

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1** to **18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in few questions in all the Sections except Section – **A**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

Question 1 to 20 are multiple choice questions of 1 mark each.

1. If $\int \frac{3ax}{b^2 + c^2x^2} dx = A \log |b^2 + c^2x^2| + K$, then the value of A is [1 mark]

- (A) $3a$
- (B) $\frac{3a}{2b^2}$
- (C) $\frac{3a}{b^2c^2}$
- (D) $\frac{3a}{2c^2}$

Correct Answer: (D) $\frac{3a}{2c^2}$

Solution:

Concept:

- For a differentiable function u , $\int u^{-1} du = \log |u| + K$, where K is an arbitrary constant.
- Choose the denominator as the new variable when its derivative is proportional to the numerator.

Step 1: Substitute the denominator Let $u = b^2 + c^2x^2$. Since b and c are constants, $du = 2c^2x dx$. Thus the numerator becomes $3ax dx = \frac{3a}{2c^2} du$.

Step 2: Integrate and compare coefficients Using the substitution,

$$\int \frac{3ax}{b^2 + c^2x^2} dx = \frac{3a}{2c^2} \int \frac{du}{u} = \frac{3a}{2c^2} \log |b^2 + c^2x^2| + K.$$

The factor comes from differentiating c^2x^2 ; the constant term b^2 contributes zero. Thus the denominators involving b^2 in (B) and (C), and the missing chain-rule factor in (A), are incorrect.

Step 3: Conclusion $A = \frac{3a}{2c^2}$. Hence, the correct option is (D).

Quick Tip to Solve: Differentiate the logarithm first: its numerator is $2c^2x$, so scale it to match $3ax$.

2. The value of $\int_{-1}^1 \frac{x^3}{x^2 + 2|x| + 1} dx$ is

[1 mark]

- (A) 0
- (B) $\log 2$
- (C) $2 \log 2$
- (D) $\frac{1}{2} \log 2$

Correct Answer: (A) 0

Solution:

Concept:

- An odd function satisfies $f(-x) = -f(x)$; its integral over a symmetric interval $[-a, a]$ is zero.
- The denominator $x^2 + 2|x| + 1 = (|x| + 1)^2$ is positive and even, so the integrand is continuous on the interval.

Step 1: Check the parity For $f(x) = \frac{x^3}{x^2 + 2|x| + 1}$, replacing x by $-x$ gives

$$f(-x) = \frac{-x^3}{x^2 + 2|x| + 1} = -f(x).$$

The absolute value in the denominator does not change sign.

Step 2: Use the symmetric limits The limits are -1 and 1 , so the negative contribution on $[-1, 0]$ cancels the positive contribution on $[0, 1]$:

$$\int_{-1}^1 f(x) dx = - \int_0^1 f(x) dx + \int_0^1 f(x) dx = 0.$$

Each of (B), (C) and (D) is positive, so none can represent this signed integral.

Step 3: Conclusion The integral equals 0. Hence, the correct option is (A).

Quick Tip to Solve: Check parity before integrating: an odd numerator divided by an even nonzero denominator is odd.

3. The area bounded by the curve $y = x|x|$, x -axis and the ordinates $x = -1$ and $x = 1$ is given by [1 mark]

- (A) 0
- (B) $\frac{1}{3}$
- (C) $\frac{2}{3}$
- (D) $\frac{4}{3}$

Correct Answer: (C) $\frac{2}{3}$

Solution:

Concept:

- Geometric area between $y = f(x)$ and the x -axis is $\int |f(x)| dx$, so regions below the axis contribute positively.
- Here $|x| \geq 0$ and $|x|^2 = x^2$, giving $|x|x$ a negative sign on $x < 0$ but $|x|x$ a positive sign on $x > 0$.

Step 1: Split the curve at zero For $-1 \leq x \leq 0$, $y = -x^2$, whereas for $0 \leq x \leq 1$, $y = x^2$. The two regions have equal positive areas, even though their signed integrals have opposite signs.

Step 2: Add the positive areas The required area is

$$\begin{aligned} S &= \int_{-1}^0 x^2 dx + \int_0^1 x^2 dx \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

Option (A) gives the signed integral, (B) counts only one region, and (D) doubles the total area once too often.

Step 3: Conclusion The bounded area is $\frac{2}{3}$ square units. Hence, the correct option is (C).

Quick Tip to Solve: When a curve crosses the axis, integrate its absolute height; symmetry doubles one lobe instead of cancelling it.

4. The integrating factor of differential equation $R \frac{dx}{dy} + Px = Q$ where P, Q, R are functions of y is [1 mark]

- (A) $e^{\int \frac{P}{Q} dy}$
(B) $e^{\int P dy}$
(C) $e^{\int \frac{P}{R} dy}$
(D) $e^{\int \frac{P}{R} dx}$

Correct Answer: (C) $e^{\int \frac{P}{R} dy}$

Solution:

Concept:

- A linear equation $\frac{dx}{dy} + p(y)x = q(y)$ has integrating factor $e^{\int p(y) dy}$.
- Normalize the coefficient of the derivative to one before reading off $p(y)$; here y is the independent variable.

Step 1: Write the standard linear form On an interval where $R(y) \neq 0$, divide every term by R :

$$\frac{dx}{dy} + \frac{P}{R}x = \frac{Q}{R}.$$

The coefficient of the dependent variable x is therefore $p(y) = P/R$, not P or P/Q .

Step 2: Choose the correct variable of integration The integrating factor is

$$\text{IF} = e^{\int (P/R) dy}.$$

Option (A) incorrectly divides by Q , (B) omits division by R , and (D) integrates with respect to x even though the equation is linear in x as a function of y .

Step 3: Conclusion The integrating factor is $e^{\int \frac{P}{R} dy}$. Hence, the correct option is (C).

Quick Tip to Solve: Divide by the derivative coefficient first, then integrate the new coefficient of x with respect to y .

5. The order and degree of the differential equation $\frac{d}{dx}(\sin y) = y^2$ respectively are [1 mark]

- (A) 1, 1
(B) 2, 1
(C) 2, 2
(D) 1, 2

Correct Answer: (A) 1, 1

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.

Quick Tip to Solve: Expand the derivative of $\sin y$ first; count the order and power of dy/dx , not the exponent on y .

- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives; powers of y do not determine degree.

Step 1: Expand the derivative Apply the chain rule to the left side:

$$\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}.$$

Hence the equation is $\cos y \frac{dy}{dx} - y^2 = 0$. Only the first derivative occurs; there is no second derivative.

Step 2: Identify the derivative power The equation is already polynomial in dy/dx , and this derivative occurs to the first power. The factor $\cos y$ is a coefficient, while y^2 contains no derivative. Thus (B) and (C) incorrectly assign order two, and (D) mistakes the power of y for the degree.

Step 3: Conclusion The order is 1 and the degree is 1. Hence, the correct option is (A).

Quick Tip to Solve: Expand the derivative of $\sin y$ first; count the order and power of dy/dx , not the exponent on y .

6. The value of p for which vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $2\hat{i} - p\hat{j} + \hat{k}$ are perpendicular to each other is [1 mark]

- (A) 0
(B) 1
(C) $\frac{5}{2}$
(D) $-\frac{5}{2}$

Correct Answer: (C) $\frac{5}{2}$

Solution:

Concept:

- Two nonzero vectors are perpendicular exactly when their scalar product is zero.
- For Cartesian components, $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.

Step 1: Form the scalar product The vectors have components $(1, 2, 3)$ and $(2, -p, 1)$. Their dot product is

$$1(2) + 2(-p) + 3(1) = 5 - 2p.$$

The $\hat{j}$ component of the second vector is $-p$, so its contribution carries a minus sign.

Step 2: Impose perpendicularity Set the dot product equal to zero:

$$5 - 2p = 0 \implies p = \frac{5}{2}.$$

Testing the other options, $p = 0, 1$ and $-5/2$ give dot products 5, 3 and 10, respectively; none is zero.

Step 3: Conclusion $p = \frac{5}{2}$. Hence, the correct option is (C).

Quick Tip to Solve: Use the dot product for perpendicularity, and retain the minus sign in the component $-p$.

7. The value of m for which the points with position vectors $-\hat{i} - \hat{j} + 2\hat{k}$, $2\hat{i} + m\hat{j} + 5\hat{k}$ and $3\hat{i} + 11\hat{j} + 6\hat{k}$ are collinear, is [1 mark]

- (A) 8
(B) -8
(C) 2
(D) $\frac{5}{2}$

Correct Answer: (A) 8

Solution:

Concept:

- Points A, B, C are collinear when the displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are proportional.
- Subtract position vectors using the same initial point before comparing their components.

Step 1: Compute the displacement vectors Write $A = (-1, -1, 2)$, $B = (2, m, 5)$ and $C = (3, 11, 6)$. Then

$$\overrightarrow{AB} = (3, m + 1, 3), \quad \overrightarrow{AC} = (4, 12, 4).$$

The first and third components both fix the proportionality factor as $3/4$.

Step 2: Compare the middle components Collinearity therefore requires

$$m + 1 = \frac{3}{4}(12) = 9 \implies m = 8.$$

For $m = -8, 2, 5/2$, the middle components of $\overrightarrow{AB}$ would be $-7, 3, 7/2$, respectively, rather than the required 9; this excludes (B), (C) and (D).

Step 3: Conclusion $m = 8$. Hence, the correct option is (A).

Quick Tip to Solve: Use two displacement vectors from the same point; the fixed first and third coordinates give the ratio immediately.

8. If $|\vec{a}| = 8$, $|\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, then the value of $|\vec{a} \cdot \vec{b}|$ is [1 mark]

- (A) $6\sqrt{3}$
(B) $8\sqrt{3}$
(C) $12\sqrt{3}$
(D) $3\sqrt{12}$

Correct Answer: (C) $12\sqrt{3}$

Solution:

Concept:

- For vectors with angle θ , $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.
- The identity $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$ follows from $\sin^2 \theta + \cos^2 \theta = 1$.

Step 1: Substitute in the identity The product of the magnitudes is $8 \times 3 = 24$, and the cross-product

Quick Tip to Solve: Square the dot-cross identity directly; this avoids having to choose between an acute and an obtuse angle.

magnitude is 12. Thus

$$(\vec{a} \cdot \vec{b})^2 = 24^2 - 12^2 = 576 - 144 = 432.$$

Step 2: Take the nonnegative square root The question asks for an absolute value, so

$$|\vec{a} \cdot \vec{b}| = \sqrt{432} = \sqrt{144 \times 3} = 12\sqrt{3}.$$

The squared values of (A), (B) and (D) are 108, 192 and 108, respectively, not 432. The sign of the dot product need not be known to find its magnitude.

Step 3: Conclusion $|\vec{a} \cdot \vec{b}| = 12\sqrt{3}$. Hence, the correct option is (C).

Quick Tip to Solve: Square the dot-cross identity directly; this avoids having to choose between an acute and an obtuse angle.

9. The length of perpendicular drawn from the point $(1, 2, 3)$ on line $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ is [1 mark]

- (A) 2
- (B) 6
- (C) $\sqrt{10}$
- (D) $\sqrt{14}$

Correct Answer: (C) $\sqrt{10}$

Solution:

Concept:

- A line with direction ratios (a, b, c) through the origin has parametric form $(x, y, z) = (at, bt, ct)$.
- The shortest distance to a line is the length of the perpendicular from the given point. For the y -axis, it is $\sqrt{x^2 + z^2}$.

Step 1: Identify the line The zero direction ratios mean that the printed symmetric form is interpreted parametrically: $x = 0, y = t, z = 0$. Thus the line is the y -axis. Zero denominators here specify zero direction components; they do not mean ordinary division by zero.

Step 2: Locate the perpendicular foot For $P = (1, 2, 3)$, the foot is $H = (0, 2, 0)$. The vector $\vec{HP} = (1, 0, 3)$ has zero dot product with $(0, 1, 0)$, confirming perpendicularity. Hence

$$PH = \sqrt{(1-0)^2 + (2-2)^2 + (3-0)^2} = \sqrt{10}.$$

The value $\sqrt{14}$ measures distance from the origin, not from the line; neither 2 nor 6 equals the perpendicular length.

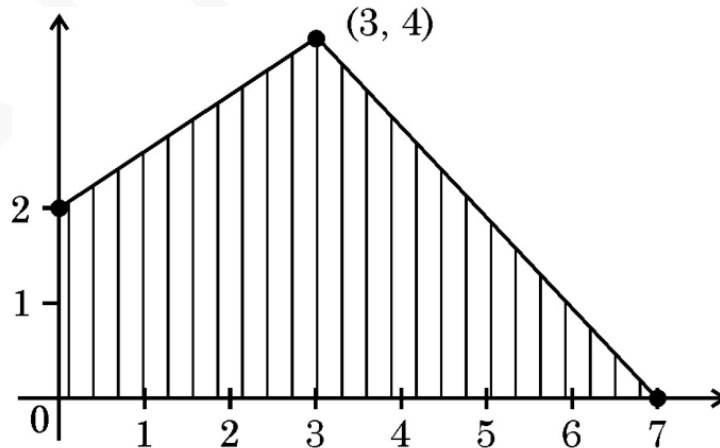
Step 3: Conclusion The perpendicular length is $\sqrt{10}$. Hence, the correct option is (C).

Quick Tip to Solve: For distance to the y -axis, omit the y -coordinate and take $\sqrt{x^2 + z^2}$.

10. The feasible region of a linear programming problem with objective function $Z = 5x + 7y$ is shown below:

The maximum value of Z - minimum value of Z is

[1 mark]



- (A) 8
- (B) 29
- (C) 35
- (D) 43

Correct Answer: (D) 43

Solution:

Concept:

- A linear objective function on a bounded polygonal feasible region attains its maximum and minimum at corner points.
- For each feasible vertex (x, y) , evaluate the given objective $Z = 5x + 7y$, then subtract its least value from its greatest value.

Step 1: Read all the vertices The shaded region includes the origin and has vertices $(0, 0)$, $(0, 2)$, $(3, 4)$ and $(7, 0)$. The lower boundary is the x -axis, so the origin must be included when finding the minimum.

Step 2: Compare objective values Substitution gives

$$\begin{aligned} Z(0, 0) &= 0, & Z(0, 2) &= 14, \\ Z(3, 4) &= 15 + 28 = 43, & Z(7, 0) &= 35. \end{aligned}$$

Thus the maximum is 43 and the minimum is 0. The option 8 subtracts the value at $(7, 0)$ from the maximum, 29 subtracts the value at $(0, 2)$, and 35 is only another vertex value.

Step 3: Conclusion The required difference is $43 - 0 = \boxed{43}$. Hence, the correct option is (D).

Quick Tip to Solve: Include the origin whenever it belongs to the shaded feasible polygon; with positive objective coefficients it gives the minimum zero.

11. The degree of an objective function of a linear programming problem is

[1 mark]

- (A) 0
- (B) 1
- (C) 2
- (D) Any natural number

Correct Answer: (B) 1

Solution:

Concept:

- The objective function of a linear programming problem is a linear polynomial in the decision variables, typically $Z = ax + by$.
- The degree of a polynomial is the largest total exponent of its variables in a term with a nonzero coefficient.

Step 1: Identify the powers of the variables In $Z = ax + by$, the constants a and b are coefficients. Each variable appears to the first power. No term such as x^2 , y^2 or xy occurs in a linear objective.

Step 2: Eliminate the other degrees The defining nonconstant linear objective therefore has degree 1. Degree 0 describes a constant expression, whereas degree 2 describes a quadratic expression. Allowing any natural-number degree would also allow nonlinear functions, contrary to linear programming.

Step 3: Conclusion The degree of the objective function is 1. Hence, the correct option is (B).

Quick Tip to Solve: In an LPP, check that each decision variable occurs only to the first power and is never multiplied by another decision variable.

12. If $\tan^{-1} x = 3y$, then

[1 mark]

- (A) $-\frac{\pi}{2} < y < \frac{\pi}{2}$
- (B) $-\frac{3\pi}{2} < y < \frac{3\pi}{2}$
- (C) $-\frac{\pi}{6} < y < \frac{\pi}{6}$
- (D) $-\frac{\pi}{6} \leq y \leq \frac{\pi}{6}$

Correct Answer: (C) $-\frac{\pi}{6} < y < \frac{\pi}{6}$

Solution:

Concept:

- The principal value of $\tan^{-1} x$, for real x , lies strictly in $(-\pi/2, \pi/2)$.
- Dividing an inequality by a positive number preserves its direction and whether its endpoints are included.

Step 1: Apply the principal-value range Since $\tan^{-1} x = 3y$, substitute $3y$ into the principal range:

$$-\frac{\pi}{2} < 3y < \frac{\pi}{2}.$$

Quick Tip to Solve: First write the principal range for the entire inverse-tangent expression, then divide the bounds by the coefficient of y .

The endpoints are excluded because tangent is undefined at $\pm\pi/2$.

Step 2: Scale both bounds Divide all three parts by 3:

$$-\frac{\pi}{6} < y < \frac{\pi}{6}.$$

Option (A) does not give the exact scaled range, and (B) multiplies the bounds instead of dividing them. Option (D) incorrectly includes the endpoint values, which cannot occur for any real x .

Step 3: Conclusion The exact range is $-\pi/6 < y < \pi/6$. Hence, the correct option is (C).

Quick Tip to Solve: First write the principal range for the entire inverse-tangent expression, then divide the bounds by the coefficient of y .

13. Which of the following cannot be an order of a column-matrix?

[1 mark]

- (A) 1×2
- (B) 2×1
- (C) 1×1
- (D) $m \times 1$, where $m \in \mathbb{N}$

Correct Answer: (A) 1×2

Solution:

Concept:

- The order of a matrix is written as number of rows $\times$ number of columns.
- A column matrix has exactly one column; its general order is $m \times 1$, where m is a positive integer.

Step 1: Check the column count The order 1×2 means one row and two columns. Since the second number is 2, this is a row matrix and does not satisfy the requirement of exactly one column.

Step 2: Verify the remaining options A matrix of order 2×1 has two rows and one column. A 1×1 matrix has one row and one column, so it is both a row and a column matrix. Finally, $m \times 1$ explicitly describes any column matrix. Thus each of options (B), (C) and (D) is a possible order.

Step 3: Conclusion The impossible order for a column matrix is 1×2 . Hence, the correct option is (A).

Quick Tip to Solve: Look only at the second number in the order: a column matrix requires that number to be exactly 1.

14. Which of the following properties is/are true for two matrices of suitable orders?

(i) $(A + B)' = A' + B'$

(ii) $(A - B)' = B' - A'$

(iii) $(AB)' = A'B'$

(iv) $(kAB)' = kB'A' \text{ (} k \text{ is a scalar)}$

[1 mark]

(A) (i) only

(B) (i), (ii) and (iii)

(C) (i) and (ii)

(D) (i) and (iv)

Correct Answer: (D) (i) and (iv)

Solution:

Concept:

- Transpose distributes over addition and subtraction without changing the order of the two terms:
 $(A \pm B)' = A' \pm B'$.
- Transpose reverses the order in a matrix product: $(AB)' = B'A'$; a scalar factor is unchanged.

Step 1: Test addition and subtraction Property (i), $(A + B)' = A' + B'$, is valid. Property (ii) is not valid in general: the transpose of $A - B$ is $A' - B'$, not $B' - A'$.

Step 2: Test the product identities Property (iii) omits reversal of the factor order, so it is not a general transpose identity. Property (iv) correctly states

$$(kAB)' = k(AB)' = kB'A'.$$

Therefore (iv) holds. Option (A) leaves out this true property, while (B) and (C) include the false subtraction property.

Step 3: Conclusion Exactly (i) and (iv) are valid general properties. Hence, the correct option is (D).

Quick Tip to Solve: Reverse the order of factors when transposing a product, but retain the order of terms when transposing a difference.

15. If $\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{vmatrix}$, then

[1 mark]

- (A) $\Delta_1 = 2\Delta_2$
 (B) $\Delta_2 = -2\Delta_1$
 (C) $\Delta_1 = \Delta_2$
 (D) $\Delta_2 = -\Delta_1$

Correct Answer: (B) $\Delta_2 = -2\Delta_1$

Solution:

Concept:

- The determinant of a diagonal matrix is the product of its diagonal entries.
- Interchanging two rows changes the sign of a determinant; alternatively, expand along a row or column containing two zeros.

Step 1: Evaluate the first determinant The first determinant is diagonal, so

$$\Delta_1 = 1 \cdot 2 \cdot 3 = 6.$$

Its off-diagonal entries vanish and hence contribute no additional terms.

Step 2: Evaluate and compare the second determinant Expand Δ_2 along its third column. The nonzero entry 6 has positive cofactor sign:

$$\Delta_2 = 6 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 6(0 - 2) = -12.$$

Consequently $\Delta_2/\Delta_1 = -2$. The proposed equality in (A) would give $6 = -24$; (C) would give $6 = -12$, and (D) would give $-12 = -6$, all false.

Step 3: Conclusion The values satisfy $\Delta_2 = -2\Delta_1$. Hence, the correct option is (B).

Quick Tip to Solve: For Δ_2 , expand down the third column; remember that the remaining 2×2 determinant is $ad - bc$, hence negative here.

16. One of the values of x for which $\begin{vmatrix} \cos x & \sin x \\ -\cos x & \sin x \end{vmatrix} = 1$ is

[1 mark]

- (A) 0
- (B) $\frac{\pi}{4}$
- (C) $\frac{\pi}{3}$
- (D) $\frac{\pi}{2}$

Correct Answer: (B) $\frac{\pi}{4}$

Solution:

Concept:

- A 2×2 determinant is evaluated as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- The double-angle identity $2 \sin x \cos x = \sin 2x$ converts the resulting product into a single trigonometric function.

Step 1: Expand the determinant The negative sign in the lower-left entry is essential. On expanding,

$$\begin{vmatrix} \cos x & \sin x \\ -\cos x & \sin x \end{vmatrix} = \cos x \sin x - \sin x(-\cos x) = \sin 2x.$$

Thus the given equation becomes $\sin 2x = 1$.

Step 2: Check the listed angles For $x = \pi/4$, we obtain $\sin 2x = \sin(\pi/2) = 1$. For $x = 0$ or $x = \pi/2$, the value is 0. For $x = \pi/3$, it is $\sin(2\pi/3) = \sqrt{3}/2$, which is less than 1. Only one of the listed angles satisfies the equation.

Step 3: Conclusion One required value is $x = \pi/4$. Hence, the correct option is (B).

Quick Tip to Solve: Expand the determinant before substituting angles: subtracting its negative cross-product gives $2 \sin x \cos x$.

17. If A and B are symmetric matrices of same order, then which of the following matrices is a skew-symmetric matrix?

[1 mark]

- (A) $A^2 + B^2$
- (B) $A^2 - B^2$
- (C) $AB + BA$
- (D) $AB - BA$

Correct Answer: (D) $AB - BA$

Solution:

Concept:

- A matrix M is skew-symmetric if $M^T = -M$, while symmetry gives $A^T = A$ and $B^T = B$.

Quick Tip to Solve: Reverse product order when taking a transpose: the minus sign makes the commutator change sign.

- Transposition reverses the order of a product: $(AB)^T = B^T A^T$. Matrices need not commute.

Step 1: Transpose the difference of products Using the symmetry of both matrices,

$$(AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -(AB - BA).$$

Thus the difference satisfies the defining transpose identity for all symmetric matrices of the same order.

Step 2: Check the other expressions The transpose of A^2 is $(A^T)^2 = A^2$, and similarly for B^2 . Hence $A^2 + B^2$ and $A^2 - B^2$ are symmetric. Also $(AB + BA)^T = BA + AB = AB + BA$, so this sum is symmetric. Such matrices are not generally skew-symmetric; only a special zero result would be both.

Step 3: Conclusion $AB - BA$ is necessarily skew-symmetric. Hence, the correct option is (D).

Quick Tip to Solve: Reverse product order when taking a transpose: the minus sign makes the commutator change sign.

18. The absolute maximum value of $f(x) = x^2 + 1$ in $[-5, 2]$ is

[1 mark]

- (A) 26
- (B) 1
- (C) 5
- (D) 2

Correct Answer: (A) 26

Solution:

Concept:

- For a continuous function on a closed interval, compare the values at its endpoints and all interior critical points to find the absolute maximum.
- A differentiable function has an interior critical point where $f'(x) = 0$; this point need not be a maximum.

Step 1: Find the interior critical point The function is $f(x) = x^2 + 1$, with derivative $f'(x) = 2x$. Thus $f'(x) = 0$ gives $x = 0$, which lies in $[-5, 2]$. Its value is $f(0) = 1$. Since $f''(x) = 2 > 0$, this stationary point is a minimum.

Step 2: Compare with the endpoint values Evaluate the function at both endpoints:

$$f(-5) = 25 + 1 = 26, \quad f(2) = 4 + 1 = 5.$$

Of the candidate values 26, 1, 5, the greatest is 26. Option (B) gives the minimum, option (C) gives only the right-endpoint value, and option (D) confuses the endpoint coordinate with the required function value.

Step 3: Conclusion The absolute maximum is 26, attained at $x = -5$. Hence, the correct option is (A).

Quick Tip to Solve: For $x^2 + 1$, the endpoint farthest from zero gives the largest value; here it is -5 .

19. Assertion (A): Lines given by $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular to each other when $pp' + rr' = 1$.

Reason (R): Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are perpendicular to each other if $\vec{b}_1 \cdot \vec{b}_2 = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true and Reason (R) is false.
(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- Direction vectors are obtained from the coefficients of a line's parameter, independently of its position vector.
- Two line directions are perpendicular when their nonzero direction vectors have zero dot product: $\vec{b}_1 \cdot \vec{b}_2 = 0$.

Step 1: Evaluate the Assertion Taking $y = t$, the first line is $(x, y, z) = (q, 0, s) + t(p, 1, r)$. The second similarly has direction vector $(p', 1, r')$. Their dot product is

$$(p, 1, r) \cdot (p', 1, r') = pp' + 1 + rr'.$$

Under the asserted condition $pp' + rr' = 1$, the dot product equals 2, not zero. The Assertion is therefore false; perpendicularity requires $pp' + rr' = -1$.

Step 2: Evaluate the Reason The vectors $\vec{b}_1$ and $\vec{b}_2$ in the vector equations are the direction vectors. Since $\vec{b}_1 \cdot \vec{b}_2 = |\vec{b}_1||\vec{b}_2|\cos\theta$, a zero dot product gives a right angle between the line directions. The Reason is true.

Step 3: Check the explanation and options Applying the true dot-product rule disproves the stated Assertion rather than establishing it. Options (A), (B) and (C) all require the Assertion to be true, so none fits these truth values.

Step 4: Conclusion The Assertion is false and the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: With y as parameter, retain its direction component 1; its product contributes the crucial +1.

20. Assertion (A): In an experiment of throwing an unbiased die, the probability of getting a prime number given that number appearing on the die being odd is $\frac{2}{3}$.

Reason (R): For any two events A and B , $P(A|B) = \frac{P(A \cup B)}{P(B)}$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true and Reason (R) is false.
(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- For events A and B with $P(B) > 0$, conditional probability is $P(A | B) = P(A \cap B) / P(B)$.
- Conditioning on odd outcomes of a fair die restricts the equally likely sample space to $\{1, 3, 5\}$; the number 1 is not prime.

Step 1: Evaluate the Assertion Let $A = \{2, 3, 5\}$ be the event of a prime outcome and $B = \{1, 3, 5\}$ the event of an odd outcome. Among the three equally likely odd outcomes, two are prime. Equivalently,

$$P(A | B) = \frac{P(\{3, 5\})}{P(\{1, 3, 5\})} = \frac{2/6}{3/6} = \frac{2}{3}.$$

Thus the Assertion is true.

Step 2: Evaluate the Reason as printed The printed Reason places $A \cup B$ in the numerator. Conditional probability requires the intersection $A \cap B$, since an outcome must satisfy both events. Here $A \cup B = \{1, 2, 3, 5\}$, so the printed expression gives $(4/6)/(3/6) = 4/3$, which is not even a possible probability. The Reason is false.

Step 3: Decide whether the Reason explains the Assertion The false union formula cannot explain the true Assertion. Options (A) and (B) require a true Reason, while option (D) requires a false Assertion, so they are excluded.

Step 4: Conclusion The Assertion is true and the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: In conditional probability use the overlap $A \cap B$, not the union; restrict outcomes to the given event first.

Section B (10 marks)

This section comprises Very Short Answer (VSA) type questions of 2 marks each.

21. A vector $\vec{a}$ of magnitude 14 has direction ratios $\langle 2, 3, -6 \rangle$. Find the projection of the vector $\vec{a}$ on $\hat{i}$. [2 marks]

Solution:

Concept:

- Direction ratios (l, m, n) give a vector in the prescribed direction. Divide by $\sqrt{l^2 + m^2 + n^2}$ to obtain the corresponding unit vector.
- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $(\vec{a} \cdot \vec{b})/|\vec{b}|$. Since $\hat{i}$ is a unit vector, this reduces to $\vec{a} \cdot \hat{i}$.

Step 1: Construct the vector of magnitude fourteen The direction ratios give the direction vector $\vec{d} = 2\hat{i} + 3\hat{j} - 6\hat{k}$. Its magnitude is

$$|\vec{d}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7.$$

The unit vector along the given direction is $\vec{d}/7$. Multiplying it by the required magnitude yields

$$\vec{a} = 14 \frac{2\hat{i} + 3\hat{j} - 6\hat{k}}{7} = 4\hat{i} + 6\hat{j} - 12\hat{k}.$$

Step 2: Project onto the positive x direction Use $\hat{i} \cdot \hat{i} = 1$ and $\hat{j} \cdot \hat{i} = \hat{k} \cdot \hat{i} = 0$. The scalar projection is

$$\frac{\vec{a} \cdot \hat{i}}{|\hat{i}|} = (4\hat{i} + 6\hat{j} - 12\hat{k}) \cdot \hat{i} = 4.$$

Only the x component survives the dot product. Its positive sign shows that the component points along the positive x direction.

Step 3: Conclusion The projection of $\vec{a}$ on $\hat{i}$ is 4. The corresponding vector projection is $4\hat{i}$.

Quick Tip to Solve: Multiply the magnitude 14 by the x direction cosine $2/7$; this gives the projection directly.

22. Vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{k}$ represent the two adjacent sides of a parallelogram. Find the vectors representing its diagonals and hence find their lengths. [2 marks]

Solution:

Concept:

- If two adjacent sides from a common vertex are $\vec{a}$ and $\vec{b}$, the parallelogram's diagonals can be represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, with suitable orientations.
- The length of $x\hat{i} + y\hat{j} + z\hat{k}$ is $\sqrt{x^2 + y^2 + z^2}$. Reversing a diagonal's direction changes its vector's sign but leaves its length unchanged.

Quick Tip to Solve: Add adjacent side vectors for one diagonal and subtract them for the other; compute lengths only after combining components.

Step 1: Form the vectors of the diagonals Take the common vertex as the origin, and let the ends of the adjacent sides have position vectors $\vec{a}$ and $\vec{b}$. The opposite vertex then has position vector $\vec{a} + \vec{b}$. Thus one diagonal runs from the origin to that vertex, and the other may be directed from the end of $\vec{b}$ to the end of $\vec{a}$:

$$\begin{aligned}\vec{d}_1 &= \vec{a} + \vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}, \\ \vec{d}_2 &= \vec{a} - \vec{b} = 2\hat{i} - 2\hat{j}.\end{aligned}$$

The k components cancel in the second diagonal because both side vectors have k component 2.

Step 2: Calculate each diagonal's length Apply the magnitude formula separately to the two diagonal vectors:

$$\begin{aligned}|\vec{d}_1| &= \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6, \\ |\vec{d}_2| &= \sqrt{2^2 + (-2)^2 + 0^2} = \sqrt{8} = 2\sqrt{2}.\end{aligned}$$

As a check, the squared lengths add to $36 + 8 = 44$, which equals $2(|\vec{a}|^2 + |\vec{b}|^2) = 2(17 + 5)$, as required by the parallelogram identity.

Step 3: Conclusion The diagonals can be represented by $4\hat{i} - 2\hat{j} + 4\hat{k}$ and $2\hat{i} - 2\hat{j}$; their respective lengths are 6 and $2\sqrt{2}$.

Quick Tip to Solve: Add adjacent side vectors for one diagonal and subtract them for the other; compute lengths only after combining components.

23(a). Simplify: $\tan^{-1}\left(\frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x}\right)$, $0 < x < \frac{\pi}{4}$.

[2 marks]

Solution:

Concept:

- The angle-difference identities give $\sin(A - B) = \sin A \cos B - \cos A \sin B$ and $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
- The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$. Thus $\tan^{-1}(\tan t) = t$ only when t belongs to this interval.

Step 1: Express the ratio as one tangent Use the angle $t = \pi/4 - 2x$. Since $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$,

$$\tan\left(\frac{\pi}{4} - 2x\right) = \frac{\cos 2x - \sin 2x}{\cos 2x + \sin 2x}.$$

The given denominator is positive: $0 < 2x < \pi/2$ makes both sine and cosine positive. The ratio is therefore defined throughout the stated interval.

Step 2: Verify the principal-value interval Multiply the given bounds by two and subtract from $\pi/4$:

$$0 < 2x < \frac{\pi}{2} \implies -\frac{\pi}{4} < \frac{\pi}{4} - 2x < \frac{\pi}{4}.$$

This entire interval lies in the principal range of inverse tangent, so no addition or subtraction of π is required when applying $\tan^{-1}$.

Quick Tip to Solve: Recognise the sine and cosine of $\pi/4 - 2x$ directly; this avoids dividing by $\cos 2x$. Always check the inverse tangent's principal range.

Step 3: Conclusion The simplified expression is $\boxed{\frac{\pi}{4} - 2x}$, for $0 < x < \pi/4$.

Quick Tip to Solve: Recognise the sine and cosine of $\pi/4 - 2x$ directly; this avoids dividing by $\cos 2x$. Always check the inverse tangent's principal range.

OR

23(b). Evaluate: $\tan \left[\sin^{-1} 1 - \cos^{-1} \left(-\frac{1}{2} \right) \right]$.

[2 marks]

Solution:

Concept:

- The principal range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$, while that of $\cos^{-1}$ is $[0, \pi]$. Choose angles from these ranges before evaluating a composition.
- Tangent is an odd function: $\tan(-\theta) = -\tan \theta$, and $\tan(\pi/6) = 1/\sqrt{3}$.

Step 1: Evaluate the inverse trigonometric values Since sine equals one at $\pi/2$, its principal inverse value is

$$\sin^{-1} 1 = \frac{\pi}{2}.$$

Cosine equals $-1/2$ at $2\pi/3$ in its principal interval. Therefore,

$$\cos^{-1} \left(-\frac{1}{2} \right) = \frac{2\pi}{3}.$$

In particular, $-\pi/3$ cannot be used here: it is outside the inverse cosine range and its cosine is positive.

Step 2: Subtract the angles and apply tangent Using a common denominator gives

$$\frac{\pi}{2} - \frac{2\pi}{3} = \frac{3\pi - 4\pi}{6} = -\frac{\pi}{6}.$$

Hence the expression becomes $\tan(-\pi/6)$. The negative angle makes the tangent negative; its magnitude is the standard value $1/\sqrt{3}$.

Step 3: Conclusion The required value is $\boxed{-\frac{1}{\sqrt{3}}}$, equivalently $-\sqrt{3}/3$.

Quick Tip to Solve: Write both principal inverse values first: $\pi/2 - 2\pi/3$ is negative, so the final tangent must also be negative.

24(a). Check whether function $f(x)$ defined as $f(x) = \begin{cases} \frac{|x-3|}{2(x-3)}, & x < 3 \\ \frac{x-6}{6}, & x \geq 3 \end{cases}$ is continuous at $x = 3$ or not. [2 marks]

Solution:

Concept:

- A function is continuous at $x = a$ when its left-hand limit, right-hand limit and value $f(a)$ all exist and are equal.
- Absolute value depends on sign: for $x < 3$, $|x - 3| = -(x - 3)$. Each one-sided limit must use the branch valid on that side.

Step 1: Evaluate the left-hand limit When $x < 3$, the number $x - 3$ is negative and nonzero, so

$$f(x) = \frac{-(x-3)}{2(x-3)} = -\frac{1}{2}.$$

This branch is constant to the left of 3. Consequently,

$$\lim_{x \rightarrow 3^-} f(x) = -\frac{1}{2}.$$

The apparent zero denominator at 3 does not affect this limit, because the left branch is used only for $x < 3$.

Step 2: Evaluate the right-hand limit and function value For $x \geq 3$, the applicable branch is the linear function $(x - 6)/6$, whose limit can be obtained by substitution:

$$\lim_{x \rightarrow 3^+} f(x) = \frac{3-6}{6} = -\frac{1}{2}.$$

Equality is included in this second branch, so the defined value is

$$f(3) = \frac{3-6}{6} = -\frac{1}{2}.$$

Step 3: Conclusion Since $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) = -1/2$, the function is continuous at $x = 3$.

Quick Tip to Solve: On the left of 3, replace $|x - 3|$ by $-(x - 3)$, not by $x - 3$; then compare both limits with the branch that includes equality.

24(b). If $\sqrt{3}(x^2 + y^2) = 4xy$, then find $\frac{dy}{dx}$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

[2 marks]

Solution:

Concept:

- In implicit differentiation, regard y as a function of x . The chain rule gives $d(y^2)/dx = 2y dy/dx$.
- The product rule gives $d(xy)/dx = y + x dy/dx$. Collect the terms containing the derivative before substituting the given point.

Step 1: Differentiate the relation implicitly Write $y' = dy/dx$. Differentiating both sides of $\sqrt{3}(x^2 + y^2) = 4xy$ with respect to x , we obtain

$$\sqrt{3}(2x + 2yy') = 4(y + xy').$$

Move the derivative terms to the left and the remaining terms to the right:

$$(2\sqrt{3}y - 4x)y' = 4y - 2\sqrt{3}x.$$

Thus, wherever the coefficient of y' is nonzero,

$$y' = \frac{2y - \sqrt{3}x}{\sqrt{3}y - 2x}.$$

Step 2: Substitute the coordinates At $x = 1/2$, $y = \sqrt{3}/2$, the numerator and denominator are

$$\begin{aligned} 2y - \sqrt{3}x &= \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}, \\ \sqrt{3}y - 2x &= \frac{3}{2} - 1 = \frac{1}{2}. \end{aligned}$$

The denominator is nonzero, so the derivative exists there and is the ratio of these two values. The point also satisfies the original relation, since both sides equal $\sqrt{3}$.

Step 3: Conclusion Therefore, $\boxed{\frac{dy}{dx}\bigg|_{(1/2, \sqrt{3}/2)} = \sqrt{3}}.$

Quick Tip to Solve: Differentiate xy using both product-rule terms. Substitute the point after collecting dy/dx , which makes the sign and denominator easy to check.

25(a). Simplify: $\cot^{-1}\left(\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

[2 marks]

Solution:

Concept:

- The double-angle identities are $1 + \cos 2x = 2 \cos^2 x$ and $1 - \cos 2x = 2 \sin^2 x$. Also, $\sqrt{u^2} = |u|$ for real u .
- The principal range of $\cot^{-1}$ is $(0, \pi)$. Therefore $\cot^{-1}(\cot x) = x$ when x lies in this range.

Step 1: Simplify the square-root argument Apply the two double-angle identities to the numerator and denominator:

$$\sqrt{\frac{1+\cos 2x}{1-\cos 2x}} = \sqrt{\frac{2\cos^2 x}{2\sin^2 x}} = \sqrt{\cot^2 x} = |\cot x|.$$

The denominator is nonzero because $0 < x < \pi/2$ ensures $\sin x > 0$. Retaining the absolute value at this stage respects the nonnegative square root.

Step 2: Use the sign and principal range On $(0, \pi/2)$, both sine and cosine are positive, hence $\cot x > 0$. Thus $|\cot x| = \cot x$, and the expression reduces to $\cot^{-1}(\cot x)$. The given interval is wholly contained in $(0, \pi)$, the principal range of inverse cotangent. The angle selected by $\cot^{-1}$ is therefore exactly the original angle x .

Step 3: Conclusion The given expression simplifies to $\boxed{x}$, for $0 < x < \pi/2$.

Quick Tip to Solve: First write $\sqrt{\cot^2 x} = |\cot x|$; the first-quadrant restriction removes the absolute value and also confirms the correct inverse-cotangent branch.

OR

25(b). Evaluate: $\sin(\tan^{-1}(-\sqrt{3}) - \sec^{-1} 2)$.

[2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1} t$ is $(-\pi/2, \pi/2)$. A negative argument therefore gives a negative principal angle.
- For $\sec^{-1} t$, use the principal range $[0, \pi] \setminus \{\pi/2\}$; $\sec \theta = 2$ is equivalent to $\cos \theta = 1/2$.

Step 1: Evaluate the two principal angles Since $\tan(-\pi/3) = -\sqrt{3}$ and $-\pi/3$ belongs to the principal range of inverse tangent,

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}.$$

Also, $\cos(\pi/3) = 1/2$, so $\sec(\pi/3) = 2$. The angle $\pi/3$ belongs to the principal range of inverse secant. Thus,

$$\sec^{-1} 2 = \frac{\pi}{3}.$$

Quick Tip to Solve: Fix each principal inverse-trigonometric angle first; here the negative tangent argument makes the angle difference $-2\pi/3$, not zero.

Step 2: Substitute and simplify the sine Subtract the principal angles before applying sine:

$$\sin\left(-\frac{\pi}{3} - \frac{\pi}{3}\right) = \sin\left(-\frac{2\pi}{3}\right).$$

Sine is an odd function, and $2\pi/3$ lies in the second quadrant, where sine is positive. Hence its reference angle is $\pi/3$, with $\sin(2\pi/3) = \sin(\pi/3) = \sqrt{3}/2$.

Step 3: Conclusion The required value is $\boxed{-\frac{\sqrt{3}}{2}}$.

Quick Tip to Solve: Fix each principal inverse-trigonometric angle first; here the negative tangent argument makes the angle difference $-2\pi/3$, not zero.

Section C (18 marks)

This section comprises Short Answer (SA) type questions of 3 marks each.

26. If $I_1 = \int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x}$ and $I_2 = \int_{-1/2}^{1/2} |x| dx$, then show that $I_1 - 4I_2 = 0$.

[3 marks]

Solution:

Concept:

- The double-angle identity $1 + \cos 2x = 2 \cos^2 x$ converts the first integrand into $\frac{1}{2} \sec^2 x$, whose antiderivative is $\frac{1}{2} \tan x$.
- The function $|x|$ is even. Thus $\int_{-a}^a |x| dx = 2 \int_0^a x dx$ for $a > 0$; on the negative half of the interval, $|x| = -x$.

Step 1: Simplify and evaluate the first integral On $[-\pi/4, \pi/4]$, $\cos x$ never vanishes, so the identity may be used throughout the integration interval. Therefore,

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x dx \\ &= \frac{1}{2} [\tan x]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2} \{1 - (-1)\} = 1. \end{aligned}$$

In applying the limits, the lower-limit tangent is negative; subtracting it adds one.

Step 2: Use symmetry for the absolute-value integral We cannot replace $|x|$ by x over the entire interval, since they differ for $x < 0$. Instead, the two halves contribute equally because $|-x| = |x|$.

Quick Tip to Solve: Use the double-angle identity for I_1 , and evenness for I_2 ; integrating x directly from $-1/2$ to $1/2$ would incorrectly give zero.

Hence,

$$\begin{aligned} I_2 &= 2 \int_0^{1/2} x \, dx \\ &= 2 \left[\frac{x^2}{2} \right]_0^{1/2} \\ &= \left(\frac{1}{2} \right)^2 = \frac{1}{4}. \end{aligned}$$

Both integrands are nonnegative on their respective intervals, consistent with the positive values obtained.

Step 3: Form the required difference Substitute the separately evaluated quantities into the stated expression:

$$I_1 - 4I_2 = 1 - 4 \left(\frac{1}{4} \right).$$

The coefficient four makes the second contribution equal to the first.

Step 4: Conclusion Since $I_1 = 1$ and $I_2 = \frac{1}{4}$, we obtain $I_1 - 4I_2 = 0$, as required.

Quick Tip to Solve: Use the double-angle identity for I_1 , and evenness for I_2 ; integrating x directly from $-1/2$ to $1/2$ would incorrectly give zero.

27(a). Find the general solution of the differential equation: $y^2 dx + (x^2 - xy + y^2) dy = 0$. [3 marks]

Solution:

Concept:

- If dx/dy depends only on x/y , use $x = vy$, where v is a function of y . Then $dx/dy = v + y dv/dy$.
- Separate the transformed variables and use $\int dv/(1 + v^2) = \tan^{-1} v$ and $\int dy/y = \ln |y|$. Division by y requires $y \neq 0$.

Step 1: Express the equation in homogeneous form Treat x as a function of y . The coefficients of dx and dy are homogeneous polynomials of degree two. For $y \neq 0$, the given equation becomes

$$\frac{dx}{dy} = -\frac{x^2 - xy + y^2}{y^2} = -\left(\frac{x}{y}\right)^2 + \frac{x}{y} - 1.$$

Its right-hand side depends only on x/y , which suggests substituting that ratio as a new variable.

Step 2: Substitute and separate the variables Put $x = vy$, so that $dx/dy = v + y dv/dy$. Substituting gives

$$\begin{aligned} v + y \frac{dv}{dy} &= -v^2 + v - 1, \\ y \frac{dv}{dy} &= -(1 + v^2), \\ \frac{dv}{1 + v^2} &= -\frac{dy}{y}. \end{aligned}$$

Quick Tip to Solve: Use $x = vy$ here: the resulting linear terms in v cancel immediately, leaving $dv/(1 + v^2) = -dy/y$.

The terms linear in v cancel. Moreover, $1 + v^2 > 0$ for real v , so dividing by this factor loses no real constant-ratio solution.

Step 3: Integrate and restore the original variables Integrating both sides yields

$$\tan^{-1} v = -\ln |y| + C.$$

Restoring $v = x/y$ gives $\tan^{-1}(x/y) + \ln |y| = C$. As a check, its differential is

$$\frac{y dx - x dy}{x^2 + y^2} + \frac{dy}{y} = 0.$$

Multiplying by $y(x^2 + y^2)$ reproduces the original equation. The line $y = 0$, excluded by the substitution, also satisfies the original differential equation since $dy = 0$ along it.

Step 4: Conclusion The general integral is $\tan^{-1}\left(\frac{x}{y}\right) + \ln |y| = C$ for $y \neq 0$, with $y = 0$ as an additional solution.

Quick Tip to Solve: Use $x = vy$ here: the resulting linear terms in v cancel immediately, leaving $dv/(1 + v^2) = -dy/y$.

OR

27(b). Find the particular solution of the differential equation $\frac{dy}{dx} = y \tan x$, given that $y = 2$ if $x = 0$. [3 marks]

Solution:

Concept:

- A first-order linear equation $dy/dx + P(x)y = Q(x)$ has integrating factor $IF = e^{\int P(x) dx}$. Multiplying by it turns the left side into $d(y IF)/dx$.
- An initial condition determines the arbitrary constant. For $P(x) = -\tan x$, we use $\int -\tan x dx = \ln |\cos x|$.

Step 1: Identify the linear equation and integrating factor Rewrite the given equation as

$$\frac{dy}{dx} - y \tan x = 0.$$

Comparing with the linear form gives $P(x) = -\tan x$ and $Q(x) = 0$. The interval containing the initial point $x = 0$ on which $\tan x$ is continuous is $(-\pi/2, \pi/2)$. On this interval $\cos x > 0$, so a suitable integrating factor is

$$IF = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x.$$

Step 2: Multiply through and integrate Multiplying every term by $\cos x$, and using $\tan x \cos x = \sin x$, yields

$$\cos x \frac{dy}{dx} - y \sin x = 0.$$

Quick Tip to Solve: With the term $-y \tan x$, use integrating factor $\cos x$; its product with y is constant, fixed here by $y(0) = 2$.

By the product rule, the left side equals the derivative of $y \cos x$. Hence,

$$\frac{d}{dx}(y \cos x) = 0 \implies y \cos x = C.$$

This form retains the arbitrary constant until the initial condition is applied.

Step 3: Apply the initial condition and verify At $x = 0$, the given value is $y = 2$. Therefore $C = 2 \cos 0 = 2$. Solving $y \cos x = 2$ for y gives $y = 2 \sec x$. Differentiation verifies the equation:

$$\frac{dy}{dx} = 2 \sec x \tan x = y \tan x.$$

Also, $y(0) = 2 \sec 0 = 2$, so the same expression satisfies the prescribed initial value.

Step 4: Conclusion The required particular solution is $y = 2 \sec x$, on $-\pi/2 < x < \pi/2$.

Quick Tip to Solve: With the term $-y \tan x$, use integrating factor $\cos x$; its product with y is constant, fixed here by $y(0) = 2$.

28. Solve the following linear programming problem graphically:

Minimize $Z = 13x - 15y$

Subject to constraints

$$x + y \leq 7,$$

$$2x - 3y + 6 \geq 0,$$

$$x \geq 0, y \geq 0.$$

[3 marks]

Solution:

Concept:

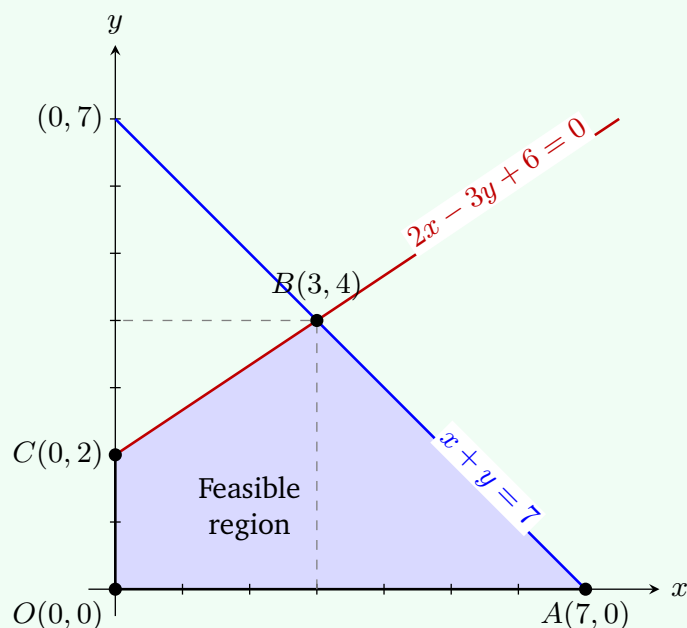
- Each linear inequality represents a closed half-plane. The feasible region is the common part of all these half-planes, restricted here to the first quadrant.
- For a nonempty bounded polygonal feasible region, a linear objective function attains its minimum at a corner point. Evaluate $Z = 13x - 15y$ at every vertex.

Step 1: Plot the boundary lines and select the half-planes The line $x + y = 7$ passes through $(7, 0)$ and $(0, 7)$. The inequality $x + y \leq 7$ selects its side containing the origin. Rearrange the second constraint as

$$2x - 3y + 6 \geq 0 \iff y \leq \frac{2}{3}x + 2.$$

Its boundary passes through $(0, 2)$ and $(3, 4)$, and its permitted side also contains the origin. Include both boundary lines, since equality is allowed. With $x, y \geq 0$, shade their common region in the first quadrant as shown.

Quick Tip to Solve: Dividing the second inequality by a negative coefficient reverses its sign: shade below $y = 2x/3 + 2$, then compare all four vertices, including the origin.



The shaded quadrilateral is the feasible region; both boundary lines are included.

Step 2: Find all vertices of the feasible region On the horizontal axis the feasible segment runs from $O(0,0)$ to $A(7,0)$; on the vertical axis it runs from O to $C(0,2)$. The other vertex B is the intersection of the two oblique boundaries:

$$\begin{aligned} x + y &= 7, & 2x - 3y &= -6, \\ 2x - 3(7 - x) &= -6, & 5x &= 15. \end{aligned}$$

Thus $B = (3, 4)$. The shaded quadrilateral $OABC$ is bounded and contains all feasible points.

Step 3: Compare the objective function at the corner points Substitute the coordinates of each vertex into the same objective function:

Vertex (x, y)	$Z = 13x - 15y$
$O(0, 0)$	0
$A(7, 0)$	91
$B(3, 4)$	$39 - 60 = -21$
$C(0, 2)$	-30

All four candidates have been checked; the corner-point theorem therefore determines the global minimum over the entire shaded region.

Step 4: Conclusion The minimum value is $Z_{\min} = -30$, attained at $(x, y) = (0, 2)$.

Quick Tip to Solve: Dividing the second inequality by a negative coefficient reverses its sign: shade below $y = 2x/3 + 2$, then compare all four vertices, including the origin.

29(a). Out of two bags, bag I contains 3 red and 4 white balls and bag II contains 8 red and 6 white balls. A die is thrown. If it shows a number less than 3 then a ball is drawn at random from bag I, otherwise a ball is drawn at random from bag II. Find the probability that the ball drawn from one of the bags is a red ball. [3 marks]

Solution:

Concept:

- If B_1, B_2 are mutually exclusive and exhaustive events and R is the required event, the total probability theorem gives $P(R) = P(B_1)P(R | B_1) + P(B_2)P(R | B_2)$.
- A fair die has six equally likely outcomes. Within a selected bag, random drawing makes each ball equally likely, so the probability of red is the number of red balls divided by the total number of balls in that bag.

Step 1: Determine the probabilities of selecting the bags Let B_1 denote selecting bag I, B_2 selecting bag II, and R drawing a red ball. Outcomes 1, 2 select bag I, whereas 3, 4, 5, 6 select bag II. Therefore,

$$P(B_1) = \frac{2}{6} = \frac{1}{3}, \quad P(B_2) = \frac{4}{6} = \frac{2}{3}.$$

Exactly one of these bags is selected on every throw, so these events form the partition needed for total probability. The two bag choices are not equally likely.

Step 2: Calculate the conditional red-ball probabilities Bag I contains $3 + 4 = 7$ balls, of which 3 are red. Bag II contains $8 + 6 = 14$ balls, of which 8 are red. Hence,

$$P(R | B_1) = \frac{3}{7}, \quad P(R | B_2) = \frac{8}{14} = \frac{4}{7}.$$

These fractions are conditioned on which bag was selected, so they must be weighted by the corresponding die probabilities.

Step 3: Add the two mutually exclusive red-ball paths A red ball can come either from bag I or from bag II. Multiplying along each path and adding gives

$$\begin{aligned} P(R) &= P(B_1)P(R | B_1) + P(B_2)P(R | B_2) \\ &= \frac{1}{3} \cdot \frac{3}{7} + \frac{2}{3} \cdot \frac{4}{7} \\ &= \frac{3}{21} + \frac{8}{21} = \frac{11}{21}. \end{aligned}$$

Step 4: Conclusion The probability that the ball drawn is red is $\boxed{\frac{11}{21}}$.

Quick Tip to Solve: Count the die outcomes before combining bag probabilities: bag II receives weight $2/3$, so an unweighted average of the two red-ball fractions would be incorrect.

29(b). The probability of simultaneous occurrence of at least one of the two events X and Y is a . If the probability that exactly one of the events X, Y occurs is b , prove that $P(X') + P(Y') = 2 - 2a + b$.

[3 marks]

Solution:**Concept:**

- The event that at least one of X, Y occurs is $X \cup Y$. The addition theorem states $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$.
- Exactly one event occurs in the disjoint union $(X \cap Y') \cup (X' \cap Y)$. Also, the complement rule gives $P(X') = 1 - P(X)$ and $P(Y') = 1 - P(Y)$.

Step 1: Express the at-least-one condition Write $p = P(X)$, $q = P(Y)$, and $r = P(X \cap Y)$. The condition that at least one event occurs gives

$$a = P(X \cup Y) = p + q - r.$$

The subtraction removes the overlap counted in both p and q . No assumption of independence or mutual exclusivity is needed.

Step 2: Express the exactly-one condition Exactly one event means either X occurs without Y , or Y occurs without X . These two possibilities cannot happen together. Their probabilities are respectively $p - r$ and $q - r$, hence

$$b = (p - r) + (q - r) = p + q - 2r.$$

Subtracting this equation from the preceding equation yields

$$a - b = (p + q - r) - (p + q - 2r) = r.$$

This also follows by partitioning the union into exactly-one and both-events regions: the probability that both occur is $a - b$.

Step 3: Apply the complement rule Substitute $r = a - b$ into $a = p + q - r$ to obtain

$$p + q = a + r = a + (a - b) = 2a - b.$$

Now calculate the required sum using the individual complement probabilities:

$$\begin{aligned} P(X') + P(Y') &= (1 - p) + (1 - q) \\ &= 2 - (p + q) \\ &= 2 - (2a - b). \end{aligned}$$

Notice that the requested expression is a sum of two probabilities, so there is no need to replace it by the probability of a union of complements.

Step 4: Conclusion Therefore $\boxed{P(X') + P(Y') = 2 - 2a + b}$, as required.

Quick Tip to Solve: Subtract the exactly-one probability from the at-least-one probability to get $P(X \cap Y) = a - b$, then use the addition theorem and the complement rule.

30. Evaluate: $\int_0^1 x \sin^{-1} x \, dx$.

[3 marks]

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. Choose the inverse trigonometric function as u , because its derivative is algebraic.
- For $0 \leq x < 1$, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. The substitution $x = \sin \theta$ removes this square root on $0 \leq \theta \leq \pi/2$.

Step 1: Apply integration by parts Let $I = \int_0^1 x \sin^{-1} x \, dx$. Take $u = \sin^{-1} x$ and $dv = x \, dx$, so that $du = dx/\sqrt{1-x^2}$ and $v = x^2/2$. Therefore

$$I = \left[\frac{x^2}{2} \sin^{-1} x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx = \frac{\pi}{4} - \frac{1}{2} J,$$

where J denotes the remaining integral. The boundary term at zero vanishes, while at one it is $(1/2)(\pi/2)$. The auxiliary integral has an integrable endpoint singularity; these operations may be performed up to $b < 1$, followed by $b \rightarrow 1^-$.

Step 2: Evaluate the remaining definite integral Put $x = \sin \theta$, so $dx = \cos \theta \, d\theta$ and $\sqrt{1-x^2} = \cos \theta$ on the chosen interval. The limits $x = 0, 1$ become $\theta = 0, \pi/2$, respectively. Hence

$$J = \int_0^{\pi/2} \sin^2 \theta \, d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{4}.$$

Here the identity $\sin^2 \theta = (1 - \cos 2\theta)/2$ makes the endpoint evaluation immediate, because the sine term vanishes at both limits.

Step 3: Conclusion Substituting the value of J ,

$$\boxed{\int_0^1 x \sin^{-1} x \, dx = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}}.$$

Quick Tip to Solve: Differentiate the inverse sine in integration by parts, then use $x = \sin \theta$ and change both definite limits immediately.

31(a). Find: $\int \sqrt{\frac{x+2}{x-2}} \, dx$.

[3 marks]

Solution:

Concept:

- Rationalising the radicand converts the integrand into a sum involving $x/\sqrt{x^2-a^2}$ and $1/\sqrt{x^2-a^2}$, whose antiderivatives are standard.

Quick Tip to Solve: Rewrite the positive-interval integrand as $(x+2)/\sqrt{x^2-4}$; retain the opposite sign when working on $x < -2$.

- The formulae are $\int x/\sqrt{x^2 - a^2} dx = \sqrt{x^2 - a^2}$ and $\int dx/\sqrt{x^2 - a^2} = \ln|x + \sqrt{x^2 - a^2}| + C$, for $|x| > a$.

Step 1: Rewrite the integrand on the positive interval The radicand is nonnegative for $x \leq -2$ or $x > 2$; the point $x = 2$ is excluded. First work on the open interval $x > 2$. Since both $x + 2$ and $x - 2$ are positive,

$$\sqrt{\frac{x+2}{x-2}} = \frac{x+2}{\sqrt{(x-2)(x+2)}} = \frac{x+2}{\sqrt{x^2-4}}.$$

Thus the required integral separates as

$$I = \int \frac{x}{\sqrt{x^2-4}} dx + 2 \int \frac{dx}{\sqrt{x^2-4}}.$$

Splitting the numerator is useful because the first term contains the derivative of the expression under the square root.

Step 2: Integrate the two standard forms For the first integral take $u = x^2 - 4$, giving $du = 2x dx$, and obtain $\sqrt{x^2 - 4}$. For the second integral the logarithmic formula with $a = 2$ gives $\ln|x + \sqrt{x^2 - 4}|$. Consequently, on $x > 2$,

$$I = \sqrt{x^2 - 4} + 2 \ln|x + \sqrt{x^2 - 4}| + C.$$

Differentiating this expression produces $(x + 2)/\sqrt{x^2 - 4}$, which agrees with the original positive square root on this interval.

Step 3: Account for the negative interval On $x < -2$, the numerator $x + 2$ is negative, so the positive square root instead equals $-(x + 2)/\sqrt{x^2 - 4}$. The antiderivative is therefore the negative of the expression obtained above, with an independent constant on this disconnected interval. This sign adjustment follows from $\sqrt{(x+2)^2} = |x+2|$; dropping the absolute value would give the wrong derivative for negative x .

Step 4: Conclusion On $x > 2$, the standard answer is

$$\boxed{\sqrt{x^2 - 4} + 2 \ln|x + \sqrt{x^2 - 4}| + C}.$$

On $x < -2$, the answer is

$$\boxed{-\sqrt{x^2 - 4} - 2 \ln|x + \sqrt{x^2 - 4}| + C}.$$

Quick Tip to Solve: Rewrite the positive-interval integrand as $(x + 2)/\sqrt{x^2 - 4}$; retain the opposite sign when working on $x < -2$.

31(b). Find: $\int \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx$.

[3 marks]

Solution:

Concept:

- Partial fractions separate a rational function into simpler terms. Here the denominator has distinct quadratic factors $x^2 + 9$ and $x^2 + 16$, and the even numerator permits constant numerators.
- The standard integral $\int dx/(x^2 + a^2) = (1/a) \tan^{-1}(x/a) + C$, with $a > 0$, supplies the antiderivatives after decomposition.

Step 1: Determine the partial-fraction constants Write

$$\frac{x^2}{(x^2 + 9)(x^2 + 16)} = \frac{A}{x^2 + 9} + \frac{B}{x^2 + 16}.$$

Multiplying by the common denominator gives the polynomial identity

$$x^2 = A(x^2 + 16) + B(x^2 + 9) = (A + B)x^2 + 16A + 9B.$$

Comparing the coefficients of x^2 and the constant terms yields $A + B = 1$ and $16A + 9B = 0$. Substituting $B = 1 - A$ in the second equation gives $7A + 9 = 0$. Hence $A = -9/7$ and $B = 16/7$. The negative coefficient belongs to the factor $x^2 + 9$, which is essential when integrating.

Step 2: Integrate each quadratic denominator Using the decomposition,

$$\begin{aligned} I &= -\frac{9}{7} \int \frac{dx}{x^2 + 3^2} + \frac{16}{7} \int \frac{dx}{x^2 + 4^2} \\ &= -\frac{9}{7} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \frac{16}{7} \cdot \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + C. \end{aligned}$$

The scale factors $1/3$ and $1/4$ arise from the chain rule for the inverse tangent. As a check, differentiating the two terms reproduces $-9/[7(x^2 + 9)] + 16/[7(x^2 + 16)]$, whose common numerator is $7x^2$, so it reduces to the given integrand.

Step 3: Conclusion Therefore,

$$I = \frac{4}{7} \tan^{-1}\left(\frac{x}{4}\right) - \frac{3}{7} \tan^{-1}\left(\frac{x}{3}\right) + C.$$

Quick Tip to Solve: Find the constant numerators before integrating, then divide each coefficient by the square root of its denominator's constant term.

Section D (20 marks)

This section comprises Long Answer (LA) type questions of 5 marks each.

32. If $x = 3 \sin t - \sin 3t$, $y = 3 \cos t - \cos 3t$, find $\frac{dy}{dx}$ and prove that $\frac{d^2y}{dx^2} = \frac{-\operatorname{cosec}^3 2t \cdot \operatorname{cosec} t}{3}$. [5 marks]

Solution:

Concept:

- For parametric functions $x = x(t)$, $y = y(t)$, the slope is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- The second derivative requires another application of the chain rule: $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) / \frac{dx}{dt}$.

Step 1: Differentiate with respect to the parameter Apply the chain rule to the terms containing $3t$. Each such differentiation contributes a factor of 3, giving

$$\begin{aligned}\frac{dx}{dt} &= 3 \cos t - 3 \cos 3t = 3(\cos t - \cos 3t), \\ \frac{dy}{dt} &= -3 \sin t + 3 \sin 3t = 3(\sin 3t - \sin t).\end{aligned}$$

Step 2: Simplify the ratio using trigonometric identities Substituting the angle pairs into the sum-to-product identities gives

$$\begin{aligned}\cos t - \cos 3t &= -2 \sin 2t \sin(-t) = 2 \sin 2t \sin t, \\ \sin 3t - \sin t &= 2 \cos 2t \sin t.\end{aligned}$$

Hence $dx/dt = 6 \sin 2t \sin t$ and $dy/dt = 6 \cos 2t \sin t$. On any interval where $dx/dt \neq 0$, both denominator factors are nonzero, so cancellation is valid:

$$\frac{dy}{dx} = \frac{6 \cos 2t \sin t}{6 \sin 2t \sin t} = \frac{\cos 2t}{\sin 2t} = \cot 2t.$$

Step 3: Find the second derivative First differentiate the slope with respect to t , remembering the factor 2 from the angle $2t$:

$$\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} (\cot 2t) = -2 \operatorname{cosec}^2 2t.$$

To convert this derivative into differentiation with respect to x , divide by dx/dt , not by dy/dt :

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-2 \operatorname{cosec}^2 2t}{6 \sin 2t \sin t} \\ &= -\frac{1}{3} \operatorname{cosec}^2 2t \cdot \operatorname{cosec} 2t \cdot \operatorname{cosec} t.\end{aligned}$$

Step 4: Conclusion Therefore, wherever $dx/dt \neq 0$,

$$\boxed{\frac{dy}{dx} = \cot 2t}, \quad \boxed{\frac{d^2y}{dx^2} = -\frac{\operatorname{cosec}^3 2t \operatorname{cosec} t}{3}}.$$

Quick Tip to Solve: Reduce the first derivative to $\cot 2t$ before differentiating again. For a parametric second derivative, always divide the derivative of the slope by dx/dt .

This proves the required identity.

Quick Tip to Solve: Reduce the first derivative to $\cot 2t$ before differentiating again. For a parametric second derivative, always divide the derivative of the slope by dx/dt .

33. Prove that the line through points $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through points $C(3, 9, 4)$ and $D(-4, 4, 4)$. Hence, write the equation of line passing through the point of intersection of lines AB and CD as well as origin. [5 marks]

Solution:

Concept:

- A line through position vectors $\vec{a}$ and $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\lambda \in \mathbb{R}$.
- Two lines intersect if independent parameters give the same three coordinates on both. A line through the origin and $P(p, q, r)$ has direction ratios p, q, r .

Step 1: Write parametric equations for both lines Subtracting coordinates gives the direction vectors:

$$\begin{aligned}\overrightarrow{AB} &= (4 - 0, 5 + 1, 1 + 1) = (4, 6, 2), \\ \overrightarrow{CD} &= (-4 - 3, 4 - 9, 4 - 4) = (-7, -5, 0).\end{aligned}$$

With independent real parameters λ and μ , the line equations are

$$\begin{aligned}AB: \quad (x, y, z) &= (4\lambda, -1 + 6\lambda, -1 + 2\lambda), \\ CD: \quad (x, y, z) &= (3 - 7\mu, 9 - 5\mu, 4).\end{aligned}$$

Step 2: Solve and check the coordinate equations For a common point, equate the coordinates:

$$4\lambda = 3 - 7\mu, \quad -1 + 6\lambda = 9 - 5\mu, \quad -1 + 2\lambda = 4.$$

The third equation fixes λ ; substitute into the first to obtain μ :

$$\begin{aligned}2\lambda = 5 &\Rightarrow \lambda = \frac{5}{2}, \\ 10 = 3 - 7\mu &\Rightarrow \mu = -1.\end{aligned}$$

The unused second equation must also hold. Its two sides are

$$-1 + 6\left(\frac{5}{2}\right) = 14, \quad 9 - 5(-1) = 14.$$

All three equations are consistent, so the lines intersect. Substituting $\lambda = 5/2$ into AB gives $P(10, 14, 4)$; substituting $\mu = -1$ into CD also gives $(3 + 7, 9 + 5, 4) = (10, 14, 4)$.

Step 3: Form the line through the intersection and origin The vector joining $O(0, 0, 0)$ to P is

$$\overrightarrow{OP} = 10\hat{i} + 14\hat{j} + 4\hat{k} = 2(5\hat{i} + 7\hat{j} + 2\hat{k}).$$

Quick Tip to Solve: Start with $z = 4$ on CD to fix the parameter on AB . Check the remaining coordinate before claiming an intersection in space.

Cancelling the common factor gives direction ratios $(5, 7, 2)$. Hence the vector equation is $\vec{r} = s(5\hat{i} + 7\hat{j} + 2\hat{k})$, $s \in \mathbb{R}$. It contains O at $s = 0$ and P at $s = 2$.

Step 4: Conclusion The intersection is $P(10, 14, 4)$, and the required line is

$$\frac{x}{5} = \frac{y}{7} = \frac{z}{2}.$$

Quick Tip to Solve: Start with $z = 4$ on CD to fix the parameter on AB . Check the remaining coordinate before claiming an intersection in space.

34(a). A relation R is defined on $\mathbb{Z}$, the set of integers, as

$$R = \{(x, y) : |x - y| \text{ is divisible by a prime number } p, x, y \in \mathbb{Z}\}.$$

Check whether R is an equivalence relation or not.

[5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive. Each property must hold for every element to which it applies.
- For the prime number p appearing in the definition, divisibility means an integer multiple: $p \mid n$ if $n = kp$ for some $k \in \mathbb{Z}$. In particular, $p \mid 0$.
- Since a number and its negative have the same divisors, $p \mid |x - y|$ is equivalent to $p \mid (x - y)$. Thus the same fixed p can be used in all three property checks.

Step 1: Verify reflexivity Let x be any integer. The difference of x from itself is zero, so

$$|x - x| = 0 = 0 \cdot p.$$

Since the multiplier 0 is an integer, zero is divisible by p . Therefore $(x, x) \in R$ for every $x \in \mathbb{Z}$, which proves that R is reflexive. Divisibility does not require the multiple to be positive.

Step 2: Verify symmetry Let $x, y \in \mathbb{Z}$ and suppose $(x, y) \in R$. By the definition of the relation, $p \mid |x - y|$. But absolute differences satisfy

$$|y - x| = |-(x - y)| = |x - y|.$$

Consequently $p \mid |y - x|$, giving $(y, x) \in R$. This implication holds for every pair in R , so the relation is symmetric.

Step 3: Verify transitivity using signed differences Let $x, y, z \in \mathbb{Z}$, with $(x, y) \in R$ and $(y, z) \in R$. Then $p \mid |x - y|$ and $p \mid |y - z|$. Removing the absolute values only changes possible signs, so there are integers m, n such that

$$x - y = mp, \quad y - z = np.$$

Quick Tip to Solve: Replace divisibility of the absolute difference by divisibility of the signed difference; then transitivity follows immediately by adding the two integer multiples of the same prime p .

Adding these equalities cancels y :

$$x - z = (x - y) + (y - z) = (m + n)p.$$

The number $m + n$ is an integer; hence $p \mid (x - z)$, and therefore $p \mid |x - z|$. It follows that $(x, z) \in R$, proving transitivity. This argument uses signed differences because absolute differences cannot in general be added as equalities.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive on $\mathbb{Z}$. Hence R is an equivalence relation. It relates precisely the integers having the same remainder upon division by p .

Quick Tip to Solve: Replace divisibility of the absolute difference by divisibility of the signed difference; then transitivity follows immediately by adding the two integer multiples of the same prime p .

OR

34(b). A function $f : \mathbb{R} - \left\{\frac{3}{5}\right\} \rightarrow \mathbb{R} - \left\{\frac{3}{5}\right\}$ is defined as $f(x) = \frac{3x+2}{5x-3}$. Show that f is one-one and onto. [5 marks]

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ forces $a = b$ for all elements a, b of its domain.
- A function is onto if every element y of its stated codomain has a preimage x in its domain satisfying $f(x) = y$.
- For a rational function, check all excluded denominator values before cross multiplication or constructing a preimage. Here both domain and codomain are $A = \mathbb{R} \setminus \{3/5\}$.

Step 1: Check that the function takes values in the stated codomain For every $x \in A$, the denominator $5x - 3$ is nonzero. Also $f(x)$ cannot equal $3/5$, since that equality would imply

$$\frac{3x+2}{5x-3} = \frac{3}{5} \implies 15x+10 = 15x-9,$$

which is impossible. Hence the formula gives a real value in A for every input in A , so the stated mapping is well-defined.

Step 2: Prove that the function is one-one Take any $a, b \in A$ and suppose $f(a) = f(b)$. Both denominators are nonzero, so cross multiplication gives

$$\begin{aligned}\frac{3a+2}{5a-3} &= \frac{3b+2}{5b-3}, \\ (3a+2)(5b-3) &= (3b+2)(5a-3), \\ 15ab-9a+10b-6 &= 15ab-9b+10a-6, \\ 19b &= 19a.\end{aligned}$$

Quick Tip to Solve: When solving for the preimage of a rational function, check that the resulting input avoids the excluded domain value. Here the preimage formula is the same as the original function.

Thus $a = b$. Since the two inputs were arbitrary, equal outputs always have equal inputs; therefore f is one-one.

Step 3: Find and verify a preimage for every codomain element Let $y \in A$ be arbitrary. Solving $y = (3x + 2)/(5x - 3)$ for x gives

$$5xy - 3y = 3x + 2, \quad x(5y - 3) = 3y + 2, \quad x = \frac{3y + 2}{5y - 3}.$$

This is a real number because $y \neq 3/5$. Moreover,

$$x - \frac{3}{5} = \frac{5(3y + 2) - 3(5y - 3)}{5(5y - 3)} = \frac{19}{5(5y - 3)} \neq 0.$$

Hence this proposed preimage belongs to the domain A . To verify its image, substitute it into the function:

$$3x + 2 = \frac{19y}{5y - 3}, \quad 5x - 3 = \frac{19}{5y - 3}, \quad f(x) = \frac{19y/(5y - 3)}{19/(5y - 3)} = y.$$

Thus every allowed y has an allowed preimage, proving that f is onto the stated codomain.

Step 4: Conclusion The function $f(x) = \frac{3x + 2}{5x - 3}$ is both one-one and onto from $\mathbb{R} \setminus \{3/5\}$ to itself, and therefore is bijective.

Quick Tip to Solve: When solving for the preimage of a rational function, check that the resulting input avoids the excluded domain value. Here the preimage formula is the same as the original function.

35(a). If $A = \begin{pmatrix} 0 & 2 & 1 \\ -2 & -1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$, find A^{-1} and use it to solve the following system of equations:
 $-2y + z = 7, 2x - y - z = 8, x - 2y = 10.$ [5 marks]

Solution:

Concept:

- For an invertible square matrix, $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- The coefficient matrix of the given equations is A^T . Use $(A^T)^{-1} = (A^{-1})^T$ to solve $A^T X = B$, where $X = (x, y, z)^T$ and $B = (7, 8, 10)^T$.

Step 1: Evaluate the determinant Expand along the first row, taking care of the alternating signs:

$$\begin{aligned} |A| &= 0 - 2 \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} \\ &= -2(2) + (2 + 1) = -1. \end{aligned}$$

The determinant is nonzero, so the inverse exists. This also ensures that the given linear system has a

Quick Tip to Solve: Compare the coefficient matrix with both A and A^T before multiplying: here the equations use the columns of A , so transpose the computed inverse.

unique solution because its coefficient matrix has the same determinant.

Step 2: Calculate the cofactors and the inverse Delete each entry's row and column, take the remaining two-by-two determinant, and multiply by $(-1)^{i+j}$. For example,

$$C_{11} = (-1)(0) - (-2)(-1) = -2, \quad C_{12} = -\{(-2)(0) - (-2)(1)\} = -2.$$

The complete cofactor matrix and its transpose are

$$C = \begin{pmatrix} -2 & -2 & 3 \\ -1 & -1 & 2 \\ -3 & -2 & 4 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -2 \\ 3 & 2 & 4 \end{pmatrix}.$$

Dividing the adjugate by $|A| = -1$ gives

$$A^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ -3 & -2 & -4 \end{pmatrix}.$$

Step 3: Express the equations using the transpose Arrange each equation in the order x, y, z . The coefficient matrix is

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & -1 & -1 \\ 1 & -2 & 0 \end{pmatrix} = A^T.$$

Thus the equation to solve is $A^T X = B$. Multiplying on the left by $(A^{-1})^T$, rather than by A^{-1} , gives

$$\begin{aligned} X &= (A^{-1})^T B \\ &= \begin{pmatrix} 2 & 2 & -3 \\ 1 & 1 & -2 \\ 3 & 2 & -4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} = \begin{pmatrix} 14 + 16 - 30 \\ 7 + 8 - 20 \\ 21 + 16 - 40 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}. \end{aligned}$$

Step 4: Verify all three equations Substitution checks both the signs and the use of the transpose:

$$\begin{aligned} -2(-5) + (-3) &= 7, \\ 2(0) - (-5) - (-3) &= 8, \\ 0 - 2(-5) &= 10. \end{aligned}$$

All three left-hand sides equal their specified right-hand sides.

Step 5: Conclusion

$$A^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ -3 & -2 & -4 \end{pmatrix}, \quad \boxed{x = 0, \quad y = -5, \quad z = -3}.$$

Quick Tip to Solve: Compare the coefficient matrix with both A and A^T before multiplying: here the equations use the columns of A , so transpose the computed inverse.

35(b). If $\begin{pmatrix} 3 & -1 & \sin 3x \\ -7 & 4 & \cos 2x \\ -11 & 7 & 2 \end{pmatrix}$ is a singular matrix, then find all values of x where $x \in \left[0, \frac{\pi}{2}\right]$. [5 marks]

Solution:

Concept:

- A square matrix is singular exactly when its determinant is zero. Expanding a determinant along a row uses the cofactor signs $+, -, +$.
- Use $1 - \cos 2x = 2 \sin^2 x$ and $\sin 3x = 3 \sin x - 4 \sin^3 x$ to express the condition entirely in $\sin x$.
- On $0 \leq x \leq \pi/2$, the value of $\sin x$ lies in $[0, 1]$ and increases from 0 to 1. Every algebraic root must satisfy this restriction.

Step 1: Set the determinant equal to zero Write $s = \sin 3x$ and $c = \cos 2x$ temporarily. Expanding along the first row gives

$$\begin{aligned} 0 &= 3 \begin{vmatrix} 4 & c \\ 7 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -7 & c \\ -11 & 2 \end{vmatrix} + s \begin{vmatrix} -7 & 4 \\ -11 & 7 \end{vmatrix} \\ &= 3(8 - 7c) + (-14 + 11c) + s(-49 + 44) \\ &= 10 - 10c - 5s. \end{aligned}$$

The middle contribution is added because the matrix entry is -1 and its cofactor sign is negative. The last minor is -5 , so its contribution has a negative sign.

Step 2: Convert the condition into a polynomial in sine Divide the determinant equation by 5 and restore the trigonometric expressions:

$$2 - 2 \cos 2x - \sin 3x = 0.$$

By the double-angle and triple-angle identities, this becomes

$$\begin{aligned} 4 \sin^2 x - (3 \sin x - 4 \sin^3 x) &= 0, \\ \sin x (4 \sin^2 x + 4 \sin x - 3) &= 0, \\ \sin x (2 \sin x - 1)(2 \sin x + 3) &= 0. \end{aligned}$$

Factoring retains the possibility $\sin x = 0$; dividing through by $\sin x$ would lose a permitted endpoint solution.

Step 3: Apply the stated interval The algebraic possibilities are

$$\sin x = 0, \quad \sin x = \frac{1}{2}, \quad \sin x = -\frac{3}{2}.$$

The last value is outside the range of the sine function and must be rejected. Within $[0, \pi/2]$, sine is one-to-one, so the first two values correspond respectively to $x = 0$ and $x = \pi/6$. There are no further angles in the interval, and both endpoints have been considered: $x = \pi/2$ has sine 1 and does not satisfy any of the factors.

Quick Tip to Solve: Factor out $\sin x$ instead of dividing by it: this preserves $x = 0$. Then use $0 \leq \sin x \leq 1$ to discard impossible roots immediately.

Step 4: Check the retained angles in the determinant condition At $x = 0$, $\cos 2x = 1$ and $\sin 3x = 0$, giving $10 - 10(1) - 5(0) = 0$. At $x = \pi/6$, $\cos 2x = 1/2$ and $\sin 3x = 1$, giving $10 - 10(1/2) - 5(1) = 0$. Thus each retained angle makes the original matrix singular; the substitutions introduced no extraneous roots.

Step 5: Conclusion All the required values in the given interval are $x = 0$ or $x = \frac{\pi}{6}$.

Quick Tip to Solve: Factor out $\sin x$ instead of dividing by it: this preserves $x = 0$. Then use $0 \leq \sin x \leq 1$ to discard impossible roots immediately.

Section E (12 marks)

This section comprises of 3 case study based questions of 4 marks each.

36(i). In an online jackpot, there is one first prize of ₹3,00,000, two second prizes of ₹2,00,000 each and three third prizes of ₹50,000 each. A total of 1,00,000 jackpot tickets each costing ₹100 were sold thereby raising a fund of ₹1,00,00,000. Rohan bought one ticket.

What are the possible amounts, the person can win?

[1 mark]

Solution:

Concept:

- The possible values of a prize variable are the distinct amounts payable to a ticket holder.
- A ticket may win a listed prize or may be a non-winning ticket, for which the amount received is zero.

Step 1: List the positive prize amounts The first prize pays Rs.3,00,000, each second prize pays Rs.2,00,000, and each third prize pays Rs.50,000. Repeated prizes change the number of winning tickets, not the distinct prize amounts.

Step 2: Include the non-winning outcome Since only six of the 1,00,000 tickets receive prizes, a ticket can also receive Rs.0. The ticket price is an expense; it is not subtracted when listing the amounts won.

Step 3: Conclusion The possible prize amounts are Rs.50,000, Rs.2,00,000 and Rs.3,00,000; including no prize, the possible receipts are $\{0, 50,000, 2,00,000, 3,00,000\}$ rupees.

Quick Tip to Solve: List each distinct prize once, and include zero if listing all possible ticket receipts.

36(ii)(a). What is the probability that the person wins atleast ₹2,00,000?

[2 marks]

Solution:

Concept:

- For equally likely tickets, $P(E) = n(E)/n(S)$, where $n(E)$ is the number of favourable tickets and $n(S)$ is the total number sold.

Quick Tip to Solve: For an “at least” prize threshold, count the prizes equal to the threshold as well as all larger prizes.

- At least Rs.2,00,000 includes both an amount equal to this threshold and any larger amount.

Step 1: Identify all qualifying prizes Let E be the event that Rohan's ticket wins at least Rs.2,00,000. The one first prize of Rs.3,00,000 and the two second prizes of Rs.2,00,000 each qualify. The three third prizes of Rs.50,000 each are below the required amount and must be excluded. Hence,

$$n(E) = 1 + 2 = 3.$$

Step 2: Divide by the total number of tickets Rohan holds one of the 1,00,000 equally likely tickets. Thus the probability is the fraction of tickets attached to qualifying prizes:

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{1,00,000} = 0.00003.$$

Prize values determine which tickets qualify, but probabilities are calculated by counting tickets, not by dividing prize money by the total fund.

Step 3: Conclusion The probability of winning at least Rs.2,00,000 is $\frac{3}{1,00,000}$.

Quick Tip to Solve: For an “at least” prize threshold, count the prizes equal to the threshold as well as all larger prizes.

OR

36(ii)(b). What is the probability that the person does not win any amount?

[2 marks]

Solution:

Concept:

- For equally likely tickets, the probability of an event equals its favourable ticket count divided by the total ticket count.
- If W is the event of winning any prize, its complement W' is winning no prize, and $P(W') = 1 - P(W)$.

Step 1: Count all prize-winning tickets A winning ticket can receive the first prize, a second prize or a third prize. There is one first prize, two second prizes and three third prizes, so the total number of prize-winning tickets is

$$n(W) = 1 + 2 + 3 = 6.$$

With 1,00,000 tickets sold and one ticket held by Rohan, each ticket has the same chance of selection. Therefore $P(W) = 6/1,00,000$.

Step 2: Use the complementary event Every ticket either wins one of these prizes or wins nothing. Subtract the six winning tickets from the total to count the non-winning tickets:

$$n(W') = 1,00,000 - 6 = 99,994.$$

Quick Tip to Solve: Count all six winning tickets first; subtracting their probability from one gives the no-prize probability immediately.

Hence,

$$P(W') = 1 - \frac{6}{1,00,000} = \frac{99,994}{1,00,000} = 0.99994.$$

The Rs.100 ticket cost does not alter the event of receiving no prize.

Step 3: Conclusion The probability that Rohan wins no amount is $\frac{99,994}{1,00,000}$, equivalently $\frac{49,997}{50,000}$.

Quick Tip to Solve: Count all six winning tickets first; subtracting their probability from one gives the no-prize probability immediately.

36(iii). In another jackpot, Rohan also bought a ticket having a prize money of ₹5,00,000. The chances of winning the jackpot are 1 in 1,00,000. Find the probability that on exactly one of tickets he wins the jackpot. [1 mark]

Solution:

Concept:

- For independent events A and B , $P(A \cap B) = P(A)P(B)$.
- Exactly one win consists of the disjoint events $A \cap B'$ and $A' \cap B$; hence its probability is $P(A)P(B') + P(A')P(B)$.

Step 1: Write the two winning probabilities Let A mean winning a prize in the first jackpot and B mean winning the second jackpot. The first has $1 + 2 + 3 = 6$ winning tickets, so

$$P(A) = \frac{6}{1,00,000}, \quad P(B) = \frac{1}{1,00,000}.$$

The two separate jackpot draws are independent.

Step 2: Add the two exclusive cases Rohan must win the first and lose the second, or lose the first and win the second. Thus

$$\begin{aligned} P(\text{exactly one win}) &= \frac{6}{100000} \frac{99999}{100000} + \frac{99994}{100000} \frac{1}{100000} \\ &= \frac{599994 + 99994}{10^{10}} = \frac{699988}{10^{10}}. \end{aligned}$$

Step 3: Conclusion The required probability is $\frac{699988}{10^{10}} = 0.0000699988$.

Quick Tip to Solve: Use $p(1 - q) + (1 - p)q$ for exactly one win; simply adding $p + q$ also counts the outcome where both tickets win.

37(i). Roundabouts are often made on busy roads to ease the traffic and avoid red lights. One such round-about is made such that equation representing its boundary is given by $C_1 : x^2 + y^2 = 64$. There is a circular pond with a fountain in the middle of the roundabout whose equation is given by $C_2 : x^2 + y^2 = 4$.

Represent the given equations C_1 and C_2 with the help of a diagram.

[1 mark]

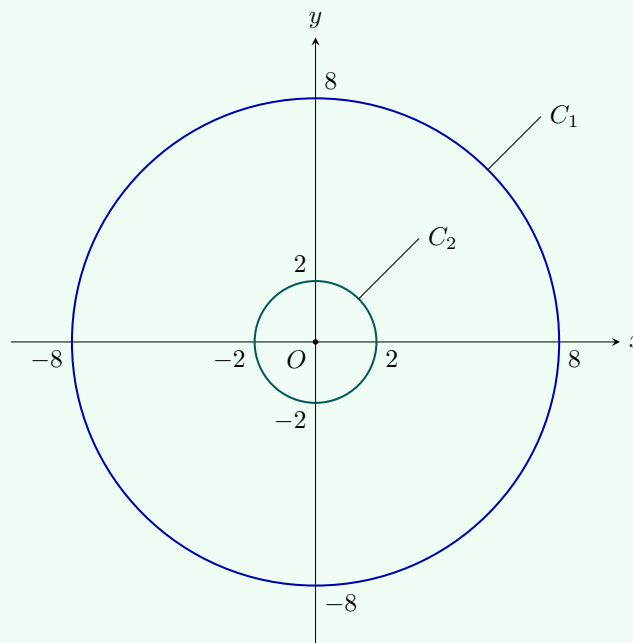
Solution:

Concept:

- The equation $x^2 + y^2 = r^2$ represents a circle centred at the origin with radius r .
- Circles with the same centre are concentric; their axis intercepts occur at distances equal to their respective radii.

Step 1: Identify the centres and radii Comparing $C_1 : x^2 + y^2 = 64$ with the standard equation gives centre $O(0, 0)$ and radius 8. Similarly, $C_2 : x^2 + y^2 = 4$ has the same centre and radius 2.

Step 2: Draw the concentric circles Draw coordinate axes through O . Draw the outer circle through $(\pm 8, 0)$ and $(0, \pm 8)$, and the inner circle through $(\pm 2, 0)$ and $(0, \pm 2)$. The inner radius is one quarter of the outer radius, as shown.



Concentric circles representing the roundabout boundary and the circular pond.

Step 3: Conclusion The diagram represents C_1 and C_2 as concentric circles centred at $O(0, 0)$, with radii 8 and 2 units respectively.

Quick Tip to Solve: Take the square roots of 64 and 4 before drawing: these give radii 8 and 2, not 64 and 4.

37(ii). Express y as a function of x , ($y = f(x)$), for both C_1 and C_2 .

[1 mark]

Solution:

Concept:

- From $x^2 + y^2 = r^2$, solving for y gives the two branches $y = \pm\sqrt{r^2 - x^2}$.
- The positive branch is the upper semicircle and the negative branch is the lower semicircle. Real values require $-r \leq x \leq r$.

Step 1: Express the branches of the outer circle For C_1 , isolate y^2 and take both square roots:

$$y^2 = 64 - x^2 \Rightarrow y = \pm\sqrt{64 - x^2}, \quad -8 \leq x \leq 8.$$

Step 2: Express the branches of the inner circle Similarly, the equation of C_2 gives

$$y^2 = 4 - x^2 \Rightarrow y = \pm\sqrt{4 - x^2}, \quad -2 \leq x \leq 2.$$

A whole circle is not a single-valued function of x ; each choice of sign defines one semicircular function. In particular, choose the positive sign when working with the upper halves.

Step 3: Conclusion C_1 has branches $y = \pm\sqrt{64 - x^2}$, and C_2 has branches $y = \pm\sqrt{4 - x^2}$, on their respective domains $[-8, 8]$ and $[-2, 2]$.

Quick Tip to Solve: Keep both signs for the complete circle; use only the positive square root for the upper semicircle.

37(iii)(a). Using integration find the area of region covered by the roundabout.

[2 marks]

Solution:

Concept:

- The area enclosed by the roundabout boundary $x^2 + y^2 = 64$ is the area of the complete disk of radius 8, including its central pond.
- Symmetry about both coordinate axes gives four equal quadrants. In the first quadrant, the area under $y = \sqrt{64 - x^2}$ is $\int_0^8 y \, dx$.

Step 1: Set up the area integral The upper semicircle is $y = \sqrt{64 - x^2}$, and its first-quadrant arc extends from $x = 0$ to $x = 8$. A vertical strip has area $y \, dx$, so the entire enclosed area is

$$A = 4 \int_0^8 \sqrt{64 - x^2} \, dx.$$

The question asks for the region covered by the roundabout, rather than only the region between its outer boundary and the pond.

Step 2: Evaluate using a trigonometric substitution Put $x = 8 \sin \theta$. Then $dx = 8 \cos \theta \, d\theta$, and the

Quick Tip to Solve: Integrate over one quadrant and multiply by four; use the outer radius 8 for the complete roundabout.

limits become 0 and $\pi/2$. Throughout this interval, $\sqrt{64 - x^2} = 8 \cos \theta$. Hence

$$\begin{aligned} A &= 256 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= 128 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}. \end{aligned}$$

The sine term vanishes at both limits, leaving the contribution from θ .

Step 3: Conclusion The area covered by the roundabout is 64π square units.

Quick Tip to Solve: Integrate over one quadrant and multiply by four; use the outer radius 8 for the complete roundabout.

OR

37(iii)(b). Using integration, find the area of region covered by circular pond.

[2 marks]

Solution:

Concept:

- The pond boundary $x^2 + y^2 = 4$ is a circle centred at the origin with radius 2. Its upper half is $y = \sqrt{4 - x^2}$.
- Definite integration adds vertical strips to obtain area. Symmetry allows the first-quadrant integral to be multiplied by four.

Step 1: Express the pond area as an integral In the first quadrant, the pond extends from $x = 0$ to its intercept $x = 2$. The height of a strip above the x -axis is $\sqrt{4 - x^2}$, so the required total area is

$$A = 4 \int_0^2 \sqrt{4 - x^2} \, dx.$$

Only the smaller circle is used here: the outer roundabout boundary does not change the area enclosed by the pond.

Step 2: Substitute and integrate Take $x = 2 \sin \theta$, giving $dx = 2 \cos \theta \, d\theta$. The limits $x = 0, 2$ correspond to $\theta = 0, \pi/2$, where the square root becomes $2 \cos \theta$. Thus

$$\begin{aligned} A &= 16 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= 8 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}. \end{aligned}$$

Evaluate the upper and lower limits; both sine terms are zero. This calculation obtains the full disk area directly by integration.

Step 3: Conclusion The circular pond covers 4π square units.

Quick Tip to Solve: For the pond use $x^2 + y^2 = 4$, so the integration limit is the radius 2, not 4.

38(i). An online delivery company in a city has 5000 subscribers and collects annual subscription fees of ₹300 per subscriber for unlimited free deliveries. The company wishes to increase the annual subscription fee. It is predicted that, for every increase of ₹1, ten subscribers will discontinue. Assume that the company increased the annual fee by ₹ x .

How many subscribers will discontinue after an increase of ₹ x in annual fee?

[1 mark]

Solution:

Concept:

- With a constant loss of ten subscribers for each rupee added to the fee, subscriber loss is directly proportional to the fee increase.
- If $L(x)$ denotes the number leaving after an increase of x rupees, then $L(x) = (\text{loss per rupee})x$.

Step 1: Identify the change being measured The increase is x rupees, not the new total annual fee $300 + x$. For each unit of this increase, ten subscribers discontinue.

Step 2: Apply the constant loss rate Multiply the increase by the number leaving for each rupee:

$$L(x) = 10 \times x.$$

The initial 5000 subscribers would be needed to calculate those remaining, but the question asks only for those who discontinue.

Step 3: Conclusion $10x$ subscribers will discontinue.

Quick Tip to Solve: Multiply x by ten for subscribers leaving; subtract this loss from 5000 only when asked for subscribers remaining.

38(ii). If $R(x)$ denotes the total revenue collected after the increase of ₹ x in subscription fee, express $R(x)$ as a function of x .

[1 mark]

Solution:

Concept:

- Total subscription revenue equals the annual fee per subscriber multiplied by the number of subscribers paying that fee.
- An increase of x rupees adds x to the original fee and causes $10x$ of the original 5000 subscribers to leave.

Step 1: Write the new fee and subscriber count The revised annual fee is $300 + x$ rupees. After $10x$ subscribers discontinue, the paying subscriber count becomes $5000 - 10x$.

Step 2: Multiply the two factors Thus the annual revenue, measured in rupees, is

$$\begin{aligned} R(x) &= (300 + x)(5000 - 10x) \\ &= 1500000 + 5000x - 3000x - 10x^2. \end{aligned}$$

Both factors change with x : keeping the original subscriber count would ignore the predicted loss of customers.

Quick Tip to Solve: First write revenue as $(300 + x)(5000 - 10x)$; expanding afterward avoids losing the customer-loss term.

Step 3: Conclusion $R(x) = -10x^2 + 2000x + 1500000$, in rupees.

Quick Tip to Solve: First write revenue as $(300 + x)(5000 - 10x)$; expanding afterward avoids losing the customer-loss term.

38(iii)(a). Find the value of x for which $R(x)$ is maximum.

[2 marks]

Solution:

Concept:

- Revenue is $R(x) = (300 + x)(5000 - 10x) = -10x^2 + 2000x + 1500000$, where x is the increase in the annual fee, in rupees.
- An interior maximum of a differentiable function must satisfy $R'(x) = 0$. A negative second derivative establishes that the stationary point is a maximum.

Step 1: Find the stationary fee increase Differentiate the revenue polynomial with respect to the increase x :

$$R'(x) = -20x + 2000.$$

Set this derivative equal to zero to locate the point where revenue stops rising:

$$-20x + 2000 = 0 \implies x = 100.$$

The variable represents the amount added to the fee; it does not represent the full annual subscription.

Step 2: Confirm that the stationary point maximises revenue Differentiate once more:

$$R''(x) = -20 < 0.$$

The quadratic is concave down everywhere, so its unique stationary point is its global maximum. It is feasible: at this increase, the fee is 400 rupees and 4000 subscribers remain. As a check, writing $R(x) = 1600000 - 10(x - 100)^2$ shows that no other increase can give a greater revenue.

Step 3: Conclusion The revenue is maximum for $x = 100$, an increase of 100 rupees in the annual subscription fee.

Quick Tip to Solve: After solving $R'(x) = 0$, check $R''(x) < 0$; report the increase 100, not the revised fee 400.

OR

38(iii)(b). Find the sub-intervals of $(0, 5000)$ in which $R(x)$ is increasing and decreasing.

[2 marks]

Solution:

Concept:

- For the revenue polynomial $R(x) = -10x^2 + 2000x + 1500000$, the sign of $R'(x)$ determines its intervals of increase and decrease.

Quick Tip to Solve: Factor the derivative as $20(100 - x)$; its sign changes only at 100, so split the stated interval there.

- A positive derivative gives an increasing function, while a negative derivative gives a decreasing function. The interval is split at any point where the derivative vanishes.

Step 1: Differentiate and find the sign-change point The derivative of the revenue is

$$R'(x) = 2000 - 20x = 20(100 - x).$$

Its only zero is $x = 100$. Since this lies inside the specified interval $(0, 5000)$, test the derivative separately on $(0, 100)$ and $(100, 5000)$.

Step 2: Determine the sign on each sub-interval For $0 < x < 100$, we have $100 - x > 0$, so $R'(x) > 0$. For $100 < x < 5000$, we have $100 - x < 0$, so $R'(x) < 0$. At $x = 100$ itself the derivative is zero, and the change from positive to negative identifies a maximum. These signs describe the given polynomial throughout the requested interval. In the subscription model, a nonnegative subscriber count restricts feasible fee increases to $0 \leq x \leq 500$; this does not change the polynomial's signs on the printed interval.

Step 3: Conclusion On $(0, 5000)$, $R(x)$ is increasing on $(0, 100)$ and decreasing on $(100, 5000)$.

Quick Tip to Solve: Factor the derivative as $20(100 - x)$; its sign changes only at 100, so split the stated interval there.