is given that $r = 1$ mm when $t = 1$ hour, find the value of k . Hence, find t for $r = 0$ mm. [2 marks]

Solution:

Concept:

- The supplied differential equation $dr/dt = 2k/3$ and initial value $r(0) = 5$ give $r(t) = 5 + (2k/3)t$.
- The rate is constant: a second radius determines k , after which setting $r = 0$ determines the evaporation time.

Step 1: Use the observation at one hour For this alternative, the given radius at $t = 1$ is 1 mm. Substitute this value into the initial-value solution, measuring r in mm and t in hours:

$$1 = 5 + \frac{2k}{3}(1), \quad \frac{2k}{3} = -4, \quad k = -6.$$

The negative value represents evaporation because the printed equation uses $dV/dt = kS$ and the volume decreases while the surface area remains positive.

Step 2: Set the radius equal to zero With this value of k , the radius is $r(t) = 5 - 4t$. Therefore,

$$0 = 5 - 4t, \quad 4t = 5, \quad t = \frac{5}{4}.$$

This time is measured from the initial instant, not from the observation at one hour. As a check, at one hour the equation gives $r = 1$; another quarter hour at a decrease of 4 mm per hour removes the remaining radius. The radius model stops at complete evaporation.

Step 3: Conclusion $k = -6 \text{ mm h}^{-1}$, and $r = 0$ at $t = \frac{5}{4}$ hours, or 1 hour 15 minutes.

Quick Tip to Solve: In this alternative the radius falls by 4 mm in one hour; keep the initial radius 5 mm when finding the total time.

37. Case Study - 2: Based upon the results of regular medical check-ups in a hospital, it was found that out of 1000 people, 700 were very healthy, 200 maintained average health and 100 had a poor health record.

Let A_1 : People with good health,

A_2 : People with average health,

and A_3 : People with poor health.

During a pandemic, the data expressed that the chances of people contracting the disease from category A_1 , A_2 and A_3 are 25%, 35% and 50%, respectively.

Based upon the above information, answer the following questions:

[4 marks]

37(i). A person was tested randomly. What is the probability that he/she has contracted the disease?

[2 marks]

Solution:

Concept:

- The health categories A_1, A_2, A_3 form a partition: they are mutually exclusive and together contain everyone.
- For the event D of contracting the disease, total probability gives $P(D) = \sum_{i=1}^3 P(A_i)P(D | A_i)$.

Step 1: Identify the prior and conditional probabilities A random selection from the 1000 people gives

$$P(A_1) = \frac{700}{1000} = 0.7, \quad P(A_2) = 0.2, \quad P(A_3) = 0.1.$$

The disease probabilities within these respective groups are

$$P(D | A_1) = 0.25, \quad P(D | A_2) = 0.35, \quad P(D | A_3) = 0.50.$$

The different sizes of the three categories must be used as weights.

Step 2: Apply the theorem of total probability The three ways of contracting the disease, one through each health category, are disjoint. Hence their probabilities add:

$$\begin{aligned} P(D) &= (0.7)(0.25) + (0.2)(0.35) + (0.1)(0.50) \\ &= 0.175 + 0.070 + 0.050 \\ &= 0.295. \end{aligned}$$

Taking a simple unweighted average of the three disease rates would incorrectly treat the category sizes as equal.

Step 3: Conclusion The probability that the randomly tested person has contracted the disease is

$$0.295 = \frac{59}{200}.$$

Quick Tip to Solve: Weight each disease rate by its group's population share before adding the three contributions.

37(ii). Given that the person has not contracted the disease, what is the probability that the person is from category A_2 ? [2 marks]

Solution:

Concept:

- Let D denote contracting the disease and D^c denote not contracting it. Within any category, $P(D^c | A_i) = 1 - P(D | A_i)$.
- Bayes' theorem gives $P(A_2 | D^c) = P(A_2)P(D^c | A_2)/P(D^c)$.

Step 1: Calculate the probability of no disease The category probabilities are 0.7, 0.2, 0.1. The corresponding probabilities of not contracting the disease are 0.75, 0.65, 0.50. By total probability,

$$\begin{aligned}P(D^c) &= (0.7)(0.75) + (0.2)(0.65) + (0.1)(0.50) \\&= 0.525 + 0.130 + 0.050 = 0.705.\end{aligned}$$

This also equals $1 - 0.295$, agreeing with the result for contracting the disease in part (i).

Step 2: Condition on the absence of disease The question asks for category A_2 , which is the average-health category. Its contribution to the probability of no disease is

$$P(A_2 \cap D^c) = (0.2)(0.65) = 0.13.$$

Divide this contribution by the probability of the conditioning event:

$$P(A_2 | D^c) = \frac{0.13}{0.705} = \frac{130}{705} = \frac{26}{141}.$$

The denominator includes disease-free people from all three categories, since the person's original health category is unknown.

Step 3: Conclusion The required probability is $P(A_2 | D^c) = \frac{26}{141}$.

Quick Tip to Solve: Complement each disease rate first, then use the average-health group's disease-free contribution as the Bayes numerator.

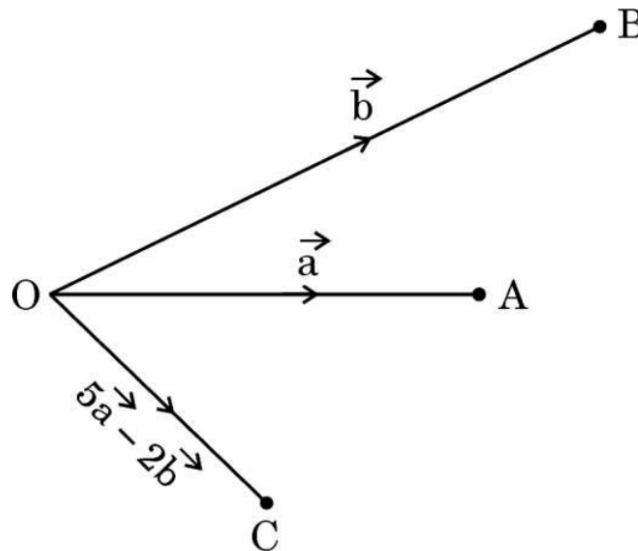
38. Three friends A , B and C move out from the same location O at the same time in three different directions to reach their destinations. They move out on straight paths and decide that A and B after reaching their destinations will meet up with C at his predecided destination, following straight paths from A to C and B to C in such a way that $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = 5\vec{a} - 2\vec{b}$ respectively.

Based upon the above information, answer the following questions:

[4 marks]

38(i). Complete the given figure to explain their entire movement plan along the respective vectors.

[1 mark]



Solution:

Concept:

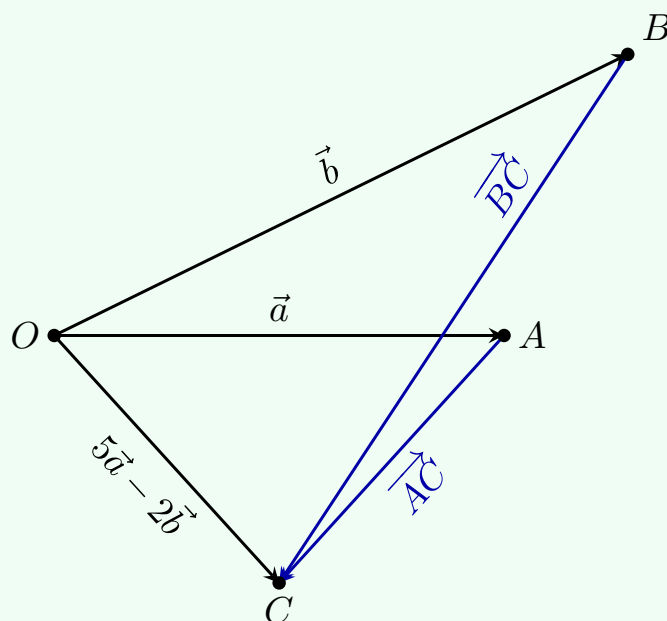
- A directed line segment represents a displacement vector: its arrow points from the starting location towards the destination.
- For position vectors measured from the same origin, $\vec{PQ} = \vec{OQ} - \vec{OP}$.

Step 1: Retain the three initial movements The initial arrows are $O \rightarrow A$, $O \rightarrow B$ and $O \rightarrow C$, labelled $\vec{a}$, $\vec{b}$ and $5\vec{a} - 2\vec{b}$, respectively. These show the three friends leaving the common starting point along their own straight paths. The completed diagram below is schematic, following the supplied figure.

Step 2: Add the two meeting paths Join A to C and B to C with straight arrows pointing towards C . These are the later displacements of the first two friends; no return arrow from C is required. Their labels correspond to

$$\vec{AC} = 4\vec{a} - 2\vec{b}, \quad \vec{BC} = 5\vec{a} - 3\vec{b}.$$

Quick Tip to Solve: Complete a movement diagram by adding arrows from each friend's present destination to the common meeting point, here C .



Completed movement plan; the arrows on both added paths point towards C .

Step 3: Conclusion The completed figure contains all five directed paths: $O \rightarrow A$, $O \rightarrow B$, $O \rightarrow C$, $A \rightarrow C$ and $B \rightarrow C$, with the last two ending at the meeting point C .

Quick Tip to Solve: Complete a movement diagram by adding arrows from each friend's present destination to the common meeting point, here C .

38(ii). Find vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

[1 mark]

Solution:

Concept:

- The displacement from a point P to a point Q is the final position vector minus the initial position vector: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.
- Vector addition combines coefficients of the same vector, while subtraction reverses the entire subtracted vector.

Step 1: Calculate the path from A to C Use A as the initial point and C as the final point. Subtract $\overrightarrow{OA} = \vec{a}$ from $\overrightarrow{OC} = 5\vec{a} - 2\vec{b}$:

$$\overrightarrow{AC} = (5\vec{a} - 2\vec{b}) - \vec{a} = 4\vec{a} - 2\vec{b}.$$

Step 2: Calculate the path from B to C The destination is still C , but the initial position is now $\overrightarrow{OB} = \vec{b}$. Therefore

$$\overrightarrow{BC} = (5\vec{a} - 2\vec{b}) - \vec{b} = 5\vec{a} - 3\vec{b}.$$

Both differences point towards the common meeting point; reversing either subtraction would describe the return journey.

Quick Tip to Solve: Read the letters in $\overrightarrow{AC}$ as start A , finish C , then subtract the start vector from the finish vector.

Step 3: Conclusion $\overrightarrow{AC} = 4\vec{a} - 2\vec{b}$ and $\overrightarrow{BC} = 5\vec{a} - 3\vec{b}$.

Quick Tip to Solve: Read the letters in $\overrightarrow{AC}$ as start A , finish C , then subtract the start vector from the finish vector.

38(iii)(a). If $\vec{a} \cdot \vec{b} = 1$, distance of O to A is 1 km and that from O to B is 2 km, then find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Also, find $|\vec{a} \times \vec{b}|$. [2 marks]

Solution:

Concept:

- The scalar product is $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where $\theta \in [0, \pi]$ is the angle between the vectors.
- The magnitude of the vector product is $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$. For displacement vectors its units are units of area.

Step 1: Use the distances as vector magnitudes Since $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, their magnitudes are $|\vec{a}| = 1$ and $|\vec{b}| = 2$ when lengths are measured in kilometres. With the given scalar product $\vec{a} \cdot \vec{b} = 1$, measured consistently in square kilometres,

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{1}{1 \times 2} = \frac{1}{2}.$$

The angle between two vectors lies between 0° and 180° , so $\theta = 60^\circ$.

Step 2: Evaluate the vector-product magnitude The angle is acute, and $\sin 60^\circ = \sqrt{3}/2$. Hence

$$|\vec{a} \times \vec{b}| = 1 \times 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \text{ km}^2.$$

As an algebraic check, its square equals $|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = 4 - 1 = 3$. This agrees with the magnitude obtained from the sine formula.

Step 3: Conclusion The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is 60° , and $|\vec{a} \times \vec{b}| = \sqrt{3} \text{ km}^2$.

Quick Tip to Solve: Find the cosine from the dot product first; alternatively, obtain the cross-product magnitude directly as $\sqrt{1^2 \cdot 2^2 - 1^2}$.

OR

38(iii)(b). If $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, then find a unit vector perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$. [2 marks]

Solution:

Concept:

- The cross product of two non-parallel vectors is perpendicular to each of them.

Quick Tip to Solve: Remove any common numerical factor from a cross product before normalizing; here dividing the normal by 2 makes the magnitude calculation shorter.

- A nonzero vector $\vec{n}$ becomes a unit vector on division by its magnitude: $\hat{n} = \vec{n}/|\vec{n}|$. Its negative is the opposite unit normal.

Step 1: Form the two given directions Substitute $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, keeping each component in its correct position:

$$\vec{u} = \vec{a} + \vec{b} = 2\hat{i} + 3\hat{k}, \quad \vec{v} = \vec{a} - \vec{b} = 2\hat{i} - 2\hat{j} + 5\hat{k}.$$

Step 2: Take the cross product and normalize A vector perpendicular to both directions is

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 3 \\ 2 & -2 & 5 \end{vmatrix} \\ &= 6\hat{i} - (10 - 6)\hat{j} - 4\hat{k} \\ &= 2(3\hat{i} - 2\hat{j} - 2\hat{k}). \end{aligned}$$

The simpler normal $\vec{n} = 3\hat{i} - 2\hat{j} - 2\hat{k}$ has magnitude $\sqrt{9 + 4 + 4} = \sqrt{17}$. Division by $\sqrt{17}$ therefore gives a unit normal. Verify perpendicularity by taking scalar products:

$$\vec{n} \cdot \vec{u} = 6 - 6 = 0, \quad \vec{n} \cdot \vec{v} = 6 + 4 - 10 = 0.$$

Its normalized magnitude is $\sqrt{17}/\sqrt{17} = 1$, so both requirements are satisfied.

Step 3: Conclusion One required unit vector is $\boxed{\frac{3\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{17}}}$. Its negative is also a valid answer.

Quick Tip to Solve: Remove any common numerical factor from a cross product before normalizing; here dividing the normal by 2 makes the magnitude calculation shorter.