

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-2-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. All questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1 to 18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36 to 38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in few questions in all the Sections except Section – **A**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

This section comprises 20 Multiple Choice Questions (MCQs) of 1 mark each.

1. A relation R on set $A = \{1, 2, 3\}$ defined as $R = \{(1, 1), (2, 2), (1, 2)\}$ is

[1 mark]

- (A) Reflexive only
- (B) Reflexive and Transitive
- (C) Symmetric and Transitive
- (D) Transitive only

Correct Answer: (D) Transitive only

Solution:

Concept:

- A relation on A is reflexive if every (a, a) , with $a \in A$, belongs to it; it is symmetric if every pair has its reversed pair.
- Transitivity requires that $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$.

Step 1: Test reflexivity and symmetry Here $3 \in A$, but $(3, 3) \notin R$, so the relation is not reflexive. Also $(1, 2) \in R$, whereas $(2, 1) \notin R$; therefore it is not symmetric. Options (A) and (B) require reflexivity, and option (C) requires symmetry, so all three fail.

Step 2: Check every possible composition The possible chains of pairs are $(1, 1), (1, 1), (1, 1), (1, 2)$, $(1, 2), (2, 2)$, and $(2, 2), (2, 2)$. Their required results are respectively $(1, 1)$, $(1, 2)$, $(1, 2)$, and $(2, 2)$, all of which belong to R . There are no chains involving 3, since no pair contains it.

Step 3: Conclusion The relation is transitive, but neither reflexive nor symmetric. Hence, the correct option is (D).

Quick Tip to Solve: First check for a missing diagonal pair and a missing reversed pair; then list only the chains that actually occur in the relation.

2. If A and B are square matrices of same order, then which of the following statements is/are always true?

(i) $(A + B)(A - B) = A^2 - B^2$

(ii) $AB = BA$

(iii) $(A + B)^2 = A^2 + AB + BA + B^2$

(iv) $AB = 0 \Rightarrow A = 0$ or $B = 0$

[1 mark]

(A) Only (i) and (iii)

(B) Only (ii) and (iii)

(C) Only (iii)

(D) Only (iii) and (iv)

Correct Answer: (C) Only (iii)

Solution:

Concept:

- Matrix multiplication is distributive and associative, but in general $AB \neq BA$.
- A zero matrix product need not have a zero factor; ordinary cancellation laws for numbers cannot be assumed for matrices.

Step 1: Expand the two algebraic expressions Preserving the order of the factors gives

$$(A + B)(A - B) = A^2 - AB + BA - B^2.$$

Thus (i) holds only when $AB = BA$, which is not guaranteed; statement (ii) is not universally valid either. On the other hand, distributivity always gives

$$(A + B)^2 = A^2 + AB + BA + B^2,$$

so (iii) is always true.

Step 2: Test the zero-product claim Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both matrices are nonzero, but $AB = 0$; this disproves (iv). Consequently options (A) and (B) each include an invalid general claim, and option (D) includes the false zero-product rule.

Step 3: Conclusion Only statement (iii) is always true. Hence, the correct option is (C).

Quick Tip to Solve: Expand products in their original order; never combine AB and BA unless commutativity is explicitly given.

3. If $A = \begin{pmatrix} 1 & a & b \\ -1 & 2 & c \\ 0 & 5 & 3 \end{pmatrix}$ is a symmetric matrix, then the value of $3a + b + c$ is

[1 mark]

- (A) 2
- (B) 6
- (C) 4
- (D) 0

Correct Answer: (A) 2

Solution:

Concept:

- A matrix is symmetric when $A^T = A$, so entries reflected across the main diagonal are equal.
- For a symmetric matrix, $a_{ij} = a_{ji}$; diagonal entries do not constrain the off-diagonal unknowns.

Step 1: Match the reflected entries Compare the first-row second-column entry with the second-row first-column entry, and similarly for the other two off-diagonal pairs:

$$a = -1, \quad b = 0, \quad c = 5.$$

The diagonal entries 1, 2, 3 remain unchanged under transposition and require no additional condition.

Step 2: Substitute into the requested expression Use the matched values without changing their signs:

$$3a + b + c = 3(-1) + 0 + 5 = -3 + 5 = 2.$$

The other listed values, 6, 4 and 0, do not agree with this sum. In particular, replacing a by 1 would ignore the negative entry opposite it.

Step 3: Conclusion The value of $3a + b + c$ is 2. Hence, the correct option is (A).

Quick Tip to Solve: Match the three off-diagonal pairs first; the diagonal values have no role in this particular calculation.

4. If $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ and $A + A' = I$, then the value of $x \in \left[0, \frac{\pi}{2}\right]$ is [1 mark]

- (A) 0
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

Correct Answer: (C) $\frac{\pi}{3}$

Solution:

Concept:

- The transpose exchanges rows and columns: the upper-right and lower-left entries of A are interchanged in A' .
- In $A + A' = I$, diagonal entries equal 1 and off-diagonal entries equal 0.

Step 1: Find the transpose and add The printed opposite signs on the sine entries are essential:

$$A' = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}.$$

Therefore

$$A + A' = \begin{pmatrix} 2\cos x & 0 \\ 0 & 2\cos x \end{pmatrix}.$$

The sine terms cancel automatically, so they impose no additional restriction.

Step 2: Use the diagonal condition and interval Comparison with I gives $2\cos x = 1$, or $\cos x = 1/2$. On $[0, \pi/2]$, cosine decreases from 1 to 0 and takes the value $1/2$ only at $\pi/3$. The other candidates $0, \pi/4, \pi/2$ give cosine values $1, 1/\sqrt{2}, 0$, respectively, and fail the equation.

Step 3: Conclusion The required value is $x = \frac{\pi}{3}$. Hence, the correct option is (C).

Quick Tip to Solve: For this rotation matrix, adding its transpose cancels sine terms; read the answer directly from $2\cos x = 1$.

5. For a square matrix A , $(3A)^{-1}$ equals

[1 mark]

(A) $3A^{-1}$

(B) $9A^{-1}$

(C) $\frac{1}{3}A^{-1}$

(D) $\frac{1}{9}A^{-1}$

Correct Answer: (C) $\frac{1}{3}A^{-1}$

Solution:

Concept:

- An inverse matrix satisfies $AA^{-1} = A^{-1}A = I$. The appearance of $(3A)^{-1}$ presupposes that A is invertible.
- For a nonzero scalar k , the inverse of kA is obtained by multiplying A^{-1} by $1/k$.

Step 1: Find the scalar needed in the inverse Let the proposed inverse be cA^{-1} , where c is a scalar. Multiplication gives

$$(3A)(cA^{-1}) = 3c AA^{-1} = 3cI.$$

For this product to be I , we require $3c = 1$, hence $c = 1/3$.

Step 2: Verify both products and reject the other factors The reverse product also works:

$$\left(\frac{1}{3}A^{-1}\right)(3A) = A^{-1}A = I.$$

The factors in options (A), (B), and (D) instead yield $9I$, $27I$, and $I/3$ when multiplied by $3A$. None is the identity, so none can be its inverse.

Step 3: Conclusion $(3A)^{-1} = \frac{1}{3}A^{-1}$. Hence, the correct option is (C).

Quick Tip to Solve: Check a scalar-matrix inverse by multiplying it with the original matrix; the scalar factors must multiply to 1.

6. If $\begin{vmatrix} -1 & -2 & 5 \\ -2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = -86$, then the sum of all possible values of a is

[1 mark]

- (A) 4
- (B) 5
- (C) -4
- (D) 9

Correct Answer: (C) -4

Solution:

Concept:

- Expand a determinant along a row using the cofactor sign pattern $+, -, +$.
- For a quadratic $pa^2 + qa + r = 0$, the sum of its roots is $-q/p$.

Step 1: Expand the determinant along its first row Using the printed entries,

$$\begin{aligned} D &= (-1)(2a^2 + 4) - (-2)(-4a) + 5(-8) \\ &= -2a^2 - 4 - 8a - 40 \\ &= -2a^2 - 8a - 44. \end{aligned}$$

In particular, the second minor is $(-2)(2a) - (-1)(0) = -4a$, and the third minor is $(-2)(4) - a(0) = -8$.

Step 2: Form the quadratic and find the sum Set the determinant equal to -86 :

$$-2a^2 - 8a - 44 = -86 \implies a^2 + 4a - 21 = 0.$$

This factors as $(a + 7)(a - 3) = 0$, so the possible values are -7 and 3 , with sum -4 . The positive choices 4 , 5 and 9 do not equal this sum; the negative sign follows directly from the coefficient of a .

Step 3: Conclusion The sum of all possible values of a is -4 . Hence, the correct option is (C).

Quick Tip to Solve: After expanding, use the negative coefficient-to-leading-coefficient ratio to find the root sum without solving the quadratic.

7. If $e^{-x} + e^{-y} = 2$, then $\frac{dy}{dx}$ is

[1 mark]

- (A) e^{x-y}
- (B) e^{y-x}
- (C) $-e^{x-y}$
- (D) $-e^{y-x}$

Correct Answer: (D) $-e^{y-x}$

Solution:

Concept:

- Implicit differentiation treats y as a function of x and applies the chain rule.
- The derivatives are $d(e^{-x})/dx = -e^{-x}$ and $d(e^{-y})/dx = -e^{-y}(dy/dx)$.

Step 1: Differentiate the printed equation Differentiating both sides of $e^{-x} + e^{-y} = 2$ with respect to x gives

$$-e^{-x} - e^{-y} \frac{dy}{dx} = 0.$$

The derivative of the constant 2 is zero, and the second term includes the chain-rule factor because y varies with x .

Step 2: Isolate the derivative and simplify Move the first term to the other side and divide by the positive quantity e^{-y} :

$$\frac{dy}{dx} = -\frac{e^{-x}}{e^{-y}} = -e^{y-x}.$$

Options (A) and (B) have the wrong sign; option (C) has the exponent reversed. Subtraction of exponents in this quotient gives $-x - (-y) = y - x$.

Step 3: Conclusion $\frac{dy}{dx} = -e^{y-x}$. Hence, the correct option is (D).

Quick Tip to Solve: Keep both chain-rule minus signs before isolating the derivative; in the exponential quotient subtract the denominator exponent.

8. For $f(x) = x + \frac{1}{x}$, ($x \neq 0$)

[1 mark]

- (A) local maximum value is 2
- (B) local minimum value is -2
- (C) local maximum value is -2
- (D) local minimum value $<$ local maximum value

Correct Answer: (C) local maximum value is -2

Solution:

Concept:

- For a differentiable function, local extrema can occur where $f'(x) = 0$. The point $x = 0$ is excluded from this function's domain.

Quick Tip to Solve: Distinguish the stationary x-values from the function values; substitute $x = -1$ and $x = 1$ only after classifying them.

- At a stationary point, $f'' < 0$ indicates a local maximum and $f'' > 0$ indicates a local minimum.

Step 1: Find and classify the stationary points For $f(x) = x + 1/x$,

$$f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

The equation $f'(x) = 0$ gives $x = \pm 1$. Since $f''(-1) = -2 < 0$, -1 is a local-maximum point; since $f''(1) = 2 > 0$, 1 is a local-minimum point.

Step 2: Evaluate the extrema and compare the choices The values are

$$f(-1) = -1 - 1 = -2, \quad f(1) = 1 + 1 = 2.$$

Thus option (A) assigns the minimum value to the maximum, while (B) assigns the maximum value to the minimum. Option (D) is false because $2 < -2$ is false. The two extrema lie on different components of the domain, so a local maximum can be below a local minimum.

Step 3: Conclusion The local maximum value is -2 . Hence, the correct option is (C).

Quick Tip to Solve: Distinguish the stationary x-values from the function values; substitute $x = -1$ and $x = 1$ only after classifying them.

9. If $\int_0^{2a} \frac{1}{1+4x^2} dx = \frac{\pi}{6}$, then the value of a is

[1 mark]

- (A) $\frac{\sqrt{3}}{4}$
- (B) $\frac{\sqrt{3}}{2}$
- (C) $\sqrt{3}$
- (D) $2\sqrt{3}$

Correct Answer: (A) $\frac{\sqrt{3}}{4}$

Solution:

Concept:

- The standard antiderivative is $\int dx/(1+k^2x^2) = (1/k) \tan^{-1}(kx) + C$, for positive k .
- A definite integral is evaluated by subtracting the antiderivative at the lower limit from its value at the upper limit.

Step 1: Evaluate at the printed limits Here $k = 2$, the lower limit is 0 and the upper limit is $2a$. Hence

$$\int_0^{2a} \frac{dx}{1+4x^2} = \left[\frac{1}{2} \tan^{-1}(2x) \right]_0^{2a} = \frac{1}{2} \tan^{-1}(4a).$$

The argument at the upper limit is $2(2a) = 4a$, not $2a$.

Quick Tip to Solve: Account for both factors of 2: one gives the antiderivative factor $1/2$, and the other turns the upper limit into $4a$.

Step 2: Solve the inverse-trigonometric equation The given integral value yields

$$\frac{1}{2} \tan^{-1}(4a) = \frac{\pi}{6} \implies \tan^{-1}(4a) = \frac{\pi}{3}.$$

This angle is within the principal range of arctangent, so taking its tangent gives $4a = \sqrt{3}$. The other options make $4a$ equal to $2\sqrt{3}$, $4\sqrt{3}$, or $8\sqrt{3}$, none of which satisfies this equation.

Step 3: Conclusion $a = \frac{\sqrt{3}}{4}$. Hence, the correct option is (A).

Quick Tip to Solve: Account for both factors of 2: one gives the antiderivative factor $1/2$, and the other turns the upper limit into $4a$.

10. Which of the following expressions will give the area of region bounded by the curve $y = x^2$ and line $y = 16$? [1 mark]

(A) $\int_0^4 x^2 dx$

(B) $2 \int_0^4 x^2 dx$

(C) $\int_0^{16} \sqrt{y} dy$

(D) $2 \int_0^{16} \sqrt{y} dy$

Correct Answer: (D) $2 \int_0^{16} \sqrt{y} dy$

Solution:

Concept:

- For integration with respect to y , the area of a horizontal strip is $(x_{\text{right}} - x_{\text{left}}) dy$.
- The parabola $y = x^2$ has branches $x = \sqrt{y}$ and $x = -\sqrt{y}$, symmetric about the y -axis.

Step 1: Find the strip width and limits The line $y = 16$ intersects the parabola at $x = \pm 4$. The enclosed region extends from $y = 0$ to $y = 16$. At a fixed height y , the strip extends from $-\sqrt{y}$ to $\sqrt{y}$, so its full width is $2\sqrt{y}$.

Step 2: Compare all four expressions The area is therefore

$$A = \int_0^{16} (\sqrt{y} - (-\sqrt{y})) dy = 2 \int_0^{16} \sqrt{y} dy.$$

Option (C) includes only the right half of the region. Options (A) and (B) integrate the height x^2 above the x -axis, instead of the height $16 - x^2$ between the line and the parabola; their values are respectively $64/3$ and $128/3$, whereas the required area is $256/3$.

Step 3: Conclusion The required expression is $2 \int_0^{16} \sqrt{y} dy$. Hence, the correct option is (D).

Quick Tip to Solve: For a sideways strip between the two branches of $y = x^2$, use width $2\sqrt{y}$, not just $\sqrt{y}$.

11. The general solution of the differential equation $\frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x}}$ is

[1 mark]

(A) $\log \sqrt{y} = \log \sqrt{x} + C$

(B) $\sqrt{y} + \sqrt{x} = C$

(C) $\sqrt{y} - \sqrt{x} = C$

(D) $\log \sqrt{y} + \log \sqrt{x} = C$

Correct Answer: (C) $\sqrt{y} - \sqrt{x} = C$

Solution:

Concept:

- A separable equation allows all factors involving y to be placed with dy , and those involving x with dx .
- The power rule gives $\int t^{-1/2} dt = 2\sqrt{t} + C$.

Step 1: Separate and integrate On an interval with $x > 0$ and $y > 0$, divide by $\sqrt{y}$ to obtain

$$\frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt{x}}.$$

Integrating both sides gives $2\sqrt{y} = 2\sqrt{x} + C_1$. Absorb the factor $1/2$ into an arbitrary constant C .

Step 2: Check the signs and alternatives The resulting family is $\sqrt{y} - \sqrt{x} = C$. Differentiation gives $y'/(2\sqrt{y}) = 1/(2\sqrt{x})$, which reproduces the stated equation. Option (B), with a plus sign, would give a negative derivative. Option (A) would give $y' = y/x$, and option (D) would give $y' = -y/x$; neither is the required equation. The constant solution $y = 0$ also satisfies the equation but is separate from the positive-solution family obtained by division.

Step 3: Conclusion The general positive-solution family is $\sqrt{y} - \sqrt{x} = C$. Hence, the correct option is (C).

Quick Tip to Solve: Integrate reciprocal square roots by the power rule; logarithms arise from reciprocal first powers, not square roots.

12. The integrating factor of the differential equation $2x \frac{dy}{dx} - y = 3$ is

[1 mark]

(A) $\sqrt{x}$

(B) $\frac{1}{\sqrt{x}}$

(C) e^x

(D) e^{-x}

Correct Answer: (B) $\frac{1}{\sqrt{x}}$

Solution:

Concept:

- For $dy/dx + P(x)y = Q(x)$, the integrating factor is $e^{\int P(x) dx}$.

Quick Tip to Solve: Keep the factor 2 in $2x$: it makes the exponent $-1/2$, rather than -1 .

- Divide the entire equation by the coefficient of dy/dx before identifying $P(x)$.

Step 1: Put the printed equation in linear form The printed equation is $2x \, dy/dx - y = 3$. For $x \neq 0$, divide every term by $2x$:

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{3}{2x}.$$

Thus $P(x) = -1/(2x)$. The constant on the right does not enter the formula for the integrating factor.

Step 2: Integrate the coefficient and test the options On the positive domain used by the printed square-root options,

$$\text{IF} = e^{\int -\frac{1}{2x} dx} = e^{-\frac{1}{2} \log x} = x^{-1/2} = \frac{1}{\sqrt{x}}.$$

On an interval with $x < 0$, one may instead write $|x|^{-1/2}$. Option (A) reverses the sign of the exponent. Options (C) and (D) correspond to constant coefficients $P = 1$ and $P = -1$, respectively, which are absent here.

Step 3: Conclusion The integrating factor is $\frac{1}{\sqrt{x}}$ on $x > 0$. Hence, the correct option is (B).

Quick Tip to Solve: Keep the factor 2 in $2x$: it makes the exponent $-1/2$, rather than -1 .

13. If $|\vec{a}| = 5$ and $-2 \leq \lambda \leq 1$, then the sum of greatest and the smallest value of $|\lambda \vec{a}|$ is [1 mark]

- (A) -5
- (B) 5
- (C) 10
- (D) 15

Correct Answer: (C) 10

Solution:

Concept:

- The magnitude of a scalar multiple is $|\lambda \vec{a}| = |\lambda| |\vec{a}|$, and is always non-negative.
- An interval containing zero can give minimum magnitude zero even if neither endpoint gives zero.

Step 1: Find the minimum Here $|\lambda \vec{a}| = 5|\lambda|$ with $-2 \leq \lambda \leq 1$. Since $\lambda = 0$ belongs to this interval, the least possible value of $|\lambda|$ is zero. Hence the smallest magnitude is 0, not the magnitude at either endpoint.

Step 2: Find the maximum and compare the options The largest distance of λ from zero in this interval is 2, attained at $\lambda = -2$. Thus the greatest magnitude is $5 \times 2 = 10$, and the requested sum is $10 + 0 = 10$. Option (A) is impossible for a sum of magnitudes. Option (B) gives only the value at $\lambda = 1$. Option (D) incorrectly adds the endpoint magnitudes 10 and 5, overlooking the interior minimum at zero.

Step 3: Conclusion The sum of the greatest and smallest magnitudes is 10. Hence, the correct option is (C).

Quick Tip to Solve: When a scalar interval crosses zero, check zero for the minimum and the endpoint farthest from zero for the maximum.

14. Vector of magnitude 3 making equal angles with x and y axes and perpendicular to z axis is

[1 mark]

(A) $\hat{i} + 2\sqrt{2}\hat{j}$

(B) $3\hat{k}$

(C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$

(D) $\sqrt{3}\hat{i} + \sqrt{3}\hat{j} + \sqrt{3}\hat{k}$

Correct Answer: (C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$

Solution:

Concept:

- For $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, the magnitude is $\sqrt{a^2 + b^2 + c^2}$.
- Equal angles with the positive x - and y -axes give equal direction cosines, so $a = b$; perpendicularity to the z -axis gives $c = 0$.

Step 1: Use the direction and magnitude conditions Write $\vec{v} = a\hat{i} + a\hat{j}$. The condition $|\vec{v}| = 3$ gives $2a^2 = 9$, so $a = \pm 3/\sqrt{2}$. The positive-component vector among the printed alternatives is therefore $(3\sqrt{2}/2)\hat{i} + (3\sqrt{2}/2)\hat{j}$. The negative of this vector also satisfies the wording but is not among the choices.

Step 2: Test every printed vector Option (C) has equal components and zero $\hat{k}$ component; its magnitude is $\sqrt{9/2 + 9/2} = 3$. Option (A) also has magnitude 3, but its components 1 and $2\sqrt{2}$ are unequal, so the angles with the two axes differ. Option (B) lies along the z -axis and is not perpendicular to it. Option (D) has magnitude 3, but its nonzero $\hat{k}$ component violates perpendicularity.

Step 3: Conclusion The listed vector satisfying all conditions is $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$. Hence, the correct option is (C).

Quick Tip to Solve: Check the zero $\hat{k}$ component and equality of the other two components before computing the magnitude.

15. Direction cosines of the line given by equations $\frac{2x-1}{4} = \frac{1-y}{3} = \frac{-z}{6}$ are

[1 mark]

- (A) $2, -3, -6$
(B) $\frac{2}{7}, \frac{-3}{7}, \frac{-6}{7}$
(C) $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$
(D) $\frac{4}{\sqrt{61}}, \frac{-3}{\sqrt{61}}, \frac{-6}{\sqrt{61}}$

Correct Answer: (B) $\frac{2}{7}, \frac{-3}{7}, \frac{-6}{7}$

Solution:

Concept:

- Direction ratios are the coefficients of a parameter in the coordinate equations of a line.
- Divide direction ratios (a, b, c) by $\sqrt{a^2 + b^2 + c^2}$ to obtain a set of direction cosines.

Step 1: Read the parameter coefficients correctly Set the common ratio equal to t . Then $2x - 1 = 4t$, $1 - y = 3t$ and $-z = 6t$, giving

$$x = \frac{1}{2} + 2t, \quad y = 1 - 3t, \quad z = -6t.$$

A direction vector therefore has ratios $(2, -3, -6)$. The coefficient 2 multiplying x must be accounted for; the first ratio is not 4.

Step 2: Normalise and eliminate the wrong choices The length of this direction vector is $\sqrt{4 + 9 + 36} = 7$. Thus a set of direction cosines is $(2/7, -3/7, -6/7)$, whose squared entries sum to one. Option (A) gives unnormalised direction ratios. Option (C) changes only the last sign, so it is not a parallel direction. Option (D) normalises the incorrect ratios $(4, -3, -6)$. Reversing all three signs would give the opposite orientation of the same line.

Step 3: Conclusion The direction cosines are $\frac{2}{7}, \frac{-3}{7}, \frac{-6}{7}$. Hence, the correct option is (B).

Quick Tip to Solve: Introduce a common parameter to handle coefficients and minus signs in symmetric line equations before normalising.

16. In a linear programming problem, the linear function which has to be maximized or minimized is called

[1 mark]

- (A) a feasible function
- (B) an objective function
- (C) an optimal function
- (D) a constraint

Correct Answer: (B) an objective function

Solution:

Concept:

- An objective function is the linear quantity whose value is maximised or minimised, commonly written $Z = ax + by$.
- Constraints describe the restrictions on the decision variables; points satisfying all restrictions form the feasible region.

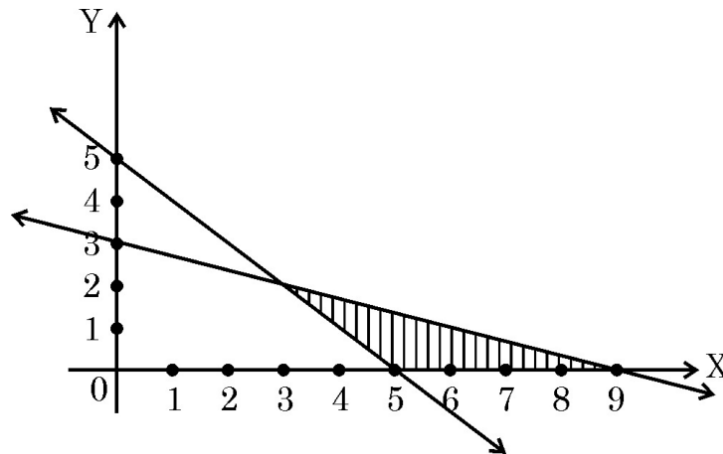
Step 1: Identify the role of the stated function The question describes the quantity being optimised in a linear programming problem. For example, if a manufacturer seeks to maximise profit $Z = 40x + 30y$, this linear profit expression is the objective function. Its coefficients specify the contribution of each decision variable to the quantity of interest.

Step 2: Distinguish the remaining terms Option (B) is exactly the standard term for the function to be maximised or minimised. Feasibility concerns whether a point obeys all constraints, so "a feasible function" in option (A) is not the term sought. An optimal value or optimal solution is obtained after optimisation; "an optimal function" in option (C) does not name the input expression. A constraint in option (D), such as $x + y \leq 10$, restricts the allowed choices rather than specifying their objective value.

Step 3: Conclusion The linear function to be maximised or minimised is an objective function. Hence, the correct option is (B).

Quick Tip to Solve: Ask what is being optimised: that expression is the objective; inequalities describing resources are constraints.

17. For the feasible region shown below, the non-trivial constraints of the linear programming problem are [1 mark]



- (A) $x + y \leq 5$, $x + 3y \leq 9$
- (B) $x + y \leq 5$, $x + 3y \geq 9$
- (C) $x + y \geq 5$, $x + 3y \leq 9$
- (D) $x + y \geq 5$, $3x + y \leq 9$

Correct Answer: (C) $x + y \geq 5$, $x + 3y \leq 9$

Solution:

Concept:

- A line with axis intercepts a and b has equation $x/a + y/b = 1$.
- To choose the inequality sign, substitute an interior point of the shaded feasible region into each boundary expression.

Step 1: Identify the two boundary lines The steeper line passes through $(0, 5)$ and $(5, 0)$, so its equation is $x + y = 5$. The shallower line passes through $(0, 3)$ and $(9, 0)$, giving $x/9 + y/3 = 1$, or $x + 3y = 9$. In particular, the second boundary is not $3x + y = 9$, which would have the intercepts reversed.

Step 2: Determine the shaded sides Choose the interior point $(6, 1/2)$ in the shaded triangle. It gives $x + y = 6.5 > 5$ and $x + 3y = 7.5 < 9$. Since the drawn boundaries are included, the non-trivial constraints are $x + y \geq 5$ and $x + 3y \leq 9$. Options (A) and (B) choose the wrong side of the first line. Option (D) gives the wrong second boundary. The first-quadrant restrictions $x \geq 0$, $y \geq 0$ are the non-negativity constraints and are not the non-trivial constraints requested.

Step 3: Conclusion The constraints are $x + y \geq 5$ and $x + 3y \leq 9$. Hence, the correct option is (C).

Quick Tip to Solve: Read each line from its intercepts, then test one point strictly inside the shaded triangle to choose both signs.

18. For two events A and B such that $P(A) \neq 0$ and $P(B) \neq 1$, $P(A'/B') =$

[1 mark]

- (A) $1 - P(A/B)$
- (B) $1 - P(A'/B)$
- (C) $\frac{1 - P(A \cap B)}{P(B')}$
- (D) $\frac{1 - P(A \cup B)}{P(B')}$

Correct Answer: (D) $\frac{1 - P(A \cup B)}{P(B')}$

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$ when $P(F) > 0$.
- De Morgan's law gives $A' \cap B' = (A \cup B)'$, so its probability is $1 - P(A \cup B)$.

Step 1: Apply the conditional probability definition Because $P(B) \neq 1$, the complement has probability $P(B') = 1 - P(B) > 0$. Hence

$$P(A' | B') = \frac{P(A' \cap B')}{P(B')}.$$

Both complements must appear in the intersection in the numerator; the conditioning event is B' , not B .

Step 2: Use the complement identity and check alternatives The event $A' \cap B'$ means neither A nor B occurs. It is precisely the complement of $A \cup B$. Therefore the numerator equals $1 - P(A \cup B)$, giving option (D). Option (C) uses the complement of an intersection, which means at least one event fails and is a different event. Options (A) and (B) keep conditioning on B , so the usual complement identities there do not provide the requested probability conditioned on B' .

Step 3: Conclusion $P(A'/B') = \frac{1 - P(A \cup B)}{P(B')}$. Hence, the correct option is (D).

Quick Tip to Solve: Write the conditional fraction first; "neither A nor B" is the complement of their union, not their intersection.

19. For two vectors $\vec{a}$ and $\vec{b}$:

Assertion (A): $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

Reason (R): $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b}) \tan \theta, \left(\theta \neq \frac{\pi}{2}\right)$

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- If θ is the angle between vectors, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.
- The cross-product magnitude is $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$; use $\sin^2 \theta + \cos^2 \theta = 1$.

Step 1: Verify the assertion Substitute the dot and cross-product formulae and add their squares:

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2.$$

Thus A is true, including the perpendicular case.

Step 2: Verify the reason For the printed restriction $\theta \neq \pi/2$, cosine is nonzero, so

$$(\vec{a} \cdot \vec{b}) \tan \theta = |\vec{a}| |\vec{b}| \cos \theta \frac{\sin \theta}{\cos \theta} = |\vec{a} \times \vec{b}|.$$

R is true; for an obtuse angle both the dot product and tangent are negative, making their product nonnegative.

Step 3: Test the explanatory connection Squaring R and adding the dot-product square gives

$$(\vec{a} \cdot \vec{b})^2 (1 + \tan^2 \theta) = |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta \sec^2 \theta = |\vec{a}|^2 |\vec{b}|^2.$$

This derives A wherever R is defined. At $\theta = \pi/2$, A follows directly because the dot product vanishes and the cross-product magnitude is $|\vec{a}| |\vec{b}|$. Thus R gives the correct explanatory relation. Option B denies that connection; C and D incorrectly classify one of the statements as false.

Step 4: Conclusion Both A and R are true, and R correctly explains A. Hence, the correct option is (A).

Quick Tip to Solve: Substitute the sine and cosine forms first; the restriction in R prevents division by zero in the tangent formula.

20. Assertion (A): A line can have direction cosines $\langle 1, 1, 1 \rangle$.

Reason (R): $\cos \theta = 1$ is possible for $\theta = 0$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- Direction cosines l, m, n are the cosines of the angles a line makes with the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$, because they are the components of a unit direction vector.

Step 1: Test the assertion Substitute the proposed values into the necessary identity:

$$1^2 + 1^2 + 1^2 = 3 \neq 1.$$

Hence $\langle 1, 1, 1 \rangle$ cannot be a triple of direction cosines. It can give direction ratios, whose normalized direction cosines would be $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$. A is false.

Step 2: Test the reason The elementary trigonometric value is $\cos 0 = 1$. Therefore it is possible for the cosine of an individual angle to equal 1, and R is true.

Step 3: Compare the statements R concerns one angle, whereas a triple of direction cosines must satisfy the joint sum-of-squares condition. A line along one positive axis has one direction cosine equal to 1 and the other two equal to 0; it cannot make angle 0 with all three mutually perpendicular axes. Options A and B require a true assertion, while C requires a false reason, so none of them fits.

Step 4: Conclusion Assertion A is false and Reason R is true. Hence, the correct option is (D).

Quick Tip to Solve: Test any proposed direction-cosine triple by adding its squares; normalize direction ratios before treating them as cosines.

Section B (10 marks)

This section comprises Very Short Answer (VSA) type questions carrying 2 marks each.

21(a). Check whether $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x-2}{x-3}$ is onto or not.

[2 marks]

Solution:

Concept:

- A function is onto when every element of its stated codomain has a preimage in its domain.
- To determine the range, set $y = f(x)$ and solve for an allowed input x ; check any values excluded

Quick Tip to Solve: Rewrite $(x-2)/(x-3)$ as $1 + 1/(x-3)$; the missing range value 1 becomes immediate.

during division.

Step 1: Test whether the codomain value 1 is attained The codomain is all of $\mathbb{R}$, so the value 1 must be attained if the function is onto. Suppose $f(x) = 1$, with $x \neq 3$. Then

$$\frac{x-2}{x-3} = 1 \implies x-2 = x-3,$$

which implies $-2 = -3$, an impossibility. Thus 1 has no preimage. Equivalently, $f(x) = 1 + 1/(x-3)$, and the second term never equals zero for any finite allowed input.

Step 2: Determine the other values in the range For $y \neq 1$, solving the equation gives

$$\begin{aligned} y(x-3) &= x-2, \\ (y-1)x &= 3y-2, \\ x &= \frac{3y-2}{y-1} = 3 + \frac{1}{y-1}. \end{aligned}$$

This is real and is never 3, so it is an allowed input. Substitution into f gives the desired value y . Hence every real number other than 1 is attained, and the exact range is $\mathbb{R} \setminus \{1\}$, strictly smaller than the codomain.

Step 3: Conclusion The function is not onto $\mathbb{R}$, since the codomain element 1 has no preimage.

Quick Tip to Solve: Rewrite $(x-2)/(x-3)$ as $1 + 1/(x-3)$; the missing range value 1 becomes immediate.

OR

21(b). Check whether $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) defined as $f(x, y) = (2y, 3x)$ is injective or not. [2 marks]

Solution:

Concept:

- A function is injective if equal images force equal inputs: $f(u) = f(v) \Rightarrow u = v$.
- Ordered pairs are equal precisely when their corresponding first and second components are equal.

Step 1: Equate the images of two arbitrary inputs Let (x_1, y_1) and (x_2, y_2) be any two pairs in $\mathbb{Z} \times \mathbb{Z}$, and assume they have the same image. The definition of the function gives

$$f(x_1, y_1) = f(x_2, y_2) \implies (2y_1, 3x_1) = (2y_2, 3x_2).$$

Equality of ordered pairs requires equality in each position. Thus the first components yield $2y_1 = 2y_2$, and the second components yield $3x_1 = 3x_2$. The interchange of the coordinates does not affect this test.

Step 2: Recover both input coordinates Since 2 and 3 are nonzero, cancel them in the respective equalities:

$$y_1 = y_2, \quad x_1 = x_2.$$

Quick Tip to Solve: Equate ordered-pair outputs component by component, then cancel each nonzero multiplier to recover the original input pair.

Consequently $(x_1, y_1) = (x_2, y_2)$. These pairs were arbitrary, so no two distinct integer pairs can have the same image. Also, the function is well defined into $\mathbb{Z} \times \mathbb{Z}$, because multiplying integer coordinates by 2 and 3 keeps them integers. The fact that some codomain pairs are not attained concerns surjectivity and does not invalidate the injectivity argument.

Step 3: Conclusion The given function $f(x, y) = (2y, 3x)$ is injective.

Quick Tip to Solve: Equate ordered-pair outputs component by component, then cancel each nonzero multiplier to recover the original input pair.

22. If $x = a \sin^3 t$, $y = b \cos^3 t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

[2 marks]

Solution:

Concept:

- For parametric coordinates $x(t), y(t)$, the derivative is $dy/dx = (dy/dt)/(dx/dt)$ wherever $dx/dt \neq 0$.
- Apply the chain rule to powers of trigonometric functions: differentiate the outer power, then the sine or cosine inside it.

Step 1: Differentiate both parametric expressions Treat a and b as constants. Differentiating each coordinate with respect to the common parameter t , we obtain

$$\begin{aligned}\frac{dx}{dt} &= 3a \sin^2 t \cos t, \\ \frac{dy}{dt} &= -3b \cos^2 t \sin t.\end{aligned}$$

The negative sign in the second expression comes from $d(\cos t)/dt = -\sin t$. We must divide these derivatives rather than differentiating y directly with respect to the independent parameter and calling it dy/dx .

Step 2: Form the ratio and evaluate at the printed parameter Taking the ratio and cancelling the common nonzero trigonometric factors near $t = \pi/4$ gives

$$\frac{dy}{dx} = \frac{-3b \cos^2 t \sin t}{3a \sin^2 t \cos t} = -\frac{b}{a} \cot t.$$

At the required value, $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, so $\cot(\pi/4) = 1$. In particular $dx/dt = 3a/(2\sqrt{2})$ is nonzero for the usual nonzero parameter a ; the ratio is therefore valid at this point.

Step 3: Conclusion

$$\left. \frac{dy}{dx} \right|_{t=\pi/4} = -\frac{b}{a}.$$

Quick Tip to Solve: Keep the minus sign from differentiating cosine, simplify the derivative ratio first, and then use $\cot(\pi/4) = 1$.

23(a). Find the absolute maximum value of $f(x) = \cos x + \sin^2 x$, $x \in [0, \pi]$.

[2 marks]

Solution:

Concept:

- A continuous function on a closed interval attains its absolute extrema; compare all interior critical points with both endpoints.
- A critical point is an interior point where $f'(x) = 0$ or the derivative does not exist. Here the function is differentiable throughout.

Step 1: Find the interior critical point Differentiate $f(x) = \cos x + \sin^2 x$ and factor:

$$f'(x) = -\sin x + 2 \sin x \cos x = \sin x(2 \cos x - 1).$$

In the open interval $(0, \pi)$, $\sin x > 0$, so an interior critical point must satisfy $2 \cos x - 1 = 0$. Therefore $\cos x = 1/2$, giving the unique interior critical point $x = \pi/3$. The zeros of $\sin x$ occur only at the endpoints, which are checked separately.

Step 2: Compare the critical and endpoint values Evaluate the function at each candidate:

$$\begin{aligned} f(0) &= 1, \\ f(\pi/3) &= \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{5}{4}, \\ f(\pi) &= -1. \end{aligned}$$

Since the function is continuous on $[0, \pi]$, this comparison identifies its absolute maximum. As an algebraic check, using $\sin^2 x = 1 - \cos^2 x$ yields

$$f(x) = \frac{5}{4} - \left(\cos x - \frac{1}{2}\right)^2 \leq \frac{5}{4},$$

with equality at the same interior point.

Step 3: Conclusion The absolute maximum value is $\frac{5}{4}$, attained at $x = \frac{\pi}{3}$.

Quick Tip to Solve: Complete the square in $\cos x$: $f(x) = 5/4 - (\cos x - 1/2)^2$ gives the bound and its equality point directly.

23(b). If the volume of a solid hemisphere increases at a uniform rate, prove that its surface area varies inversely as its radius. [2 marks]

Solution:

Concept:

- A hemisphere of radius $r > 0$ has volume $V = \frac{2}{3}\pi r^3$ and total surface area $S = 3\pi r^2$, including its circular base.
- Uniform increase means $\frac{dV}{dt} = k$, where $k > 0$ is constant. The chain rule relates the changing radius to the rates of volume and area.

Step 1: Express the radius growth rate Both volume and radius depend on time t . Differentiating the volume formula gives

$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt} = k, \quad \frac{dr}{dt} = \frac{k}{2\pi r^2}.$$

Thus the radius increases more slowly as the hemisphere becomes larger.

Step 2: Differentiate the surface area Apply the chain rule to $S = 3\pi r^2$, and substitute the radius growth rate:

$$\frac{dS}{dt} = 6\pi r \frac{dr}{dt} = 6\pi r \frac{k}{2\pi r^2} = \frac{3k}{r}.$$

Since $3k$ is constant, the rate of increase of surface area is inversely proportional to radius. If surface area means curved area $S_c = 2\pi r^2$, the same calculation gives $\frac{dS_c}{dt} = \frac{2k}{r}$, with the same inverse dependence.

Step 3: Interpret the printed wording precisely The surface area itself is proportional to r^2 , so it cannot be inversely proportional to r . The uniform-volume condition proves an inverse relation for the rate of change of area, which is the related-rates interpretation of the question.

Step 4: Conclusion $\frac{dS}{dt} = \frac{3k}{r} \propto \frac{1}{r}$. The rate of increase of surface area varies inversely as the radius; the printed statement about the area itself is not literally true.

Quick Tip to Solve: Differentiate both volume and area with respect to time, then eliminate dr/dt ; distinguish an area from its rate of increase.

24. If $\vec{AB} = \hat{j} + \hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$ represent the two vectors along the sides AB and AC of $\triangle ABC$, prove that the median $\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2}$, where D is midpoint of BC . Hence, find the length of median AD . [2 marks]

Solution:

Concept:

- For consecutive displacements, $\vec{AD} = \vec{AB} + \vec{BD}$. A midpoint divides its side into two equal vector displacements.
- In the orthonormal basis $\hat{i}, \hat{j}, \hat{k}$, the length of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$.

Quick Tip to Solve: Average the two side vectors from the same vertex first, then take the magnitude; averaging the side lengths does not give a median's length.

Step 1: Prove the vector formula for the median Since D is the midpoint of BC , $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$. Also, traversing from B to A and then to C gives $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$. Therefore,

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) \\ &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).\end{aligned}$$

This argument uses displacements from the common vertex A , so no choice of origin is needed.

Step 2: Substitute the given vectors and calculate the length Add corresponding components before dividing by two:

$$\overrightarrow{AD} = \frac{(\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k})}{2} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}.$$

The $\hat{j}$ components cancel. Its magnitude is

$$|\overrightarrow{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}.$$

Step 3: Conclusion $\overrightarrow{AD} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}$, and the length of median AD is $\boxed{\frac{\sqrt{34}}{2}}$ units.

Quick Tip to Solve: Average the two side vectors from the same vertex first, then take the magnitude; averaging the side lengths does not give a median's length.

25. Find the co-ordinates of the point on the line $\vec{r} = -\hat{j} + 3\hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ such that the sum of co-ordinates is 3. [2 marks]

Solution:

Concept:

- A line in vector form $\vec{r} = \vec{a} + \lambda\vec{b}$ represents all points reached from position vector $\vec{a}$ in direction $\vec{b}$, with real parameter λ .
- The coefficients of $\hat{i}, \hat{j}, \hat{k}$ in a position vector are the Cartesian coordinates x, y, z . A coordinate condition becomes an equation in λ .

Step 1: Read the coordinates from the vector equation Expand the given line and collect each basis vector:

$$\vec{r} = 2\lambda\hat{i} + (-1 - 2\lambda)\hat{j} + (3 + \lambda)\hat{k}.$$

Consequently a general point on this line has

$$x = 2\lambda, \quad y = -1 - 2\lambda, \quad z = 3 + \lambda.$$

The constant position vector supplies no $\hat{i}$ component, so the constant term in x is zero.

Quick Tip to Solve: Sum the base-point coordinates and the direction-vector coordinates separately: here $x + y + z = 2 + \lambda$, so the parameter follows immediately.

Step 2: Apply the condition on the coordinate sum Substitute these expressions into $x + y + z = 3$:

$$2\lambda + (-1 - 2\lambda) + (3 + \lambda) = 3 \implies 2 + \lambda = 3 \implies \lambda = 1.$$

The 2λ and -2λ terms cancel. Since the remaining coefficient of λ is nonzero, there is exactly one point on the line satisfying the condition. Substitution gives $x = 2$, $y = -3$, and $z = 4$. As a check, $2 - 3 + 4 = 3$.

Step 3: Conclusion The required point on the given line is $\boxed{(2, -3, 4)}$.

Quick Tip to Solve: Sum the base-point coordinates and the direction-vector coordinates separately: here $x + y + z = 2 + \lambda$, so the parameter follows immediately.

Section C (18 marks)

This section comprises Short Answer (SA) type questions of 3 marks each.

26. Find: $\int \frac{x+2}{\sqrt{9x-x^2}} dx.$

[3 marks]

Solution:

Concept:

- For an integral involving $\sqrt{f(x)}$, separate a multiple of $f'(x)$ from the numerator to use the substitution $u = f(x)$.
- Completing the square reduces $\int dx/\sqrt{a^2 - (x-b)^2}$ to $\sin^{-1}((x-b)/a) + C$, where $a > 0$ and C is a constant of integration.

Step 1: Split the numerator using the derivative of the radicand Let I denote the required integral. The expression under the root is $f(x) = 9x - x^2$, whose derivative is $9 - 2x$. Write

$$x + 2 = -\frac{1}{2}(9 - 2x) + \frac{13}{2}.$$

Thus

$$I = -\frac{1}{2} \int \frac{9 - 2x}{\sqrt{9x - x^2}} dx + \frac{13}{2} \int \frac{dx}{\sqrt{9x - x^2}}.$$

The first numerator is now exactly the derivative needed for a direct substitution.

Step 2: Integrate the derivative term Put $u = 9x - x^2$, so $du = (9 - 2x) dx$. Therefore,

$$-\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} = -\sqrt{9x - x^2}.$$

Real-valued working takes place on $0 < x < 9$, where the denominator is nonzero and the square root is positive.

Quick Tip to Solve: Match part of $x + 2$ to $9 - 2x$ before completing the square; the split produces one square-root term and one inverse-sine term.

Step 3: Complete the square for the remaining term Since

$$9x - x^2 = \left(\frac{9}{2}\right)^2 - \left(x - \frac{9}{2}\right)^2,$$

use the standard inverse-sine formula to get

$$\int \frac{dx}{\sqrt{9x - x^2}} = \sin^{-1} \left(\frac{2x - 9}{9} \right).$$

For verification, differentiating this inverse sine yields $1/\sqrt{9x - x^2}$. The derivative of the first term is $-(9 - 2x)/(2\sqrt{9x - x^2})$; adding the second term's derivative restores the numerator $x + 2$.

Step 4: Conclusion

$$\int \frac{x + 2}{\sqrt{9x - x^2}} dx = -\sqrt{9x - x^2} + \frac{13}{2} \sin^{-1} \left(\frac{2x - 9}{9} \right) + C.$$

Quick Tip to Solve: Match part of $x + 2$ to $9 - 2x$ before completing the square; the split produces one square-root term and one inverse-sine term.

27(a). Evaluate: $\int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}$

[3 marks]

Solution:

Concept:

- The reflection property of a definite integral is $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$. It follows by substituting $u = a + b - x$ and reversing the limits.
- For $0 < x < \pi/2$, $\cot(\pi/2 - x) = \tan x$ and $\sqrt{\tan x} = 1/\sqrt{\cot x}$. Positivity ensures that the square roots obey this reciprocal identity.

Step 1: Reflect the integral about the midpoint of its interval Write

$$I = \int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}.$$

The endpoints add to $\pi/2$. Under $u = \pi/2 - x$, the lower endpoint becomes $5\pi/12$, the upper endpoint becomes $\pi/12$, and $dx = -du$. Reversing the limits and renaming u as x gives

$$I = \int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\tan x}}.$$

This transformation retains exactly the same interval and turns cotangent into tangent.

Step 2: Rewrite the reflected integrand Throughout this interval both tangent and cotangent are

Quick Tip to Solve: Check the sum of the limits first: it is $\pi/2$, so reflection swaps tangent and cotangent and makes the paired integrands sum to one.

positive. Let $s = \sqrt{\cot x} > 0$; then $\sqrt{\tan x} = 1/s$. Hence

$$\frac{1}{1 + \sqrt{\tan x}} = \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}}.$$

The original integrand and its reflected form therefore add to one for every point of the interval.

Step 3: Add the two forms and integrate the constant Adding the equal expressions for I yields

$$\begin{aligned} 2I &= \int_{\pi/12}^{5\pi/12} \left(\frac{1}{1 + \sqrt{\cot x}} + \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} \right) dx \\ &= \int_{\pi/12}^{5\pi/12} 1 \, dx = \frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}. \end{aligned}$$

The functions are continuous on this closed interval, so the property and addition of the integrals apply directly.

Step 4: Conclusion Dividing by two, the required definite integral is $I = \frac{\pi}{6}$.

Quick Tip to Solve: Check the sum of the limits first: it is $\pi/2$, so reflection swaps tangent and cotangent and makes the paired integrands sum to one.

OR

27(b). Evaluate: $\int_{-\pi/6}^{\pi/2} (\sin |x| + \cos |x|) \, dx$

[3 marks]

Solution:

Concept:

- The modulus satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$, so an integral crossing zero can be split there.
- The function $f(x) = \sin |x| + \cos |x|$ is even: $f(-x) = f(x)$. Hence its integral from $-a$ to zero equals its integral from zero to a .
- On the nonnegative axis, an antiderivative of $\sin x + \cos x$ is $-\cos x + \sin x$.

Step 1: Split at the point where the modulus changes form Write the required integral as I . Since its negative and positive limits have different magnitudes, doubling the integral over the entire positive interval would be incorrect. Instead,

$$\begin{aligned} I &= \int_{-\pi/6}^0 (\sin |x| + \cos |x|) \, dx + \int_0^{\pi/2} (\sin x + \cos x) \, dx \\ &= \int_0^{\pi/6} (\sin x + \cos x) \, dx + \int_0^{\pi/2} (\sin x + \cos x) \, dx. \end{aligned}$$

The second equality follows by putting $t = -x$ in the negative interval, or directly by evenness.

Step 2: Evaluate both nonnegative intervals Using the antiderivative and the exact trigonometric

Quick Tip to Solve: Split a modulus integral at zero first; use evenness only on matching negative and positive intervals.

values,

$$\begin{aligned}\int_0^{\pi/6} (\sin x + \cos x) dx &= [-\cos x + \sin x]_0^{\pi/6} \\ &= -\frac{\sqrt{3}}{2} + \frac{1}{2} + 1 = \frac{3 - \sqrt{3}}{2}, \\ \int_0^{\pi/2} (\sin x + \cos x) dx &= [-\cos x + \sin x]_0^{\pi/2} = 1 - (-1) = 2.\end{aligned}$$

Both contributions are positive, as expected because the integrand is positive on these intervals.

Step 3: Conclusion Therefore, $I = \frac{3 - \sqrt{3}}{2} + 2 = \boxed{\frac{7 - \sqrt{3}}{2}}$.

Quick Tip to Solve: Split a modulus integral at zero first; use evenness only on matching negative and positive intervals.

28. If $\frac{d}{dx}(F(x)) = \frac{1}{e^x + 1}$, then find $F(x)$ given that $F(0) = \log \frac{1}{2}$.

[3 marks]

Solution:

Concept:

- If $F'(x) = f(x)$, then $F(x) = \int f(x) dx$; the arbitrary constant is determined by the given value of F .
- The logarithmic derivative of $1 + e^x$ is $e^x/(1 + e^x)$; recognising this form makes the integral immediate.

Step 1: Rewrite the derivative for direct integration Starting from the given derivative, use $1 = (1 + e^x) - e^x$ to obtain

$$F'(x) = \frac{1}{1 + e^x} = 1 - \frac{e^x}{1 + e^x}.$$

The numerator in the second fraction is exactly the derivative of its denominator. Therefore no expansion or infinite series is needed; this fraction integrates to a logarithm. Also, $1 + e^x > 0$ for every real x , so the logarithm is well-defined without an absolute-value sign.

Step 2: Integrate and impose the initial condition Integrating both terms gives

$$F(x) = x - \log(1 + e^x) + C.$$

Now substitute $x = 0$. As $e^0 = 1$, the prescribed condition yields

$$\begin{aligned}F(0) &= -\log 2 + C = \log \frac{1}{2} = -\log 2, \\ C &= 0.\end{aligned}$$

Thus the condition selects one particular antiderivative from the family.

Quick Tip to Solve: Replace $1/(1 + e^x)$ by $1 - e^x/(1 + e^x)$, then use the initial value to fix the constant.

Step 3: Verify the selected function Differentiating the resulting expression,

$$\frac{d}{dx}[x - \log(1 + e^x)] = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x},$$

which agrees with the given derivative. Its value at zero is $-\log 2$, so the initial condition is also satisfied.

Step 4: Conclusion The required function is

$$F(x) = x - \log(1 + e^x).$$

Quick Tip to Solve: Replace $1/(1 + e^x)$ by $1 - e^x/(1 + e^x)$, then use the initial value to fix the constant.

29(a). Solve the following differential equation: $x \frac{dy}{dx} = y - x \sin^2\left(\frac{y}{x}\right)$, **given that** $y(1) = \frac{\pi}{6}$. [3 marks]

Solution:

Concept:

- A first-order differential equation depending on the ratio y/x is reduced by the homogeneous substitution $y = vx$, where v is a function of x .
- The product rule gives $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Also, $\int \csc^2 v \, dv = -\cot v$.
- An initial condition fixes the integration constant and selects the local branch of the implicit solution.

Step 1: Apply the homogeneous substitution Put $v = y/x$, so $y = vx$. Substituting in the printed equation,

$$x \left(v + x \frac{dv}{dx} \right) = xv - x \sin^2 v.$$

Cancel the xv terms and divide by x , which is nonzero on an interval containing the initial point $x = 1$:

$$x \frac{dv}{dx} = -\sin^2 v.$$

The equation is now separable. At the initial point, $v = \pi/6$, so $\sin v \neq 0$ and separation is valid locally.

Step 2: Separate variables and integrate Rearranging and integrating gives

$$\begin{aligned} \csc^2 v \, dv &= -\frac{dx}{x}, \\ -\cot v &= -\log|x| + C_1, \\ \cot v &= \log|x| + C. \end{aligned}$$

Quick Tip to Solve: For a ratio y/x inside a trigonometric function, put $y = vx$ and cancel the common xv terms first.

Restoring $v = y/x$, the implicit family is

$$\cot\left(\frac{y}{x}\right) = \log|x| + C.$$

Constant-ratio solutions $v = n\pi$ lost when dividing by $\sin^2 v$ do not satisfy the prescribed initial value.

Step 3: Determine the constant using the given value At $x = 1$, we have $y = \pi/6$. Hence

$$\cot\frac{\pi}{6} = \log 1 + C, \quad C = \sqrt{3}.$$

Since the interval containing 1 has $x > 0$, $\log|x| = \log x$ there. The branch through $y/x = \pi/6$ is chosen continuously when expressing y explicitly.

Step 4: Conclusion The solution satisfying the initial condition is $\cot\left(\frac{y}{x}\right) = \log x + \sqrt{3}, \quad x > 0$, with the branch through $y(1) = \pi/6$.

Quick Tip to Solve: For a ratio y/x inside a trigonometric function, put $y = vx$ and cancel the common xv terms first.

OR

29(b). Find the general solution of the differential equation: $y \log y \frac{dx}{dy} + x = \frac{2}{y}$

[3 marks]

Solution:

Concept:

- Treat x as a function of y . A linear differential equation has form $\frac{dx}{dy} + P(y)x = Q(y)$.
- An integrating factor is $\mu(y) = e^{\int P(y) dy}$; multiplication by it makes the left side the derivative of μx .

Step 1: Put the equation in linear form The printed equation is

$$y \log y \frac{dx}{dy} + x = \frac{2}{y}.$$

Divide by $y \log y$ on an interval with $y > 0$ and $y \neq 1$:

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{2}{y^2 \log y}.$$

Thus $P(y) = 1/(y \log y)$ and $Q(y) = 2/(y^2 \log y)$. The independent variable is y , so all integrations below are with respect to y .

Step 2: Find the integrating factor With $t = \log y$, we have $dt = dy/y$, hence

$$\int \frac{dy}{y \log y} = \log |\log y|.$$

Quick Tip to Solve: When the equation contains dx/dy , use y as the independent variable; the integrating factor here is $\log y$.

The exponential gives $|\log y|$. On either interval $(0, 1)$ or $(1, \infty)$, its sign is fixed, so an equivalent nonzero integrating factor is $\log y$. Multiplying the equation by this factor gives

$$\log y \frac{dx}{dy} + \frac{x}{y} = \frac{2}{y^2}.$$

Step 3: Integrate the product derivative By the product rule the left side is $\frac{d}{dy}(x \log y)$. Consequently,

$$\begin{aligned} \frac{d}{dy}(x \log y) &= \frac{2}{y^2}, \\ x \log y &= \int 2y^{-2} dy = -\frac{2}{y} + C. \end{aligned}$$

Differentiating this relation gives $\log y \, dx/dy + x/y = 2/y^2$; multiplying by y recovers the original equation and checks the sign of the antiderivative.

Step 4: Conclusion The general solution is $x \log y + \frac{2}{y} = C$, for $y > 0, y \neq 1$.

Quick Tip to Solve: When the equation contains dx/dy , use y as the independent variable; the integrating factor here is $\log y$.

30. Solve the following linear programming problem graphically:

Maximize $Z = 10500x + 9000y$

Subject to constraints

$$x + y \leq 50$$

$$2x + y \leq 80$$

$$x, y \geq 0$$

[3 marks]

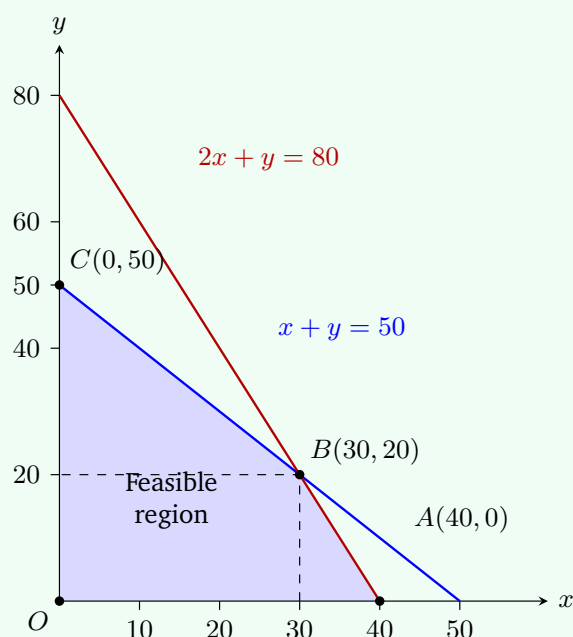
Solution:

Concept:

- The feasible region is the common region satisfying all the constraints. Since $x, y \geq 0$, it lies in the first quadrant.
- For a linear objective function on a bounded polygonal feasible region, a maximum occurs at a vertex. Evaluate $Z = 10500x + 9000y$ at every vertex.

Step 1: Plot the boundary lines and identify the feasible region The line $x + y = 50$ passes through $(50, 0)$ and $(0, 50)$. The line $2x + y = 80$ passes through $(40, 0)$ and $(0, 80)$. For each inequality the origin satisfies the constraint, so take the side of its line containing the origin. Together with the coordinate axes, the common shaded region is the quadrilateral $OABC$ in the graph. In particular, on the x -axis the stricter bound is $x \leq 40$, while on the y -axis it is $y \leq 50$.

Quick Tip to Solve: On each axis choose the smaller intercept, then include the intersection of the two lines before comparing objective values.



Shaded feasible region for the two constraints in the first quadrant.

Step 2: Calculate the intersection and all corner points At the intersection of the two sloping boundaries,

$$x + y = 50,$$

$$2x + y = 80.$$

Subtracting the first equation from the second gives $x = 30$; substituting back gives $y = 20$. Thus the vertices are $O(0, 0)$, $A(40, 0)$, $B(30, 20)$, and $C(0, 50)$. Each lies in the first quadrant and satisfies both given inequalities.

Step 3: Evaluate the objective function at the vertices The corner-point values are

Vertex (x, y)	$Z = 10500x + 9000y$
$(0, 0)$	0
$(40, 0)$	420000
$(30, 20)$	$315000 + 180000 = 495000$
$(0, 50)$	450000

The feasible region is closed and bounded, so the corner-point method applies. The largest tabulated value is attained at the intersection of the two constraint lines.

Step 4: Conclusion The maximum value is $Z_{\max} = 495000$, attained at $x = 30, y = 20$.

Quick Tip to Solve: On each axis choose the smaller intercept, then include the intersection of the two lines before comparing objective values.

31(a). The probability of hitting the target by a trained sniper is three times the probability of not hitting the target on a stormy day due to high wind speed. The sniper fired two shots on the target on a stormy day when wind speed was very high. Find the probability that

(i) target is hit

(ii) atleast one shot misses the target.

[3 marks]

Solution:

Concept:

- For one shot, hitting and missing are complementary outcomes. If their probabilities are p and q , then $p + q = 1$.
- Treat the two shots as independent trials with the same hit probability. Products give joint probabilities, and $P(A) = 1 - P(A^c)$ gives the probability of at least one occurrence.

Step 1: Determine the probabilities for one shot Let $p = P(\text{hit})$ and $q = P(\text{miss})$ on the stated stormy day. The condition gives $p = 3q$. Since a shot either hits or misses,

$$p + q = 1 \implies 3q + q = 1.$$

Thus $q = \frac{1}{4}$ and $p = \frac{3}{4}$. The wind conditions are already incorporated in these single-shot probabilities; no further adjustment is needed.

Step 2: Find the probability that the target is hit In the two-shot experiment, the target is hit if at least one of the shots hits it. The only outcome in which it is not hit is that both shots miss. By independence,

$$P(\text{both miss}) = q^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

Therefore

$$P(\text{target is hit}) = 1 - P(\text{both miss}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

This includes hit-hit, hit-miss, and miss-hit, whose probabilities are respectively $9/16$, $3/16$, and $3/16$.

Step 3: Find the probability that at least one shot misses The complement of at least one miss is that neither shot misses, that is, both shots hit. Hence

$$P(\text{both hit}) = p^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16},$$

and

$$P(\text{at least one miss}) = 1 - \frac{9}{16} = \frac{7}{16}.$$

Equivalently, add the disjoint outcomes hit-miss, miss-hit and miss-miss: $3/16 + 3/16 + 1/16 = 7/16$. This event overlaps the event in part (i), since a pair containing one hit and one miss belongs to both.

Step 4: Conclusion (i) $P(\text{target is hit}) = \frac{15}{16}$. (ii) $P(\text{at least one shot misses}) = \frac{7}{16}$.

Quick Tip to Solve: For two shots, at least one hit excludes only two misses; at least one miss excludes only two hits.

31(b). Mother, Father and Son line up at random for a family picture. Let events E : Son on one end and F : Father in the middle. Find $P(E/F)$. [3 marks]

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, where $P(F) > 0$. The condition restricts the sample space to outcomes in F .
- Distinct people arranged at random in a line have equally likely permutations. If an event is guaranteed by the condition, its conditional probability is 1.

Step 1: List the equally likely arrangements Write M for Mother, D for Father, and S for Son, reserving F for the event stated in the question. Read each arrangement from left to right. There are $3! = 6$ equally likely outcomes:

$$\Omega = \{MDS, MSD, DMS, DSM, SMD, SDM\}.$$

An end position is either the first or the third position, whereas the middle is the second position. Thus both possible ends must be counted in event E .

Step 2: Identify the conditioning event and its intersection with E The event that Son is on one end is

$$E = \{MDS, DMS, SMD, SDM\}.$$

The event that Father is in the middle is

$$F = \{MDS, SDM\}.$$

In each outcome of F , the two remaining positions are the ends, occupied by Mother and Son. Consequently every outcome in F is also in E , and

$$E \cap F = \{MDS, SDM\} = F.$$

Therefore $P(F) = 2/6 = 1/3$, and $P(E \cap F) = 2/6 = 1/3$.

Step 3: Apply the conditional-probability formula The notation $P(E/F)$ means $P(E | F)$. As $P(F) = 1/3 > 0$,

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/3}{1/3} = 1.$$

This agrees with direct counting in the restricted sample space: once Father is known to be in the middle, only MDS and SDM remain possible, and both place Son on an end. The unconditional probability $P(E) = 4/6 = 2/3$ does not apply after imposing the condition.

Step 4: Conclusion $P(E/F) = 1$. Father being in the middle guarantees that Son is on one end.

Quick Tip to Solve: First check whether the condition forces the desired event: here occupying the middle leaves only end positions for the other two people.

Section D (20 marks)

This section comprises Long Answer (LA) type questions of 5 marks each.

32(a). If $P = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, find (QP) and hence solve the following system of equations using matrices:

$$x - y = 3, 2x + 3y + 4z = 17, y + 2z = 7$$

[5 marks]

Solution:

Concept:

- Matrix multiplication is performed by taking the dot product of each row of the first matrix with each column of the second. The order matters: here the required product is QP .
- A system of linear equations has the matrix form $PX = B$, where P contains its coefficients, X is the column of unknowns and B is the column of constants.
- If $QP = kI$ with $k \neq 0$, multiplying $PX = B$ on the left by Q gives $kX = QB$. Thus $X = k^{-1}QB$; equivalently, $P^{-1} = k^{-1}Q$.

Step 1: Calculate the required matrix product Use the rows of Q and columns of P . In particular, the first row gives $2(1) + 2(2) - 4(0) = 6$, $2(-1) + 2(3) - 4(1) = 0$, and $2(0) + 2(4) - 4(2) = 0$. Computing the other entries in the same order gives

$$QP = \begin{pmatrix} 2+4+0 & -2+6-4 & 0+8-8 \\ -4+4+0 & 4+6-4 & 0+8-8 \\ 2-2+0 & -2-3+5 & 0-4+10 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I.$$

This scalar identity matrix allows us to solve the system directly without calculating cofactors separately.

Step 2: Express the equations as a matrix equation Keep the unknowns in the order x, y, z . The first equation has coefficient zero for z , and the last has coefficient zero for x . Therefore the given equations become

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}.$$

Write this as $PX = B$, with $X = (x, y, z)^T$ and $B = (3, 17, 7)^T$. Since $QP = 6I$, P is invertible and the system has a unique solution.

Step 3: Multiply by Q to determine the unknowns Multiplying both sides on the left by Q preserves the order of the factors:

$$QPX = QB \implies 6X = QB.$$

Quick Tip to Solve: Once QP is a nonzero multiple of the identity, multiply the constants column by Q and divide by that scalar; keep the order QP throughout.

Calculate the column on the right:

$$QB = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix}.$$

Dividing every entry by 6 gives $X = (2, -1, 4)^T$.

Step 4: Verify the values in all three equations The values obtained give $x - y = 2 - (-1) = 3$, $2x + 3y + 4z = 4 - 3 + 16 = 17$, and $y + 2z = -1 + 8 = 7$. Each right-hand side matches the printed system, which also checks the signs and ordering of the matrix calculation.

Step 5: Conclusion $QP = 6I$, and the unique solution of the given system is $x = 2, y = -1, z = 4$.

Quick Tip to Solve: Once QP is a nonzero multiple of the identity, multiply the constants column by Q and divide by that scalar; keep the order QP throughout.

OR

32(b). Obtain the value of $\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$ in terms of x, y and z . Further, if $\Delta = 0$ and x, y, z are non-zero real numbers, prove that $x^{-1} + y^{-1} + z^{-1} = -1$. [5 marks]

Solution:

Concept:

- Adding a multiple of one row to another row leaves the value of a determinant unchanged. This can create zeros and simplify the calculation.
- Expanding a determinant along its first row uses the alternating signs $+, -, +$. Each coefficient is multiplied by the determinant of the remaining two-by-two submatrix.
- Division by xyz is valid here because the second part explicitly states that x, y, z are nonzero. Also, $xy/(xyz) = 1/z$, with analogous identities for the other products.

Step 1: Simplify the determinant by row operations Begin with

$$\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}.$$

Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, using the original first row in both operations. Neither operation changes the determinant. The second row becomes $(-x, y, 0)$ and the third becomes $(-x, 0, z)$. Hence

$$\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix}.$$

Quick Tip to Solve: Subtract the first row from the other two to create zeros, then divide the resulting zero-determinant equation by xyz only after checking all three variables are nonzero.

The two zeros now make expansion along the first row especially straightforward. No division has been used, so this calculation is valid for every real x, y, z , even when one is zero.

Step 2: Expand and collect the terms Expanding with the signs $+, -, +$, we obtain

$$\begin{aligned}\Delta &= (1+x) \begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} - \begin{vmatrix} -x & 0 \\ -x & z \end{vmatrix} \\ &\quad + \begin{vmatrix} -x & y \\ -x & 0 \end{vmatrix} \\ &= (1+x)yz - (-xz) + xy \\ &= yz + xyz + xz + xy.\end{aligned}$$

In the middle minor, the determinant is $(-x)z - 0 = -xz$; its cofactor sign changes this contribution to $+xz$. The last minor is $(-x)0 - y(-x) = xy$. Thus the expression is symmetric in the three variables, as expected from the original arrangement.

Step 3: Use the zero determinant condition For the second part, the hypothesis $\Delta = 0$ yields

$$xyz + xy + xz + yz = 0.$$

Because $x, y, z \neq 0$, their product is nonzero. Divide the entire equation by xyz , rather than dropping any of its terms:

$$\frac{xyz}{xyz} + \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} = 0.$$

Cancelling the nonzero factors in each fraction gives

$$1 + \frac{1}{z} + \frac{1}{y} + \frac{1}{x} = 0.$$

Subtracting 1 and rearranging the reciprocal terms proves the required identity. The nonzero condition is essential at this step: without it, a reciprocal might not be defined.

Step 4: Conclusion The determinant is $\Delta = xyz + xy + xz + yz$. When $\Delta = 0$ and all three variables are nonzero, the required identity is

$$x^{-1} + y^{-1} + z^{-1} = -1.$$

Quick Tip to Solve: Subtract the first row from the other two to create zeros, then divide the resulting zero-determinant equation by xyz only after checking all three variables are nonzero.

33(a). Find the sub intervals in which $f(x) = \cot^{-1}(\sin x + \cos x)$, $x \in (0, \pi)$ is increasing and decreasing.

[5 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where its derivative is positive and decreasing on an interval where its derivative is negative. The derivative must be tested throughout the given domain.
- For a differentiable real function $u(x)$, the chain rule for the principal inverse cotangent gives $\frac{d}{dx} \cot^{-1} u(x) = -\frac{u'(x)}{1 + u(x)^2}$. The denominator is strictly positive for every real $u(x)$.
- A stationary point satisfies $f'(x) = 0$. Such points divide the domain into intervals on which the sign of the derivative can be checked; a zero derivative alone does not determine increasing or decreasing behaviour.

Step 1: Differentiate the composite function Put $u(x) = \sin x + \cos x$. Then $u'(x) = \cos x - \sin x$. Applying the chain rule,

$$\begin{aligned} f'(x) &= -\frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} \\ &= \frac{\sin x - \cos x}{1 + (\sin x + \cos x)^2}. \end{aligned}$$

Since $1 + (\sin x + \cos x)^2 \geq 1$, the derivative exists throughout $(0, \pi)$. Its sign is therefore exactly the sign of $\sin x - \cos x$.

Step 2: Locate the only stationary point in the domain Use the identity

$$\sin x - \cos x = \sqrt{2} \sin \left(x - \frac{\pi}{4} \right).$$

Thus $f'(x) = 0$ when $x - \pi/4 = k\pi$, where k is an integer. Of the resulting values $x = \pi/4 + k\pi$, only $x = \pi/4$ belongs to $(0, \pi)$. This splits the given domain into $(0, \pi/4)$ and $(\pi/4, \pi)$.

Step 3: Determine the derivative's sign on each subinterval If $0 < x < \pi/4$, then $-\pi/4 < x - \pi/4 < 0$; the sine of this angle is negative. Hence $f'(x) < 0$ throughout this subinterval. If $\pi/4 < x < \pi$, then $0 < x - \pi/4 < 3\pi/4$; the sine is positive. Hence $f'(x) > 0$ throughout this subinterval. The derivative changes from negative to positive at $\pi/4$, so the stationary point is a minimum, with value $f(\pi/4) = \cot^{-1} \sqrt{2}$. There are no other sign changes in the stated domain.

Step 4: Conclusion The function is decreasing on $(0, \pi/4)$ and increasing on $(\pi/4, \pi)$. The stationary point $x = \pi/4$ separates these two intervals.

Quick Tip to Solve: The inverse cotangent reverses monotonicity: after differentiating, ignore the positive denominator and test $\sin x - \cos x = \sqrt{2} \sin(x - \pi/4)$ on the stated domain.

33(b). A rectangle of perimeter 36 cm is revolved around one of its sides to sweep out a cylinder of maximum volume. Find the dimensions of the rectangle. [5 marks]

Solution:

Concept:

- When a rectangle revolves about one of its sides, that side becomes the height h of the cylinder and the perpendicular side becomes its radius r . Thus the swept volume is $V = \pi r^2 h$.
- The perimeter constraint is $2(r + h) = 36$, so $h = 18 - r$. Positive side lengths require $0 < r < 18$, allowing the volume to be expressed as a function of one variable.
- An interior maximum of a differentiable function is sought among its stationary points. A derivative changing from positive to negative establishes a maximum; checking the whole admissible interval establishes that it is the greatest possible volume.

Step 1: Express the cylinder's volume using the perimeter Let the side about which the rectangle revolves be h cm, and let the other side be r cm. The resulting cylinder has radius r and height h . The perimeter condition gives

$$2(r + h) = 36 \implies h = 18 - r.$$

Consequently,

$$V(r) = \pi r^2(18 - r) = \pi(18r^2 - r^3), \quad 0 < r < 18.$$

Step 2: Find the admissible stationary point Differentiating with respect to r ,

$$\begin{aligned} V'(r) &= \pi(36r - 3r^2) \\ &= 3\pi r(12 - r). \end{aligned}$$

Setting $V'(r) = 0$ gives $r = 0$ or $r = 12$. The value $r = 0$ does not represent a rectangle with positive side lengths, so the only stationary point within the permissible interval is $r = 12$. The corresponding axis side has length $h = 18 - 12 = 6$ cm.

Step 3: Verify the maximum over all allowed dimensions For $0 < r < 12$, both r and $12 - r$ are positive, so $V'(r) > 0$. For $12 < r < 18$, the second factor is negative, so $V'(r) < 0$. Therefore the volume increases up to $r = 12$ and decreases thereafter; this is the maximum throughout the admissible interval. Also,

$$V''(r) = \pi(36 - 6r), \quad V''(12) = -36\pi < 0.$$

At either degenerate limit, $r \rightarrow 0$ or $r \rightarrow 18$, the volume tends to zero. The stationary configuration gives $V(12) = \pi(12)^2(6) = 864\pi \text{ cm}^3$; its perimeter is $2(12 + 6) = 36$ cm.

Step 4: Conclusion The rectangle has dimensions $12 \text{ cm} \times 6 \text{ cm}$, and it must revolve about its 6 cm side. The resulting cylinder has radius 12 cm, height 6 cm, and maximum volume $864\pi \text{ cm}^3$.

Quick Tip to Solve: Name the axis side as the height first. For a fixed sum $r + h = 18$, differentiate $r^2(18 - r)$; its maximum occurs at $r = 2h$, so the shorter side is the axis.

34. Find the domain of $g(x) = \cos^{-1}(x^2 - 1)$. Hence, find the value of x for which $g(x) = \frac{\pi}{3}$. Also, write the range of $\cos^{-1} x$ other than its principal branch. [5 marks]

Solution:

Concept:

- The real inverse cosine $\cos^{-1} u$ is defined exactly when its input satisfies $-1 \leq u \leq 1$. Its principal values lie in $[0, \pi]$.
- For a composite function, apply this input restriction to the entire inner expression. Here the input to inverse cosine is $u = x^2 - 1$, not x .
- An inverse branch is obtained by restricting cosine to an interval on which it is one-to-one and covers $[-1, 1]$. For example, cosine decreases from 1 to -1 on $[2\pi, 3\pi]$.

Step 1: Apply the real-input restriction Since the argument of inverse cosine must belong to $[-1, 1]$, impose both inequalities together:

$$-1 \leq x^2 - 1 \leq 1.$$

Add 1 throughout to obtain

$$0 \leq x^2 \leq 2.$$

The lower inequality holds for every real x . The upper inequality is equivalent to $|x| \leq \sqrt{2}$. The endpoints give argument 1, and $x = 0$ gives argument -1 ; all are allowed.

Step 2: Solve the prescribed inverse-cosine equation The given angle $\pi/3$ belongs to the principal-value interval $[0, \pi]$, so applying cosine recovers the argument without any branch ambiguity:

$$\begin{aligned}\cos^{-1}(x^2 - 1) &= \frac{\pi}{3}, \\ x^2 - 1 &= \cos \frac{\pi}{3} = \frac{1}{2}, \\ x^2 &= \frac{3}{2}, \\ x &= \pm \sqrt{\frac{3}{2}} = \pm \frac{\sqrt{6}}{2}.\end{aligned}$$

Retain both signs: the original function depends on x^2 . Both values lie in the domain and give argument $\pi/3$, as required.

Step 3: Specify a different inverse-cosine branch To obtain a branch other than the principal branch, restrict cosine to $[2\pi, 3\pi]$. On this interval cosine is one-to-one, taking every value from 1 to -1 . Its inverse therefore has input domain $[-1, 1]$ and range $[2\pi, 3\pi]$. In terms of the principal inverse, this branch is

$$h(u) = 2\pi + \cos^{-1} u, \quad -1 \leq u \leq 1.$$

Its endpoints are $h(1) = 2\pi$ and $h(-1) = 3\pi$, giving a valid non-principal range.

Step 4: Conclusion The domain of g is $[-\sqrt{2}, \sqrt{2}]$. The equation $g(x) = \pi/3$ holds for $x = \pm \sqrt{3/2}$. One range for a non-principal inverse-cosine branch is $[2\pi, 3\pi]$.

Quick Tip to Solve: First restrict the entire argument $x^2 - 1$ to $[-1, 1]$; after taking cosine, retain both square roots and check them against that domain.

35. A line passing through the points $A(1, 2, 3)$ and $B(5, 8, 11)$ intersects the line $\vec{r} = 4\hat{i} + \hat{j} + \lambda(5\hat{i} + 2\hat{j} + \hat{k})$. Find the co-ordinates of the point of intersection. Hence, write the equation of a line passing through the point of intersection and perpendicular to both the lines. [5 marks]

Solution:

Concept:

- A line through points with position vectors $\vec{a}$ and $\vec{b}$ has direction $\vec{b} - \vec{a}$, so its equation is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, with a real parameter t .
- At the intersection of two lines, all three coordinates must agree. Their parameters are independent, so solve for them separately and check every coordinate.
- The cross product of two non-parallel direction vectors is perpendicular to both. A line through their intersection with this cross product as its direction is the required common perpendicular.

Step 1: Write the line through the two given points The vector from $A(1, 2, 3)$ to $B(5, 8, 11)$ is

$$\vec{AB} = (5 - 1, 8 - 2, 11 - 3) = (4, 6, 8).$$

A nonzero scalar multiple has the same direction, so use the simpler direction vector $\vec{d}_1 = (2, 3, 4)$. The first line becomes

$$\vec{r} = (1, 2, 3) + t(2, 3, 4).$$

The other line has direction $\vec{d}_2 = (5, 2, 1)$. Use separate parameters t and λ for the two lines.

Step 2: Solve for the intersection parameters Equating corresponding coordinates gives

$$1 + 2t = 4 + 5\lambda,$$

$$2 + 3t = 1 + 2\lambda,$$

$$3 + 4t = \lambda.$$

Substitute the last equation into the second:

$$2 + 3t = 1 + 2(3 + 4t),$$

$$-5t = 5, \quad t = -1,$$

$$\lambda = 3 + 4(-1) = -1.$$

Verify the unused first equation: $1 + 2(-1) = -1$ and $4 + 5(-1) = -1$. Both lines therefore give $P = (-1, -1, -1)$.

Step 3: Find a direction perpendicular to both lines Compute the cross product of the two line directions:

$$\begin{aligned} \vec{d}_1 \times \vec{d}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} \\ &= -5\hat{i} + 18\hat{j} - 11\hat{k}. \end{aligned}$$

Quick Tip to Solve: Simplify $\vec{AB}$ before taking the cross product; use the easy equation $\lambda = 3 + 4t$ first and verify the remaining coordinate afterward.

Equivalently use $\vec{n} = (5, -18, 11)$. Check:

$$\vec{n} \cdot \vec{d}_1 = 10 - 54 + 44 = 0,$$

$$\vec{n} \cdot \vec{d}_2 = 25 - 36 + 11 = 0.$$

Combining this nonzero direction with the position vector of P gives the required line through the intersection.

Step 4: Conclusion The intersection point is $(-1, -1, -1)$. A required line is

$$\vec{r} = -\hat{i} - \hat{j} - \hat{k} + s(5\hat{i} - 18\hat{j} + 11\hat{k}), \quad s \in \mathbb{R}.$$

Equivalently, $\frac{x+1}{5} = \frac{y+1}{-18} = \frac{z+1}{11}.$

Quick Tip to Solve: Simplify $\vec{AB}$ before taking the cross product; use the easy equation $\lambda = 3 + 4t$ first and verify the remaining coordinate afterward.

Section E (12 marks)

This section comprises of 3 case study based questions of 4 marks each.

36. Smoking increases the risk of lung problems.

A study revealed that 170 in 1000 males who smoke develop lung complications, while 120 out of 1000 females who smoke develop lung related problems. In a colony, 50 people were found to be smokers of which 30 are males. A person is selected at random from these 50 people and tested for lung related problems.

Based on the given information, answer the following questions:

[4 marks]

36(i). What is the probability that selected person is a female?

[1 mark]

Solution:

Concept:

- For equally likely choices, the probability of an event is the number of favourable outcomes divided by the total number of outcomes.
- Let F be the event that the selected smoker is female. Male and female smokers together constitute the stated group of 50 people.

Step 1: Count the female smokers There are 50 smokers altogether, of whom 30 are males. Thus the number of females in this group is

$$50 - 30 = 20.$$

Step 2: Use the full group as the sample space The selection is from all 50 smokers and no information about lung complications is imposed in this part. Each of these people is equally likely to be selected, so

$$P(F) = \frac{20}{50} = \frac{2}{5}.$$

Quick Tip to Solve: For this part, count females first; the lung-problem rates are not needed.

The complication rates do not affect this unconditional selection probability.

Step 3: Conclusion The probability that the selected person is female is $\boxed{\frac{2}{5}}$.

Quick Tip to Solve: For this part, count females first; the lung-problem rates are not needed.

36(ii). If a male person is selected, what is the probability that he will not be suffering from lung problems? [1 mark]

Solution:

Concept:

- Conditional probability restricts the sample space to the event already specified. Here the selected person is known to be male.
- If L denotes lung problems and M denotes a male smoker, the complementary probability satisfies $P(L^c | M) = 1 - P(L | M)$.

Step 1: Identify the given male complication rate The study reports lung complications for 170 out of every 1000 male smokers. Hence

$$P(L | M) = \frac{170}{1000} = \frac{17}{100}.$$

Step 2: Take the complement within male smokers For a selected male, suffering from lung problems and not suffering from them are mutually exclusive and exhaustive possibilities. Therefore

$$P(L^c | M) = 1 - \frac{17}{100} = \frac{83}{100}.$$

Multiplication by 30/50 is unnecessary because the selection is already known to be male.

Step 3: Conclusion The required probability is $\boxed{\frac{83}{100}}$, or 0.83.

Quick Tip to Solve: Once male selection is given, complement the male illness rate without multiplying by the male selection probability.

36(iii)(a). A person selected at random is detected with lung complications. Find the probability that selected person is a female. [2 marks]

Solution:

Concept:

- Let M , F and L denote the events that the selected smoker is male, is female and has lung complications, respectively.
- The events M and F partition the population. The total probability formula is $P(L) = P(M)P(L | M) + P(F)P(L | F)$.
- Bayes theorem reverses the conditioning: $P(F | L) = \frac{P(F)P(L | F)}{P(L)}$.

Quick Tip to Solve: Weight each complication rate by its gender proportion before applying Bayes theorem.

Step 1: Record the selection probabilities and complication rates Of the 50 smokers, 30 are male and 20 are female. Thus

$$P(M) = \frac{3}{5}, \quad P(F) = \frac{2}{5}.$$

The reported rates give

$$P(L | M) = \frac{17}{100}, \quad P(L | F) = \frac{12}{100}.$$

Step 2: Calculate the total probability of lung complications Complications may occur for either a male or a female. These cases are disjoint, so

$$\begin{aligned} P(L) &= \frac{3}{5} \cdot \frac{17}{100} + \frac{2}{5} \cdot \frac{12}{100} \\ &= \frac{51 + 24}{500} = \frac{75}{500}. \end{aligned}$$

Step 3: Condition on the positive finding The probability that the selected smoker is both female and has lung complications is $24/500$. Among all selections resulting in complications, the female proportion is therefore

$$P(F | L) = \frac{24/500}{75/500} = \frac{24}{75} = \frac{8}{25}.$$

The rate $12/100$ alone is the reverse conditional probability and is not the quantity asked for.

Step 4: Conclusion The probability that a person detected with lung complications is female is $\boxed{\frac{8}{25}}$.

Quick Tip to Solve: Weight each complication rate by its gender proportion before applying Bayes theorem.

OR

36(iii)(b). A person selected at random is not having lung problems, find the probability that the person is a male. [2 marks]

Solution:

Concept:

- Let M , F and L^c denote male selection, female selection and absence of lung problems, respectively.
- Complement the complication rates within each gender: $P(L^c | M) = 1 - P(L | M)$ and $P(L^c | F) = 1 - P(L | F)$.
- Bayes theorem gives $P(M | L^c) = \frac{P(M)P(L^c | M)}{P(M)P(L^c | M) + P(F)P(L^c | F)}$.

Step 1: Find the priors and the probabilities of no lung problems The gender probabilities are $P(M) = 30/50 = 3/5$ and $P(F) = 20/50 = 2/5$. Using the study rates,

$$P(L^c | M) = \frac{83}{100}, \quad P(L^c | F) = \frac{88}{100}.$$

Step 2: Combine the two ways of having no lung problems The person can be a male without lung

Quick Tip to Solve: Complement both gender-specific rates first, then use their weighted sum as the Bayes denominator.

problems or a female without lung problems. The total probability is

$$\begin{aligned} P(L^c) &= \frac{3}{5} \cdot \frac{83}{100} + \frac{2}{5} \cdot \frac{88}{100} \\ &= \frac{249 + 176}{500} = \frac{425}{500}. \end{aligned}$$

Step 3: Find the male share among those without lung problems The joint probability of male selection and absence of lung problems is $249/500$. Conditioning on no lung problems therefore gives

$$P(M | L^c) = \frac{249/500}{425/500} = \frac{249}{425}.$$

The denominator includes both genders, because the absence of lung problems does not establish that the person is male.

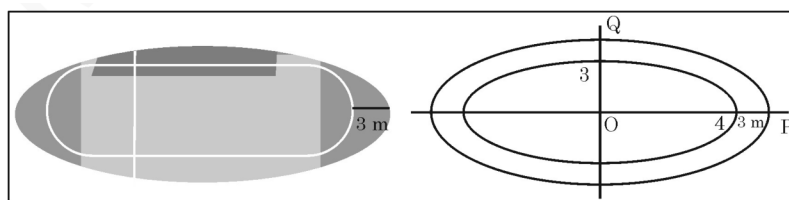
Step 4: Conclusion The required probability that the person is male is $\boxed{\frac{249}{425}}$.

Quick Tip to Solve: Complement both gender-specific rates first, then use their weighted sum as the Bayes denominator.

37. A racing track is built around an elliptical ground whose equation is given by $9x^2 + 16y^2 = 144$. The width of the track is 3 m as shown below:

Based on given information, answer the following questions:

[4 marks]



37(i). Express y as a function of x from the given equation of ellipse.

[1 mark]

Solution:

Concept:

- An ellipse symmetric about the coordinate axes has an upper branch with $y \geq 0$ and a lower branch with $y \leq 0$.
- Taking square roots gives both signs: if $y^2 = f(x)$, then $y = \pm\sqrt{f(x)}$, wherever $f(x) \geq 0$.

Step 1: Isolate the square of y From the given equation, $16y^2 = 144 - 9x^2 = 9(16 - x^2)$. Dividing by 16, we obtain

$$y^2 = \frac{9}{16}(16 - x^2).$$

Step 2: Select the branch needed for integration Taking the square root gives $y = \pm\frac{3}{4}\sqrt{16 - x^2}$. For real coordinates, $16 - x^2 \geq 0$, so $-4 \leq x \leq 4$. The positive sign describes the upper half of the ellipse, used to compute its area by symmetry; the negative sign describes the lower half. A single positive

Quick Tip to Solve: Factor $144 - 9x^2$ as $9(16 - x^2)$ before taking square roots, and retain both signs for the whole ellipse.

square root alone does not represent the entire ellipse.

Step 3: Conclusion

$$y = \pm \frac{3}{4} \sqrt{16 - x^2}, \quad -4 \leq x \leq 4.$$

In the upper half-plane, $y = \frac{3}{4} \sqrt{16 - x^2}$.

Quick Tip to Solve: Factor $144 - 9x^2$ as $9(16 - x^2)$ before taking square roots, and retain both signs for the whole ellipse.

37(ii). Integrate the function obtained in (i) with respect to x .

[1 mark]

Solution:

Concept:

- The upper branch obtained in part (i) is $y = \frac{3}{4} \sqrt{16 - x^2}$.
- The standard antiderivative is $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$, where $a > 0$ and C is an arbitrary constant.

Step 1: Apply the square-root integration formula For the upper branch, take the constant outside the integral:

$$\int y dx = \frac{3}{4} \int \sqrt{16 - x^2} dx.$$

Comparing with $\sqrt{a^2 - x^2}$, we have $a = 4$ and $a^2 = 16$. Hence

$$\int y dx = \frac{3}{4} \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right] + C.$$

Step 2: Simplify and account for the lower branch Multiplying the two terms by $3/4$ gives coefficients $3/8$ and 6 , respectively. On the lower half of the ellipse the ordinate has the opposite sign, so the nonconstant part of the antiderivative also changes sign. In either branch the integration constant remains arbitrary; it must be included for this indefinite integral.

Step 3: Conclusion For the upper branch,

$$\int y dx = \frac{3x}{8} \sqrt{16 - x^2} + 6 \sin^{-1} \left(\frac{x}{4} \right) + C.$$

For the lower branch, take the negative of both displayed nonconstant terms, plus C .

Quick Tip to Solve: In the standard formula use $a = 4$, so the inverse-sine coefficient before multiplying by $3/4$ is $a^2/2 = 8$.

37(iii)(a). Find the area of the region enclosed within the elliptical ground excluding the track using integration. [2 marks]

Solution:

Concept:

- The area between the upper curve $y = f(x) \geq 0$ and the x -axis on $[u, v]$ is $\int_u^v f(x) dx$.
- The ellipse $9x^2 + 16y^2 = 144$ is symmetric about both axes. Its total enclosed area is four times the area in the first quadrant. The surrounding track is outside this ground.

Step 1: Set the bounds for the ground Dividing the equation by 144 gives

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Thus the inner ground meets the positive axes at $(4, 0)$ and $(0, 3)$. In the first quadrant its upper boundary is $y = \frac{3}{4}\sqrt{16 - x^2}$, with $0 \leq x \leq 4$. The stated track width does not alter these bounds because the requested area excludes the track.

Step 2: Evaluate the area integral Using symmetry and the antiderivative from part (ii),

$$\begin{aligned} A &= 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} dx \\ &= 3 \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right]_0^4. \end{aligned}$$

At $x = 4$, the square-root term is zero and $\sin^{-1}(1) = \pi/2$; at $x = 0$, both terms vanish. Therefore $A = 3(8 \cdot \pi/2) = 12\pi$. This is the whole inner region, as the multiplier four includes all quadrants.

Step 3: Conclusion

Area of the elliptical ground excluding the track = $12\pi \text{ m}^2$.

Quick Tip to Solve: Integrate only from 0 to the inner x -intercept 4, then multiply by four; the track width is irrelevant to the enclosed ground area.

OR

37(iii)(b). Write the co-ordinates of the points P and Q where the outer edge of the track cuts x axis and y axis in first quadrant and find the area of the triangle formed by points P , O , Q using integration. [2 marks]

Solution:

Concept:

- The inner ellipse $x^2/16 + y^2/9 = 1$ has positive axis intercepts 4 and 3. The track extends each of these outward by its width 3 m, as shown in the figure.

Quick Tip to Solve: Add the track width to each positive inner intercept, then integrate the intercept-form straight line joining the two outer points.

- A line with positive intercepts a, b has equation $x/a + y/b = 1$. The area beneath this line from $x = 0$ to $x = a$ is $\int_0^a b(1 - x/a) dx$.

Step 1: Find the outer intercepts and the line PQ The inner positive x -intercept is 4, so the outer intercept is $4 + 3 = 7$. Similarly, the inner positive y -intercept is 3, so the outer intercept is $3 + 3 = 6$. Thus $P = (7, 0)$, $Q = (0, 6)$, and $O = (0, 0)$. The straight side joining P and Q has equation

$$\frac{x}{7} + \frac{y}{6} = 1, \quad y = 6 - \frac{6x}{7}.$$

Step 2: Integrate over the triangular region Triangle POQ lies above the x -axis and below the line PQ , between $x = 0$ and $x = 7$. Consequently its area is

$$\begin{aligned} A &= \int_0^7 \left(6 - \frac{6x}{7} \right) dx \\ &= \left[6x - \frac{3x^2}{7} \right]_0^7 \\ &= 42 - 21 = 21. \end{aligned}$$

The integral uses the straight segment PQ , since the required region is a triangle. Its value also agrees with half the product of the perpendicular sides $OP = 7$ and $OQ = 6$.

Step 3: Conclusion

$$P = (7, 0), \quad Q = (0, 6), \quad \text{area of } \triangle POQ = 21 \text{ m}^2.$$

Quick Tip to Solve: Add the track width to each positive inner intercept, then integrate the intercept-form straight line joining the two outer points.

38. Sports car racing is a form of motorsport which uses sports car prototypes. The competition is held on special tracks designed in various shapes. The equation of one such track is given as follows:

$$f(x) = \begin{cases} x^4 - 4x^2 + 4, & 0 \leq x < 3 \\ x^2 + 40, & x \geq 3 \end{cases}$$

Based on given information, answer the following questions:

[4 marks]

38(i). Find $f'(x)$ for $0 < x < 3$.

[1 mark]

Solution:

Concept:

- On an open interval, differentiate the formula valid throughout that interval.
- The power rule is $\frac{d}{dx} x^n = nx^{n-1}$; a constant has derivative zero.

Step 1: Select the appropriate branch For every x in $0 < x < 3$, the track function is $f(x) = x^4 - 4x^2 + 4$. The branch $x^2 + 40$ applies only when $x \geq 3$, so it is not used here. Each point of the requested open

Quick Tip to Solve: Select the piece valid on the requested interval before applying the power rule; do not differentiate both pieces there.

interval has a neighbourhood within the first branch.

Step 2: Differentiate each term Apply the power rule to the polynomial and retain its coefficients:

$$\frac{d}{dx}(x^4 - 4x^2 + 4) = 4x^3 - 4(2x) + 0.$$

The constant term contributes nothing, while the coefficient of x^2 gives the negative linear term.

Step 3: Conclusion $f'(x) = 4x^3 - 8x$ for $0 < x < 3$.

Quick Tip to Solve: Select the piece valid on the requested interval before applying the power rule; do not differentiate both pieces there.

38(ii). Find $f'(4)$.

[1 mark]

Solution:

Concept:

- A derivative at an interior point of a piecewise branch is found by differentiating that branch.
- For $f(x) = x^2 + c$, the power rule gives $f'(x) = 2x$, since a constant c has derivative zero.

Step 1: Identify the branch containing four Since $4 > 3$, the applicable formula is $f(x) = x^2 + 40$. Points sufficiently close to 4 are also greater than 3, so this same polynomial controls the two-sided derivative at 4. There is no need to compare the two branches at their joining point for this part.

Step 2: Differentiate and substitute Differentiating the applicable polynomial gives

$$f'(x) = \frac{d}{dx}(x^2 + 40) = 2x + 0.$$

Now substitute the requested value $x = 4$:

$$f'(4) = 2(4).$$

The number $56 = f(4)$ is the function value, not its derivative.

Step 3: Conclusion $f'(4) = 8$.

Quick Tip to Solve: At four, differentiate the second piece first and then substitute four; evaluating the original piece gives the wrong quantity.

38(iii)(a). Test for continuity of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- Continuity at $x = a$ requires $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$.
- Polynomials are continuous, so their one-sided limits are evaluated by direct substitution in the

Quick Tip to Solve: At the joining point, use the first formula for the left limit and the second for both the right limit and the actual function value.

relevant branch.

Step 1: Evaluate the left-hand limit For points approaching 3 from below, the applicable expression is $x^4 - 4x^2 + 4$. Hence

$$\begin{aligned}\lim_{x \rightarrow 3^-} f(x) &= 3^4 - 4(3^2) + 4 \\ &= 81 - 36 + 4 = 49.\end{aligned}$$

This is a limit obtained from values below the joining point; the first branch itself excludes $x = 3$.

Step 2: Evaluate the right-hand limit and function value The expression on the right of 3 is $x^2 + 40$, which also defines the function at 3, because its domain includes $x \geq 3$. Therefore

$$\lim_{x \rightarrow 3^+} f(x) = 3^2 + 40 = 49, \quad f(3) = 3^2 + 40 = 49.$$

Both one-sided limits agree, so the two-sided limit exists. It also equals the defined value of the function; consequently all three quantities required by the continuity criterion agree.

Step 3: Conclusion $\lim_{x \rightarrow 3} f(x) = f(3) = 49$. Thus f is continuous at $x = 3$.

Quick Tip to Solve: At the joining point, use the first formula for the left limit and the second for both the right limit and the actual function value.

OR

38(iii)(b). Test for differentiability of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- Differentiability at $x = a$ requires equal finite left-hand and right-hand difference-quotient limits $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$.
- For a piecewise function, use the formula valid on each side separately. Continuity alone does not ensure equal derivatives.

Step 1: Determine the value at the joining point Since the second branch includes 3, $f(3) = 3^2 + 40 = 49$. The limiting value of the first branch at 3 is also $81 - 36 + 4 = 49$. Thus the pieces join continuously, but their rates of change must still be compared.

Step 2: Compute the left-hand derivative For $h < 0$ sufficiently close to zero, use the first branch:

$$\begin{aligned}\frac{f(3+h) - f(3)}{h} &= \frac{(3+h)^4 - 4(3+h)^2 + 4 - 49}{h} \\ &= 84 + 50h + 12h^2 + h^3.\end{aligned}$$

Taking $h \rightarrow 0^-$ gives the left-hand derivative 84.

Quick Tip to Solve: Even when the pieces meet at the same height, compare their slopes: unequal one-sided derivatives rule out differentiability.

Step 3: Compute the right-hand derivative For $h > 0$, use the second branch:

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 + 40 - 49}{h} = 6 + h.$$

As $h \rightarrow 0^+$, this tends to 6. Although both one-sided derivatives exist, $84 \neq 6$, so the two-sided derivative does not exist.

Step 4: Conclusion The left-hand derivative is 84 and the right-hand derivative is 6. Therefore f is not differentiable at $x = 3$.

Quick Tip to Solve: Even when the pieces meet at the same height, compare their slopes: unequal one-sided derivatives rule out differentiability.