CBSE Class 12 Mathematics Question Paper 2025 with Solution PDF-Set 65-5-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All questions are compulsory.**

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculator is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then A^3 is: [1 mark]

(A) $3\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(B) $\begin{pmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{pmatrix}$

(C) $\begin{pmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

(D) $\begin{pmatrix} 5^3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{pmatrix}$

Solution:

Concept:

- A scalar matrix has the form $A = kI$, where I is the identity matrix.
- Matrix powers mean repeated multiplication: $A^3 = AAA$, and $I^2 = I$.

Step 1: Identify the scalar matrix All three diagonal entries are 5 and every off-diagonal entry is zero. Thus $A = 5I_3$. Multiplication by this matrix multiplies each coordinate by 5.

Step 2: Compute the third power and compare

$$A^3 = (5I_3)(5I_3)(5I_3) = 125I_3.$$

Options (A) and (C) give $3A = 15I_3$, which is scalar multiplication rather than cubing. Option (D) cubes only one diagonal entry, although repeated matrix multiplication cubes all three diagonal entries.

Step 3: Conclusion $A^3 = \begin{pmatrix} 125 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 125 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: For a diagonal matrix, raise every diagonal entry to the required power; keep all off-diagonal entries zero.

2. If $P(A \cup B) = 0.9$ and $P(A \cap B) = 0.4$, then $P(\overline{A}) + P(\overline{B})$ is: [1 mark]

(A) 0.3

(B) 1

(C) 1.3

(D) 0.7

Correct Answer: (D) 0.7

Solution:

Concept:

- The addition theorem is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- For any event E, its complement satisfies $P(\overline{E}) = 1 - P(E)$.

Step 1: Find the sum of the event probabilities Rearranging the addition theorem gives

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) = 0.9 + 0.4 = 1.3.$$

The intersection is added here because it was subtracted when the union was formed.

Step 2: Use the complement rule

$$P(\overline{A}) + P(\overline{B}) = 2 - [P(A) + P(B)] = 2 - 1.3 = 0.7.$$

Option (C) is the sum for the original events, not their complements. Option (A) incorrectly subtracts the intersection, and option (B) does not follow from the given probabilities.

Step 3: Conclusion $P(\overline{A}) + P(\overline{B}) = 0.7$. Hence, the correct option is (D).

Quick Tip to Solve: Directly use $P(\overline{A}) + P(\overline{B}) = 2 - P(A \cup B) - P(A \cap B)$.

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3. If $A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ -1 & 2 \\ 0 & 5 \end{pmatrix}$, then the correct statement is: [1 mark]

(A) Only AB is defined.

(B) Only BA is defined.

(C) AB and BA, both are defined.

(D) AB and BA, both are not defined.

Correct Answer: (C) AB and BA, both are defined.

Solution:

Concept:

- A product of an $m \times n$ matrix and an $n \times p$ matrix exists and has order $m \times p$.
- Test each order of multiplication separately by comparing the inner dimensions.

Step 1: Check the product AB The matrix A has two rows and three columns, so its order is 2×3.

Quick Tip to Solve: Write the matrix orders beside each other and check the two inner numbers before doing any multiplication.

The matrix B has three rows and two columns, so its order is 3×2. For AB, the inner dimensions are both 3, and the resulting matrix has order 2×2.

Step 2: Check the product BA For BA, the dimensions are $(3 \times 2)(2 \times 3)$. The inner dimensions are both 2, so this product also exists and has order 3×3. Thus (A) and (B) each wrongly exclude one valid product, while (D) wrongly excludes both. The different result sizes do not prevent either product from existing.

Step 3: Conclusion AB and BA are both defined. Hence, the correct option is (C).

Quick Tip to Solve: Write the matrix orders beside each other and check the two inner numbers before doing any multiplication.

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4. If $\begin{vmatrix} 2x & 5 \\ 12 & x \end{vmatrix} = \begin{vmatrix} 6 & -5 \\ 4 & 3 \end{vmatrix}$**, then the value of** x **is:** [1 mark]

(A) 3

(B) 7

(C) ± 7

(D) ± 3

Correct Answer: (C) ± 7

Solution:

Concept:

- For a 2×2 matrix, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- An equation $x^2 = k$, with $k > 0$, has both roots $x = \pm\sqrt{k}$.

Step 1: Expand both determinants The left determinant is $(2x)(x) - (5)(12) = 2x^2 - 60$. The right determinant is $(6)(3) - (-5)(4) = 18 + 20 = 38$. The negative entry in the right determinant makes the second term positive after subtraction.

Step 2: Solve and test the options Equating the values gives

$$2x^2 - 60 = 38 \implies 2x^2 = 98 \implies x^2 = 49.$$

Both 7 and -7 satisfy the original equality. Option (B) omits the negative root; options (A) and (D) give $x^2 = 9$, which does not satisfy it.

Step 3: Conclusion $x = \pm 7$. Hence, the correct option is (C).

Quick Tip to Solve: Expand each determinant as diagonal product minus off-diagonal product, then retain both square roots.

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5. If $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **is continuous at** $x = 0$**, then the value of** a **is:** [1 mark]

(A) 1

(B) -1

(C) ± 1

(D) 0

Correct Answer: (C) ± 1

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \to 0} f(x) = f(0)$.
- The standard trigonometric limit is $\lim_{u \to 0} \sin u/u = 1$, with angles in radians.

Step 1: Evaluate the limiting value For $a \neq 0$, rewrite

$$\frac{\sin^2(ax)}{x^2} = a^2 \left(\frac{\sin(ax)}{ax} \right)^2 .$$

As $x \to 0$, the bracket tends to 1, so the limit is a^2. For $a = 0$, the expression is zero for every nonzero x, giving the same formula for the limit.

Step 2: Impose continuity and compare options The defined value is $f(0) = 1$. Therefore continuity requires $a^2 = 1$, which admits $a = 1$ and $a = -1$. Options (A) and (B) each list just one of the two permissible values. Option (D) gives a limit of zero, unequal to the defined value.

Step 3: Conclusion $a = \pm 1$. Hence, the correct option is (C).

Quick Tip to Solve: Extract the factor a^2 before using the sine limit; the square preserves both positive and negative values of a.

6. If $A = [a_{ij}]$ **is a** 3×3 **diagonal matrix such that** $a_{11} = 1$, $a_{22} = 5$ **and** $a_{33} = -2$**, then** $|A|$ **is:** [1 mark]

(A) 0

(B) -10

(C) 10

(D) 1

Correct Answer: (B) -10

Solution:

Concept:

- The determinant of a diagonal matrix equals the product of its diagonal entries.
- All off-diagonal entries of a diagonal matrix are zero.

Quick Tip to Solve: For a diagonal determinant, multiply the diagonal entries and count negative factors to check the sign.

Step 1: Write the given diagonal matrix Using the stated entries,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since only the diagonal entries are nonzero, the determinant expansion has just the diagonal product as its nonzero term.

Step 2: Multiply and check the sign

$$|A| = 1 \cdot 5 \cdot (-2) = -10.$$

The product is negative because exactly one factor is negative. Option (A) would require a zero diagonal factor, but none is zero. Option (C) loses the negative sign, and option (D) uses only the first entry instead of all three.

Step 3: Conclusion $|A| = -10$. Hence, the correct option is (B).

Quick Tip to Solve: For a diagonal determinant, multiply the diagonal entries and count negative factors to check the sign.

7. The principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is: [1 mark]

(A) $-\frac{\pi}{3}$

(B) $-\frac{2\pi}{3}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Correct Answer: (D) $\frac{2\pi}{3}$

Solution:

Concept:

- The principal value range for $\cot^{-1} x$ is $(0, \pi)$.
- Within this interval, cotangent is negative in the second quadrant.

Step 1: Find the reference angle Let $\theta = \cot^{-1}(-1/\sqrt{3})$. Then $\cot\theta = -1/\sqrt{3}$. Since $\cot(\pi/3) = 1/\sqrt{3}$, the reference angle is $\pi/3$. The negative sign puts the principal angle in the second quadrant.

Step 2: Choose the principal angle and test options The required angle is $\theta = \pi - \pi/3 = 2\pi/3$, and $\cot(2\pi/3) = -1/\sqrt{3}$. Options (A) and (B) are outside $(0, \pi)$; in addition, (B) has positive cotangent. Option (C) lies in the range but has positive cotangent, so it does not match the input.

Step 3: Conclusion The principal value is $\frac{2\pi}{3}$. Hence, the correct option is (D).

Quick Tip to Solve: For a negative inverse-cotangent input, subtract the positive reference angle from π.

8. If $\begin{pmatrix} 4+x & x-1 \\ -2 & 3 \end{pmatrix}$ is a singular matrix, then the value of x is: [1 mark]

(A) 0

(B) 1

(C) -2

(D) -4

Correct Answer: (C) -2

Solution:

Concept:

- A square matrix is singular exactly when its determinant is zero.
- For a 2×2 matrix, the determinant is the main diagonal product minus the other diagonal product.

Step 1: Set up the singularity equation The determinant of the given matrix is

$$\begin{vmatrix} 4+x & x-1 \\ -2 & 3 \end{vmatrix} = 3(4+x) - (-2)(x-1).$$

Carefully retaining the subtraction sign gives $12 + 3x + 2x - 2 = 10 + 5x$.

Step 2: Solve and check the alternatives Singularity requires $10 + 5x = 0$, hence $5x = -10$ and $x = -2$. Substituting the four listed values into the determinant gives respectively $10, 15, 0, -10$. Only the third value produces zero, so the other three options all correspond to nonsingular matrices.

Step 3: Conclusion $x = -2$. Hence, the correct option is (C).

Quick Tip to Solve: Set the determinant to zero and expand the negative off-diagonal product carefully before collecting terms.

9. If $f(x) = \{[x],\ x \in \mathbb{R}\}$ is the greatest integer function, then the correct statement is: [1 mark]

(A) f is continuous but not differentiable at $x = 2$.

(B) f is neither continuous nor differentiable at $x = 2$.

(C) f is continuous as well as differentiable at $x = 2$.

(D) f is not continuous but differentiable at $x = 2$.

Correct Answer: (B) f is neither continuous nor differentiable at $x = 2$.

Solution:

Concept:

- The greatest integer function $[x]$ is the largest integer not exceeding x.
- Continuity requires both one-sided limits to equal the function value. Differentiability implies continuity.

Step 1: Check continuity at the integer For $1 \le x < 2$, $f(x) = 1$, while for $2 \le x < 3$, $f(x) = 2$. Thus $\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = f(2) = 2$. The unequal limits show a jump discontinuity.

Quick Tip to Solve: At an integer, compare the two neighbouring constant values of the greatest integer function first.

Step 2: Check differentiability and the alternatives A differentiable function must be continuous at the point. The jump therefore rules out differentiability at $x = 2$. Options (A) and (C) incorrectly assert continuity, and option (D) incorrectly allows differentiability despite discontinuity.

Step 3: Conclusion f is neither continuous nor differentiable at $x = 2$. Hence, the correct option is (B).

Quick Tip to Solve: At an integer, compare the two neighbouring constant values of the greatest integer function first.

10. The slope of the curve $y = -x^3 + 3x^2 + 8x - 20$ is maximum at: [1 mark]

(A) $(1, -10)$

(B) $(1, 10)$

(C) $(10, 1)$

(D) $(-10, 1)$

Correct Answer: (A) $(1, -10)$

Solution:

Concept:

- The slope of a curve $y = f(x)$ is $m(x) = f\prime(x)$.
- A quadratic with a negative coefficient of x^2 has its global maximum at its vertex.

Step 1: Maximise the slope function Differentiating gives $m(x) = -3x^2 + 6x + 8$. Completing the square,

$$m(x) = 11 - 3(x - 1)^2.$$

Since a square is non-negative, the largest slope occurs at $x = 1$.

Step 2: Find the corresponding point At $x = 1$, $y = -1 + 3 + 8 - 20 = -10$. Option (B) has the correct abscissa but the wrong ordinate. Options (C) and (D) have abscissae different from 1, so neither can give the largest slope. The slope value 11 is not the required ordinate.

Step 3: Conclusion The slope is maximum at $(1, -10)$. Hence, the correct option is (A).

Quick Tip to Solve: Maximise the first derivative, then substitute its abscissa in the original curve to find the point.

11. $\int \sqrt{1+\sin x}\,dx$ **is equal to:** [1 mark]

(A) $2\left(-\sin\frac{x}{2}+\cos\frac{x}{2}\right)+C$

(B) $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+C$

(C) $-2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+C$

(D) $2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)+C$

Correct Answer: (B) $2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+C$

Solution:

Concept:

- The half-angle identity is $1+\sin x=(\sin\frac{x}{2}+\cos\frac{x}{2})^2$.
- The square root of a square is an absolute value; an antiderivative is chosen on an interval of fixed sign.

Step 1: Reduce the square root Write

$$\sqrt{1+\sin x}=\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|.$$

On an interval where the expression inside is non-negative, as in the usual half-angle evaluation, the absolute value equals $\sin\frac{x}{2}+\cos\frac{x}{2}$.

Step 2: Integrate and compare the options Integration on such an interval gives

$$2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)+C.$$

Differentiating option (B) returns the positive half-angle sum. Option (A) returns its negative, and (C) and (D) return differences rather than this sum. On intervals where the half-angle sum is negative, the antiderivative has the opposite sign; no single displayed expression works globally on all real x.

Step 3: Conclusion The standard intervalwise choice is $2(\sin\frac{x}{2}-\cos\frac{x}{2})+C$. Hence, the intended correct option is (B).

Quick Tip to Solve: Use the half-angle square identity, but check the sign before dropping the absolute value.

12. $\int_0^{\pi/2} \cos x \cdot e^{\sin x}\, dx$ **is equal to:** [1 mark]

(A) 0

(B) $1 - e$

(C) $e - 1$

(D) e

Correct Answer: (C) $e - 1$

Solution:

Concept:

- For $u = \sin x$, the differential is $du = \cos x\, dx$.
- When substituting in a definite integral, transform both limits into values of the new variable.

Step 1: Change the variable and the limits Let $u = \sin x$. At $x = 0$, $u = 0$, and at $x = \pi/2$, $u = 1$. Therefore the integral becomes

$$\int_0^1 e^u\, du.$$

The cosine factor is exactly the derivative needed for this substitution.

Step 2: Evaluate the exponential integral The antiderivative of e^u is e^u, so the value is $[e^u]_0^1 = e - 1$. It is positive, ruling out options (A) and (B). Option (D) omits the lower-limit contribution $e^0 = 1$.

Step 3: Conclusion The value is $e - 1$. Hence, the correct option is (C).

Quick Tip to Solve: Recognise the cosine as the derivative of the sine in the exponent and subtract both endpoint values.

13. The area of the region enclosed between the curve $y = x|x|$, x-axis, $x = -2$ and $x = 2$ is: [1 mark]

(A) $\frac{8}{3}$

(B) $\frac{16}{3}$

(C) 0

(D) 8

Correct Answer: (B) $\frac{16}{3}$

Solution:

Concept:

- Geometric area is $\int |y|\, dx$, so portions below the axis contribute positively.
- For $y = x|x|$, one has $y = -x^2$ for $x < 0$ and $y = x^2$ for $x \geq 0$.

Step 1: Split at the change of sign The curve lies below the axis from -2 to 0 and above it from 0 to 2. Thus

$$A = -\int_{-2}^{0} (-x^2)\, dx + \int_0^2 x^2\, dx.$$

Quick Tip to Solve: For an odd curve crossing the axis, add the absolute areas of the two lobes instead of cancelling them.

Both lobes must be included with positive area.

Step 2: Add the two equal areas Each integral of the positive height x^2 contributes $8/3$. Hence $A = 8/3 + 8/3 = 16/3$ square units. Option (A) counts just one lobe; (C) confuses signed integral with area; (D) does not equal the integral of the quadratic heights.

Step 3: Conclusion The enclosed area is $\frac{16}{3}$ square units. Hence, the correct option is (B).

Quick Tip to Solve: For an odd curve crossing the axis, add the absolute areas of the two lobes instead of cancelling them.

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14. The integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ **is:** [1 mark]

(A) $e^{-1/\sqrt{x}}$

(B) $e^{2/\sqrt{x}}$

(C) $e^{2\sqrt{x}}$

(D) $e^{-2\sqrt{x}}$

Correct Answer: (C) $e^{2\sqrt{x}}$

Solution:

Concept:

- A linear equation $dy/dx + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$.
- The coefficient $P(x)$ must be identified after the coefficient of dy/dx is made one.

Step 1: Express the equation in linear form From the printed equation,

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}.$$

Hence

$$\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}, \qquad x > 0.$$

The independent variable is x, despite the original use of dx/dy.

Step 2: Integrate the coefficient of y Here $P(x) = x^{-1/2}$, so $\int P(x)\,dx = 2\sqrt{x}$. Thus the integrating factor is $e^{2\sqrt{x}}$. Option (D) has the wrong sign; options (A) and (B) use reciprocal square roots, which do not result from integrating $x^{-1/2}$.

Step 3: Conclusion The integrating factor is $e^{2\sqrt{x}}$. Hence, the correct option is (C).

Quick Tip to Solve: Rewrite using dy/dx first; integrate the coefficient of y, not the exponential on the right-hand side.

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15. The sum of the order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$ is: [1 mark]

(A) 2
(B) $\frac{5}{2}$
(C) 3
(D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The order is the order of the highest derivative present.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The printed equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}.$$

Its highest derivative is d^2y/dx^2, so its order is 2. The exponent outside the bracket does not change the derivative order.

Step 2: Find the degree and compare the choices The equation already is polynomial in its derivatives. Expanding the left side gives $1 + 3(y\prime)^2 + 3(y\prime)^4 + (y\prime)^6$. The highest-order derivative $y\prime\prime$ has power 1, so degree is 1. The sum is $2 + 1 = 3$. Option (A) counts only the order; (B) gives a fractional degree; (D) incorrectly uses degree 2.

Step 3: Conclusion The sum of order and degree is 3. Hence, the correct option is (C).

Quick Tip to Solve: Look at the power of the highest-order derivative, not the largest power of any derivative in the equation.

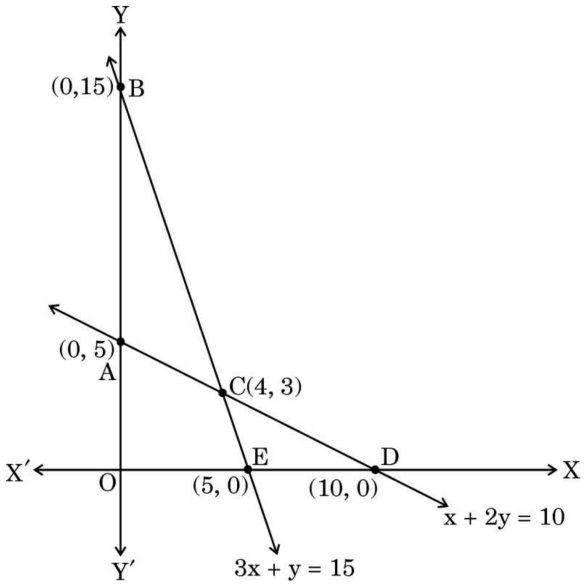
16. For a Linear Programming Problem (LPP), the given objective function $Z = 3x + 2y$ is subject to constraints:

$x + 2y \leq 10$

$3x + y \leq 15$

$x, y \geq 0$

The correct feasible region is: [1 mark]



(A) ABC

(B) $AOEC$

(C) CED

(D) Open unbounded region BCD

Correct Answer: (B) $AOEC$

Solution:

Concept:

- The feasible region is the common part of all constraint half-planes, with $x, y \geq 0$.
- Test the origin to select each half-plane; the objective function does not affect feasibility.

Step 1: Identify the common half-planes At the origin, both $x + 2y \leq 10$ and $3x + y \leq 15$ hold. Therefore take the side containing O for each line and restrict to the first quadrant. The first line meets the positive y-axis at $A(0, 5)$, and the second meets the positive x-axis at $E(5, 0)$.

Step 2: Find the intersection and boundary Solving $x + 2y = 10$ and $3x + y = 15$ gives $x = 4, y = 3$, the printed point C. The common region has vertices A, O, E, C. Triangle ABC includes $B(0, 15)$, which violates the first inequality. Triangle CED includes $D(10, 0)$, which violates the second. The unbounded region BCD cannot satisfy both upper bounds in the first quadrant.

Step 3: Conclusion The feasible region is $AOEC$, including its boundary: option (B).

Quick Tip to Solve: Test the origin in both inequalities, then retain only their common sides in the first quadrant.

17. Let $\vec{a}$ be a position vector whose tip is the point $(2, -3)$. If $\overrightarrow{AB} = \vec{a}$, where coordinates of A are $(-4, 5)$, then the coordinates of B are: [1 mark]

(A) $(-2, -2)$

(B) $(2, -2)$

(C) $(-2, 2)$

(D) $(2, 2)$

Correct Answer: (C) $(-2, 2)$

Solution:

Concept:

- A position vector with tip (u, v) has components $u\hat{i} + v\hat{j}$, measured from the origin.
- For points A and B, $\overrightarrow{AB} = \vec{OB} - \vec{OA}$; hence the endpoint equals the starting point plus the displacement.

Step 1: Translate the vector into components The position vector ending at $(2, -3)$ is $\vec{a} = 2\hat{i} - 3\hat{j}$. Write $B = (x, y)$. Since $A = (-4, 5)$, the vector from A to B is $\overrightarrow{AB} = (x + 4)\hat{i} + (y - 5)\hat{j}$.

Step 2: Equate components and check options Using $\overrightarrow{AB} = \vec{a}$, we obtain $x + 4 = 2$ and $y - 5 = -3$, so $x = -2$, $y = 2$. Option (A) gives vertical displacement -7, while (B) and (D) give horizontal displacement 6; none equals the specified vector.

Step 3: Conclusion $B = (-2, 2)$. Hence, the correct option is (C).

Quick Tip to Solve: Add the displacement components to the coordinates of A; do not subtract A from the displacement.

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18. The respective values of $|\vec{a}|$ and $|\vec{b}|$, if given $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 512$ and $|\vec{a}| = 3|\vec{b}|$, are: [1 mark]

(A) 48 and 16

(B) 3 and 1

(C) 24 and 8

(D) 6 and 2

Correct Answer: (C) 24 and 8

Solution:

Concept:

- The scalar product satisfies $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$, because the mixed products cancel.
- Vector magnitudes are nonnegative; use the positive square root when solving for a magnitude.

Step 1: Expand the scalar product By distributivity and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, the given equation becomes $|\vec{a}|^2 - |\vec{b}|^2 = 512$. Set $|\vec{b}| = t \geq 0$; the given ratio then yields $|\vec{a}| = 3t$.

Step 2: Solve and test the alternatives

$$9t^2 - t^2 = 512 \quad \Longrightarrow \quad t^2 = 64 \quad \Longrightarrow \quad t = 8.$$

Quick Tip to Solve: Cancel the mixed dot products first, then substitute the ratio into the difference of squared magnitudes.

Thus $|\vec{a}| = 24$. All four pairs have ratio $3 : 1$, but their squared differences are respectively $2048, 8, 512, 32$. Only the third pair meets both conditions.

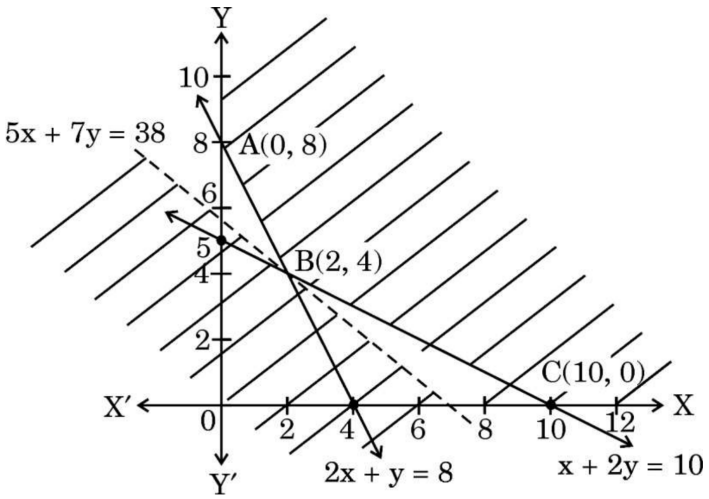
Step 3: Conclusion $|\vec{a}| = 24$ and $|\vec{b}| = 8$. Hence, the correct option is (C).

Quick Tip to Solve: Cancel the mixed dot products first, then substitute the ratio into the difference of squared magnitudes.

19. Assertion (A): The shaded portion of the graph represents the feasible region for the given Linear Programming Problem (LPP).

Min $Z = 50x + 70y$ subject to constraints $2x + y \geq 8$, $x + 2y \geq 10$, $x, y \geq 0$. $Z = 50x + 70y$ has a minimum value $= 380$ at $B(2, 4)$.

Reason (R): The region representing $50x + 70y < 380$ does not have any point common with the feasible region. [1 mark]



(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The feasible region is the intersection of all constraint half-planes, including $x, y \geq 0$.
- For an unbounded feasible region, a candidate minimum must also be checked by ruling out feasible points with smaller objective values.

Quick Tip to Solve: For an unbounded region, verify that the strict lower-objective half-plane has no feasible points.

Step 1: Judge Assertion (A) The shaded region lies above both boundary lines in the first quadrant, as required by the two inequalities. Their intersection solves $2x + y = 8$, $x + 2y = 10$, giving $B = (2, 4)$. At the boundary vertices $A = (0, 8)$, $B = (2, 4)$, $C = (10, 0)$, the objective values are $560, 380, 500$, respectively. Thus the stated candidate minimum is 380 at B.

Step 2: Judge Reason (R) Multiply $2x + y \geq 8$ by 10, and $x + 2y \geq 10$ by 30. Adding gives

$$50x + 70y \geq 80 + 300 = 380.$$

Therefore no point satisfying the constraints can lie in the region $50x + 70y < 380$, so Reason (R) is true.

Step 3: Check whether the reason explains the assertion Point $B = (2, 4)$ is feasible and attains $Z = 380$. Reason (R) excludes every smaller value throughout the unbounded feasible region. It therefore establishes the claimed global minimum, and Assertion (A) is true with Reason (R) as its correct explanation.

Step 4: Conclusion Both Assertion (A) and Reason (R) are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: For an unbounded region, verify that the strict lower-objective half-plane has no feasible points.

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20. Assertion (A): Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. If $f : A \to A$ is defined as $f(x) = x^2$, then f is not an onto function.

Reason (R): If $y = -1 \in A$, then $x = \pm\sqrt{-1} \notin A$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- A function $f : A \to A$ is onto only if every element of its codomain has a preimage in its domain.
- For real x, $x^2 \geq 0$, and for $-1 \leq x \leq 1$, also $x^2 \leq 1$.

Step 1: Judge Assertion (A) Here $A = [-1, 1]$. Squaring every allowed input gives the range $[0, 1]$, which is a proper subset of the codomain $[-1, 1]$. The negative elements of the codomain are missing from the range. Therefore f is not onto, and Assertion (A) is true.

Step 2: Judge Reason (R) The element $y = -1$ belongs to the codomain. Its preimage would have to satisfy $x^2 = -1$, formally giving $x = \pm\sqrt{-1}$. Neither value is real, so neither lies in A. Equivalently,

Quick Tip to Solve: To disprove onto, choose one codomain value and show that its preimage equation has no allowed solution.

no real square equals -1. Thus Reason (R) is true.

Step 3: Check the explanatory connection Reason (R) identifies a specific codomain element with no preimage. One such element suffices to disprove surjectivity, so it directly explains why the function is not onto. The two statements are true, and the explanation is valid.

Step 4: Conclusion Both Assertion (A) and Reason (R) are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: To disprove onto, choose one codomain value and show that its preimage equation has no allowed solution.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Find the domain of the function $f(x) = \cos^{-1}(x^2 - 4)$. [2 marks]

Solution:

Concept:

- The real inverse cosine $\cos^{-1} u$ is defined exactly for $-1 \le u \le 1$.
- For $a > 0$, $x^2 \ge a$ means $|x| \ge \sqrt{a}$, while $x^2 \le a$ means $|x| \le \sqrt{a}$.

Step 1: Impose the inverse cosine condition The polynomial $x^2 - 4$ itself exists for every real x, but its value must be in the input interval of inverse cosine. Consequently,

$$-1 \le x^2 - 4 \le 1.$$

Adding 4 to all three terms preserves both inequality directions and gives

$$3 \le x^2 \le 5.$$

Step 2: Solve the simultaneous inequalities The lower bound requires $x \le -\sqrt{3}$ or $x \ge \sqrt{3}$. The upper bound requires $-\sqrt{5} \le x \le \sqrt{5}$. Taking their intersection gives

$$-\sqrt{5} \le x \le -\sqrt{3} \quad \text{or} \quad \sqrt{3} \le x \le \sqrt{5}.$$

The endpoints are included: the inner expression is 1 or -1, and inverse cosine is defined at both. Values between $-\sqrt{3}$ and $\sqrt{3}$ give an inner expression less than -1; values outside the outer endpoints give one greater than 1.

Step 3: Conclusion The domain is $\boxed{[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}$.

Quick Tip to Solve: Solve the inverse cosine input inequality first; remember both positive and negative intervals when taking square roots.

22. Surface area of a balloon (spherical), when air is blown into it, increases at a rate of $5\ \text{mm}^2/\text{s}$. When the radius of the balloon is 8 mm, find the rate at which the volume of the balloon is increasing.

[2 marks]

Solution:

Concept:

- For a sphere of radius r, surface area is $S = 4\pi r^2$ and volume is $V = \frac{4}{3}\pi r^3$.
- When r varies with time t, the chain rule gives $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$ and $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$.

Step 1: Find the instantaneous radius growth The given area growth is $\frac{dS}{dt} = 5\,\text{mm}^2/\text{s}$, and the radius at the instant of interest is $r = 8\,\text{mm}$. Differentiating the area formula with respect to time,

$$5 = 8\pi(8)\frac{dr}{dt}, \qquad \frac{dr}{dt} = \frac{5}{64\pi}\ \text{mm/s}.$$

This is the instantaneous rate; the problem does not require it to remain constant at other radii.

Step 2: Use the volume derivative Substitute that same instantaneous radius rate into the derivative of volume:

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt}\\ &= 4\pi(8)^2\frac{5}{64\pi}\\ &= 20\,\text{mm}^3/\text{s}.\end{aligned}$$

The positive sign agrees with air being blown into the balloon. As a check, eliminating dr/dt directly gives $dV/dt = (r/2)(dS/dt)$, which also yields $(8/2)5 = 20$.

Step 3: Conclusion The volume is increasing at $\boxed{20\,\text{mm}^3/\text{s}}$.

Quick Tip to Solve: Divide the volume and surface-area derivatives to get dV/dt = (r/2)dS/dt without computing the radius rate.

23(a). Differentiate $\frac{\sin x}{\sqrt{\cos x}}$ with respect to x. [2 marks]

Solution:

Concept:

- The product rule is $(uv)\prime = u\prime v + uv\prime$, and the chain rule differentiates a composite function by multiplying the outer derivative by the inner derivative.
- Use $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$. For this real-valued quotient, work where $\cos x > 0$.

Step 1: Rewrite and differentiate the factors Let $y = \sin x(\cos x)^{-1/2}$. The reciprocal square-root factor has derivative

$$\frac{d}{dx}(\cos x)^{-1/2} = -\frac{1}{2}(\cos x)^{-3/2}(-\sin x) = \frac{\sin x}{2(\cos x)^{3/2}}.$$

Quick Tip to Solve: Write the denominator as a power before differentiating; the two minus signs in its derivative cancel.

The two negative signs cancel; retaining this sign is essential.

Step 2: Apply the product rule and simplify Therefore

$$\frac{dy}{dx} = \cos x(\cos x)^{-1/2} + \sin x \frac{\sin x}{2(\cos x)^{3/2}}$$
$$= \frac{2\cos^2 x + \sin^2 x}{2(\cos x)^{3/2}}.$$

Using $\sin^2 x + \cos^2 x = 1$, the numerator becomes $1 + \cos^2 x$. Both terms have been combined over the same positive denominator on the domain.

Step 3: Conclusion

$$\boxed{\frac{d}{dx}\left(\frac{\sin x}{\sqrt{\cos x}}\right) = \frac{1 + \cos^2 x}{2(\cos x)^{3/2}}}.$$

Quick Tip to Solve: Write the denominator as a power before differentiating; the two minus signs in its derivative cancel.

OR

23(b). **If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.** [2 marks]

Solution:

Concept:

- The derivatives are $\frac{d}{dx}\cos x = -\sin x$ and $\frac{d}{dx}\sin x = \cos x$. Constants multiply their respective derivatives.
- Differentiating either sine or cosine twice gives the negative of the original function. Apply this to each term and then add the original y.

Step 1: Calculate the first derivative For the given function $y = 5\cos x - 3\sin x$, differentiate each term with respect to x:

$$\frac{dy}{dx} = 5(-\sin x) - 3(\cos x) = -5\sin x - 3\cos x.$$

The coefficient of the cosine term is negative because the original sine term already has coefficient -3.

Step 2: Calculate the second derivative and verify the identity Differentiate the first derivative once more:

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x.$$

Quick Tip to Solve: For any constant linear combination of sine and cosine, the second derivative is the negative of the function.

Now substitute the given y into the required expression:

$$\frac{d^2y}{dx^2} + y = (-5\cos x + 3\sin x) + (5\cos x - 3\sin x)$$
$$= (-5+5)\cos x + (3-3)\sin x = 0.$$

Thus the sine coefficients cancel separately, as do the cosine coefficients, for every real x.

Step 3: Conclusion

$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$

Hence the required identity is proved.

Quick Tip to Solve: For any constant linear combination of sine and cosine, the second derivative is the negative of the function.

24(a). Find a vector of magnitude 5 which is perpendicular to both the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $4\hat{i} + 3\hat{j} - 2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Its direction is well-defined here because the two vectors are not parallel.
- A vector of magnitude m in the direction of a nonzero vector $\vec{n}$ is $m\vec{n}/|\vec{n}|$. Reversing its sign gives the other perpendicular direction.

Step 1: Find a common perpendicular direction Let $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} + 3\hat{j} - 2\hat{k}$. Then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$
$$= (4-3)\hat{i} - (-6-4)\hat{j} + (9+8)\hat{k}$$
$$= \hat{i} + 10\hat{j} + 17\hat{k}.$$

In particular, this cross product is nonzero.

Step 2: Scale to the required magnitude The magnitude of the cross product is

$$\sqrt{1^2 + 10^2 + 17^2} = \sqrt{390}.$$

Multiplication by $5/\sqrt{390}$ gives magnitude 5. As a direct check on the direction, its dot products with the original vectors have numerators $3 - 20 + 17 = 0$ and $4 + 30 - 34 = 0$, respectively. Scaling preserves these zero dot products, so the resulting vector meets both perpendicularity conditions.

Quick Tip to Solve: Find the cross product first, then divide by its magnitude before multiplying by the required length.

Step 3: Conclusion One required vector is

$$\boxed{\frac{5}{\sqrt{390}}(\hat{i}+10\hat{j}+17\hat{k})}.$$

Its negative is also a valid answer.

Quick Tip to Solve: Find the cross product first, then divide by its magnitude before multiplying by the required length.

OR

24(b). Let $\vec{a},\vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$. Show that $\vec{b}=\vec{c}$. [2 marks]

Solution:

Concept:

- Dot and cross products distribute over subtraction. Introduce $\vec{d}=\vec{b}-\vec{c}$ to combine the two given equalities.
- For nonzero $\vec{a}$, the equality $\vec{a}\times\vec{d}=\vec{0}$ says $\vec{d}$ is a scalar multiple of $\vec{a}$. Also, $\vec{a}\cdot\vec{a}=|\vec{a}|^2>0$.

Step 1: Subtract the given equalities Put $\vec{d}=\vec{b}-\vec{c}$. From $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, distributivity gives $\vec{a}\cdot\vec{d}=0$. Similarly, from $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, we obtain $\vec{a}\times\vec{d}=\vec{0}$. Thus the same difference vector satisfies both conditions.

Step 2: Use the nonzero vector hypothesis Since $\vec{a}\neq\vec{0}$ and $\vec{a}\times\vec{d}=\vec{0}$, write $\vec{d}=\lambda\vec{a}$ for some real scalar λ; this includes the possibility $\vec{d}=\vec{0}$. Substitution into the dot-product condition gives

$$0=\vec{a}\cdot\vec{d}=\lambda(\vec{a}\cdot\vec{a})=\lambda|\vec{a}|^2.$$

Because $|\vec{a}|^2>0$, division by this number is valid and yields $\lambda=0$. Therefore the difference vector vanishes.

Step 3: Conclusion

$$\vec{b}-\vec{c}=\vec{0} \quad\Longrightarrow\quad \boxed{\vec{b}=\vec{c}}.$$

Quick Tip to Solve: Subtract first: a difference vector that is both parallel and perpendicular to a nonzero vector must vanish.

25. A man needs to hang two lanterns on a straight wire whose end points have coordinates $A(4, 1, -2)$ and $B(6, 2, -3)$. Find the coordinates of the points where he hangs the lanterns such that these points trisect the wire AB. [2 marks]

Solution:

Concept:

- The internal section formula for a point dividing AB in the ratio $m : n$ is $P = (nA + mB)/(m + n)$, applied coordinate by coordinate.
- The two trisection points divide AB in ratios $1 : 2$ and $2 : 1$, measured from A. Equivalently they lie one-third and two-thirds of the way from A to B.

Step 1: Find the lantern point nearer A Let the first point be P, so $AP : PB = 1 : 2$. With $A = (4, 1, -2)$ and $B = (6, 2, -3)$, the internal section formula gives

$$\begin{aligned} P &= \frac{2A + B}{3} \\ &= \left(\frac{8+6}{3}, \frac{2+2}{3}, \frac{-4-3}{3}\right) \\ &= \left(\frac{14}{3}, \frac{4}{3}, -\frac{7}{3}\right). \end{aligned}$$

The greater weight on A places this point nearer A.

Step 2: Find the lantern point nearer B and check the spacing For the second point Q, $AQ : QB = 2 : 1$. Hence

$$Q = \frac{A + 2B}{3} = \left(\frac{4+12}{3}, \frac{1+4}{3}, \frac{-2-6}{3}\right) = \left(\frac{16}{3}, \frac{5}{3}, -\frac{8}{3}\right).$$

To check the spacing, $B - A = (2, 1, -1)$. Each of $P - A$, $Q - P$, and $B - Q$ equals $(2/3, 1/3, -1/3)$. Thus the three successive portions have the same direction and length.

Step 3: Conclusion The lanterns should be hung at

$$\boxed{\left(\frac{14}{3}, \frac{4}{3}, -\frac{7}{3}\right) \text{ and } \left(\frac{16}{3}, \frac{5}{3}, -\frac{8}{3}\right)}.$$

Quick Tip to Solve: Use A plus one-third and two-thirds of the displacement B minus A to avoid reversing section-formula weights.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. Find the value of a for which $f(x) = \sqrt{3}\sin x - \cos x - 2ax + 6$ is decreasing in $\mathbb{R}$. [3 marks]

Solution:

Concept:

- A differentiable function is decreasing on an interval when its derivative is nonpositive throughout that interval. Isolated zeros of the derivative do not prevent strict decrease.
- The expression $A\cos x + B\sin x$ has maximum $\sqrt{A^2 + B^2}$. The parameter must control this maximum, since the condition is required for every real x.

Step 1: Differentiate the function The derivative of the constant is zero, while the derivative of the linear term is constant. Thus

$$f\prime(x) = \sqrt{3}\cos x + \sin x - 2a.$$

We must compare this derivative with zero for every real argument; checking its sign at only one arbitrarily chosen point would not establish decrease on the whole real line.

Step 2: Find the largest possible derivative Using the addition formula,

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right).$$

Since the cosine takes every value from -1 to 1, the maximum derivative is $2 - 2a$, attained when $x = \pi/6 + 2n\pi$, where $n \in \mathbb{Z}$. Therefore the necessary and sufficient derivative condition is

$$2 - 2a \leq 0 \quad \Longleftrightarrow \quad a \geq 1.$$

If $a < 1$, the derivative is positive at these points and, by continuity, on a nearby interval, so the function cannot be decreasing everywhere.

Step 3: Check the boundary value For $a > 1$, the derivative is strictly negative everywhere. For $a = 1$, it is nonpositive and vanishes only at isolated points. On any interval of positive length it is negative except at these isolated points, so the change in f, obtained by integrating the derivative, is strictly negative. Thus the boundary value also gives a decreasing function.

Step 4: Conclusion

$$\boxed{a \in [1, \infty)}.$$

Quick Tip to Solve: Combine the sine and cosine into a single wave; compare its maximum amplitude with the constant derivative term.

27(a). Find: $\displaystyle\int \frac{2x}{(x^2+3)(x^2-5)}\,dx$. [3 marks]

Solution:

Concept:

- When a rational integrand contains x^2 and its derivative $2x$, substitute $t = x^2$, so $dt = 2x\,dx$.
- Decompose a rational function with distinct linear factors into partial fractions, then use $\int dt/(t-c) = \ln|t-c| + C$.

Step 1: Substitute the square of the variable Let $t = x^2$. Then $dt = 2x\,dx$, and the integral becomes

$$I = \int \frac{dt}{(t+3)(t-5)}.$$

The original integrand is defined away from $x = \pm\sqrt{5}$. An antiderivative is understood separately on each interval not crossing these singularities.

Step 2: Determine the partial fractions Write

$$\frac{1}{(t+3)(t-5)} = \frac{A}{t+3} + \frac{B}{t-5}.$$

Multiplication by the denominator gives $1 = A(t-5) + B(t+3)$. Setting $t = -3$ yields $A = -1/8$; setting $t = 5$ yields $B = 1/8$. Hence

$$I = \frac{1}{8}\int \left(\frac{1}{t-5} - \frac{1}{t+3}\right) dt.$$

The signs are essential: reversing them would differentiate to the negative of the required integrand.

Step 3: Integrate and restore the original variable Integration gives

$$I = \frac{1}{8}\big(\ln|t-5| - \ln|t+3|\big) + C.$$

Returning to $t = x^2$, note that $x^2 + 3$ is always positive. As a check, differentiation gives

$$\frac{1}{8}\left(\frac{2x}{x^2-5} - \frac{2x}{x^2+3}\right) = \frac{2x}{(x^2+3)(x^2-5)},$$

exactly the printed integrand.

Step 4: Conclusion

$$\boxed{\int \frac{2x\,dx}{(x^2+3)(x^2-5)} = \frac{1}{8}\ln\left|\frac{x^2-5}{x^2+3}\right| + C}.$$

Quick Tip to Solve: Use the numerator as the differential of x^2; the two denominator factors then differ by the constant 8.

OR

27(b). Evaluate: $\int_1^4 \left(|x-2|+|x-4|\right) dx$. [3 marks]

Solution:

Concept:

- The absolute value satisfies $|u| = u$ for $u \geq 0$ and $|u| = -u$ for $u < 0$. Split an integral at each sign change inside its limits.
- Definite integrals are additive over adjacent intervals: $\int_a^b g(x)dx = \int_a^c g(x)dx + \int_c^b g(x)dx$.

Step 1: Remove the moduli on the correct intervals The expressions inside the moduli vanish at $x = 2$ and $x = 4$. The point 2 lies inside the interval, whereas 4 is its upper endpoint. For $1 \leq x \leq 2$, both expressions are nonpositive, so

$$|x-2| + |x-4| = (2-x) + (4-x) = 6 - 2x.$$

For $2 \leq x \leq 4$, the first expression is nonnegative and the second nonpositive, giving

$$|x-2| + |x-4| = (x-2) + (4-x) = 2.$$

At the shared boundary these formulae both give 2, as expected from continuity.

Step 2: Evaluate the first part The required integral is therefore

$$I = \int_1^2 (6-2x)\, dx + \int_2^4 2\, dx.$$

For the first integral,

$$\int_1^2 (6-2x)\, dx = [6x - x^2]_1^2 = (12-4) - (6-1) = 3.$$

Removing both moduli with a single sign over the entire interval would be incorrect because $x - 2$ changes sign.

Step 3: Evaluate the second part and add The integrand is constant on the remaining interval, so

$$\int_2^4 2\, dx = [2x]_2^4 = 8 - 4 = 4.$$

Adding the contributions gives $I = 3 + 4$. Both are positive, consistent with the fact that the original integrand is a sum of absolute values and hence cannot be negative.

Step 4: Conclusion

$$\boxed{\int_1^4 \left(|x-2|+|x-4|\right) dx = 7}.$$

Quick Tip to Solve: On the interval between 2 and 4 the two distances add to the constant 2; only the segment from 1 to 2 needs linear integration.

28. Find the particular solution of the differential equation $\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx+x\,dy=0$, given that $y=\frac{\pi}{4}$ when $x=1$. [3 marks]

Solution:

Concept:

- A differential equation involving y/x is simplified by $v=y/x$, or $y=vx$; the product rule gives $dy/dx=v+x\,dv/dx$.
- After separation, use $\int \csc^2 v\,dv=-\cot v$ and $\int dx/x=\ln|x|+C$. The given initial point determines the constant and the relevant local branch.

Step 1: Express the equation in derivative form Dividing the printed equation by $x\,dx$, for $x\neq 0$, gives

$$\frac{dy}{dx}=\frac{y}{x}-\sin^2\left(\frac{y}{x}\right).$$

Its right-hand side depends only on the ratio y/x, so the homogeneous substitution is appropriate. The initial point has $x=1$, and hence belongs to the positive x domain.

Step 2: Make the homogeneous substitution and separate Put $y=vx$. Substituting the product-rule derivative yields

$$v+x\frac{dv}{dx}=v-\sin^2 v,\qquad x\frac{dv}{dx}=-\sin^2 v.$$

Near the initial value $v=\pi/4$, the sine is nonzero, so division by $\sin^2 v$ is valid. Thus

$$\csc^2 v\,dv=-\frac{dx}{x}.$$

Integrating both sides gives $-\cot v=-\ln|x|+C_1$, equivalently

$$\cot v=\ln|x|+C.$$

Constant-ratio solutions lost in separation have $\sin v=0$, and none passes through the specified initial point.

Step 3: Apply the initial condition and verify At $x=1$, $y=\pi/4$, so $v=\pi/4$ and $\cot v=1$. Since $\ln 1=0$, the constant is $C=1$. On the positive domain containing the initial point, $\ln|x|=\ln x$. Differentiating the resulting implicit relation yields

$$-\csc^2\left(\frac{y}{x}\right)\frac{xy\prime-y}{x^2}=\frac{1}{x},$$

or $xy\prime-y=-x\sin^2(y/x)$, which is exactly the original equation rearranged.

Step 4: Conclusion The required particular solution, on the branch through the initial point, is

$$\boxed{\cot\left(\frac{y}{x}\right)=\ln x+1,\qquad x>0}.$$

Quick Tip to Solve: After substituting $y=vx$, the two terms v cancel; retain the minus sign when integrating $\csc^2 v$.

29. In the Linear Programming Problem (LPP), find the point/points giving maximum value for $Z = 5x + 10y$ subject to constraints $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$. [3 marks]

Solution:

Concept:

- A feasible point satisfies every constraint simultaneously. A linear objective on a bounded polygon attains a maximum at a vertex or along an entire boundary segment.
- Here $Z = 5x + 10y = 5(x + 2y)$. Thus the constraint $x + 2y \leq 120$ immediately gives the upper bound $Z \leq 600$.

Step 1: Identify the boundary on which equality is possible To attain the upper bound, a feasible point must satisfy $x + 2y = 120$. Write $x = 120 - 2y$ on this line and impose the remaining conditions. The constraint $x + y \geq 60$ becomes $120 - y \geq 60$, so $y \leq 60$. The constraint $x - 2y \geq 0$ becomes $120 - 4y \geq 0$, so $y \leq 30$. Together with nonnegativity, this gives $0 \leq y \leq 30$; throughout this range $x = 120 - 2y \geq 60 > 0$. Thus the equality boundary contains a whole feasible segment.

Step 2: Find and verify the endpoints At $y = 0$, the point is $A = (120, 0)$. At $y = 30$, it is $B = (60, 30)$. Both satisfy $x + y \geq 60$, $x - 2y \geq 0$, and nonnegativity. For any point $(120 - 2t, t)$ with $0 \leq t \leq 30$,

$$Z = 5(120 - 2t) + 10t = 600.$$

Since every feasible point has $Z \leq 600$, these values prove attainment of the global maximum, rather than merely comparing two candidate points. Conversely, a point attaining 600 must lie on $x + 2y = 120$; the preceding constraint analysis then places it on precisely this segment.

Step 3: Conclusion

$$\boxed{Z_{\max} = 600.}$$

Every point on the closed line segment joining $\boxed{(120, 0)}$ and $\boxed{(60, 30)}$ is optimal, equivalently $(x, y) = (120 - 2t, t)$ for $0 \leq t \leq 30$.

Quick Tip to Solve: Factor the objective as 5(x+2y); then check exactly which portion of x+2y=120 is feasible.

30(a). If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ such that $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$. [3 marks]

Solution:

Concept:

- The angle θ between two nonzero vectors lies in $[0, \pi]$, and their scalar product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.
- The squared magnitude of a sum follows from distributivity: $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$. A vector and its negative have equal magnitudes.

Step 1: Use the zero-sum condition Given $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, we have $\vec{a}+\vec{b} = -\vec{c}$. Therefore $|\vec{a}+\vec{b}| = |\vec{c}| = 7$. Squaring the magnitude, rather than adding the individual magnitudes, retains the information about the angle between the two vectors. Taking the scalar product of the sum with itself gives

$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}.$$

Quick Tip to Solve: When three vectors sum to zero, square the magnitude of the sum of two to obtain the angle between them.

Step 2: Find the scalar product and angle Substitute $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{a} + \vec{b}| = 7$:

$$49 = 9 + 25 + 2\vec{a} \cdot \vec{b}, \qquad \vec{a} \cdot \vec{b} = \frac{15}{2}.$$

Since both vectors are nonzero, division by their magnitudes is valid. Hence

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{15/2}{3 \times 5} = \frac{1}{2}.$$

In the interval $[0, \pi]$, this cosine value determines a unique angle. The positive scalar product also confirms that the angle is acute.

Step 3: Conclusion The angle between $\vec{a}$ and $\vec{b}$ is $\boxed{60^\circ}$.

Quick Tip to Solve: When three vectors sum to zero, square the magnitude of the sum of two to obtain the angle between them.

OR

30(b). If $\vec{a}$ and $\vec{b}$ are unit vectors inclined with each other at an angle θ, then prove that $\frac{1}{2}|\vec{a} - \vec{b}| = \sin\frac{\theta}{2}$. [3 marks]

Solution:

Concept:

- For unit vectors, $|\vec{a}| = |\vec{b}| = 1$, so $\vec{a} \cdot \vec{b} = \cos\theta$, where $0 \le \theta \le \pi$ is their angle.
- The squared difference satisfies $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$. The half-angle identity is $1 - \cos\theta = 2\sin^2(\theta/2)$.

Step 1: Expand the squared magnitude Take the scalar product of $\vec{a} - \vec{b}$ with itself and use the commutativity of the scalar product:

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} \\ &= 1 + 1 - 2\cos\theta \\ &= 2(1 - \cos\theta). \end{aligned}$$

The minus sign in the mixed term is essential because we are finding the magnitude of a difference.

Step 2: Apply the half-angle identity and take the root Using $1 - \cos\theta = 2\sin^2(\theta/2)$, the preceding result becomes

$$|\vec{a} - \vec{b}|^2 = 4\sin^2\frac{\theta}{2}.$$

Magnitudes are nonnegative, so taking square roots initially gives $|\vec{a} - \vec{b}| = 2|\sin(\theta/2)|$. Since the angle between the vectors is in $[0, \pi]$, its half lies in $[0, \pi/2]$; consequently $\sin(\theta/2) \ge 0$. The absolute value can therefore be removed. This reasoning includes the limiting cases of parallel and opposite unit vectors.

Quick Tip to Solve: Expand the squared vector difference first; remove the square-root absolute value only after checking the half-angle range.

Step 3: Conclusion Thus $\boxed{\frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}}$, as required.

Quick Tip to Solve: Expand the squared vector difference first; remove the square-root absolute value only after checking the half-angle range.

31(a). The probability that a student buys a colouring book is 0.7 and that she buys a box of colours is 0.2. The probability that she buys a colouring book, given that she buys a box of colours, is 0.3. Find the probability that the student:

(i) Buys both the colouring book and the box of colours.

(ii) Buys a box of colours given that she buys the colouring book. [3 marks]

Solution:

Concept:

- Let A be the event that the student buys a colouring book, and B the event that she buys a box of colours. Thus $P(A) = 0.7$, $P(B) = 0.2$ and $P(A \mid B) = 0.3$.
- For a conditioning event of positive probability, $P(A \mid B) = P(A \cap B)/P(B)$. The intersection describes both purchases, while reversing the conditional probability changes its denominator.

Step 1: Find the probability of both purchases Part (i) asks for $P(A \cap B)$. The given conditional probability concerns buying the book among students who buy the box of colours. Multiplying it by the probability of buying the box gives

$$P(A \cap B) = P(A \mid B)P(B) = 0.3 \times 0.2 = 0.06.$$

This is at most either individual probability, as required for an intersection. We do not multiply $P(A)P(B)$, because independence is not given; indeed the supplied conditional probability differs from $P(A)$.

Step 2: Reverse the conditioning event Part (ii) asks for the probability of buying a box of colours among students who buy a colouring book. The numerator is the same intersection as in part (i), but the conditioning event is now A. Since $P(A) = 0.7 > 0$,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.06}{0.7} = \frac{6}{70} = \frac{3}{35}.$$

The reversal does not preserve the numerical value of a conditional probability, because the populations on which it is conditioned have different probabilities.

Step 3: Conclusion The required probabilities are $\boxed{\text{(i) } 0.06, \qquad \text{(ii) } \frac{3}{35}}$.

Quick Tip to Solve: Find the intersection using the given conditional probability, then divide by the probability of the newly given event.

OR

31(b). A person has a fruit box that contains 6 apples and 4 oranges. He picks out a fruit three times, one after the other, after replacing the previous one in the box. Find:

(i) The probability distribution of the number of oranges he draws.

(ii) The expectation of the random variable (number of oranges). [3 marks]

Solution:

Concept:

- Let X denote the number of oranges obtained in three draws. Each draw gives an orange with probability $p = 4/10 = 2/5$ and an apple with probability $1 - p = 3/5$.
- Replacement restores the original composition before each draw, making the draws independent with the same success probability. Thus X has a binomial distribution: $P(X = x) = \binom{3}{x} p^x (1-p)^{3-x}$, for $x = 0, 1, 2, 3$.

Step 1: Calculate the probability distribution For exactly x oranges, choose which x of the three draws are oranges and multiply the probabilities of the independent outcomes. Therefore

$$P(X = 0) = \left(\frac{3}{5}\right)^3 = \frac{27}{125},$$
$$P(X = 1) = 3\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^2 = \frac{54}{125},$$
$$P(X = 2) = 3\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right) = \frac{36}{125},$$
$$P(X = 3) = \left(\frac{2}{5}\right)^3 = \frac{8}{125}.$$

The factors of three for one and two oranges count the possible positions of those outcomes. As a check, the sum is $(27 + 54 + 36 + 8)/125 = 1$, so the distribution is normalized.

Step 2: Find the expectation The expected number of oranges is the weighted average of the possible values of X:

$$E(X) = \sum_{x=0}^{3} xP(X = x)$$
$$= \frac{0 \times 27 + 1 \times 54 + 2 \times 36 + 3 \times 8}{125}$$
$$= \frac{150}{125} = \frac{6}{5}.$$

This agrees with the binomial mean $np = 3(2/5)$. An expectation need not itself be one of the integer outcomes; it represents the average number over many repetitions of this three-draw experiment.

Step 3: Conclusion $\boxed{P(X = 0) = \frac{27}{125},\ P(X = 1) = \frac{54}{125},\ P(X = 2) = \frac{36}{125},\ P(X = 3) = \frac{8}{125}}$. The expectation is $\boxed{E(X) = \frac{6}{5}}$.

Quick Tip to Solve: Replacement signals a binomial model: use three trials with orange probability 2/5 and check that the four probabilities sum to one.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

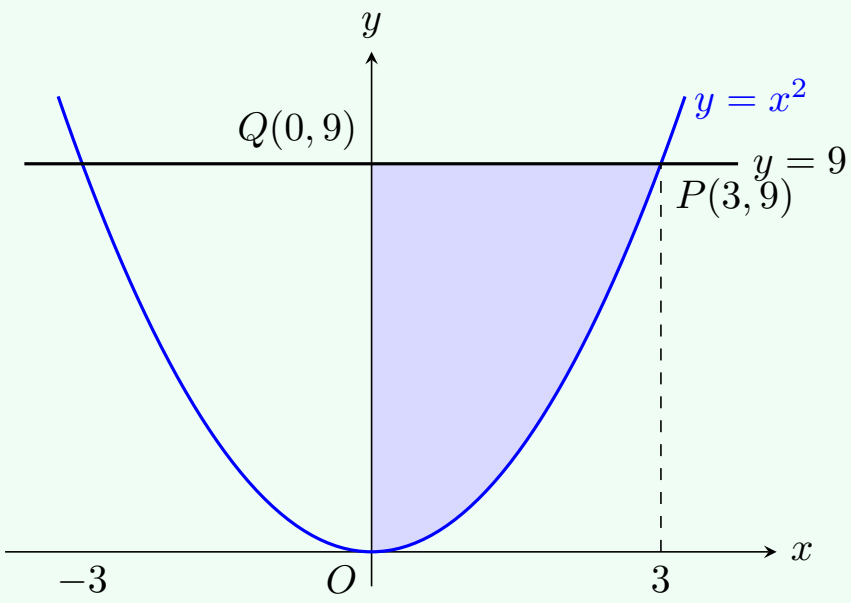
32. Sketch a graph of $y = x^2$. Using integration, find the area of the region bounded by $y = 9$, $x = 0$ and $y = x^2$. [5 marks]

Solution:

Concept:

- The curve $y = x^2$ is an upward-opening parabola with vertex $(0, 0)$ and axis of symmetry the y-axis. The line $y = 9$ is horizontal, and $x = 0$ is the y-axis.
- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$ is $A = \int_a^b [f(x) - g(x)]\, dx$. A vertical strip has height equal to the difference of their ordinates.
- Symmetry about the y-axis gives two equal regions. The region on either side, enclosed by all three specified boundaries, has the same area.

Step 1: Sketch and identify the region Plot the vertex $O(0, 0)$ and points $(\pm 1, 1)$, $(\pm 2, 4)$, $(\pm 3, 9)$. Join them smoothly to draw the parabola. Draw the line $y = 9$, parallel to the x-axis. Select the region to the right of the y-axis, shaded in the sketch. Its boundary goes from O along the parabola to $P(3, 9)$, then along the horizontal line to $Q(0, 9)$, and down the y-axis back to O. Selecting the reflected left-hand region would give the same area.



Graph of $y = x^2$ with the right-hand bounded region shaded

Step 2: Determine the limits and strip height The line meets the parabola where $x^2 = 9$, giving $x = \pm 3$. On the selected right-hand region the relevant intersection is $P(3, 9)$. Therefore the integration interval is $0 \le x \le 3$. On this interval $x^2 \le 9$, so the upper boundary is $y = 9$ and the lower boundary is $y = x^2$. A vertical strip of width dx has nonnegative height $9 - x^2$ and area $(9 - x^2)\, dx$.

Quick Tip to Solve: Use the y-axis as one boundary, so integrate from 0 to 3 with upper curve 9 minus lower curve x squared.

Step 3: Evaluate the area integral Adding the areas of all vertical strips,

$$\begin{aligned} A &= \int_0^3 (9 - x^2)\, dx \\ &= \left[9x - \frac{x^3}{3}\right]_0^3 \\ &= 27 - 9 - 0 = 18. \end{aligned}$$

For verification, the rectangle between $x = 0$, $x = 3$, $y = 0$ and $y = 9$ has area 27. The area below the parabola within that rectangle is $\int_0^3 x^2\, dx = 9$. Removing it leaves precisely the bounded region. Integrating from -3 to 3 instead would combine both symmetric regions and therefore double the area required for either one.

Step 4: Conclusion The required area is $\boxed{18 \text{ square units}}$. The bounded region on the other side of the y-axis has the same area.

Quick Tip to Solve: Use the y-axis as one boundary, so integrate from 0 to 3 with upper curve 9 minus lower curve x squared.

33. A furniture workshop produces three types of furniture, chairs, tables and beds each day. On a particular day the total number of furniture pieces produced is 45. It was also found that production of beds exceeds that of chairs by 8, while the total production of beds and chairs together is twice the production of tables. Determine the units produced of each type of furniture, using matrix method. [5 marks]

Solution:

Concept:

- Translate each statement into a linear equation using one variable per furniture type. The statement that beds exceed chairs by eight means beds minus chairs equals eight.
- A system of three linear equations is written $AX = B$, where A is the coefficient matrix, X the column of unknown quantities and B the column of constants.
- When $|A| \neq 0$, the inverse exists and the unique solution is $X = A^{-1}B$, with $A^{-1} = \text{adj}\, A/|A|$.

Step 1: Form the three equations Let x, y, z denote the numbers of chairs, tables and beds produced on the given day, respectively. The total production gives $x + y + z = 45$. Since beds exceed chairs by eight, $-x + z = 8$. Since beds and chairs together are twice the number of tables, $x + z = 2y$, or $x - 2y + z = 0$. All three conditions must hold simultaneously.

Step 2: Write the system in matrix form and check invertibility The system is $AX = B$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 45 \\ 8 \\ 0 \end{pmatrix}.$$

Quick Tip to Solve: Keep the unknowns in the fixed order chairs, tables, beds in every matrix row; transpose the cofactor matrix before using the inverse formula.

Expanding the determinant along the first row,

$$|A| = 1(0+2) - 1(-1-1) + 1(2-0) = 6 \neq 0.$$

Thus this coefficient matrix is invertible, so the furniture quantities are determined uniquely by the three conditions.

Step 3: Calculate the inverse using cofactors Taking signed minors of the entries gives the cofactor matrix

$$C = \begin{pmatrix} 2 & 2 & 2 \\ -3 & 0 & 3 \\ 1 & -2 & 1 \end{pmatrix}.$$

For example, $C_{21} = -(1\cdot 1 - 1\cdot(-2)) = -3$ and $C_{32} = -(1\cdot 1 - 1\cdot(-1)) = -2$. Transpose the cofactor matrix to obtain the adjugate. Therefore

$$A^{-1} = \frac{1}{6}C^T = \frac{1}{6}\begin{pmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Step 4: Multiply by the constants and verify the production conditions Multiplying both sides of $AX = B$ by A^{-1},

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{6}\begin{pmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 45 \\ 8 \\ 0 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 90-24 \\ 90 \\ 90+24 \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \\ 19 \end{pmatrix}. \end{aligned}$$

The quantities are nonnegative integers, as required for furniture pieces. Checking the original statements gives $11 + 15 + 19 = 45$, $19 - 11 = 8$, and $19 + 11 = 30 = 2(15)$. Thus every stated production condition is satisfied.

Step 5: Conclusion The workshop produced $\boxed{11 \text{ chairs},\ 15 \text{ tables},\ 19 \text{ beds}}$ on the given day.

Quick Tip to Solve: Keep the unknowns in the fixed order chairs, tables, beds in every matrix row; transpose the cofactor matrix before using the inverse formula.

34(a). For a positive constant a, differentiate $a^{t+\frac{1}{t}}$ with respect to $\left(t+\frac{1}{t}\right)^a$, where t is a non-zero real number. [5 marks]

Solution:

Concept:

- For functions Y and X of a common parameter, $\dfrac{dY}{dX} = \dfrac{dY/dt}{dX/dt}$, provided the denominator is non-zero.
- For a positive constant a, $\dfrac{d}{du}a^u = a^u \ln a$, whereas $\dfrac{d}{du}u^a = au^{a-1}$, wherever the real power is differentiable.

Step 1: Introduce the common inner expression Let $u = t + \dfrac{1}{t}$, $Y = a^u$, and $X = u^a$. The derivative requested is dY/dX. Since $t \neq 0$,

$$\frac{du}{dt} = 1 - \frac{1}{t^2}.$$

Both functions depend on the same inner expression. Consequently the chain-rule factor du/dt occurs in both parameter derivatives and will cancel when their quotient is formed.

Step 2: Differentiate the exponential and the power The base a is constant in Y, so use the exponential rule. The exponent a is constant in X, so use the power rule. This distinction is essential: the logarithmic factor occurs in the exponential derivative only. Applying the chain rule gives

$$\frac{dY}{dt} = a^{t+1/t} \ln a \left(1 - \frac{1}{t^2}\right),$$
$$\frac{dX}{dt} = a\left(t + \frac{1}{t}\right)^{a-1}\left(1 - \frac{1}{t^2}\right).$$

Step 3: Divide the derivatives and simplify Where $dX/dt \neq 0$, the derivative of the first quantity with respect to the second is

$$\frac{dY}{dX} = \frac{a^{t+1/t}\ln a\,(1 - 1/t^2)}{a(t+1/t)^{a-1}(1-1/t^2)}$$
$$= a^{t+1/t-1}\left(t + \frac{1}{t}\right)^{1-a} \ln a.$$

Alternatively, taking the quotient $(dY/du)/(dX/du)$ directly gives the same expression without cancelling a zero parameter derivative at $t = \pm 1$. The result applies wherever the stated real powers and derivatives are defined; for an arbitrary positive real exponent, the usual real-power calculation uses $u > 0$. When $a = 1$, the first quantity is the constant 1, and the formula correctly gives zero since $\ln 1 = 0$.

Step 4: Conclusion The required derivative is

$$\boxed{a^{t+1/t-1}\left(t + \frac{1}{t}\right)^{1-a} \ln a.}$$

Quick Tip to Solve: Set $u = t + 1/t$ first, then divide the derivatives of a^u and u^a with respect to u.

OR

34(b). Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$, where a and b are constants. [5 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x, so differentiating any expression involving y introduces $y' = dy/dx$.
- For positive functions f and g, logarithmic differentiation gives $\frac{d}{dx}(f^g) = f^g\left(g' \ln f + g\frac{f'}{f}\right)$.

Step 1: Differentiate the first variable power Work in the real differentiable domain $x > 0$, $y > 0$, and write $y' = dy/dx$. For $U = y^x$, taking logarithms gives $\ln U = x \ln y$. Differentiating using the product rule,

$$\frac{U'}{U} = \ln y + \frac{x}{y}y', \qquad \frac{d}{dx}(y^x) = y^x\left(\ln y + \frac{x}{y}y'\right).$$

Here both the exponent x and the base y vary. Keeping only the ordinary power-rule term would omit the derivative of the exponent.

Step 2: Differentiate the other two variable powers Similarly, $\ln(x^y) = y \ln x$. Its derivative includes both $y' \ln x$ and y/x. For x^x, both base and exponent are x, yielding

$$\frac{d}{dx}(x^y) = x^y\left(y' \ln x + \frac{y}{x}\right),$$
$$\frac{d}{dx}(x^x) = x^x(\ln x + 1).$$

The extra 1 in the last expression follows from $x \cdot (1/x)$ when differentiating $x \ln x$.

Step 3: Differentiate the equation and collect the derivative terms Since a and b are constants, a^b is constant and its derivative is zero. Thus

$$y^x\left(\ln y + \frac{x}{y}y'\right) + x^y\left(y' \ln x + \frac{y}{x}\right) + x^x(\ln x + 1) = 0.$$

Place the two terms containing y' together and move all other terms to the right:

$$\left(xy^{x-1} + x^y \ln x\right) y' = -\left(y^x \ln y + yx^{y-1} + x^x(\ln x + 1)\right).$$

The coefficients follow from $y^x(x/y) = xy^{x-1}$ and $x^y(y/x) = yx^{y-1}$. Dividing by the coefficient of y' gives the derivative whenever that coefficient is non-zero, as required for this explicit formula.

Step 4: Conclusion Therefore,

$$\boxed{\frac{dy}{dx} = -\frac{y^x \ln y + yx^{y-1} + x^x(\ln x + 1)}{xy^{x-1} + x^y \ln x}.}$$

Quick Tip to Solve: For each variable power, differentiate its logarithm first; then collect every dy/dx term before dividing.

35(a). Find the foot of the perpendicular drawn from the point $(1, 1, 4)$ on the line $\frac{x+2}{5} = \frac{y+1}{2} = \frac{-z+4}{-3}$. [5 marks]

Solution:

Concept:

- A point on a line through A, with direction vector $\mathbf{d}$, has position $H = A + t\mathbf{d}$ for a real parameter t.
- The perpendicular foot satisfies $(H - P) \cdot \mathbf{d} = 0$, because the dot product of two perpendicular vectors is zero.
- Equivalently, $t = ((P - A) \cdot \mathbf{d})/(\mathbf{d} \cdot \mathbf{d})$. This projection parameter is unique when the direction vector is nonzero.

Step 1: Parameterise the printed line Set each of the three ratios equal to t. Then $x + 2 = 5t$, $y + 1 = 2t$ and $-z + 4 = -3t$. The third equation gives $z = 4 + 3t$, so the direction vector has positive third component. Therefore

$$H = (5t - 2, 2t - 1, 3t + 4), \qquad A = (-2, -1, 4), \qquad \mathbf{d} = (5, 2, 3).$$

This parameterisation supplies all points of the line. The point A is obtained when $t = 0$. In reading the direction ratios, the negative numerator and negative denominator of the last fraction cancel: it is also $(z - 4)/3$.

Step 2: Determine the projection parameter For the given point $P = (1, 1, 4)$, the vector joining it to the candidate foot is $\overrightarrow{PH} = (5t - 3, 2t - 2, 3t)$. Enforce perpendicularity to the line by taking its dot product with $\mathbf{d}$:

$$\begin{aligned}(5t - 3)5 + (2t - 2)2 + 3t \cdot 3 &= 0,\\ 25t - 15 + 4t - 4 + 9t &= 0,\\ 38t - 19 &= 0.\end{aligned}$$

Thus $t = 1/2$. This agrees with the projection formula because $P - A = (3, 2, 0)$, giving numerator $15 + 4 = 19$ and denominator $25 + 4 + 9 = 38$. Since the denominator is positive, there is exactly one point satisfying the perpendicularity condition.

Step 3: Calculate and check the coordinates Using $t = 1/2$ gives $x = 5/2 - 2 = 1/2$, $y = 1 - 1 = 0$, and $z = 3/2 + 4 = 11/2$. Check directly in the original equation:

$$\frac{1/2 + 2}{5} = \frac{0 + 1}{2} = \frac{-11/2 + 4}{-3} = \frac{1}{2}.$$

Thus the point lies on the printed line. Its displacement from P is $(-1/2, -1, 3/2)$, whose dot product with the direction vector is

$$-\frac{1}{2} \cdot 5 - 1 \cdot 2 + \frac{3}{2} \cdot 3 = -\frac{5}{2} - 2 + \frac{9}{2} = 0.$$

The joining segment is therefore perpendicular to the line. Both defining conditions for a perpendicular foot are satisfied.

Quick Tip to Solve: First rewrite the last ratio as $(z-4)/3$, then obtain the foot using one dot-product equation and verify its line membership.

Step 4: Conclusion The required foot of the perpendicular is $\boxed{\left(\frac{1}{2}, 0, \frac{11}{2}\right)}$.

Quick Tip to Solve: First rewrite the last ratio as $(z-4)/3$, then obtain the foot using one dot-product equation and verify its line membership.

OR

35(b). Find the point on the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3}$ at a distance of $2\sqrt{2}$ units from the point $(-1, -1, 2)$. [5 marks]

Solution:

Concept:

- A line in symmetric form can be parameterised by equating its three ratios to one real parameter. The direction ratios here are $(3, 2, 3)$.
- The distance between $P(x, y, z)$ and $A(x_0, y_0, z_0)$ satisfies $PA^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$. Squaring a specified positive distance introduces no change in the distance condition.

Step 1: Parameterise the line Let the common ratio be t. Then

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3} = t, \qquad P = (1+3t, -1+2t, 4+3t).$$

Every real value of t gives one point on the line, and every point of this line has this form. Relative to the given point $A = (-1, -1, 2)$, the three coordinate differences are $2+3t$, $2t$, and $2+3t$, respectively.

Step 2: Impose and solve the distance condition The required distance is $2\sqrt{2}$, whose square is 8. Thus

$$\begin{aligned}(2+3t)^2 + (2t)^2 + (2+3t)^2 &= 8,\\ 2(4+12t+9t^2) + 4t^2 &= 8,\\ 22t^2 + 24t &= 0,\\ 2t(11t+12) &= 0.\end{aligned}$$

Hence $t = 0$ or $t = -12/11$. Both roots must be retained: the distance condition describes a sphere centred at A, and the line can meet this sphere at two points even though the question uses the singular word point.

Step 3: Substitute and verify both candidates For $t = 0$, the point is $P_1 = (1, -1, 4)$. For $t = -12/11$,

$$P_2 = \left(1 - \frac{36}{11}, -1 - \frac{24}{11}, 4 - \frac{36}{11}\right) = \left(-\frac{25}{11}, -\frac{35}{11}, \frac{8}{11}\right).$$

For P_1, the squared distance from A is $2^2 + 0^2 + 2^2 = 8$. For P_2, the coordinate differences from A are $-14/11, -24/11, -14/11$, so

$$AP_2^2 = \frac{196 + 576 + 196}{121} = \frac{968}{121} = 8.$$

Quick Tip to Solve: Keep both roots of the distance equation; a singular wording does not remove a valid second intersection.

Each candidate also satisfies all three line ratios with its corresponding common value of t.

Step 4: Conclusion The required points are $\boxed{(1,-1,4)}$ and $\boxed{\left(-\frac{25}{11},-\frac{35}{11},\frac{8}{11}\right)}$, each at distance $2\sqrt{2}$ units from $(-1,-1,2)$.

Quick Tip to Solve: Keep both roots of the distance equation; a singular wording does not remove a valid second intersection.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study – 1. A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.

On the basis of the above information, answer the following questions: [4 marks]

36(i). Taking length $=$ breadth $= x$ m and height $= y$ m, express the surface area (S) of the box in terms of x and its volume (V), which is constant. [1 mark]

Solution:

Concept:

- For a closed cuboid with square base of side x and height y, volume is $V = x^2y$, while all six faces contribute to surface area.
- The two square faces contribute $2x^2$, and the four rectangular faces contribute $4xy$.

Step 1: Express height using fixed volume Since $V = x^2y$ and $x > 0$, we obtain $y = V/x^2$. This removes the second variable while preserving the prescribed volume.

Step 2: Substitute into the surface area The box is closed, so both its top and bottom must be counted. Substituting the height gives

$$S = 2x^2 + 4xy = 2x^2 + 4x\left(\frac{V}{x^2}\right) = 2x^2 + \frac{4V}{x}.$$

Both terms have units of square metres when x is measured in metres and V in cubic metres.

Step 3: Conclusion $\boxed{S(x) = 2x^2 + \frac{4V}{x}}$, $x > 0$, where V is constant.

Quick Tip to Solve: For a closed square-base box, count two square faces before eliminating height using the volume.

36(ii). Find $\frac{dS}{dx}$. [1 mark]

Solution:

Concept:

- From part (i), the area at fixed volume is $S(x) = 2x^2 + 4Vx^{-1}$, with $x > 0$.
- Use $\frac{d}{dx}x^n = nx^{n-1}$; the prescribed volume V is a constant during differentiation.

Step 1: Differentiate the square-face term The derivative of $2x^2$ is $4x$. This term measures the increase in top and bottom area as the base side grows.

Step 2: Differentiate the lateral-area term Writing the remaining term as $4Vx^{-1}$, its derivative is $-4Vx^{-2}$. Therefore

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2}.$$

The negative contribution reflects that the height decreases as the base increases at fixed volume; treating the height as constant would describe a different constraint.

Step 3: Conclusion $\boxed{\frac{dS}{dx} = 4x - \frac{4V}{x^2}}$, with V held constant.

Quick Tip to Solve: Eliminate height first, and hold volume constant when differentiating the reciprocal term.

36(iii)(a). Find a relation between x and y such that the surface area (S) is minimum. [2 marks]

Solution:

Concept:

- A stationary value of a differentiable area function occurs where its first derivative vanishes.
- For fixed positive volume, $S = 2x^2 + 4V/x$, $S' = 4x - 4V/x^2$, and $V = x^2y$, with $x, y > 0$.

Step 1: Find the stationary relation Set the derivative to zero:

$$4x - \frac{4V}{x^2} = 0 \quad \Longrightarrow \quad x^3 = V.$$

Using the volume condition $V = x^2y$, this gives $x^3 = x^2y$. Since a physical box has $x > 0$, division by x^2 is valid and gives $y = x$.

Step 2: Verify that the area is minimum The second derivative is

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}.$$

It is positive for every $x > 0$, because $V > 0$. At the stationary point $x^3 = V$, it equals $12 > 0$, confirming a strict minimum. Since the area is strictly convex on the positive domain, this stationary point gives its global minimum.

Step 3: Conclusion Thus $\boxed{x = y}$: the minimum-area box is a cube.

Quick Tip to Solve: After solving the area derivative, replace volume by x squared times height to obtain the requested relation.

OR

36(iii)(b). If surface area (S) is constant, the volume $(V) = \frac{1}{4}(Sx - 2x^3)$, x being the edge of base. Show that volume (V) is maximum for $x = \sqrt{\frac{S}{6}}$. [2 marks]

Solution:

Concept:

- Here $S > 0$ is fixed, and the volume is the one-variable function $V(x) = \frac{1}{4}(Sx - 2x^3)$.
- At an interior maximum, $V'(x) = 0$; a negative second derivative confirms a strict local maximum. A physical box requires $x > 0$ and height $y > 0$.

Step 1: Find the admissible stationary point Differentiating while holding S constant gives

$$V'(x) = \frac{1}{4}(S - 6x^2).$$

Thus $V'(x) = 0$ implies $x^2 = S/6$. Because x is the positive edge length, only $x = \sqrt{S/6}$ is admissible; the negative algebraic root cannot describe a box.

Step 2: Confirm maximum volume The second derivative is $V''(x) = -3x$, which is negative at $x = \sqrt{S/6} > 0$. Also, the area relation gives $y = (S - 2x^2)/(4x) > 0$, so the physical domain is $0 < x < \sqrt{S/2}$. On this domain, V' is positive before $\sqrt{S/6}$ and negative after it. Hence volume increases up to this point and decreases thereafter, establishing the global maximum among admissible boxes.

Step 3: Conclusion The volume is maximum for $\boxed{x = \sqrt{\frac{S}{6}}}$, as required.

Quick Tip to Solve: Keep surface area constant and reject the negative stationary root because x represents an edge length.

37. Case Study – 2. Let A be the set of 30 students of class XII in a school. Let $f : A \to \mathbb{N}$, $\mathbb{N}$ is a set of natural numbers such that function $f(x) =$ Roll Number of student x.

On the basis of the given information, answer the following: [4 marks]

37(i). Is f a bijective function? [1 mark]

Solution:

Concept:

- A function is bijective if it is both one-one (injective) and onto (surjective).
- The range consists of actual images of domain elements; being onto requires that this range equal the entire codomain.

Step 1: Check uniqueness of roll numbers Different students in the class have different roll numbers. Therefore, for students $x_1, x_2 \in A$, equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Hence f is one-one.

Step 2: Compare the range with the codomain There are only 30 students, so $f(A)$ contains only

Quick Tip to Solve: Check the stated codomain: 30 distinct outputs cannot cover all natural numbers.

30 natural numbers. The codomain $\mathbb{N}$ contains infinitely many natural numbers, and thus includes numbers that are not the roll number of any student in this class. Therefore $f(A) \neq \mathbb{N}$, so f is not onto.

Step 3: Conclusion No. The function $f : A \to \mathbb{N}$ is one-one but not onto, and hence is not bijective.

Quick Tip to Solve: Check the stated codomain: 30 distinct outputs cannot cover all natural numbers.

37(ii). Give reasons to support your answer to (i). [1 mark]

Solution:

Concept:

- Injectivity means that equal function values can arise only from the same input.
- Surjectivity means every member of the stated codomain has a preimage in the domain.

Step 1: Justify that the function is one-one A roll number identifies a unique student in the class. If two students $x, y \in A$ have $f(x) = f(y)$, their roll numbers coincide, so they must be the same student: $x = y$. This proves injectivity.

Step 2: Justify that the function is not onto The domain has 30 students; hence its image has only 30 elements. The codomain is the infinite set $\mathbb{N}$. Consequently, there are natural numbers outside the image, so those numbers have no preimage in A. For the usual roll numbers $1, 2, \ldots, 30$, the natural number 31 is one such example.

Step 3: Conclusion Distinct students have distinct roll numbers, but not every natural number is assigned to a student. Thus f is injective and not surjective, which supports the answer that it is not bijective.

Quick Tip to Solve: Prove uniqueness of outputs and coverage of the codomain separately; bijectivity needs both.

37(iii)(a). Let R be a relation defined by the teacher to plan the seating arrangement of students in pairs, where $R = \{(x, y) : x, y$ are Roll Numbers of students such that $y = 3x\}$. List the elements of R. Is the relation R reflexive, symmetric and transitive? Justify your answer. [2 marks]

Solution:

Concept:

- The roll-number set is $S = \{1, 2, \ldots, 30\}$. Both coordinates of an ordered pair in R must belong to S.
- Reflexivity requires $(x, x) \in R$ for every $x \in S$; symmetry requires each reverse pair; transitivity requires (x, z) whenever $(x, y), (y, z) \in R$.

Step 1: List the ordered pairs The rule is $y = 3x$. Since $1 \leq y \leq 30$, we need $1 \leq x \leq 10$. Substituting

Quick Tip to Solve: For transitivity, find a chain such as 1 to 3 to 9 and test whether the direct pair is present.

these ten integer values gives

$$R = \{(1,3),(2,6),(3,9),(4,12),(5,15),$$
$$(6,18),(7,21),(8,24),(9,27),(10,30)\}.$$

Values of x above 10 would give a second coordinate outside the roll-number set.

Step 2: Test reflexivity and symmetry The pair $(1,1)$ is absent because $1 \neq 3(1)$; hence R is not reflexive. Also $(1,3) \in R$, but $(3,1) \notin R$, since $1 \neq 3(3)$. Therefore R is not symmetric.

Step 3: Test transitivity Both $(1,3)$ and $(3,9)$ belong to R. Transitivity would require $(1,9) \in R$, but this pair does not satisfy $9 = 3(1)$. Thus R is not transitive. One counterexample suffices to disprove each universal property.

Step 4: Conclusion The relation contains the ten pairs listed above and is neither reflexive, nor symmetric, nor transitive.

Quick Tip to Solve: For transitivity, find a chain such as 1 to 3 to 9 and test whether the direct pair is present.

OR

37(iii)(b). Let R be a relation defined by $R = \{(x,y) : x, y$ are Roll Numbers of students such that $y = x^3\}$. List the elements of R. Is R a function? Justify your answer. [2 marks]

Solution:

Concept:

- A relation on the roll-number set $S = \{1, 2, \ldots, 30\}$ contains only pairs whose two coordinates lie in S.
- A function from S to S must assign exactly one image to every element of S; uniqueness without existence for every input is insufficient.

Step 1: List all pairs satisfying the rule The rule is $y = x^3$. Both x and y are positive integer roll numbers no larger than 30. The values $x = 1, 2, 3$ give $y = 1, 8, 27$, respectively. Already $4^3 = 64 > 30$, and cubes increase for positive integers, so no larger roll number can occur as a first coordinate. Therefore

$$R = \{(1,1),(2,8),(3,27)\}.$$

Step 2: Check whether every roll number has an image Each of the inputs 1, 2 and 3 has one image, but the other members of S have no image in the relation. For instance, no pair with first coordinate 4 belongs to R, because its required second coordinate 64 is not a student roll number. Thus the requirement that every element of the given roll-number set have an image fails.

Step 3: Conclusion $R = \{(1,1),(2,8),(3,27)\}$. It is not a function on the set of all 30 roll numbers, because inputs 4 through 30 have no image in that set.

Quick Tip to Solve: When testing a relation as a function on a given set, check missing inputs as well as repeated inputs.

38. Case Study – 3. A gardener wanted to plant vegetables in his garden. Hence he bought 10 seeds of brinjal plant, 12 seeds of cabbage plant and 8 seeds of radish plant. The shopkeeper assured him of germination probabilities of brinjal, cabbage and radish to be 25%, 35% and 40% respectively. But before he could plant the seeds, they got mixed up in the bag and he had to sow them randomly.

Based upon the above information, answer the following questions: [4 marks]

38(i). Calculate the probability of a randomly chosen seed to germinate. [2 marks]

Solution:

Concept:

- Let B, C, R denote the events of choosing a brinjal, cabbage and radish seed, respectively, and let G denote germination.
- For mutually exclusive and exhaustive seed types, total probability gives $P(G) = P(B)P(G \mid B) + P(C)P(G \mid C) + P(R)P(G \mid R)$.

Step 1: Find selection and conditional probabilities There are $10 + 12 + 8 = 30$ seeds in the bag. Random selection therefore gives

$$P(B) = \frac{10}{30}, \quad P(C) = \frac{12}{30}, \quad P(R) = \frac{8}{30}.$$

The stated germination probabilities are $P(G \mid B) = \frac{1}{4}$, $P(G \mid C) = \frac{7}{20}$, and $P(G \mid R) = \frac{2}{5}$. The seed types exhaust all possibilities.

Step 2: Apply the law of total probability Weight each germination probability by the chance of selecting that seed type:

$$\begin{aligned} P(G) &= \frac{10}{30} \cdot \frac{1}{4} + \frac{12}{30} \cdot \frac{7}{20} + \frac{8}{30} \cdot \frac{2}{5} \\ &= \frac{1}{12} + \frac{7}{50} + \frac{8}{75} \\ &= \frac{25 + 42 + 32}{300} = \frac{99}{300}. \end{aligned}$$

An unweighted average would be inappropriate because the three seed types occur in different numbers.

Step 3: Conclusion The probability that a randomly chosen seed germinates is $\boxed{\frac{33}{100} = 0.33}$.

Quick Tip to Solve: Use seed counts as weights when averaging the three germination probabilities.

38(ii). What is the probability that it is a cabbage seed, given that the chosen seed germinates? [2 marks]

Solution:

Concept:

- Let C be the event that the chosen seed is cabbage and G the event that it germinates.
- Bayes' theorem gives $P(C \mid G) = \dfrac{P(C)P(G \mid C)}{P(G)}$, provided $P(G) > 0$.

Quick Tip to Solve: Divide the cabbage-and-germination probability by the overall germination probability.

Step 1: Compute the joint probability for cabbage and germination Of the 30 seeds, 12 are cabbage seeds, so $P(C) = 12/30 = 2/5$. Their germination probability is $P(G \mid C) = 35/100 = 7/20$. Hence

$$P(C \cap G) = P(C)P(G \mid C) = \frac{2}{5} \cdot \frac{7}{20} = \frac{7}{50}.$$

This is the probability of both choosing cabbage and obtaining germination.

Step 2: Condition on germination using Bayes theorem The total germination probability, accounting for all three seed types, is

$$P(G) = \frac{10}{30} \cdot \frac{1}{4} + \frac{12}{30} \cdot \frac{7}{20} + \frac{8}{30} \cdot \frac{2}{5} = \frac{33}{100}.$$

Therefore

$$P(C \mid G) = \frac{7/50}{33/100} = \frac{7}{50} \cdot \frac{100}{33} = \frac{14}{33}.$$

The denominator includes germinating seeds of every type, whereas the numerator includes only germinating cabbage seeds.

Step 3: Conclusion Given that the seed germinates, the probability that it is a cabbage seed is $\boxed{\frac{14}{33}}$.

Quick Tip to Solve: Divide the cabbage-and-germination probability by the overall germination probability.