

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1** to **18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in few questions in all the Sections except Section – **A**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

Question 1 to 20 are multiple choice questions of 1 mark each.

1. If $2 \cos^{-1} x = y$, then

[1 mark]

- (A) $0 \leq y \leq \pi$
- (B) $-\pi \leq y \leq \pi$
- (C) $0 \leq y \leq 2\pi$
- (D) $-\pi \leq y \leq 0$

Correct Answer: (C) $0 \leq y \leq 2\pi$

Solution:

Concept:

- The principal value branch of $\cos^{-1}$ has domain $[-1, 1]$ and range $[0, \pi]$, so $0 \leq \cos^{-1} x \leq \pi$ for every admissible x .
- Multiplying an inequality by a positive constant keeps the direction of the inequality signs, so the range of $k \cos^{-1} x$ with $k > 0$ is $[0, k\pi]$.

Step 1: Write the range of the principal branch For $x \in [-1, 1]$, the principal value satisfies

$$0 \leq \cos^{-1} x \leq \pi.$$

The value 0 is attained at $x = 1$ and the value π is attained at $x = -1$, so both ends are included.

Step 2: Scale by 2 to get the range of y Multiplying throughout by the positive number 2,

$$0 \leq 2 \cos^{-1} x \leq 2\pi \implies 0 \leq y \leq 2\pi.$$

Step 3: Reject the other options Option (A) $0 \leq y \leq \pi$ is the range of $\cos^{-1} x$ itself, not of $2 \cos^{-1} x$; for example $x = -1$ gives $y = 2\pi > \pi$. Option (B) and option (D) allow negative values of y , which is impossible because $\cos^{-1} x$ is never negative. Option (D) even excludes all positive values, although $x = 0$ gives $y = \pi$.

Step 4: Conclusion Since $y = 2 \cos^{-1} x$ takes every value from 0 (at $x = 1$) to 2π (at $x = -1$), we get $0 \leq y \leq 2\pi$. Hence, the correct option is (C).

Quick Tip to Solve: Start from the principal range of cos inverse, $[0, \pi]$, and multiply by the coefficient: 2 times it gives $[0, 2\pi]$. No negative value is ever possible.

2. Which of the following cannot be the order of a row-matrix?

[1 mark]

- (A) 2×1
- (B) 1×2
- (C) 1×1
- (D) $1 \times n$

Correct Answer: (A) 2×1

Solution:

Concept:

- A matrix of order $m \times n$ has m rows and n columns.
- A row matrix has exactly one row, so its order is $1 \times n$ for some $n \geq 1$. A column matrix has exactly one column, with order $m \times 1$.

Step 1: Identify the form of a row matrix A row matrix looks like

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix},$$

with one row and n columns. Hence its order must have the number of rows equal to 1, that is, it is of the form $1 \times n$.

Step 2: Test each option (B) 1×2 : one row, two columns, for example $\begin{bmatrix} 3 & 5 \end{bmatrix}$; this is a row matrix.

(C) 1×1 : one row, one column, for example $\begin{bmatrix} 7 \end{bmatrix}$; this is both a row and a column matrix.

(D) $1 \times n$: the general form of a row matrix.

(A) 2×1 : two rows and one column, for example $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$; this is a column matrix, and since it has two rows it cannot be a row matrix.

Step 3: Conclusion A matrix of order 2×1 has two rows, so it cannot be a row matrix. Hence, the correct option is (A).

Quick Tip to Solve: In an order $m \times n$, the first number counts rows. A row matrix must start with 1; any order whose first entry exceeds 1, like 2×1 , is ruled out.

3. Which of the following properties is/are true for two matrices of suitable orders?

(i) $(A + B)' = A' + B'$

(ii) $(A - B)' = B' - A'$

(iii) $(AB)' = A'B'$

(iv) $(kAB)' = kB'A'$ (k is a scalar)

[1 mark]

(A) (i) only

(B) (i), (ii) and (iii)

(C) (i) and (ii)

(D) (i) and (iv)

Correct Answer: (D) (i) and (iv)

Solution:

Concept:

- For matrices of suitable orders: $(A \pm B)' = A' \pm B'$, $(kA)' = kA'$ for a scalar k , and the reversal law $(AB)' = B'A'$.

Step 1: Check statements (i) and (ii) Transposition is linear: the (i, j) entry of $(A + B)'$ is $a_{ji} + b_{ji}$, which is the (i, j) entry of $A' + B'$. So (i) is true.

In the same way $(A - B)' = A' - B'$. Statement (ii) gives $B' - A' = -(A' - B')$, which is the negative of the correct result; it fails whenever $A \neq B$ (take $A = I$, $B = O$: the left side is I , the right side is $-I$). So (ii) is false.

Step 2: Check statements (iii) and (iv) The reversal law gives $(AB)' = B'A'$, not $A'B'$. In general $A'B' \neq B'A'$; for instance, if A is 2×3 and B is 3×2 , then $A'B'$ is 3×3 while $(AB)'$ is 2×2 . So (iii) is false.

For (iv), using $(kM)' = kM'$ with $M = AB$ and then the reversal law,

$$(kAB)' = k(AB)' = kB'A'.$$

So (iv) is true.

Step 3: Match with the options The true statements are (i) and (iv). Option (A) omits (iv); options (B) and (C) include the false statement (ii), and (B) also includes the false statement (iii).

Step 4: Conclusion Only (i) and (iv) hold for all matrices of suitable orders. Hence, the correct option is (D).

Quick Tip to Solve: Sums and differences keep their order under transpose, while products reverse it: $(AB)' = B'A'$. A scalar simply comes out unchanged.

4. If $\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{vmatrix}$, then

[1 mark]

- (A) $\Delta_1 = 2\Delta_2$
(B) $\Delta_2 = -2\Delta_1$
(C) $\Delta_1 = \Delta_2$
(D) $\Delta_2 = -\Delta_1$

Correct Answer: (B) $\Delta_2 = -2\Delta_1$

Solution:

Concept:

- The determinant of a diagonal matrix is the product of its diagonal entries.
- A 3×3 determinant can be expanded along any row or column: $\Delta = \sum_j a_{1j} A_{1j}$, where $A_{1j} = (-1)^{1+j} M_{1j}$ is the cofactor.

Step 1: Evaluate the first determinant Δ_1 is diagonal, so

$$\Delta_1 = 1 \times 2 \times 3 = 6.$$

Step 2: Evaluate the second determinant Expanding Δ_2 along the first row, only the middle entry 2 is non-zero:

$$\Delta_2 = -2 \begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} = -2(6 - 0) = -12.$$

(Equivalently, interchanging R_1 and R_2 gives the diagonal determinant with entries 1, 2, 6, so $\Delta_2 = -(12) = -12$.)

Step 3: Compare with each option (A) $2\Delta_2 = -24 \neq 6$, false.

(B) $-2\Delta_1 = -12 = \Delta_2$, true.

(C) $6 \neq -12$, false.

(D) $-\Delta_1 = -6 \neq -12$, false.

Step 4: Conclusion Since $\Delta_1 = 6$ and $\Delta_2 = -12$, we have $\Delta_2 = -2\Delta_1$. Hence, the correct option is (B).

Quick Tip to Solve: Swapping R_1 and R_2 in the second determinant makes it diagonal (1, 2, 6), giving 12 with a sign change: -12, which is -2 times 6.

5. One of the values of x for which $\begin{vmatrix} \cos x & \sin x \\ -\cos x & \sin x \end{vmatrix} = 1$ is

[1 mark]

- (A) 0
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{3}$
- (D) $\frac{\pi}{2}$

Correct Answer: (B) $\frac{\pi}{4}$

Solution:

Concept:

• $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

- Double-angle identity: $2 \sin x \cos x = \sin 2x$, and $\sin \theta = 1$ when $\theta = \frac{\pi}{2} + 2n\pi$.

Step 1: Expand the determinant

$$\begin{vmatrix} \cos x & \sin x \\ -\cos x & \sin x \end{vmatrix} = \cos x \sin x - (\sin x)(-\cos x) = 2 \sin x \cos x = \sin 2x.$$

Step 2: Solve the equation

$$\sin 2x = 1 \implies 2x = \frac{\pi}{2} + 2n\pi \implies x = \frac{\pi}{4} + n\pi, \quad n \in \mathbb{Z}.$$

With $n = 0$, $x = \frac{\pi}{4}$.

Step 3: Check the other options (A) $x = 0$: $\sin 0 = 0 \neq 1$.

(C) $x = \frac{\pi}{3}$: $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \neq 1$.

(D) $x = \frac{\pi}{2}$: $\sin \pi = 0 \neq 1$.

Step 4: Conclusion The determinant equals $\sin 2x$, which is 1 at $x = \frac{\pi}{4}$. Hence, the correct option is (B).

Quick Tip to Solve: Watch the minus sign in the bottom-left entry: $ad - bc$ becomes $\cos x \sin x + \sin x \cos x = \sin 2x$, so you need $2x = \pi/2$.

6. If A and B are skew symmetric matrices of same order, then which of the following matrices is also skew symmetric? [1 mark]

- (A) AB
- (B) $AB + BA$
- (C) $(A + B)^2$
- (D) $A - B$

Correct Answer: (D) $A - B$

Solution:

Concept:

- A square matrix M is skew symmetric if $M' = -M$; here $A' = -A$ and $B' = -B$.
- Transpose rules: $(A \pm B)' = A' \pm B'$ and $(AB)' = B'A'$.

Step 1: Test the products (A) and (B)

$$(AB)' = B'A' = (-B)(-A) = BA,$$

which in general is not $-AB$; so AB need not be skew symmetric.

$$(AB + BA)' = B'A' + A'B' = BA + AB,$$

so $AB + BA$ is symmetric, not skew symmetric (unless it is the zero matrix).

Step 2: Test (C) and (D) $A + B$ is skew symmetric, say $S = A + B$ with $S' = -S$. Then $(S^2)' = (S')^2 = (-S)^2 = S^2$, so $(A + B)^2$ is symmetric.

$$(A - B)' = A' - B' = -A + B = -(A - B),$$

so $A - B$ is skew symmetric.

Step 3: Conclusion Only $A - B$ always satisfies $M' = -M$. Hence, the correct option is (D).

Quick Tip to Solve: Sums and differences of skew symmetric matrices stay skew symmetric, but products bring two minus signs that cancel, giving symmetric-type results.

7. The least value of $f(x) = x^3 - 12x$, $x \in [0, 3]$ is

[1 mark]

- (A) -16
- (B) -9
- (C) 0
- (D) 16

Correct Answer: (A) -16

Solution:

Concept:

- For a continuous f on $[a, b]$, the absolute minimum is the least of the values of f at the critical points inside (a, b) and at the end points a, b .
- Critical points are found from $f'(x) = 0$.

Step 1: Find the critical points

$$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2) = 0 \implies x = 2 \text{ or } x = -2.$$

Only $x = 2$ lies in $[0, 3]$.

Step 2: Evaluate at the critical point and the end points

$$f(0) = 0 - 0 = 0,$$

$$f(2) = 8 - 24 = -16,$$

$$f(3) = 27 - 36 = -9.$$

The smallest of these is -16 .

Step 3: Reject the other options -9 is the value at the end point $x = 3$, and 0 is the value at $x = 0$; both are larger than $f(2) = -16$. The value 16 is the maximum of f on $[-3, 0]$ (at $x = -2$) and is not even attained on $[0, 3]$.

Step 4: Conclusion The least value of f on $[0, 3]$ is $f(2) = -16$. Hence, the correct option is (A).

Quick Tip to Solve: On a closed interval, compare f at the critical point $x = 2$ with f at both end points; the interior point $x = 2$ gives -16 , smaller than -9 and 0 .

8. If $\int \frac{3ax}{b^2 + c^2x^2} dx = A \log |b^2 + c^2x^2| + K$, then the value of A is

[1 mark]

- (A) $3a$
(B) $\frac{3a}{2b^2}$
(C) $\frac{3a}{b^2c^2}$
(D) $\frac{3a}{2c^2}$

Correct Answer: (D) $\frac{3a}{2c^2}$

Solution:

Concept:

- If the numerator is a constant multiple of the derivative of the denominator, then $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$.
- Here $\frac{d}{dx}(b^2 + c^2x^2) = 2c^2x$.

Step 1: Substitute for the denominator Let $t = b^2 + c^2x^2$. Then $dt = 2c^2x dx$, so $x dx = \frac{dt}{2c^2}$.

Step 2: Integrate

$$\int \frac{3ax}{b^2 + c^2x^2} dx = \frac{3a}{2c^2} \int \frac{dt}{t} = \frac{3a}{2c^2} \log |t| + K = \frac{3a}{2c^2} \log |b^2 + c^2x^2| + K.$$

Step 3: Compare and reject the other options Comparing with $A \log |b^2 + c^2x^2| + K$ gives $A = \frac{3a}{2c^2}$. Option (A) forgets to divide by the derivative factor $2c^2$; options (B) and (C) wrongly bring b^2 into the coefficient, although the constant b^2 differentiates to zero and plays no role.

Step 4: Conclusion Therefore $A = \frac{3a}{2c^2}$. Hence, the correct option is (D).

Quick Tip to Solve: Differentiate the denominator to get $2c^2x$ and divide the numerator coefficient $3a$ by $2c^2$; the constant b^2 never enters the answer.

9. The value of $\int_{-1}^1 \frac{x^3}{x^2 + 2|x| + 1} dx$ is

[1 mark]

- (A) 0
(B) $\log 2$
(C) $2 \log 2$
(D) $\frac{1}{2} \log 2$

Correct Answer: (A) 0

Solution:

Concept:

- A function f is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$ for every x in its domain.
- Property of definite integrals: $\int_{-a}^a f(x) dx = 0$ when f is odd, and $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ when f is even.

Step 1: Test the parity of the integrand Let $f(x) = \frac{x^3}{x^2 + 2|x| + 1}$. Since $(-x)^2 = x^2$ and $|-x| = |x|$, the denominator is even, while $(-x)^3 = -x^3$. Hence

$$f(-x) = \frac{-x^3}{x^2 + 2|x| + 1} = -f(x),$$

so f is an odd function. The denominator equals $(|x| + 1)^2 \geq 1$, so f is continuous on $[-1, 1]$ and the integral exists.

Step 2: Apply the odd-function property Over the symmetric interval $[-1, 1]$ the areas on the two sides of the origin cancel exactly:

$$\int_{-1}^1 \frac{x^3}{x^2 + 2|x| + 1} dx = 0.$$

Step 3: Why the logarithmic options fail Options (B), (C) and (D) are all non-zero, but an odd integrand over a symmetric interval always integrates to 0, so none of them is possible. For reference, the integral over $[0, 1]$ alone, with $t = x + 1$, is

$$\int_1^2 \left(t - 3 + \frac{3}{t} - \frac{1}{t^2} \right) dt = 3 \log 2 - 2,$$

and the part over $[-1, 0]$ is exactly its negative, $2 - 3 \log 2$.

Step 4: Conclusion The value of the integral is 0. Hence, the correct option is (A).

Quick Tip to Solve: Before integrating over $[-a, a]$, check parity: x^3 over any even expression in x and $|x|$ is odd, so the integral is 0 at once.

10. The area bounded by the curve $y = x|x|$, x -axis and the ordinates $x = -1$ and $x = 1$ is given by [1 mark]

- (A) 0
- (B) $\frac{1}{3}$
- (C) $\frac{2}{3}$
- (D) $\frac{4}{3}$

Correct Answer: (C) $\frac{2}{3}$

Solution:

Concept:

- The area between $y = f(x)$, the x -axis and $x = a$, $x = b$ is $\int_a^b |f(x)| dx$; parts below the axis are counted as positive area.
- $x|x| = x^2$ for $x \geq 0$ and $x|x| = -x^2$ for $x < 0$.

Step 1: Split the curve at the origin For $0 \leq x \leq 1$, $y = x^2 \geq 0$ (above the axis); for $-1 \leq x < 0$, $y = -x^2 \leq 0$ (below the axis). In both cases $|y| = x^2$.

Step 2: Compute the area

$$\begin{aligned} A &= \int_{-1}^0 |-x^2| dx + \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \text{ sq. units.} \end{aligned}$$

Step 3: Why the other options fail Option (A), 0, is the value of the signed integral $\int_{-1}^1 x|x| dx$ (an odd function), not the area. Option (B) is the area on one side only, and (D) doubles the correct value.

Step 4: Conclusion The required area is $\frac{2}{3}$ square units. Hence, the correct option is (C).

Quick Tip to Solve: Area is never the signed integral: for an odd curve such as $y = x|x|$, take twice the area on $[0, 1]$, i.e. $2 \times \frac{1}{3}$.

11. The integrating factor of differential equation $R \frac{dx}{dy} + Px = Q$ where P, Q, R are functions of y is [1 mark]

- (A) $e^{\int \frac{P}{Q} dy}$
- (B) $e^{\int P dy}$
- (C) $e^{\int \frac{P}{R} dy}$
- (D) $e^{\int \frac{P}{R} dx}$

Correct Answer: (C) $e^{\int \frac{P}{R} dy}$

Solution:

Concept:

- A linear equation in x has the standard form $\frac{dx}{dy} + P_1(y)x = Q_1(y)$, with integrating factor I.F. = $e^{\int P_1 dy}$.
- The coefficient of $\frac{dx}{dy}$ must be 1 before the integrating factor is read off.

Step 1: Reduce to standard form Divide $R \frac{dx}{dy} + Px = Q$ by R (with $R \neq 0$):

$$\frac{dx}{dy} + \frac{P}{R}x = \frac{Q}{R}.$$

This is linear in x with $P_1 = \frac{P}{R}$, a function of y .

Step 2: Write the integrating factor

$$\text{I.F.} = e^{\int P_1 dy} = e^{\int \frac{P}{R} dy}.$$

Step 3: Why the other options fail (A) divides by Q , the right-hand side, which is wrong. (B) forgets to divide by R , so the coefficient of $\frac{dx}{dy}$ is not 1. (D) integrates with respect to x , but P/R is a function of y and y is the independent variable here.

Step 4: Conclusion The integrating factor is $e^{\int \frac{P}{R} dy}$. Hence, the correct option is (C).

Quick Tip to Solve: Always make the derivative's coefficient 1 first, then integrate the coefficient of the dependent variable with respect to the independent one (here y).

12. The order and degree of the differential equation $\frac{d}{dx}(e^y) = 0$ respectively are

[1 mark]

- (A) 0, 1
- (B) 1, 1
- (C) 2, 1
- (D) 1, not defined

Correct Answer: (B) 1, 1

Solution:

Concept:

- Order: the order of the highest derivative present in the equation.
- Degree: the highest power of the highest-order derivative, once the equation is a polynomial in its derivatives.

Step 1: Expand the derivative By the chain rule,

$$\frac{d}{dx}(e^y) = e^y \frac{dy}{dx} = 0.$$

Step 2: Read off order and degree The only derivative present is $\frac{dy}{dx}$, so the order is 1. The equation $e^y \frac{dy}{dx} = 0$ is a polynomial in $\frac{dy}{dx}$ whose power is 1 (the factor e^y involves y , not a derivative), so the degree is 1.

Step 3: Why the other options fail (A) ignores the derivative hidden in $\frac{d}{dx}$; (C) wrongly counts the outer $\frac{d}{dx}$ and y as a second derivative; (D) treats e^y as making the degree undefined, but the exponential is of y , not of a derivative, so the degree is defined.

Step 4: Conclusion The order is 1 and the degree is 1. Hence, the correct option is (B).

Quick Tip to Solve: Degree is undefined only when a derivative sits inside a non-polynomial function (like $e^{y'}$ or $\sin y'$); e^y multiplying y' is harmless.

13. The value of p for which vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $2\hat{i} - p\hat{j} + \hat{k}$ are perpendicular to each other is [1 mark]

- (A) 0
(B) 1
(C) $\frac{5}{2}$
(D) $-\frac{5}{2}$

Correct Answer: (C) $\frac{5}{2}$

Solution:

Concept:

- Non-zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a} \cdot \vec{b} = 0$.
- $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = a_1b_1 + a_2b_2 + a_3b_3$.

Step 1: Form the dot product

$$(\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (2\hat{i} - p\hat{j} + \hat{k}) = (1)(2) + (2)(-p) + (3)(1) = 5 - 2p.$$

Step 2: Set it to zero

$$5 - 2p = 0 \Rightarrow p = \frac{5}{2}.$$

Check: $2 - 2 \cdot \frac{5}{2} + 3 = 0$. Option (D), $-\frac{5}{2}$, comes from a sign slip with the $-p\hat{j}$ term and gives $2 + 5 + 3 = 10 \neq 0$; $p = 0$ gives 5 and $p = 1$ gives 3, both non-zero.

Step 3: Conclusion The vectors are perpendicular for $p = \frac{5}{2}$. Hence, the correct option is (C).

Quick Tip to Solve: Perpendicular means dot product zero; keep the minus sign of $-p\hat{j}$ inside the product, so $2 \cdot (-p) = -2p$.

14. The value of m for which the points with position vectors $-\hat{i} - \hat{j} + 2\hat{k}$, $2\hat{i} + m\hat{j} + 5\hat{k}$ and $3\hat{i} + 11\hat{j} + 6\hat{k}$ are collinear, is [1 mark]

- (A) 8
(B) -8
(C) 2
(D) $\frac{5}{2}$

Correct Answer: (A) 8

Solution:

Concept:

- Points A, B, C are collinear if and only if $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some scalar λ , i.e. their components are proportional.

Quick Tip to Solve: The $\hat{i}$ and $\hat{k}$ components fix the ratio 3 : 4 at once; scale the $\hat{j}$ component of $\overrightarrow{AC}$ by $\frac{3}{4}$ to get $m + 1 = 9$.

- $\vec{AB} = \vec{b} - \vec{a}$ (position vector of head minus that of tail).

Step 1: Find two joining vectors With $A(-1, -1, 2)$, $B(2, m, 5)$, $C(3, 11, 6)$:

$$\vec{AB} = 3\hat{i} + (m+1)\hat{j} + 3\hat{k},$$

$$\vec{AC} = 4\hat{i} + 12\hat{j} + 4\hat{k}.$$

Step 2: Impose proportionality

$$\frac{3}{4} = \frac{m+1}{12} = \frac{3}{4} \Rightarrow m+1 = 9 \Rightarrow m = 8.$$

Then $\vec{AB} = 3\hat{i} + 9\hat{j} + 3\hat{k} = \frac{3}{4}\vec{AC}$. For $m = -8, 2$ or $\frac{5}{2}$ the middle ratio becomes $-\frac{7}{12}, \frac{3}{12}$ or $\frac{7}{24}$, none equal to $\frac{3}{4}$.

Step 3: Conclusion The points are collinear for $m = 8$. Hence, the correct option is (A).

Quick Tip to Solve: The $\hat{i}$ and $\hat{k}$ components fix the ratio 3 : 4 at once; scale the $\hat{j}$ component of $\vec{AC}$ by $\frac{3}{4}$ to get $m+1 = 9$.

15. If $|\vec{a}| = 8$, $|\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, then the value of $|\vec{a} \cdot \vec{b}|$ is

[1 mark]

- (A) $6\sqrt{3}$
 (B) $8\sqrt{3}$
 (C) $12\sqrt{3}$
 (D) $3\sqrt{12}$

Correct Answer: (C) $12\sqrt{3}$

Solution:

Concept:

- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, with $\theta \in [0, \pi]$ the angle between the vectors.
- Lagrange's identity: $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$.

Step 1: Apply Lagrange's identity

$$(\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a} \times \vec{b}|^2 = 64 \times 9 - 144 = 576 - 144 = 432.$$

Step 2: Take the square root

$$|\vec{a} \cdot \vec{b}| = \sqrt{432} = \sqrt{144 \times 3} = 12\sqrt{3}.$$

Check by angle: $\sin\theta = \frac{12}{24} = \frac{1}{2}$, so $|\cos\theta| = \frac{\sqrt{3}}{2}$ and $|\vec{a} \cdot \vec{b}| = 24 \times \frac{\sqrt{3}}{2} = 12\sqrt{3}$. Option (D), $3\sqrt{12} = 6\sqrt{3}$, equals (A), and (B) $8\sqrt{3}$ has no basis in the data.

Step 3: Conclusion $|\vec{a} \cdot \vec{b}| = 12\sqrt{3}$. Hence, the correct option is (C).

Quick Tip to Solve: Use $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$: here $576 - 144 = 432 = 144 \times 3$.

16. The length of perpendicular drawn from point $(2, 5, 7)$ on line $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ is

[1 mark]

- (A) 2
- (B) 5
- (C) $\sqrt{74}$
- (D) $\sqrt{78}$

Correct Answer: (C) $\sqrt{74}$

Solution:

Concept:

- A line $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ passes through the origin with direction ratios a, b, c ; a zero denominator means that coordinate is constant (here $y = 0, z = 0$).
- The foot of the perpendicular M from P to the line satisfies $\overrightarrow{PM} \cdot \vec{d} = 0$, and the length is $|\overrightarrow{PM}|$.

Step 1: Identify the line Direction ratios 1, 0, 0 through the origin: the line is the x -axis. A general point is $M(\lambda, 0, 0)$.

Step 2: Find the foot of the perpendicular $\overrightarrow{PM} = (\lambda - 2)\hat{i} - 5\hat{j} - 7\hat{k}$. Perpendicular to $\hat{i}$: $\lambda - 2 = 0$, so $\lambda = 2$ and $M = (2, 0, 0)$.

Step 3: Compute the length

$$PM = \sqrt{0^2 + 5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74}.$$

Option (D), $\sqrt{78} = \sqrt{2^2 + 5^2 + 7^2}$, is the distance to the origin, not to the line; (A) and (B) are single coordinates.

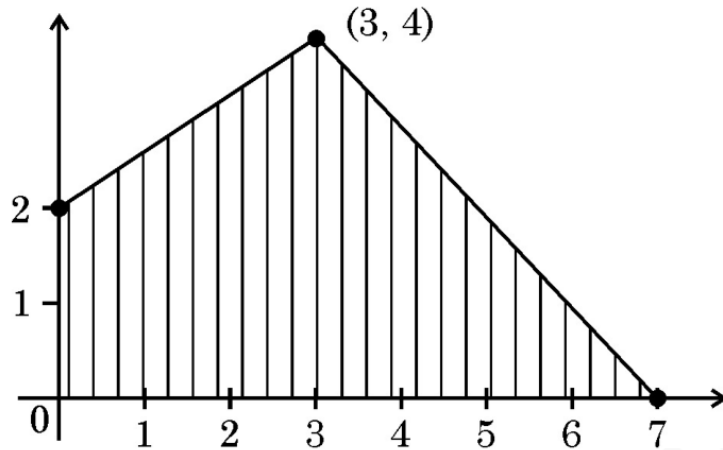
Step 4: Conclusion The length of the perpendicular is $\sqrt{74}$ units. Hence, the correct option is (C).

Quick Tip to Solve: The distance of (x_1, y_1, z_1) from the x -axis is simply $\sqrt{y_1^2 + z_1^2}$; here $\sqrt{25 + 49} = \sqrt{74}$.

17. The feasible region of a linear programming problem with objective function $Z = 5x + 7y$ is shown below:

The maximum value of Z - minimum value of Z is

[1 mark]



- (A) 8
- (B) 29
- (C) 35
- (D) 43

Correct Answer: (D) 43

Solution:

Concept:

- Corner Point Theorem: if a bounded feasible region exists, the maximum and minimum of a linear objective $Z = ax + by$ occur at corner points of the region.
- So evaluate $Z = 5x + 7y$ at every corner point, then take (largest value) – (smallest value).

Step 1: Read the corner points from the figure The shaded region is the quadrilateral bounded by the y -axis, the x -axis and two slanting lines. Its corners are

$$O(0, 0), \quad P(0, 2), \quad Q(3, 4), \quad R(7, 0).$$

The region is bounded, so both the maximum and the minimum of Z exist and occur at these corners.

Step 2: Evaluate $Z = 5x + 7y$ at each corner

$$Z(0, 0) = 0 + 0 = 0,$$

$$Z(0, 2) = 0 + 14 = 14,$$

$$Z(3, 4) = 15 + 28 = 43,$$

$$Z(7, 0) = 35 + 0 = 35.$$

So $Z_{\max} = 43$ at $(3, 4)$ and $Z_{\min} = 0$ at $(0, 0)$.

Step 3: Compute the difference and test the options $Z_{\max} - Z_{\min} = 43 - 0 = 43$. The other options come from common slips: $8 = 43 - 35$ uses the two largest values, $29 = 43 - 14$ wrongly takes $(0, 2)$

Quick Tip to Solve: Never forget the origin as a corner point when the region touches both axes at $(0, 0)$; here it gives $Z = 0$, the minimum.

as the minimum by forgetting the origin, and 35 is just the value at $(7, 0)$. The origin is a corner of the shaded region, so the minimum is 0.

Step 4: Conclusion The maximum value of Z minus the minimum value of Z is $43 - 0 = 43$. Hence, the correct option is (D).

Quick Tip to Solve: Never forget the origin as a corner point when the region touches both axes at $(0,0)$; here it gives $Z = 0$, the minimum.

18. The degree of an objective function of a linear programming problem is

[1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) Any natural number

Correct Answer: (B) 1

Solution:

Concept:

- In a linear programming problem the objective function has the form $Z = ax + by$ (more generally $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$), where the c_i are constants.
- The degree of a polynomial is the highest total power of the variables in any of its terms.

Step 1: Identify the form of the objective function The word "linear" in linear programming means that both the constraints and the objective function are linear expressions. Every term of $Z = ax + by$ is a constant times a single variable raised to the power 1; there are no terms like x^2 , xy or y^3 .

Step 2: Find the degree and reject the other options Each term ax or by has degree 1, so the degree of Z is 1. Degree 0 would make Z a constant, which cannot be optimised at all. Degree 2 (for example $x^2 + y^2$) or any higher natural number gives a non-linear (quadratic or higher) programming problem, not an LPP. So options (A), (C) and (D) fail.

Step 3: Conclusion The objective function of an LPP is linear, so its degree is 1. Hence, the correct option is (B).

Quick Tip to Solve: Linear means degree one: in an LPP both the objective function and every constraint are of the first degree in x and y .

19. Assertion (A): In an experiment of throwing an unbiased die, the probability of getting a prime number given that number appearing on the die being odd is $\frac{2}{3}$.

Reason (R): For any two events A and B , $P(A|B) = \frac{P(A \cup B)}{P(B)}$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true and Reason (R) is false.
(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- Conditional probability: for events A and B with $P(B) \neq 0$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- For equally likely outcomes, $P(A|B) = \frac{n(A \cap B)}{n(B)}$.

Step 1: Check Assertion (A) The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let $E =$ "prime number" $= \{2, 3, 5\}$ and $F =$ "odd number" $= \{1, 3, 5\}$. Then $E \cap F = \{3, 5\}$, so

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

Assertion (A) is true.

Step 2: Check Reason (R) The Reason states $P(A|B) = \frac{P(A \cup B)}{P(B)}$, with the union in the numerator. The correct definition uses the intersection $A \cap B$. The stated formula is false in general: in the die example it gives $\frac{P(E \cup F)}{P(F)} = \frac{4/6}{3/6} = \frac{4}{3} > 1$, which cannot be a probability. So Reason (R) is false.

Step 3: Choose the code Since R is false, options (A) and (B), which need R true, are ruled out, and (D) fails because A is true. Only "A true, R false" fits.

Step 4: Conclusion The Assertion is true and the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Read the Reason's formula symbol by symbol: conditional probability uses the intersection, so a union sign makes R false.

20. Assertion (A): Lines given by $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular to each other when $pp' + rr' = 1$.

Reason (R): Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are perpendicular to each other if $\vec{b}_1 \cdot \vec{b}_2 = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true and Reason (R) is false.
(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- A line written as $x = py + q$, $z = ry + s$ can be put in symmetric form $\frac{x - q}{p} = \frac{y}{1} = \frac{z - s}{r}$, so its direction ratios are $p, 1, r$.
- Lines with direction vectors $\vec{b}_1$ and $\vec{b}_2$ are perpendicular if and only if $\vec{b}_1 \cdot \vec{b}_2 = 0$, i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Step 1: Find the direction ratios of the two lines Taking $y = t$ as the parameter, the first line is $x = q + pt$, $y = t$, $z = s + rt$, so its direction vector is $\vec{b}_1 = p\hat{i} + \hat{j} + r\hat{k}$. Similarly the second line has $\vec{b}_2 = p'\hat{i} + \hat{j} + r'\hat{k}$.

Step 2: Check Assertion (A) The lines are perpendicular when

$$\vec{b}_1 \cdot \vec{b}_2 = pp' + 1 \cdot 1 + rr' = 0 \iff pp' + rr' = -1.$$

The Assertion gives the condition $pp' + rr' = 1$, which would make $\vec{b}_1 \cdot \vec{b}_2 = 2 \neq 0$. So Assertion (A) is false.

Step 3: Check Reason (R) and choose the code The angle between two lines is the angle between their direction vectors, so $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are perpendicular when $\vec{b}_1 \cdot \vec{b}_2 = 0$. Reason (R) is true. With A false and R true, options (A), (B) and (C) are ruled out.

Step 4: Conclusion The correct condition is $pp' + rr' = -1$, so the Assertion is false, while the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: For lines solved in terms of y, the y-direction ratio is 1, so the dot product carries an extra 1: the condition is $pp' + rr' + 1 = 0$.

Section B (10 marks)

This section comprises Very Short Answer (VSA) type questions of 2 marks each.

21(a). Check whether function $f(x)$ defined as $f(x) = \begin{cases} \frac{|x-3|}{2(x-3)}, & x < 3 \\ \frac{x-6}{6}, & x \geq 3 \end{cases}$ is continuous at $x = 3$ or not. [2 marks]

Solution:

Concept:

- f is continuous at $x = a$ if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$.
- Modulus: $|x - 3| = -(x - 3) = 3 - x$ when $x < 3$, and $|x - 3| = x - 3$ when $x \geq 3$.

Step 1: Value of the function at $x = 3$ For $x \geq 3$, $f(x) = \frac{x-6}{6}$, so

$$f(3) = \frac{3-6}{6} = -\frac{3}{6} = -\frac{1}{2}.$$

Step 2: Left-hand limit For $x < 3$, $x - 3 < 0$, so $|x - 3| = -(x - 3)$. Hence, for $x < 3$,

$$f(x) = \frac{-(x-3)}{2(x-3)} = -\frac{1}{2},$$

since $x - 3 \neq 0$ can be cancelled. Therefore

$$\text{LHL} = \lim_{x \rightarrow 3^-} f(x) = \lim_{h \rightarrow 0} \frac{|3-h-3|}{2(3-h-3)} = \lim_{h \rightarrow 0} \frac{h}{-2h} = -\frac{1}{2}.$$

Step 3: Right-hand limit

$$\text{RHL} = \lim_{x \rightarrow 3^+} f(x) = \lim_{h \rightarrow 0} \frac{(3+h)-6}{6} = \frac{-3}{6} = -\frac{1}{2}.$$

Step 4: Compare the three values $\text{LHL} = \text{RHL} = f(3) = -\frac{1}{2}$, so the limit of $f(x)$ as $x \rightarrow 3$ exists and equals the value of the function there.

Step 5: Conclusion Since $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) = -\frac{1}{2}$, the function f is continuous at $x = 3$.

Quick Tip to Solve: Simplify the modulus piece first: for $x < 3$ the expression $|x-3|/(2(x-3))$ is the constant $-1/2$, so only the right piece needs evaluating.

21(b). If $\sqrt{3}(x^2 + y^2) = 4xy$, then find $\frac{dy}{dx}$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

[2 marks]

Solution:

Concept:

- Implicit differentiation: differentiate both sides with respect to x , treating y as a function of x , so $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.
- Product rule: $\frac{d}{dx}(xy) = y + x \frac{dy}{dx}$.

Step 1: Differentiate both sides with respect to x Given $\sqrt{3}(x^2 + y^2) = 4xy$. Differentiating,

$$\sqrt{3} \left(2x + 2y \frac{dy}{dx} \right) = 4 \left(y + x \frac{dy}{dx} \right).$$

Step 2: Solve for $\frac{dy}{dx}$

$$\begin{aligned} 2\sqrt{3}x + 2\sqrt{3}y \frac{dy}{dx} &= 4y + 4x \frac{dy}{dx} \\ (2\sqrt{3}y - 4x) \frac{dy}{dx} &= 4y - 2\sqrt{3}x \\ \frac{dy}{dx} &= \frac{2y - \sqrt{3}x}{\sqrt{3}y - 2x}. \end{aligned}$$

Step 3: Substitute the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ The point lies on the curve: $\sqrt{3}\left(\frac{1}{4} + \frac{3}{4}\right) = \sqrt{3}$ and $4 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$.

Now

$$\begin{aligned} 2y - \sqrt{3}x &= \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}, \\ \sqrt{3}y - 2x &= \frac{3}{2} - 1 = \frac{1}{2}, \end{aligned}$$

so

$$\frac{dy}{dx} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Step 4: Conclusion $\frac{dy}{dx} = \frac{2y - \sqrt{3}x}{\sqrt{3}y - 2x}$, and at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ its value is $\sqrt{3}$.

Quick Tip to Solve: Divide through by 2 before substituting: $\frac{dy}{dx} = \frac{2y - \sqrt{3}x}{\sqrt{3}y - 2x}$ keeps the arithmetic with the surd short.

22. A room freshener bottle in the shape of an inverted cone sprays the perfume at regular intervals such that volume of the perfume in the bottle decreases at the steady rate of $1 \text{ mm}^3/\text{min}$. Find the rate at which level of perfume is dropping at an instant when level of perfume in the bottle is 10 mm , if the semi-vertical angle of conical bottle is $\frac{\pi}{6}$. [2 marks]

Solution:

Concept:

- Volume of a cone of radius r and height h : $V = \frac{1}{3}\pi r^2 h$.
- For an inverted cone with semi-vertical angle α , the radius of the liquid surface at depth h is $r = h \tan \alpha$, so V can be written in terms of h alone.
- Chain rule for related rates: $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$.

Step 1: Express the volume in terms of the level h Let h be the level (height) of perfume and r the radius of its top surface at time t . The semi-vertical angle is $\alpha = \frac{\pi}{6}$, so from the right triangle formed by the axis, the slant side and the radius,

$$\tan \frac{\pi}{6} = \frac{r}{h} \implies r = \frac{h}{\sqrt{3}}.$$

Hence the volume of perfume is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot \frac{h^2}{3} \cdot h = \frac{\pi h^3}{9}.$$

Step 2: Differentiate with respect to time Differentiating both sides with respect to t ,

$$\frac{dV}{dt} = \frac{\pi}{9} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{3} \frac{dh}{dt}.$$

The volume decreases steadily at $1 \text{ mm}^3/\text{min}$, so $\frac{dV}{dt} = -1 \text{ mm}^3/\text{min}$ (negative because V is decreasing).

Step 3: Substitute $h = 10 \text{ mm}$

$$-1 = \frac{\pi(10)^2}{3} \frac{dh}{dt} = \frac{100\pi}{3} \frac{dh}{dt} \implies \frac{dh}{dt} = -\frac{3}{100\pi} \text{ mm/min}.$$

The negative sign shows that the level is falling, and its magnitude is the rate of drop.

Step 4: Conclusion The level of perfume is dropping at the rate of $\frac{3}{100\pi} \text{ mm/min} \approx 0.0095 \text{ mm/min}$ when the level is 10 mm .

Quick Tip to Solve: Eliminate r first using $r = h \tan \alpha$; with $\alpha = 30^\circ$, $V = \pi h^3/9$, so $dh/dt = 3(dV/dt)/(\pi h^2)$ in one line.

23. Find the vector of magnitude 14 in the direction of $\vec{QP}$, where P and Q are the points $(1, 3, 2)$ and $(-1, 0, 8)$ respectively. [2 marks]

Solution:

Concept:

- The vector joining Q to P is $\vec{QP} = \vec{OP} - \vec{OQ}$ (position vector of the head minus that of the tail).
- The unit vector along $\vec{v}$ is $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$; a vector of magnitude λ in the same direction is $\lambda\hat{v}$.

Step 1: Find $\vec{QP}$ With $P(1, 3, 2)$ and $Q(-1, 0, 8)$,

$$\vec{QP} = \vec{OP} - \vec{OQ} = (1 - (-1))\hat{i} + (3 - 0)\hat{j} + (2 - 8)\hat{k} = 2\hat{i} + 3\hat{j} - 6\hat{k}.$$

Step 2: Find its magnitude and the unit vector

$$|\vec{QP}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7.$$

So the unit vector in the direction of $\vec{QP}$ is

$$\widehat{QP} = \frac{1}{7} (2\hat{i} + 3\hat{j} - 6\hat{k}).$$

Step 3: Scale to magnitude 14

$$14\widehat{QP} = \frac{14}{7} (2\hat{i} + 3\hat{j} - 6\hat{k}) = 4\hat{i} + 6\hat{j} - 12\hat{k}.$$

Check: $\sqrt{16 + 36 + 144} = \sqrt{196} = 14$, as required.

Step 4: Conclusion The required vector is $4\hat{i} + 6\hat{j} - 12\hat{k}$.

Quick Tip to Solve: Direction matters: $\vec{QP}$ is P minus Q , not Q minus P . Taking it the wrong way gives the opposite vector $-4\hat{i} - 6\hat{j} + 12\hat{k}$.

24. Vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{k}$ represent the two adjacent sides of a parallelogram. Find the vectors representing its diagonals and hence find their lengths. [2 marks]

Solution:

Concept:

- If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, its diagonals are represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ (or $\vec{b} - \vec{a}$), by the triangle law of addition.
- Length of $x\hat{i} + y\hat{j} + z\hat{k}$ is $\sqrt{x^2 + y^2 + z^2}$.

Step 1: First diagonal $\vec{a} + \vec{b}$ With $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 0\hat{j} + 2\hat{k}$,

$$\vec{a} + \vec{b} = (3 + 1)\hat{i} + (-2 + 0)\hat{j} + (2 + 2)\hat{k} = 4\hat{i} - 2\hat{j} + 4\hat{k},$$

Quick Tip to Solve: Note that $\vec{b}$ has no $\hat{j}$ component; write it as $\hat{i} + 0\hat{j} + 2\hat{k}$ before adding or subtracting to avoid sign slips.

$$|\vec{a} + \vec{b}| = \sqrt{16 + 4 + 16} = \sqrt{36} = 6.$$

Step 2: Second diagonal $\vec{a} - \vec{b}$

$$\vec{a} - \vec{b} = (3 - 1)\hat{i} + (-2 - 0)\hat{j} + (2 - 2)\hat{k} = 2\hat{i} - 2\hat{j},$$

$$|\vec{a} - \vec{b}| = \sqrt{4 + 4 + 0} = \sqrt{8} = 2\sqrt{2}.$$

The vector $\vec{b} - \vec{a} = -2\hat{i} + 2\hat{j}$ represents the same diagonal traversed in the opposite sense, with the same length.

Step 3: Conclusion The diagonals are represented by $4\hat{i} - 2\hat{j} + 4\hat{k}$ and $2\hat{i} - 2\hat{j}$, and their lengths are 6 and $2\sqrt{2}$ units respectively.

Quick Tip to Solve: Note that $\vec{b}$ has no $\hat{j}$ component; write it as $\hat{i} + 0\hat{j} + 2\hat{k}$ before adding or subtracting to avoid sign slips.

25(a). Simplify: $\tan^{-1}\left(\frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x}\right)$, $0 < x < \frac{\pi}{4}$.

[2 marks]

Solution:

Concept:

- $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan \theta}{1 + \tan \theta}$.
- $\tan^{-1}(\tan y) = y$ only when y lies in the principal value branch $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 1: Rewrite the expression in terms of $\tan 2x$ For $0 < x < \frac{\pi}{4}$ we have $0 < 2x < \frac{\pi}{2}$, so $\cos 2x > 0$. Dividing numerator and denominator by $\cos 2x$,

$$\frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x} = \frac{1 - \tan 2x}{1 + \tan 2x} = \frac{\tan \frac{\pi}{4} - \tan 2x}{1 + \tan \frac{\pi}{4} \tan 2x} = \tan\left(\frac{\pi}{4} - 2x\right).$$

Step 2: Check the principal value range Since $0 < 2x < \frac{\pi}{2}$,

$$-\frac{\pi}{4} < \frac{\pi}{4} - 2x < \frac{\pi}{4},$$

which lies inside $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore

$$\tan^{-1}\left(\frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - 2x\right)\right) = \frac{\pi}{4} - 2x.$$

Step 3: Conclusion $\tan^{-1}\left(\frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x}\right) = \frac{\pi}{4} - 2x$ for $0 < x < \frac{\pi}{4}$.

Quick Tip to Solve: The pattern $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta}$ is always $\tan\left(\frac{\pi}{4} - \theta\right)$; just confirm $\frac{\pi}{4} - \theta$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

25(b). Evaluate: $\tan \left[\sin^{-1} 1 - \cos^{-1} \left(-\frac{1}{2} \right) \right]$.

[2 marks]

Solution:

Concept:

- Principal value branches: $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $\cos^{-1} x \in [0, \pi]$.
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$ for $x \in [-1, 1]$; also $\tan(-\theta) = -\tan \theta$.

Step 1: Principal value of $\sin^{-1} 1$ The angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ whose sine is 1 is $\frac{\pi}{2}$, so $\sin^{-1} 1 = \frac{\pi}{2}$.

Step 2: Principal value of $\cos^{-1} \left(-\frac{1}{2} \right)$

$$\cos^{-1} \left(-\frac{1}{2} \right) = \pi - \cos^{-1} \frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3},$$

which lies in $[0, \pi]$ as required.

Step 3: Evaluate the tangent

$$\sin^{-1} 1 - \cos^{-1} \left(-\frac{1}{2} \right) = \frac{\pi}{2} - \frac{2\pi}{3} = \frac{3\pi - 4\pi}{6} = -\frac{\pi}{6}.$$

Hence

$$\tan \left[\sin^{-1} 1 - \cos^{-1} \left(-\frac{1}{2} \right) \right] = \tan \left(-\frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}.$$

Step 4: Conclusion $\tan \left[\sin^{-1} 1 - \cos^{-1} \left(-\frac{1}{2} \right) \right] = -\frac{1}{\sqrt{3}}.$

Quick Tip to Solve: A common slip is taking $\cos^{-1} \left(-\frac{1}{2} \right) = -\frac{\pi}{3}$; the principal range of $\cos^{-1}$ is $[0, \pi]$, so it is $\frac{2\pi}{3}$.

Section C (18 marks)

This section comprises Short Answer (SA) type questions of 3 marks each.

26. Evaluate: $\int_0^1 x \tan^{-1} x \, dx.$

[3 marks]

Solution:

Concept:

- Integration by parts for definite integrals: $\int_a^b u v \, dx = \left[u \int v \, dx \right]_a^b - \int_a^b \left(u' \int v \, dx \right) dx$, where u is the first function and v the second.
- ILATE order: an inverse trigonometric function is chosen as the first function u ahead of an algebraic function, because $\tan^{-1} x$ has a simple derivative $\frac{1}{1+x^2}$ but no elementary antiderivative of a

Quick Tip to Solve: With $\tan^{-1} x$ times x , always take $\tan^{-1} x$ as the first function; the leftover $\frac{x^2}{1+x^2}$ splits at once as $1 - \frac{1}{1+x^2}$, so the answer is ready in two lines.

convenient form.

- Standard results: $\int \frac{dx}{1+x^2} = \tan^{-1} x$, and $\tan^{-1} 1 = \frac{\pi}{4}$, $\tan^{-1} 0 = 0$.

Step 1: Choose the functions and apply integration by parts Let $I = \int_0^1 x \tan^{-1} x \, dx$. Take $u = \tan^{-1} x$ (first function) and $v = x$ (second function), so $\int x \, dx = \frac{x^2}{2}$ and $u' = \frac{1}{1+x^2}$. Then

$$I = \left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^2}{2} \, dx.$$

Step 2: Evaluate the boundary term

$$\left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 = \frac{1}{2} \cdot \frac{\pi}{4} - 0 = \frac{\pi}{8}.$$

Step 3: Simplify and evaluate the remaining integral Write the integrand as a polynomial part plus a proper fraction: $\frac{x^2}{1+x^2} = \frac{(1+x^2) - 1}{1+x^2} = 1 - \frac{1}{1+x^2}$. Hence

$$\begin{aligned} \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx &= \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) \, dx = \frac{1}{2} \left[x - \tan^{-1} x \right]_0^1 \\ &= \frac{1}{2} \left[\left(1 - \frac{\pi}{4} \right) - (0 - 0) \right] = \frac{1}{2} - \frac{\pi}{8}. \end{aligned}$$

Step 4: Combine the two parts

$$I = \frac{\pi}{8} - \left(\frac{1}{2} - \frac{\pi}{8} \right) = \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}.$$

As a check, $\frac{\pi}{4} - \frac{1}{2} \approx 0.285 > 0$, which is consistent with the integrand $x \tan^{-1} x$ being non-negative on $[0, 1]$ and at most $\frac{\pi}{4} \approx 0.785$.

Step 5: Conclusion $\int_0^1 x \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$.

Quick Tip to Solve: With $\tan^{-1} x$ times x , always take $\tan^{-1} x$ as the first function; the leftover $\frac{x^2}{1+x^2}$ splits at once as $1 - \frac{1}{1+x^2}$, so the answer is ready in two lines.

27(a). Find: $\int \sqrt{\frac{x+2}{x-2}} dx$.

[3 marks]

Solution:

Concept:

- Rationalising the numerator: $\sqrt{\frac{x+2}{x-2}} = \frac{x+2}{\sqrt{(x+2)(x-2)}} = \frac{x+2}{\sqrt{x^2-4}}$ on the domain $x > 2$, where both $x+2$ and $x-2$ are positive.
- Substitution: $\int \frac{x}{\sqrt{x^2-a^2}} dx = \sqrt{x^2-a^2} + C$ (put $t = x^2 - a^2$).
- Standard integral: $\int \frac{dx}{\sqrt{x^2-a^2}} = \log |x + \sqrt{x^2-a^2}| + C$, here with $a = 2$.

Step 1: Rewrite the integrand Multiply the numerator and the denominator inside the root by $x+2$. For $x > 2$,

$$\sqrt{\frac{x+2}{x-2}} = \sqrt{\frac{(x+2)^2}{(x+2)(x-2)}} = \frac{x+2}{\sqrt{x^2-4}}.$$

Hence

$$I = \int \sqrt{\frac{x+2}{x-2}} dx = \int \frac{x}{\sqrt{x^2-4}} dx + 2 \int \frac{dx}{\sqrt{x^2-4}} = I_1 + 2I_2.$$

Step 2: Evaluate I_1 by substitution Put $t = x^2 - 4$, so $dt = 2x dx$, i.e. $x dx = \frac{dt}{2}$. Then

$$I_1 = \int \frac{x dx}{\sqrt{x^2-4}} = \frac{1}{2} \int t^{-1/2} dt = \frac{1}{2} \cdot 2\sqrt{t} = \sqrt{x^2-4} + C_1.$$

Step 3: Evaluate I_2 with the standard form With $a = 2$,

$$I_2 = \int \frac{dx}{\sqrt{x^2-2^2}} = \log |x + \sqrt{x^2-4}| + C_2.$$

Step 4: Combine the results

$$I = I_1 + 2I_2 = \sqrt{x^2-4} + 2 \log |x + \sqrt{x^2-4}| + C,$$

where $C = C_1 + 2C_2$. Check by differentiation: $\frac{d}{dx} \sqrt{x^2-4} = \frac{x}{\sqrt{x^2-4}}$ and $\frac{d}{dx} 2 \log |x + \sqrt{x^2-4}| = \frac{2}{\sqrt{x^2-4}}$; their sum is $\frac{x+2}{\sqrt{x^2-4}}$, the integrand.

Step 5: Conclusion $\int \sqrt{\frac{x+2}{x-2}} dx = \sqrt{x^2-4} + 2 \log |x + \sqrt{x^2-4}| + C.$

Quick Tip to Solve: For $\sqrt{\frac{x+a}{x-a}}$, multiply inside the root by $x+a$: the integrand becomes $\frac{x+a}{\sqrt{x^2-a^2}}$, which splits into a direct substitution and one standard log form.

27(b). Find: $\int \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx$.

[3 marks]

Solution:

Concept:

- Partial fractions in x^2 : since only even powers of x appear, put $x^2 = y$ (only for the decomposition) and write $\frac{y}{(y + 9)(y + 16)} = \frac{A}{y + 9} + \frac{B}{y + 16}$.
- Standard integral: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$.

Step 1: Set up the partial fractions Put $x^2 = y$. Then

$$\frac{y}{(y + 9)(y + 16)} = \frac{A}{y + 9} + \frac{B}{y + 16} \implies y = A(y + 16) + B(y + 9).$$

Step 2: Find the constants Put $y = -9$: $-9 = A(7)$, so $A = -\frac{9}{7}$. Put $y = -16$: $-16 = B(-7)$, so $B = \frac{16}{7}$. Check: comparing the coefficients of y , $A + B = \frac{-9 + 16}{7} = 1$, as required. Therefore

$$\frac{x^2}{(x^2 + 9)(x^2 + 16)} = -\frac{9}{7} \cdot \frac{1}{x^2 + 9} + \frac{16}{7} \cdot \frac{1}{x^2 + 16}.$$

Step 3: Integrate each term Using $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ with $a = 3$ and $a = 4$:

$$\begin{aligned} \int \frac{x^2 dx}{(x^2 + 9)(x^2 + 16)} &= -\frac{9}{7} \int \frac{dx}{x^2 + 3^2} + \frac{16}{7} \int \frac{dx}{x^2 + 4^2} \\ &= -\frac{9}{7} \cdot \frac{1}{3} \tan^{-1} \frac{x}{3} + \frac{16}{7} \cdot \frac{1}{4} \tan^{-1} \frac{x}{4} + C \\ &= -\frac{3}{7} \tan^{-1} \frac{x}{3} + \frac{4}{7} \tan^{-1} \frac{x}{4} + C. \end{aligned}$$

Step 4: Conclusion $\int \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx = \frac{4}{7} \tan^{-1} \frac{x}{4} - \frac{3}{7} \tan^{-1} \frac{x}{3} + C$.

Quick Tip to Solve: When the integrand has only x^2 terms, decompose in $y = x^2$ but integrate in x ; each piece is then a direct $\frac{1}{a} \tan^{-1} \frac{x}{a}$ form.

28. If $I_1 = \int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x}$ and $I_2 = \int_{-1/2}^{1/2} |x| dx$, then show that $I_1 - 4I_2 = 0$.

[3 marks]

Solution:

Concept:

- Double-angle identity: $1 + \cos 2x = 2 \cos^2 x$, so $\frac{1}{1 + \cos 2x} = \frac{1}{2} \sec^2 x$, and $\int \sec^2 x dx = \tan x$.
- Even-function property: if $f(-x) = f(x)$, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- Modulus: $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$; $|x|$ is an even function.

Step 1: Evaluate I_1 On $[-\frac{\pi}{4}, \frac{\pi}{4}]$, $\cos x \neq 0$, so the identity $1 + \cos 2x = 2 \cos^2 x$ gives

$$\begin{aligned} I_1 &= \int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x} = \int_{-\pi/4}^{\pi/4} \frac{dx}{2 \cos^2 x} = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x dx \\ &= \frac{1}{2} \left[\tan x \right]_{-\pi/4}^{\pi/4} = \frac{1}{2} \left[\tan \frac{\pi}{4} - \tan \left(-\frac{\pi}{4} \right) \right] = \frac{1}{2} [1 - (-1)] = 1. \end{aligned}$$

Step 2: Evaluate I_2 Since $|-x| = |x|$, the integrand is even, so

$$I_2 = \int_{-1/2}^{1/2} |x| dx = 2 \int_0^{1/2} x dx = 2 \left[\frac{x^2}{2} \right]_0^{1/2} = \left(\frac{1}{2} \right)^2 - 0 = \frac{1}{4}.$$

(Equivalently, splitting at 0: $\int_{-1/2}^0 (-x) dx + \int_0^{1/2} x dx = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.)

Step 3: Form $I_1 - 4I_2$

$$I_1 - 4I_2 = 1 - 4 \cdot \frac{1}{4} = 1 - 1 = 0.$$

Step 4: Conclusion Since $I_1 = 1$ and $I_2 = \frac{1}{4}$, we get $I_1 - 4I_2 = 0$. Hence proved.

Quick Tip to Solve: Convert $\frac{1}{1 + \cos 2x}$ to $\frac{1}{2} \sec^2 x$ at once, and use evenness of $|x|$ to avoid splitting the modulus integral into two pieces.

29(a). Find the general solution of the following differential equation: $x^2 \frac{dy}{dx} = x^2 + xy + y^2$. [3 marks]

Solution:

Concept:

- A differential equation $\frac{dy}{dx} = F(x, y)$ is homogeneous when $F(\lambda x, \lambda y) = \lambda^0 F(x, y)$, i.e. F depends only on the ratio $\frac{y}{x}$.
- Such an equation is solved by the substitution $y = vx$, which gives $\frac{dy}{dx} = v + x \frac{dv}{dx}$ and turns the equation into one with separable variables v and x .

Quick Tip to Solve: Every term on the right has degree 2, the same as the x^2 on the left, so divide by x^2 at once and put $y = vx$; the v terms cancel and leave the standard $\tan^{-1}$ integral.

- Standard integrals used: $\int \frac{dv}{1+v^2} = \tan^{-1} v$ and $\int \frac{dx}{x} = \log |x|$.

Step 1: Rewrite the equation and check that it is homogeneous For $x \neq 0$, divide both sides of $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ by x^2 :

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 = F(x, y).$$

Replacing x, y by $\lambda x, \lambda y$ gives $F(\lambda x, \lambda y) = 1 + \frac{\lambda y}{\lambda x} + \frac{\lambda^2 y^2}{\lambda^2 x^2} = F(x, y)$, so F is homogeneous of degree zero and the equation is homogeneous.

Step 2: Substitute $y = vx$ Put $y = vx$, so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$. The equation becomes

$$\begin{aligned} v + x \frac{dv}{dx} &= 1 + v + v^2 \\ x \frac{dv}{dx} &= 1 + v^2. \end{aligned}$$

Step 3: Separate the variables and integrate Since $1 + v^2 > 0$ always, we may divide by it:

$$\frac{dv}{1+v^2} = \frac{dx}{x}.$$

Integrating both sides,

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x} \implies \tan^{-1} v = \log |x| + C,$$

where C is an arbitrary constant.

Step 4: Return to the original variables Substituting back $v = \frac{y}{x}$:

$$\tan^{-1} \left(\frac{y}{x} \right) = \log |x| + C.$$

Equivalently, $y = x \tan(\log |x| + C)$. Check: differentiating $y = x \tan(\log |x| + C)$ gives $\frac{dy}{dx} = \tan(\cdot) + \sec^2(\cdot) = 1 + \frac{y}{x} + \frac{y^2}{x^2}$, which is the given equation after multiplying by x^2 .

Step 5: Conclusion The general solution of $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ is

$$\tan^{-1} \left(\frac{y}{x} \right) = \log |x| + C,$$

where C is an arbitrary constant.

Quick Tip to Solve: Every term on the right has degree 2, the same as the x^2 on the left, so divide by x^2 at once and put $y = vx$; the v terms cancel and leave the standard $\tan^{-1}$ integral.

29(b). Find the particular solution of the differential equation $xy \frac{dy}{dx} = (x+2)(y+2)$, given that $y(1) = -1$. [3 marks]

Solution:

Concept:

- An equation of the form $g(y) dy = f(x) dx$ has separable variables; integrate each side with respect to its own variable and add one arbitrary constant.
- The particular solution is obtained by substituting the given initial condition $y(x_0) = y_0$ into the general solution to fix the constant.
- Standard integrals used: $\int \frac{dx}{x} = \log |x|$ and $\int \frac{dy}{y+2} = \log |y+2|$.

Step 1: Separate the variables Given $xy \frac{dy}{dx} = (x+2)(y+2)$. For $x \neq 0$ and $y \neq -2$, divide both sides by $x(y+2)$:

$$\frac{y}{y+2} dy = \frac{x+2}{x} dx.$$

The initial point $y(1) = -1$ has $x = 1 \neq 0$ and $y + 2 = 1 \neq 0$, so this division is valid near it.

Step 2: Simplify the integrands Write each fraction as a constant plus a simple fraction:

$$\frac{y}{y+2} = \frac{(y+2)-2}{y+2} = 1 - \frac{2}{y+2}, \quad \frac{x+2}{x} = 1 + \frac{2}{x}.$$

Step 3: Integrate both sides to get the general solution

$$\int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx$$

$$y - 2 \log |y+2| = x + 2 \log |x| + C,$$

where C is an arbitrary constant. This is the general solution.

Step 4: Use $y(1) = -1$ to find C Put $x = 1, y = -1$:

$$\begin{aligned} -1 - 2 \log |-1+2| &= 1 + 2 \log 1 + C \\ -1 - 2 \log 1 &= 1 + 0 + C \\ -1 &= 1 + C \implies C = -2. \end{aligned}$$

Step 5: Write the particular solution Substituting $C = -2$:

$$y - 2 \log |y+2| = x + 2 \log |x| - 2.$$

Collecting the logarithms, $2 \log |x| + 2 \log |y+2| = 2 \log |x(y+2)|$, so the same solution can be written as

$$y - x + 2 = 2 \log |x(y+2)|.$$

Quick Tip to Solve: Split $\frac{y}{y+2}$ as $1 - \frac{2}{y+2}$ and $\frac{x+2}{x}$ as $1 + \frac{2}{x}$ before integrating; at $(1, -1)$ both logarithms are $\log 1 = 0$, so C comes out instantly as -2 .

Check at $(1, -1)$: left side $-1 - 1 + 2 = 0$, right side $2 \log |1 \cdot 1| = 0$.

Step 6: Conclusion The required particular solution is

$$y - 2 \log |y + 2| = x + 2 \log |x| - 2, \quad \text{i.e.} \quad y - x + 2 = 2 \log |x(y + 2)|.$$

Quick Tip to Solve: Split $\frac{y}{y+2}$ as $1 - \frac{2}{y+2}$ and $\frac{x+2}{x}$ as $1 + \frac{2}{x}$ before integrating; at $(1, -1)$ both logarithms are $\log 1 = 0$, so C comes out instantly as -2 .

30. Solve the following linear programming problem graphically:

Minimize $Z = 13x - 15y$

Subject to constraints

$$x + y \leq 7,$$

$$2x - 3y + 6 \geq 0,$$

$$x \geq 0, y \geq 0.$$

[3 marks]

Solution:

Concept:

- The feasible region of an LPP is the common region of all the half-planes given by the constraints; for a bounded feasible region the optimum of a linear objective $Z = ax + by$ occurs at a corner point.
- Corner point method: find every corner of the feasible region (intersections of the boundary lines), evaluate Z at each, and pick the smallest value for a minimisation problem.

Step 1: Draw the boundary lines Treat each inequality as an equation:

- $x + y = 7$ passes through $(7, 0)$ and $(0, 7)$. Testing $(0, 0)$: $0 \leq 7$ is true, so $x + y \leq 7$ is the side containing the origin.
- $2x - 3y + 6 = 0$ passes through $(0, 2)$ and $(-3, 0)$, and also through $(3, 4)$. Testing $(0, 0)$: $6 \geq 0$ is true, so $2x - 3y + 6 \geq 0$ is the side containing the origin (below the line).
- $x \geq 0, y \geq 0$ restrict the region to the first quadrant.

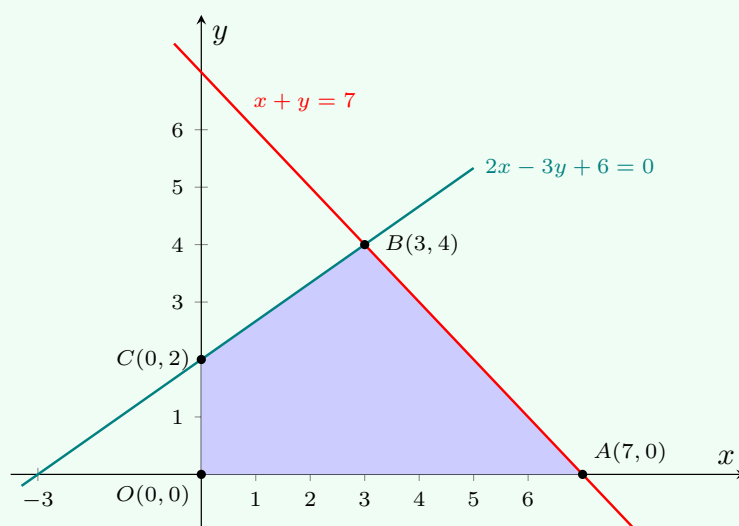
Step 2: Find the corner points of the feasible region The intersection of $x + y = 7$ and $2x - 3y = -6$: from the first, $y = 7 - x$; substituting,

$$2x - 3(7 - x) = -6 \implies 5x = 15 \implies x = 3, y = 4.$$

The feasible region is the bounded quadrilateral $OABC$ shaded in the figure, with corners

$$O(0, 0), \quad A(7, 0), \quad B(3, 4), \quad C(0, 2).$$

Quick Tip to Solve: Compare the two candidate corners quickly: going from $C(0, 2)$ to $B(3, 4)$ adds $13 \times 3 = 39$ but subtracts only $15 \times 2 = 30$, so Z rises by 9 and C gives the minimum.



Feasible region $OABC$ for $x + y \leq 7$, $2x - 3y + 6 \geq 0$, $x, y \geq 0$

Step 3: Evaluate $Z = 13x - 15y$ at each corner

Corner point	$Z = 13x - 15y$
$O(0, 0)$	0
$A(7, 0)$	91
$B(3, 4)$	$39 - 60 = -21$
$C(0, 2)$	$0 - 30 = -30$ (smallest)

Since the feasible region is bounded, the smallest of these values is the minimum of Z over the region.

Step 4: Conclusion The minimum value of Z is -30 , attained at the point $(0, 2)$, i.e. when $x = 0$ and $y = 2$.

Quick Tip to Solve: Compare the two candidate corners quickly: going from $C(0, 2)$ to $B(3, 4)$ adds $13 \times 3 = 39$ but subtracts only $15 \times 2 = 30$, so Z rises by 9 and C gives the minimum.

31(a). Out of two bags, bag I contains 3 red and 4 white balls and bag II contains 8 red and 6 white balls. A die is thrown. If it shows a number less than 3 then a ball is drawn at random from bag I, otherwise a ball is drawn at random from bag II. Find the probability that the ball drawn from one of the bags is a red ball. [3 marks]

Solution:

Concept:

- Theorem of total probability: if E_1, E_2 are mutually exclusive and exhaustive events and A is any event, then $P(A) = P(E_1)P(A | E_1) + P(E_2)P(A | E_2)$.
- For a fair die, the probability of an event is (number of favourable faces)/6; for a random draw from a bag, $P(\text{red}) = \frac{\text{number of red balls}}{\text{total number of balls}}$.

Step 1: Define the events and find their probabilities Let E_1 : the die shows a number less than 3 (i.e.

Quick Tip to Solve: The condition *less than 3* means only the faces 1 and 2, so bag I gets weight $\frac{1}{3}$, not $\frac{1}{2}$; reduce $\frac{8}{14}$ to $\frac{4}{7}$ so both terms share the denominator 21.

1 or 2), so bag I is used; E_2 : the die shows 3, 4, 5 or 6, so bag II is used; A : the ball drawn is red. E_1 and E_2 are mutually exclusive and exhaustive, and

$$P(E_1) = \frac{2}{6} = \frac{1}{3}, \quad P(E_2) = \frac{4}{6} = \frac{2}{3}.$$

Step 2: Find the conditional probabilities of drawing a red ball Bag I has 3 red and 4 white balls (7 in all); bag II has 8 red and 6 white balls (14 in all). Hence

$$P(A | E_1) = \frac{3}{7}, \quad P(A | E_2) = \frac{8}{14} = \frac{4}{7}.$$

Step 3: Apply the theorem of total probability

$$\begin{aligned} P(A) &= P(E_1)P(A | E_1) + P(E_2)P(A | E_2) \\ &= \frac{1}{3} \cdot \frac{3}{7} + \frac{2}{3} \cdot \frac{4}{7} \\ &= \frac{3}{21} + \frac{8}{21} = \frac{11}{21}. \end{aligned}$$

As a check, the probability of a white ball is $\frac{1}{3} \cdot \frac{4}{7} + \frac{2}{3} \cdot \frac{3}{7} = \frac{10}{21}$, and $\frac{11}{21} + \frac{10}{21} = 1$.

Step 4: Conclusion The probability that the ball drawn is red is $\frac{11}{21}$.

Quick Tip to Solve: The condition *less than 3* means only the faces 1 and 2, so bag I gets weight $\frac{1}{3}$, not $\frac{1}{2}$; reduce $\frac{8}{14}$ to $\frac{4}{7}$ so both terms share the denominator 21.

OR

31(b). The probability of simultaneous occurrence of atleast one of the two events X and Y is a . If the probability that exactly one of the events X, Y occurs is b , prove that $P(X') + P(Y') = 2 - 2a + b$.

[3 marks]

Solution:

Concept:

- “At least one of X, Y occurs” is the event $X \cup Y$, and the addition theorem gives $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$.
- “Exactly one of X, Y occurs” is the event $(X \cap Y') \cup (X' \cap Y)$. The two parts are mutually exclusive, so its probability is $P(X) + P(Y) - 2P(X \cap Y)$.
- Complement rule: $P(E') = 1 - P(E)$ for any event E .

Step 1: Translate the given data into equations The probability that at least one of the events occurs is a :

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = a. \quad (1)$$

“Exactly one occurs” means X occurs without Y , or Y occurs without X . Since $P(X \cap Y') =$

Quick Tip to Solve: Write “at least one” as $P(X) + P(Y) - P(X \cap Y)$ and “exactly one” as $P(X) + P(Y) - 2P(X \cap Y)$. Subtracting the two gives $P(X \cap Y)$ at once, and the complements give $2 - [P(X) + P(Y)]$.

$P(X) - P(X \cap Y)$ and $P(X' \cap Y) = P(Y) - P(X \cap Y)$, and these two events are mutually exclusive,

$$P(X \cap Y') + P(X' \cap Y) = P(X) + P(Y) - 2P(X \cap Y) = b. \quad (2)$$

Step 2: Eliminate $P(X \cap Y)$ Subtracting (2) from (1):

$$[P(X) + P(Y) - P(X \cap Y)] - [P(X) + P(Y) - 2P(X \cap Y)] = a - b \Rightarrow P(X \cap Y) = a - b.$$

Substituting this in (1):

$$P(X) + P(Y) = a + P(X \cap Y) = a + (a - b) = 2a - b. \quad (3)$$

Step 3: Use the complement rule Since $P(X') = 1 - P(X)$ and $P(Y') = 1 - P(Y)$,

$$\begin{aligned} P(X') + P(Y') &= 2 - [P(X) + P(Y)] \\ &= 2 - (2a - b) \quad \text{using (3)} \\ &= 2 - 2a + b. \end{aligned}$$

Step 4: Conclusion Since $P(X \cap Y) = a - b$ and $P(X) + P(Y) = 2a - b$, we get $P(X') + P(Y') = 2 - 2a + b$. Hence proved.

Quick Tip to Solve: Write “at least one” as $P(X) + P(Y) - P(X \cap Y)$ and “exactly one” as $P(X) + P(Y) - 2P(X \cap Y)$. Subtracting the two gives $P(X \cap Y)$ at once, and the complements give $2 - [P(X) + P(Y)]$.

Section D (20 marks)

This section comprises Long Answer (LA) type questions of 5 marks each.

32(a). A relation R is defined on $\mathbb{Z}$, the set of integers, as

$$R = \{(x, y) : |x - y| \text{ is divisible by a prime number } p, x, y \in \mathbb{Z}\}.$$

Check whether R is an equivalence relation or not.

[5 marks]

Solution:

Concept:

- A relation R on a set A is an equivalence relation if it is reflexive ($(a, a) \in R$ for all $a \in A$), symmetric ($(a, b) \in R \Rightarrow (b, a) \in R$) and transitive ($(a, b), (b, c) \in R \Rightarrow (a, c) \in R$).
- Here p is a fixed prime. For integers, $|m|$ is divisible by p exactly when $m = pk$ for some integer k . So $(x, y) \in R \iff x - y = pk$ for some $k \in \mathbb{Z}$.
- Useful facts: $|-m| = |m|$, and the sum of two multiples of p is a multiple of p . Also $0 = p \cdot 0$ is divisible by every prime.

Step 1: Restate the relation Let p be the given prime. For $x, y \in \mathbb{Z}$, $x - y$ is an integer and

Quick Tip to Solve: Read “ $|x - y|$ divisible by p ” as $x - y = pk$. Then reflexive uses $0 = p \cdot 0$, symmetric uses $y - x = p(-k)$, and transitive adds $x - y = pk_1$ and $y - z = pk_2$ to get $x - z = p(k_1 + k_2)$.

$|x - y| = \pm(x - y)$. So

$$\begin{aligned}(x, y) \in R &\iff p \mid |x - y| \iff p \mid (x - y) \\ &\iff x - y = pk \text{ for some } k \in \mathbb{Z}.\end{aligned}$$

We check the three properties one by one.

Step 2: Reflexivity Let $x \in \mathbb{Z}$ be arbitrary. Then

$$|x - x| = 0 = p \cdot 0,$$

and 0 is divisible by p . Hence $(x, x) \in R$ for every $x \in \mathbb{Z}$, so R is reflexive.

Step 3: Symmetry Let $(x, y) \in R$. Then $|x - y|$ is divisible by p , say $|x - y| = pk$ with $k \in \mathbb{Z}$. Now

$$|y - x| = |-(x - y)| = |x - y| = pk,$$

so $|y - x|$ is also divisible by p . Hence $(y, x) \in R$. Since $(x, y) \in R \Rightarrow (y, x) \in R$, R is symmetric.

Step 4: Transitivity Let $(x, y) \in R$ and $(y, z) \in R$. Then there exist integers k_1, k_2 such that

$$x - y = pk_1 \quad \text{and} \quad y - z = pk_2.$$

Adding the two equations,

$$x - z = (x - y) + (y - z) = pk_1 + pk_2 = p(k_1 + k_2).$$

Since $k_1 + k_2 \in \mathbb{Z}$, $x - z$ is a multiple of p , and therefore $|x - z| = p|k_1 + k_2|$ is divisible by p . Hence $(x, z) \in R$, and R is transitive.

Step 5: Check with an example Take $p = 3$. Then $(7, 1) \in R$ since $|7 - 1| = 6 = 3 \times 2$, and $(1, -5) \in R$ since $|1 - (-5)| = 6$. Also $(7, -5) \in R$ since $|7 - (-5)| = 12 = 3 \times 4$. This agrees with transitivity. The equivalence classes are the residue classes modulo p : $[0], [1], \dots, [p - 1]$.

Step 6: Conclusion R is reflexive, symmetric and transitive. Hence R is an equivalence relation on $\mathbb{Z}$.

Quick Tip to Solve: Read “ $|x - y|$ divisible by p ” as $x - y = pk$. Then reflexive uses $0 = p \cdot 0$, symmetric uses $y - x = p(-k)$, and transitive adds $x - y = pk_1$ and $y - z = pk_2$ to get $x - z = p(k_1 + k_2)$.

OR

32(b). A function $f : \mathbb{R} - \left\{\frac{3}{5}\right\} \rightarrow \mathbb{R} - \left\{\frac{3}{5}\right\}$ is defined as $f(x) = \frac{3x+2}{5x-3}$. Show that f is one-one and onto. [5 marks]

Solution:

Concept:

- A function $f : X \rightarrow Y$ is one-one (injective) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$.
- f is onto (surjective) if for every $y \in Y$ there is some $x \in X$ with $f(x) = y$; the candidate x is

Quick Tip to Solve: The pre-image formula $x = \frac{3y+2}{5y-3}$ has the same form as f , so f is its own inverse: $f(f(x)) = x$. In an exam, always show that $x \neq \frac{3}{5}$ and that $f(x) = y$; leaving out these checks is where students usually lose credit.

found by solving $y = f(x)$ for x , and it must be checked to lie in the domain X .

- Here $X = Y = \mathbb{R} - \{\frac{3}{5}\}$, so both the pre-image and the values of f must avoid $\frac{3}{5}$.

Step 1: Check that f is well defined into $\mathbb{R} - \{\frac{3}{5}\}$ For $x \neq \frac{3}{5}$, the denominator $5x - 3 \neq 0$, so $f(x)$ is a real number. Suppose $f(x) = \frac{3}{5}$ for some x . Then

$$\frac{3x+2}{5x-3} = \frac{3}{5} \Rightarrow 5(3x+2) = 3(5x-3) \Rightarrow 15x+10 = 15x-9 \Rightarrow 10 = -9,$$

which is impossible. Hence $f(x) \neq \frac{3}{5}$ for every x in the domain, and f really maps $\mathbb{R} - \{\frac{3}{5}\}$ into $\mathbb{R} - \{\frac{3}{5}\}$.

Step 2: Prove that f is one-one Let $x_1, x_2 \in \mathbb{R} - \{\frac{3}{5}\}$ with $f(x_1) = f(x_2)$. Then

$$\begin{aligned} \frac{3x_1+2}{5x_1-3} &= \frac{3x_2+2}{5x_2-3} \\ (3x_1+2)(5x_2-3) &= (3x_2+2)(5x_1-3) \\ 15x_1x_2 - 9x_1 + 10x_2 - 6 &= 15x_1x_2 - 9x_2 + 10x_1 - 6 \\ -9x_1 - 10x_1 &= -9x_2 - 10x_2 \\ -19x_1 &= -19x_2 \Rightarrow x_1 = x_2. \end{aligned}$$

Cross-multiplication is valid because both denominators are non-zero. Thus equal images force equal inputs, so f is one-one.

Step 3: Find a pre-image for an arbitrary y Let $y \in \mathbb{R} - \{\frac{3}{5}\}$ be arbitrary. We look for x with $f(x) = y$:

$$\begin{aligned} \frac{3x+2}{5x-3} &= y \Rightarrow 3x+2 = 5xy-3y \\ \Rightarrow 3y+2 &= 5xy-3x = x(5y-3) \\ \Rightarrow x &= \frac{3y+2}{5y-3}. \end{aligned}$$

Since $y \neq \frac{3}{5}$, we have $5y-3 \neq 0$, so this x is a real number.

Step 4: Verify that the pre-image lies in the domain and maps to y If $x = \frac{3}{5}$, then $\frac{3y+2}{5y-3} = \frac{3}{5}$, i.e. $15y+10 = 15y-9$, which is impossible. So $x \in \mathbb{R} - \{\frac{3}{5}\}$. Now

$$f(x) = \frac{3 \cdot \frac{3y+2}{5y-3} + 2}{5 \cdot \frac{3y+2}{5y-3} - 3} = \frac{9y+6+10y-6}{15y+10-15y+9} = \frac{19y}{19} = y.$$

Hence every element of the codomain has a pre-image in the domain, so f is onto.

Step 5: Conclusion Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, the function is one-one, and since every $y \in \mathbb{R} - \{\frac{3}{5}\}$ equals $f\left(\frac{3y+2}{5y-3}\right)$ with $\frac{3y+2}{5y-3} \in \mathbb{R} - \{\frac{3}{5}\}$, it is onto. Hence $f(x) = \frac{3x+2}{5x-3}$ is one-one and onto (a bijection).

Quick Tip to Solve: The pre-image formula $x = \frac{3y+2}{5y-3}$ has the same form as f , so f is its own inverse: $f(f(x)) = x$. In an exam, always show that $x \neq \frac{3}{5}$ and that $f(x) = y$; leaving out these checks is where students usually lose credit.

33(a). If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix}$, find A^{-1} and use it to solve the following system of equations: $-2y + z = 7$, $2x - y - z = 8$, $x - 2y = 10$. [5 marks]

Solution:

Concept:

- For a square matrix with $|A| \neq 0$, $A^{-1} = \frac{1}{|A|} \text{adj } A$, where $\text{adj } A$ is the transpose of the matrix of cofactors $C_{ij} = (-1)^{i+j} M_{ij}$.
- A system $MX = B$ with $|M| \neq 0$ has the unique solution $X = M^{-1}B$.
- $(A^T)^{-1} = (A^{-1})^T$, so if the coefficient matrix of the system is A^T , then $X = (A^{-1})^T B$.

Step 1: Evaluate $|A|$ Expanding along the first row,

$$\begin{aligned} |A| &= 0((-1)(0) - (-2)(-1)) - 2((-2)(0) - (-2)(1)) + 1((-2)(-1) - (-1)(1)) \\ &= 0 - 2(2) + 1(3) = -1 \neq 0. \end{aligned}$$

So A is non-singular and A^{-1} exists.

Step 2: Find the cofactors of A

$$\begin{aligned} C_{11} &= (-1)(0) - (-2)(-1) = -2, & C_{12} &= -(0 + 2) = -2, & C_{13} &= 2 + 1 = 3, \\ C_{21} &= -(2)(0) - (1)(-1) = -1, & C_{22} &= 0 - 1 = -1, & C_{23} &= -(0 - 2) = 2, \\ C_{31} &= (2)(-2) - (1)(-1) = -3, & C_{32} &= -(0 + 2) = -2, & C_{33} &= 0 + 4 = 4. \end{aligned}$$

Step 3: Form $\text{adj } A$ and A^{-1} Transposing the cofactor matrix,

$$\text{adj } A = \begin{bmatrix} -2 & -1 & -3 \\ -2 & -1 & -2 \\ 3 & 2 & 4 \end{bmatrix}, \quad A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \text{adj } A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ -3 & -2 & -4 \end{bmatrix}.$$

Check: the first row of A times the columns of A^{-1} gives $0 + 4 - 3 = 1$, $0 + 2 - 2 = 0$, $0 + 4 - 4 = 0$; the other rows similarly give $AA^{-1} = I$.

Step 4: Write the system in matrix form The equations $-2y + z = 7$, $2x - y - z = 8$, $x - 2y = 10$ give

$$\begin{bmatrix} 0 & -2 & 1 \\ 2 & -1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}, \quad \text{i.e. } A^T X = B.$$

The rows of the coefficient matrix are exactly the columns of A , so the coefficient matrix is A^T . Since $|A^T| = |A| = -1 \neq 0$, the system has a unique solution

$$X = (A^T)^{-1} B = (A^{-1})^T B, \quad (A^{-1})^T = \begin{bmatrix} 2 & 2 & -3 \\ 1 & 1 & -2 \\ 3 & 2 & -4 \end{bmatrix}.$$

Quick Tip to Solve: Compare the coefficient rows with A before multiplying: here they are the columns of A , so use the transpose of the inverse, $X = (A^{-1})^T B$. Using $A^{-1}B$ directly would give a wrong answer.

Step 5: Solve and verify

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 2 & -3 \\ 1 & 1 & -2 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 14 + 16 - 30 \\ 7 + 8 - 20 \\ 21 + 16 - 40 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}.$$

Check: $-2(-5) + (-3) = 7$, $2(0) - (-5) - (-3) = 8$, $0 - 2(-5) = 10$. All three equations hold.

Step 6: Conclusion $A^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ -3 & -2 & -4 \end{bmatrix}$, and the solution of the system is $x = 0$, $y = -5$, $z = -3$.

Quick Tip to Solve: Compare the coefficient rows with A before multiplying: here they are the columns of A , so use the transpose of the inverse, $X = (A^{-1})^T B$. Using $A^{-1}B$ directly would give a wrong answer.

OR

33(b). If $\begin{bmatrix} 3 & -1 & \sin 3x \\ -7 & 4 & \cos 2x \\ -11 & 7 & 2 \end{bmatrix}$ is a singular matrix, then find all values of x where $x \in \left[0, \frac{\pi}{2}\right]$. [5 marks]

Solution:**Concept:**

- A square matrix A is singular exactly when $|A| = 0$; so the condition gives an equation in x .
- Expansion of a 3×3 determinant along the first row: $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, where $C_{ij} = (-1)^{i+j}M_{ij}$ is the cofactor.
- Identities used: $\cos 2x = 1 - 2\sin^2 x$ and $\sin 3x = 3\sin x - 4\sin^3 x$.

Step 1: Write the singularity condition Let $A = \begin{bmatrix} 3 & -1 & \sin 3x \\ -7 & 4 & \cos 2x \\ -11 & 7 & 2 \end{bmatrix}$. Since A is singular,

$$|A| = \begin{vmatrix} 3 & -1 & \sin 3x \\ -7 & 4 & \cos 2x \\ -11 & 7 & 2 \end{vmatrix} = 0.$$

Step 2: Expand the determinant along the first row

$$\begin{aligned} |A| &= 3(4 \cdot 2 - 7 \cos 2x) - (-1)((-7)(2) - (-11) \cos 2x) + \sin 3x((-7)(7) - 4(-11)) \\ &= 3(8 - 7 \cos 2x) + (-14 + 11 \cos 2x) + \sin 3x(-49 + 44) \\ &= 24 - 21 \cos 2x - 14 + 11 \cos 2x - 5 \sin 3x \\ &= 10 - 10 \cos 2x - 5 \sin 3x. \end{aligned}$$

Quick Tip to Solve: After expanding, keep $\cos 2x$ and $\sin 3x$ together and rewrite both in $\sin x$: the cubic then factors out $\sin x$ at once, so do not lose the root $x = 0$ by dividing by $\sin x$.

Setting $|A| = 0$ and dividing by 5:

$$2 - 2 \cos 2x - \sin 3x = 0.$$

Step 3: Convert to an equation in $\sin x$ Using $1 - \cos 2x = 2 \sin^2 x$ and $\sin 3x = 3 \sin x - 4 \sin^3 x$:

$$\begin{aligned} 2(1 - \cos 2x) - \sin 3x &= 4 \sin^2 x - 3 \sin x + 4 \sin^3 x = 0 \\ \sin x (4 \sin^2 x + 4 \sin x - 3) &= 0 \\ \sin x (2 \sin x - 1)(2 \sin x + 3) &= 0. \end{aligned}$$

Check of the factorisation: $(2s - 1)(2s + 3) = 4s^2 + 6s - 2s - 3 = 4s^2 + 4s - 3$.

Step 4: Solve each factor in $\left[0, \frac{\pi}{2}\right]$

- $\sin x = 0$: in $\left[0, \frac{\pi}{2}\right]$ this gives $x = 0$.
- $2 \sin x - 1 = 0$, i.e. $\sin x = \frac{1}{2}$: in $\left[0, \frac{\pi}{2}\right]$ this gives $x = \frac{\pi}{6}$.
- $2 \sin x + 3 = 0$, i.e. $\sin x = -\frac{3}{2}$: impossible, since $-1 \leq \sin x \leq 1$.

Step 5: Verify the values At $x = 0$: $\cos 0 = 1$, $\sin 0 = 0$, so $|A| = 10 - 10 - 0 = 0$.

At $x = \frac{\pi}{6}$: $\cos \frac{\pi}{3} = \frac{1}{2}$, $\sin \frac{\pi}{2} = 1$, so $|A| = 10 - 5 - 5 = 0$.

Both values make the matrix singular.

Step 6: Conclusion The matrix is singular for $x = 0$ and $x = \frac{\pi}{6}$ in the interval $\left[0, \frac{\pi}{2}\right]$.

Quick Tip to Solve: After expanding, keep $\cos 2x$ and $\sin 3x$ together and rewrite both in $\sin x$: the cubic then factors out $\sin x$ at once, so do not lose the root $x = 0$ by dividing by $\sin x$.

34. If $x = \cos t$, $y = \cos mt$, prove that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$.

[5 marks]

Solution:

Concept:

- Parametric differentiation: if $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $\frac{dx}{dt} \neq 0$.
- To reach a relation free of t , express $\sin t$ and $\sin mt$ through x and y using $\sin^2 \theta = 1 - \cos^2 \theta$.
- Differentiate the squared relation implicitly with respect to x (product rule and chain rule), then cancel the common factor $2 \frac{dy}{dx}$.

Step 1: Differentiate the parametric equations Given $x = \cos t$ and $y = \cos mt$,

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = -m \sin mt.$$

Quick Tip to Solve: Square $\frac{dy}{dx}$ first so that $\sin^2 t = 1 - x^2$ and $\sin^2 mt = 1 - y^2$ remove t ; then one implicit differentiation and a cancellation of $2y'$ finish the proof.

Hence

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-m \sin mt}{-\sin t} = \frac{m \sin mt}{\sin t}.$$

Step 2: Eliminate t Squaring,

$$\left(\frac{dy}{dx}\right)^2 = \frac{m^2 \sin^2 mt}{\sin^2 t} = \frac{m^2(1 - \cos^2 mt)}{1 - \cos^2 t} = \frac{m^2(1 - y^2)}{1 - x^2}.$$

Therefore

$$(1 - x^2) \left(\frac{dy}{dx}\right)^2 = m^2(1 - y^2). \quad (1)$$

Step 3: Differentiate relation (1) with respect to x On the left, $\frac{d}{dx}(1 - x^2) = -2x$ and $\frac{d}{dx} \left(\frac{dy}{dx}\right)^2 = 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2}$; on the right, $\frac{d}{dx}(1 - y^2) = -2y \frac{dy}{dx}$. Using the product rule on the left and the chain rule on the right,

$$(1 - x^2) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 (-2x) = m^2 \left(-2y \frac{dy}{dx}\right).$$

Step 4: Cancel the common factor Dividing throughout by $2 \frac{dy}{dx}$ (which is not identically zero, since $\frac{dy}{dx} = \frac{m \sin mt}{\sin t}$),

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = -m^2 y.$$

Transposing,

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0.$$

Step 5: Conclusion Hence $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$, as required.

Quick Tip to Solve: Square $\frac{dy}{dx}$ first so that $\sin^2 t = 1 - x^2$ and $\sin^2 mt = 1 - y^2$ remove t ; then one implicit differentiation and a cancellation of $2y'$ finish the proof.

35. Check whether the lines given by $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are parallel or not. If parallel, find the distance between them, otherwise find their point of intersection, if the lines are intersecting. [5 marks]

Solution:

Concept:

- A line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ passes through (x_1, y_1, z_1) and has direction ratios a, b, c .

Quick Tip to Solve: Use the simplest equation (here z of L_2 gives $\mu = 4\lambda + 3$) to eliminate one parameter, solve with a second equation, and always substitute into the third to confirm the lines really meet rather than being skew.

- Two lines are parallel if and only if their direction ratios are proportional: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
- Non-parallel lines intersect if and only if there are parameter values λ, μ making the general points of both lines equal; solve two coordinate equations and check the third.
- For parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$, $\vec{r} = \vec{a}_2 + \mu\vec{b}$, the distance is $d = \frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$ (needed only if the lines turn out parallel).

Step 1: Read off points and direction ratios Line L_1 : $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ passes through $A(1, 2, 3)$ with direction ratios 2, 3, 4.

Line L_2 : $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$, i.e. $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$, passes through $B(4, 1, 0)$ with direction ratios 5, 2, 1.

Step 2: Test for parallelism Compare the ratios of corresponding direction ratios:

$$\frac{2}{5} = 0.4, \quad \frac{3}{2} = 1.5, \quad \frac{4}{1} = 4.$$

These are not equal, so the direction ratios are not proportional. Hence the lines are **not parallel**, and we must check whether they intersect.

Step 3: Write the general points of both lines Setting each ratio of L_1 equal to λ and each ratio of L_2 equal to μ :

$$P \in L_1 : (2\lambda + 1, 3\lambda + 2, 4\lambda + 3), \quad Q \in L_2 : (5\mu + 4, 2\mu + 1, \mu).$$

The lines intersect if $P = Q$ for some λ, μ , i.e.

$$2\lambda + 1 = 5\mu + 4 \quad (1)$$

$$3\lambda + 2 = 2\mu + 1 \quad (2)$$

$$4\lambda + 3 = \mu \quad (3)$$

Step 4: Solve two of the equations From (3), $\mu = 4\lambda + 3$. Substituting in (2):

$$3\lambda + 2 = 2(4\lambda + 3) + 1 = 8\lambda + 7 \Rightarrow -5\lambda = 5 \Rightarrow \lambda = -1.$$

Then $\mu = 4(-1) + 3 = -1$.

Step 5: Verify the remaining equation Check (1) with $\lambda = -1$, $\mu = -1$:

$$\text{LHS} = 2(-1) + 1 = -1, \quad \text{RHS} = 5(-1) + 4 = -1.$$

Equation (1) is satisfied, so all three equations hold together and the lines do intersect.

Step 6: Find the point of intersection Put $\lambda = -1$ in the general point of L_1 :

$$(2(-1) + 1, 3(-1) + 2, 4(-1) + 3) = (-1, -1, -1).$$

Quick Tip to Solve: Use the simplest equation (here z of L_2 gives $\mu = 4\lambda + 3$) to eliminate one parameter, solve with a second equation, and always substitute into the third to confirm the lines really meet rather than being skew.

As a check, $\mu = -1$ in L_2 gives $(5(-1) + 4, 2(-1) + 1, -1) = (-1, -1, -1)$, the same point. It also satisfies both equations directly: $\frac{-2}{2} = \frac{-3}{3} = \frac{-4}{4} = -1$ and $\frac{-5}{5} = \frac{-2}{2} = \frac{-1}{1} = -1$.

Step 7: Conclusion The direction ratios 2, 3, 4 and 5, 2, 1 are not proportional, so the lines are not parallel (no distance to compute). They intersect, and their point of intersection is $(-1, -1, -1)$.

Quick Tip to Solve: Use the simplest equation (here z of L_2 gives $\mu = 4\lambda + 3$) to eliminate one parameter, solve with a second equation, and always substitute into the third to confirm the lines really meet rather than being skew.

Section E (12 marks)

This section comprises of 3 case study based questions of 4 marks each.

36(i). An online delivery company in a city has 5000 subscribers and collects annual subscription fees of ₹300 per subscriber for unlimited free deliveries. The company wishes to increase the annual subscription fee. It is predicted that, for every increase of ₹1, ten subscribers will discontinue. Assume that the company increased the annual fee by ₹ x .

How many subscribers will discontinue after an increase of ₹ x in annual fee?

[1 mark]

Solution:

Concept:

- The loss of subscribers is proportional to the increase in fee: each ₹1 increase causes 10 subscribers to leave.
- So an increase of ₹ x causes $10 \times x$ subscribers to leave (direct proportion).

Step 1: Use the given rate of discontinuation For every increase of ₹1 in the annual fee, 10 subscribers discontinue. The number leaving grows in direct proportion to the increase, so an increase of ₹ x is x such steps of ₹1.

Step 2: Count the subscribers who leave

$$\text{Subscribers discontinuing} = 10 \times x = 10x.$$

Consequently the number of subscribers remaining is $5000 - 10x$, which will be used to build the revenue function in the next part.

Step 3: Conclusion After an increase of ₹ x in the annual fee, $10x$ subscribers will discontinue.

Quick Tip to Solve: Multiply the per-rupee loss (10) by the increase x ; then remaining subscribers are $5000 - 10x$, ready for the revenue in part (ii).

36(ii). If $R(x)$ denotes the total revenue collected after the increase of ₹ x in subscription fee, express $R(x)$ as a function of x . [1 mark]

Solution:

Concept:

- Total revenue = (fee per subscriber) $\times$ (number of subscribers).
- After an increase of ₹ x : fee = $300 + x$ and subscribers = $5000 - 10x$ (since $10x$ leave).

Step 1: New fee and new number of subscribers The annual fee becomes ₹ $(300 + x)$. From part (i), $10x$ subscribers discontinue, so the number of subscribers becomes $5000 - 10x$.

Step 2: Form and expand the revenue function

$$\begin{aligned}R(x) &= (300 + x)(5000 - 10x) \\&= 1500000 - 3000x + 5000x - 10x^2 \\&= 1500000 + 2000x - 10x^2.\end{aligned}$$

Check at $x = 0$: $R(0) = 300 \times 5000 = 1500000$, the original revenue.

Step 3: Conclusion $R(x) = (300 + x)(5000 - 10x) = 1500000 + 2000x - 10x^2$ (in rupees).

Quick Tip to Solve: Write revenue as (new price)(new quantity) before expanding, and check $R(0)$ equals the original revenue 300×5000 .

36(iii)(a). Find the value of x for which $R(x)$ is maximum. [2 marks]

Solution:

Concept:

- At a local maximum of a differentiable function, $R'(x) = 0$ (critical point).
- Second derivative test: if $R'(c) = 0$ and $R''(c) < 0$, then R has a local maximum at $x = c$.
- From part (ii), $R(x) = 1500000 + 2000x - 10x^2$.

Step 1: Differentiate and find the critical point

$$R'(x) = \frac{d}{dx} (1500000 + 2000x - 10x^2) = 2000 - 20x.$$

Setting $R'(x) = 0$:

$$2000 - 20x = 0 \Rightarrow x = 100.$$

Step 2: Second derivative test

$$R''(x) = -20 < 0 \quad \text{for all } x,$$

so in particular $R''(100) < 0$, and R has a maximum at $x = 100$. Since R'' is negative everywhere, this is the absolute maximum of R on the admissible range $0 < x < 500$ (where the number of subscribers $5000 - 10x$ stays positive).

Quick Tip to Solve: $R(x) = -10x^2 + 2000x + 1500000$ is a downward parabola, so its maximum is at the vertex $x = -\frac{b}{2a} = \frac{2000}{20} = 100$; this confirms the derivative answer instantly.

Step 3: Maximum revenue With $x = 100$: fee = $300 + 100 = ₹400$, subscribers = $5000 - 10(100) = 4000$, so

$$R(100) = 400 \times 4000 = 1600000,$$

i.e. ₹16,00,000, compared with the original ₹15,00,000.

Step 4: Conclusion $R(x)$ is maximum when $x = 100$, i.e. when the annual fee is raised by ₹100 to ₹400; the maximum revenue is ₹16,00,000.

Quick Tip to Solve: $R(x) = -10x^2 + 2000x + 1500000$ is a downward parabola, so its maximum is at the vertex $x = -\frac{b}{2a} = \frac{2000}{20} = 100$; this confirms the derivative answer instantly.

OR

36(iii)(b). Find the sub-intervals of $(0, 5000)$ in which $R(x)$ is increasing and decreasing. [2 marks]

Solution:

Concept:

- A differentiable function f is increasing on an interval where $f'(x) > 0$ and decreasing where $f'(x) < 0$; the critical points $f'(x) = 0$ split the interval into such sub-intervals.
- Revenue = (fee per subscriber) $\times$ (number of subscribers). After an increase of ₹ x , the fee is $300 + x$ and the subscribers are $5000 - 10x$.

Step 1: Write the revenue function Since ten subscribers leave for every ₹1 increase, an increase of ₹ x leaves $5000 - 10x$ subscribers, each paying ₹ $(300 + x)$. Hence

$$\begin{aligned} R(x) &= (300 + x)(5000 - 10x) \\ &= 1500000 - 3000x + 5000x - 10x^2 \\ &= 1500000 + 2000x - 10x^2. \end{aligned}$$

Step 2: Differentiate and find the critical point

$$R'(x) = 2000 - 20x = 20(100 - x).$$

Setting $R'(x) = 0$ gives $x = 100$, which lies in $(0, 5000)$. This point divides the interval into $(0, 100)$ and $(100, 5000)$.

Step 3: Sign of $R'(x)$ on each sub-interval For $0 < x < 100$: $100 - x > 0$, so $R'(x) > 0$ (for example, $R'(50) = 1000 > 0$). Hence R is increasing on $(0, 100)$.

For $100 < x < 5000$: $100 - x < 0$, so $R'(x) < 0$ (for example, $R'(200) = -2000 < 0$). Hence R is decreasing on $(100, 5000)$.

This matches the shape of R : it is a downward parabola with vertex at $x = 100$, where the revenue is greatest, $R(100) = 400 \times 4000 = ₹16,00,000$.

Step 4: Conclusion $R(x)$ is increasing in $(0, 100)$ and decreasing in $(100, 5000)$.

Quick Tip to Solve: Factor the derivative as $R'(x) = 20(100 - x)$: its sign is then read off at a glance, positive before $x = 100$ and negative after it.

37(i). In an online jackpot, there is one first prize of ₹3,00,000, two second prizes of ₹2,00,000 each and three third prizes of ₹50,000 each. A total of 1,00,000 jackpot tickets each costing ₹100 were sold thereby raising a fund of ₹1,00,00,000. Rohan bought one ticket.

What are the possible amounts, the person can win?

[1 mark]

Solution:

Concept:

- The possible values of the amount won form the range of the random variable X = amount won on one ticket.
- One ticket can win at most one prize, and a ticket that wins none of the prizes gets ₹0.

Step 1: List the prizes on offer The jackpot has one first prize of ₹3,00,000, two second prizes of ₹2,00,000 each and three third prizes of ₹50,000 each, that is, $1 + 2 + 3 = 6$ winning tickets out of 1,00,000.

Step 2: Include the case of winning nothing Rohan's single ticket either is one of the 6 winning tickets, in which case he gets the corresponding prize, or is one of the remaining 99,994 tickets, in which case he wins nothing, i.e. ₹0. The ticket cost of ₹100 is not part of the amount won.

Step 3: Conclusion The possible amounts the person can win are ₹3,00,000, ₹2,00,000, ₹50,000 and ₹0.

Quick Tip to Solve: When listing the values of a "winnings" random variable, always include 0 for the losing tickets; it carries almost all of the probability here.

37(ii)(a). What is the probability that the person wins atleast ₹2,00,000?

[2 marks]

Solution:

Concept:

- For equally likely outcomes, $P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$.
- "At least ₹2,00,000" means the amount won X satisfies $X \geq 2,00,000$, i.e. $X = 2,00,000$ or $X = 3,00,000$.

Step 1: Identify the favourable tickets The prizes worth at least ₹2,00,000 are the first prize (₹3,00,000, one ticket) and the second prizes (₹2,00,000, two tickets). The third prize of ₹50,000 does not qualify. So the number of favourable tickets is $1 + 2 = 3$.

Step 2: Compute the probability Each of the 1,00,000 tickets is equally likely to be Rohan's, so

$$\begin{aligned} P(X \geq 2,00,000) &= P(X = 3,00,000) + P(X = 2,00,000) \\ &= \frac{1}{100000} + \frac{2}{100000} = \frac{3}{100000} = 0.00003. \end{aligned}$$

Step 3: Conclusion The probability that the person wins at least ₹2,00,000 is $\frac{3}{100000} = 0.00003$.

Quick Tip to Solve: Translate "at least ₹2,00,000" into the prize list first (first and second prizes only), then simply count tickets: $1 + 2 = 3$ out of 10^5 .

37(ii)(b). What is the probability that the person does not win any amount?

[2 marks]

Solution:

Concept:

- Complement rule: $P(\text{no prize}) = 1 - P(\text{some prize})$.
- With equally likely tickets, $P(\text{some prize}) = \frac{\text{number of prize-winning tickets}}{\text{total tickets}}$.

Step 1: Count the winning tickets There is 1 first prize, 2 second prizes and 3 third prizes, so the number of winning tickets is

$$1 + 2 + 3 = 6 \quad \text{out of} \quad 1,00,000.$$

$$\text{Hence } P(\text{wins some amount}) = \frac{6}{100000}.$$

Step 2: Use the complement

$$\begin{aligned} P(\text{wins no amount}) &= 1 - \frac{6}{100000} = \frac{99994}{100000} \\ &= \frac{49997}{50000} = 0.99994. \end{aligned}$$

Equivalently, $1,00,000 - 6 = 99,994$ tickets carry no prize, giving the same fraction directly.

Step 3: Conclusion The probability that the person does not win any amount is $\frac{99994}{100000} = \frac{49997}{50000} = 0.99994$.

Quick Tip to Solve: Count the few winning tickets ($1 + 2 + 3 = 6$) and subtract from 1; counting the losers directly is the same as $10^5 - 6$.

37(iii). In another jackpot, Rohan also bought a ticket having a prize money of ₹5,00,000. The chances of winning the jackpot are 1 in 1,00,000. Find the probability that on exactly one of tickets he wins the jackpot.

[1 mark]

Solution:

Concept:

- The two jackpots are independent, so for events E_1, E_2 : $P(E_1 \cap E_2') = P(E_1)P(E_2')$ and $P(E_1' \cap E_2) = P(E_1')P(E_2)$.
- "Exactly one" is the union of the two mutually exclusive cases: $P = P(E_1)P(E_2') + P(E_1')P(E_2)$.

Step 1: Define the events and their probabilities Let E_1 : Rohan wins (a prize) on the first jackpot ticket, and E_2 : he wins the ₹5,00,000 jackpot on the second ticket. The first jackpot has 6 prize-winning tickets out of 1,00,000, and the second has chances of 1 in 1,00,000:

$$P(E_1) = \frac{6}{10^5}, \quad P(E_1') = \frac{99994}{10^5}, \quad P(E_2) = \frac{1}{10^5}, \quad P(E_2') = \frac{99999}{10^5}.$$

Quick Tip to Solve: For "exactly one of two independent events", use $p_1(1 - p_2) + (1 - p_1)p_2$; do not forget the factor of the other ticket losing.

Step 2: Add the two exclusive cases Since the two jackpots are independent,

$$\begin{aligned}P(\text{exactly one}) &= P(E_1)P(E'_2) + P(E'_1)P(E_2) \\&= \frac{6 \times 99999}{10^{10}} + \frac{99994 \times 1}{10^{10}} \\&= \frac{599994 + 99994}{10^{10}} = \frac{699988}{10^{10}} \approx 0.00007.\end{aligned}$$

Step 3: Conclusion The probability that Rohan wins on exactly one of the two tickets is $\frac{699988}{10^{10}} = 0.0000699988 \approx 0.00007$.

Quick Tip to Solve: For "exactly one of two independent events", use $p_1(1 - p_2) + (1 - p_1)p_2$; do not forget the factor of the other ticket losing.

38(i). Roundabouts are often made on busy roads to ease the traffic and avoid red lights. One such round-about is made such that equation representing its boundary is given by $C_1 : x^2 + y^2 = 64$. There is a circular pond with a fountain in the middle of the roundabout whose equation is given by $C_2 : x^2 + y^2 = 4$.

Represent the given equations C_1 and C_2 with the help of a diagram.

[1 mark]

Solution:

Concept:

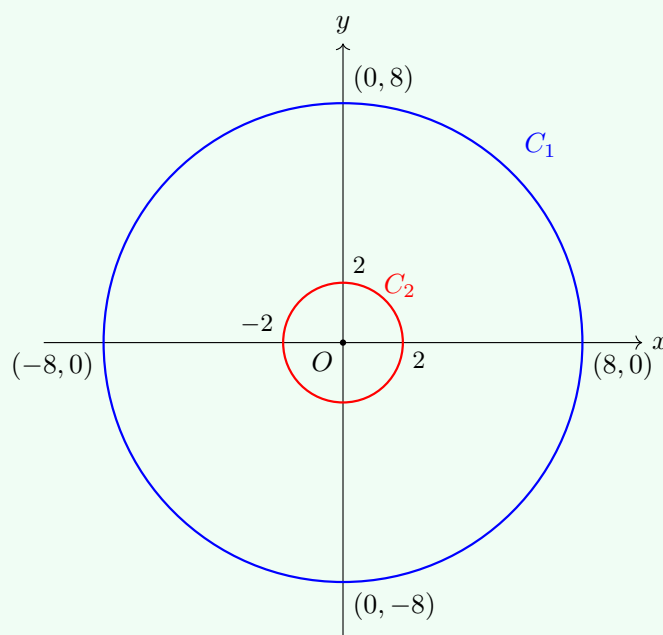
- The equation $x^2 + y^2 = r^2$ represents a circle with centre at the origin $O(0, 0)$ and radius r .
- Two circles with the same centre but different radii are concentric; the one with the smaller radius lies wholly inside the other.

Step 1: Identify the centre and radius of each circle For $C_1 : x^2 + y^2 = 64 = 8^2$, comparing with $x^2 + y^2 = r^2$ gives centre $O(0, 0)$ and radius $r_1 = 8$. It meets the axes at $(\pm 8, 0)$ and $(0, \pm 8)$.

For $C_2 : x^2 + y^2 = 4 = 2^2$, the centre is again $O(0, 0)$ and the radius is $r_2 = 2$. It meets the axes at $(\pm 2, 0)$ and $(0, \pm 2)$.

Step 2: Draw the two circles Since both circles have the same centre O and $r_2 = 2 < 8 = r_1$, they are concentric and C_2 (the pond) lies completely inside C_1 (the boundary of the roundabout). The diagram below shows both circles on the coordinate axes with their intercepts marked.

Quick Tip to Solve: Read the radius straight off the constant term: $64 = 8^2$ and $4 = 2^2$, so mark the intercepts ± 8 and ± 2 on both axes before drawing the circles.



Concentric circles $C_1 : x^2 + y^2 = 64$ and $C_2 : x^2 + y^2 = 4$

Step 3: Conclusion The diagram shows two concentric circles centred at the origin: the outer circle C_1 of radius 8 (roundabout boundary) passing through $(\pm 8, 0)$, $(0, \pm 8)$, and the inner circle C_2 of radius 2 (pond) passing through $(\pm 2, 0)$, $(0, \pm 2)$.

Quick Tip to Solve: Read the radius straight off the constant term: $64 = 8^2$ and $4 = 2^2$, so mark the intercepts ± 8 and ± 2 on both axes before drawing the circles.

38(ii). Express y as a function of x , ($y = f(x)$), for both C_1 and C_2 .

[1 mark]

Solution:

Concept:

- A circle $x^2 + y^2 = r^2$ is not the graph of a single function; solving for y gives two branches, $y = \sqrt{r^2 - x^2}$ (upper semicircle) and $y = -\sqrt{r^2 - x^2}$ (lower semicircle).
- The square root is real only when $r^2 - x^2 \geq 0$, i.e. $-r \leq x \leq r$.

Step 1: Solve C_1 for y From $x^2 + y^2 = 64$,

$$y^2 = 64 - x^2 \Rightarrow y = \pm \sqrt{64 - x^2}, \quad -8 \leq x \leq 8.$$

The upper half of C_1 is $y = \sqrt{64 - x^2}$ and the lower half is $y = -\sqrt{64 - x^2}$.

Step 2: Solve C_2 for y From $x^2 + y^2 = 4$,

$$y^2 = 4 - x^2 \Rightarrow y = \pm \sqrt{4 - x^2}, \quad -2 \leq x \leq 2.$$

The upper half of C_2 is $y = \sqrt{4 - x^2}$ and the lower half is $y = -\sqrt{4 - x^2}$.

Quick Tip to Solve: Keep both signs of the square root: the positive root $\sqrt{r^2 - x^2}$ alone is only the upper semicircle, which is what you integrate (then double) for the area.

Step 3: Conclusion $C_1 : y = \pm\sqrt{64 - x^2}$ for $x \in [-8, 8]$, and $C_2 : y = \pm\sqrt{4 - x^2}$ for $x \in [-2, 2]$; the + sign gives the upper semicircle and the – sign the lower one.

Quick Tip to Solve: Keep both signs of the square root: the positive root $\sqrt{r^2 - x^2}$ alone is only the upper semicircle, which is what you integrate (then double) for the area.

38(iii)(a). Using integration find the area of region covered by the roundabout.

[2 marks]

Solution:

Concept:

- Area under a curve $y = f(x) \geq 0$ from $x = a$ to $x = b$ is $\int_a^b f(x) dx$.
- Standard integral: $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$.
- A circle centred at the origin is symmetric about both axes, so its area is 4 times the area of the part in the first quadrant.

Step 1: Set up the integral using symmetry The roundabout is the region bounded by $C_1 : x^2 + y^2 = 64$, a circle of radius 8 centred at O . In the first quadrant its boundary is $y = \sqrt{64 - x^2}$, $0 \leq x \leq 8$. By symmetry about both axes,

$$A = 4 \int_0^8 \sqrt{64 - x^2} dx.$$

Step 2: Evaluate the integral Using the standard formula with $a = 8$:

$$\begin{aligned} \int_0^8 \sqrt{64 - x^2} dx &= \left[\frac{x}{2} \sqrt{64 - x^2} + \frac{64}{2} \sin^{-1} \frac{x}{8} \right]_0^8 \\ &= \left(\frac{8}{2} \cdot 0 + 32 \sin^{-1} 1 \right) - (0 + 32 \sin^{-1} 0) \\ &= 32 \cdot \frac{\pi}{2} = 16\pi. \end{aligned}$$

Step 3: Find the total area

$$A = 4 \times 16\pi = 64\pi \text{ square units.}$$

Check: this agrees with $\pi r^2 = \pi(8)^2 = 64\pi$.

Step 4: Conclusion The area of the region covered by the roundabout is 64π square units (about 201.06 square units).

Quick Tip to Solve: For a circle $x^2 + y^2 = a^2$, $\int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^2}{4}$ since the first term vanishes at both limits; multiply by 4 and verify against πa^2 .

38(iii)(b). Using integration, find the area of region covered by circular pond.

[2 marks]

Solution:

Concept:

- Area under a curve $y = f(x) \geq 0$ from $x = a$ to $x = b$ is $\int_a^b f(x) dx$.
- Standard integral: $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$.
- A circle centred at the origin is symmetric about both axes, so its area is 4 times the area of the part in the first quadrant.

Step 1: Set up the integral using symmetry The pond is the region bounded by $C_2 : x^2 + y^2 = 4$, a circle of radius 2 centred at O . In the first quadrant its boundary is $y = \sqrt{4 - x^2}$, $0 \leq x \leq 2$. By symmetry about both axes,

$$A = 4 \int_0^2 \sqrt{4 - x^2} dx.$$

Step 2: Evaluate the integral Using the standard formula with $a = 2$:

$$\begin{aligned} \int_0^2 \sqrt{4 - x^2} dx &= \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 \\ &= (1 \cdot 0 + 2 \sin^{-1} 1) - (0 + 2 \sin^{-1} 0) \\ &= 2 \cdot \frac{\pi}{2} = \pi. \end{aligned}$$

Step 3: Find the total area

$$A = 4 \times \pi = 4\pi \text{ square units.}$$

Check: this agrees with $\pi r^2 = \pi(2)^2 = 4\pi$.

Step 4: Conclusion The area of the region covered by the circular pond is 4π square units (about 12.57 square units).

Quick Tip to Solve: At the upper limit $x = a$ the term $\frac{x}{2} \sqrt{a^2 - x^2}$ is zero, so the quarter-circle integral is just $\frac{a^2}{2} \cdot \frac{\pi}{2}$; here $2 \cdot \frac{\pi}{2} = \pi$.