

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-2-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. All questions are **compulsory**.
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1** to **18** are Multiple Choice Questions (MCQs) and Question Number **19** & **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32** to **35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36** to **38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in few questions in all the Sections except Section – A.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

This section comprises 20 Multiple Choice Questions (MCQs) of 1 mark each.

1. $\int \frac{1}{1 + \cos x} dx$ is equal to

[1 mark]

- (A) $\frac{1}{2} \tan \frac{x}{2} + C$
- (B) $\frac{-1}{2} \cot \frac{x}{2} + C$
- (C) $-\cot \frac{x}{2} + C$
- (D) $\tan \frac{x}{2} + C$

Correct Answer: (D) $\tan \frac{x}{2} + C$

Solution:

Concept:

- Use the half-angle identity $1 + \cos x = 2 \cos^2(x/2)$.
- The substitution $u = x/2$ gives $dx = 2 du$; an antiderivative can be checked by differentiation.

Step 1: Rewrite the integrand On any interval where the integrand is defined,

$$\frac{1}{1 + \cos x} = \frac{1}{2 \cos^2(x/2)} = \frac{1}{2} \sec^2(x/2).$$

This identity exposes the derivative of a tangent function.

Step 2: Integrate and compare the options With $u = x/2$,

$$\int \frac{1}{2} \sec^2(x/2) dx = \int \sec^2 u du = \tan u + C.$$

Option (A) retains an extra factor $1/2$, so its derivative is half the required integrand. The derivatives of (B) and (C) involve $\csc^2(x/2)$, which corresponds to a denominator involving $1 - \cos x$, not the given denominator.

Step 3: Conclusion $\int \frac{dx}{1 + \cos x} = \tan \frac{x}{2} + C$. Hence, the correct option is (D).

Quick Tip to Solve: Use the half-angle identity first, and cancel the substitution factor 2 against the prefactor $1/2$.

2. For $f(x) = x + \frac{1}{x}$, ($x \neq 0$)

[1 mark]

- (A) local maximum value is 2
- (B) local minimum value is -2
- (C) local maximum value is -2
- (D) local minimum value $<$ local maximum value

Correct Answer: (C) local maximum value is -2

Solution:

Concept:

- A stationary point satisfies $f'(x) = 0$. If $f''(a) > 0$, it is a local minimum; if $f''(a) < 0$, it is a local maximum.
- The extremum value is $f(a)$, rather than the stationary coordinate a .

Step 1: Find and classify the stationary points For $f(x) = x + 1/x$, with $x \neq 0$,

$$f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

The equation $f'(x) = 0$ gives $x = \pm 1$. Since $f''(1) = 2 > 0$, $x = 1$ gives a local minimum; since $f''(-1) = -2 < 0$, $x = -1$ gives a local maximum.

Step 2: Evaluate and check the choices The local minimum value is $f(1) = 1 + 1 = 2$, and the local maximum value is $f(-1) = -1 - 1 = -2$. Thus (A) and (B) exchange these values. Option (D) is also false because $2 < -2$ is false. The domain has two disconnected intervals, so a local minimum on one interval may exceed a local maximum on the other.

Step 3: Conclusion The local maximum value is -2 . Hence, the correct option is (C).

Quick Tip to Solve: Substitute each stationary coordinate into f ; the words maximum and minimum refer locally to its own domain interval.

3. Which of the following expressions will give the area of region bounded by the curve $y = x^2$ and line $y = 16$? [1 mark]

(A) $\int_0^4 x^2 dx$

(B) $2 \int_0^4 x^2 dx$

(C) $\int_0^{16} \sqrt{y} dy$

(D) $2 \int_0^{16} \sqrt{y} dy$

Correct Answer: (D) $2 \int_0^{16} \sqrt{y} dy$

Solution:

Concept:

- For horizontal strips, area is $\int (x_{\text{right}} - x_{\text{left}}) dy$.
- The parabola $y = x^2$ has branches $x = \sqrt{y}$ and $x = -\sqrt{y}$.

Step 1: Determine the strip width and limits The line $y = 16$ meets the parabola at $x = \pm 4$. The enclosed region extends from $y = 0$ to $y = 16$. At height y , the left and right boundaries are $-\sqrt{y}$ and $\sqrt{y}$; consequently the horizontal strip width is $2\sqrt{y}$.

Step 2: Form the area integral and compare The area is

$$\int_0^{16} (\sqrt{y} - (-\sqrt{y})) dy = 2 \int_0^{16} \sqrt{y} dy.$$

Option (C) counts only the right half of the enclosed region. Options (A) and (B) integrate the height x^2 above the x-axis, whereas the vertical-strip height of the required region is $16 - x^2$. A vertical-strip check is $2 \int_0^4 (16 - x^2) dx = 256/3$.

Step 3: Conclusion The required expression is $2 \int_0^{16} \sqrt{y} dy$. Hence, the correct option is (D).

Quick Tip to Solve: When a horizontal line caps a parabola, use horizontal strips: right branch minus left branch supplies the factor 2.

4. The general solution of the differential equation $x dy - y dx = 0$ is

[1 mark]

- (A) $x^2 - y^2 = k$
- (B) $xy = k$
- (C) $x = ky$
- (D) $\log y + \log x = k$

Correct Answer: (C) $x = ky$

Solution:

Concept:

- On intervals with $y \neq 0$, $d(x/y) = (y dx - x dy)/y^2$.
- A general solution contains an arbitrary constant; differentiate a proposed family to verify the equation.

Step 1: Identify a constant ratio The equation $x dy - y dx = 0$ gives $y dx - x dy = 0$. Therefore

$$d\left(\frac{x}{y}\right) = 0, \quad \frac{x}{y} = k$$

on an interval where $y \neq 0$. Equivalently, separating $dy/dx = y/x$ gives $y = Cx$ where $x \neq 0$.

Step 2: Check the printed families For $x = ky$, differentiating gives $dx = k dy$, and substitution yields $x dy - y dx = ky dy - yk dy = 0$. Options (B) and (D) instead give $x dy + y dx = 0$. Option (A) gives $x dx - y dy = 0$, a different relation. The solution $y = 0$ also satisfies the equation directly; the printed constant-ratio family is the matching choice.

Step 3: Conclusion The printed general-solution family is $x = ky$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate each proposed relation: the required cross terms have a minus sign, so the ratio must be constant.

5. The integrating factor of the differential equation $2x \frac{dy}{dx} - y = 3$ is

[1 mark]

- (A) $\sqrt{x}$
- (B) $\frac{1}{\sqrt{x}}$
- (C) e^x
- (D) e^{-x}

Correct Answer: (B) $\frac{1}{\sqrt{x}}$

Solution:

Concept:

- For $dy/dx + P(x)y = Q(x)$, the integrating factor is $e^{\int P(x) dx}$.
- First divide by the coefficient of dy/dx , since the formula requires that coefficient to be 1.

Quick Tip to Solve: Divide by $2x$ before reading $P(x)$; the negative coefficient of y produces the reciprocal square root.

Step 1: Put the equation in standard form Dividing $2x \, dy/dx - y = 3$ by $2x$, for $x \neq 0$, gives

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{3}{2x}.$$

Thus $P(x) = -1/(2x)$. The constant on the right does not enter the integrating-factor formula.

Step 2: Evaluate the integrating factor and compare On the positive- x interval used by the printed choices,

$$\text{I.F.} = e^{\int -\frac{1}{2x} dx} = e^{-\frac{1}{2} \log x} = x^{-1/2} = \frac{1}{\sqrt{x}}.$$

More generally, on an interval excluding zero it is $|x|^{-1/2}$. Option (A) reverses the sign of the exponent; (C) and (D) would require constant coefficients $P = 1$ and $P = -1$, respectively.

Step 3: Conclusion The integrating factor in the options is $1/\sqrt{x}$. Hence, the correct option is (B).

Quick Tip to Solve: Divide by $2x$ before reading $P(x)$; the negative coefficient of y produces the reciprocal square root.

6. If $|\vec{a}| = 5$ and $-2 \leq \lambda \leq 1$, then the sum of greatest and the smallest value of $|\lambda \vec{a}|$ is [1 mark]

- (A) -5
- (B) 5
- (C) 10
- (D) 15

Correct Answer: (C) 10

Solution:

Concept:

- For a scalar λ , $|\lambda \vec{a}| = |\lambda| |\vec{a}|$; magnitude is always nonnegative.
- Find the smallest and largest absolute values over the entire allowed interval, including zero.

Step 1: Find the extrema of the scalar magnitude The scalar interval is $-2 \leq \lambda \leq 1$. Since zero belongs to it, the smallest possible $|\lambda|$ is 0. The endpoint values are $|-2| = 2$ and $|1| = 1$, so the greatest possible $|\lambda|$ is 2.

Step 2: Scale by the vector magnitude and add Using $|\vec{a}| = 5$, the smallest magnitude is $0 \times 5 = 0$, and the greatest is $2 \times 5 = 10$. Their sum is $0 + 10 = 10$. Option (A) cannot be a sum of magnitudes because it is negative. Option (B) misses the greatest value at $\lambda = -2$; (D) wrongly takes the minimum at $\lambda = 1$ instead of the interior point $\lambda = 0$.

Step 3: Conclusion The required sum is 10. Hence, the correct option is (C).

Quick Tip to Solve: For magnitude questions replace lambda by its absolute value and check whether its interval contains zero.

7. Vector of magnitude 3 making equal angles with x and y axes and perpendicular to z axis is [1 mark]

- (A) $\hat{i} + 2\sqrt{2}\hat{j}$
(B) $3\hat{k}$
(C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$
(D) $\sqrt{3}\hat{i} + \sqrt{3}\hat{j} + \sqrt{3}\hat{k}$

Correct Answer: (C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$

Solution:

Concept:

- A vector of magnitude r has components rl, rm, rn , where l, m, n are its direction cosines and $l^2 + m^2 + n^2 = 1$.
- Equal angles with the x - and y -axes imply $l = m$; perpendicularity to the z -axis implies $n = 0$.

Step 1: Apply the angle conditions With $l = m$ and $n = 0$, the direction-cosine identity gives $2l^2 = 1$. For the positive components listed in the matching choice, $l = m = 1/\sqrt{2}$. Multiplying by magnitude 3 gives components $3/\sqrt{2} = 3\sqrt{2}/2$ along both x and y , and zero along z .

Step 2: Test all listed vectors Option (C) has equal x and y components, zero z component, and squared magnitude $9/2 + 9/2 = 9$. Option (A) has magnitude 3 but unequal x and y components, so its angles are unequal. Option (B) lies along z and fails perpendicularity. Option (D) has magnitude 3 and equal components, but its nonzero z component fails the last condition. The opposite of (C) also has equal obtuse angles, but is not a printed option.

Step 3: Conclusion The vector is $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$. Hence, the correct option is (C).

Quick Tip to Solve: Test zero z -component and equal x, y -components first; then square the remaining components to check magnitude 3.

8. Direction cosines of line $x = y = 1 - z$ are

[1 mark]

- (A) 1, 1, 1
(B) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$
(C) 0, 0, 1
(D) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Correct Answer: (B) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Solution:

Concept:

- Direction ratios are proportional to the coefficients of a common line parameter.
- Direction cosines are obtained by dividing direction ratios by their Euclidean length, so their

Quick Tip to Solve: Solve $1-z=t$ as $z=1-t$ before extracting direction ratios; the constant 1 shifts the line but does not affect its direction.

squares sum to 1.

Step 1: Introduce a common parameter Put $x = y = 1 - z = t$. Then $x = t$, $y = t$ and $z = 1 - t$. As t increases, the coordinate changes are proportional to $(1, 1, -1)$, so these are direction ratios of the line.

Step 2: Normalize and compare the options The length of $(1, 1, -1)$ is $\sqrt{1+1+1} = \sqrt{3}$. Therefore one set of direction cosines is

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right).$$

Option (A) is not normalized; (C) has no change in x or y ; and (D) has a positive z -change while x and y increase, contrary to $z = 1 - t$. Reversing the orientation negates all three cosines simultaneously.

Step 3: Conclusion The direction cosines are $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$. Hence, the correct option is (B).

Quick Tip to Solve: Solve $1-z=t$ as $z=1-t$ before extracting direction ratios; the constant 1 shifts the line but does not affect its direction.

9. In a linear programming problem, the linear function which has to be maximized or minimized is called [1 mark]

- (A) a feasible function
- (B) an objective function
- (C) an optimal function
- (D) a constraint

Correct Answer: (B) an objective function

Solution:

Concept:

- A linear programming problem optimizes a linear objective $Z = ax + by$ subject to linear constraints.
- The feasible region consists of points satisfying all constraints; an optimal point gives the best allowed objective value.

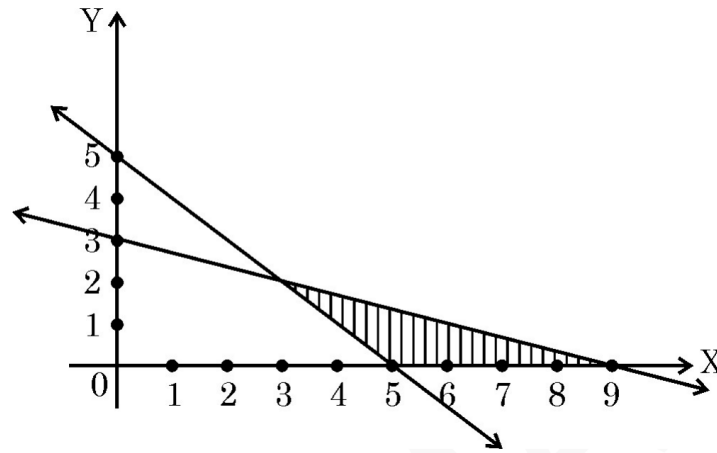
Step 1: Identify the role of the given function The wording asks for the name of the function that is maximized or minimized. In a linear programming model, a quantity such as profit or cost is represented by $Z = ax + by$, with fixed coefficients a, b and decision variables x, y . This target quantity specifies the aim of the optimization.

Step 2: Distinguish it from the other terms A constraint is an inequality or equality restricting the allowable decisions, rather than the quantity being optimized. Feasible describes a point or region satisfying those restrictions, so (A) is not the standard term for the target function. Optimal describes a best solution or value obtained after optimization; it does not name the function itself. Thus only (B) matches the definition.

Step 3: Conclusion The linear function is called the objective function. Hence, the correct option is (B).

Quick Tip to Solve: Ask what the model aims to maximize or minimize: that expression is the objective, while restrictions are constraints.

10. For the feasible region shown below, the non-trivial constraints of the linear programming problem are [1 mark]



(A) $x + y \leq 5$, $x + 3y \leq 9$

(B) $x + y \leq 5$, $x + 3y \geq 9$

(C) $x + y \geq 5$, $x + 3y \leq 9$

(D) $x + y \geq 5$, $3x + y \leq 9$

Correct Answer: (C) $x + y \geq 5$, $x + 3y \leq 9$

Solution:

Concept:

- A line with intercepts $(a, 0)$ and $(0, b)$ has equation $x/a + y/b = 1$.
- The shaded side of a boundary determines whether the corresponding linear expression is at most or at least its boundary value.

Step 1: Identify the two boundary lines The steeper line passes through $(0, 5)$ and $(5, 0)$, so its equation is $x + y = 5$. The other passes through $(0, 3)$ and $(9, 0)$, giving $x/9 + y/3 = 1$, or $x + 3y = 9$. These intercepts exclude the line $3x + y = 9$ in option (D).

Step 2: Test a point in the shaded region Choose $(6, \frac{1}{2})$, inside the shaded triangle. Here $x + y = 6.5 > 5$ and $x + 3y = 7.5 < 9$. Thus the shaded region lies on the $x + y \geq 5$ side of the first line and the $x + 3y \leq 9$ side of the second. Options (A) and (B) choose the wrong side of the first boundary. The boundaries themselves are included, as indicated by solid lines.

Step 3: Conclusion The non-trivial constraints are $x + y \geq 5$ and $x + 3y \leq 9$. Hence, the correct option is (C).

Quick Tip to Solve: Read both intercepts first, then test one point inside the shading to fix both inequality signs.

11. For two events A and B such that $P(A) \neq 0$ and $P(B) \neq 1$, $P(A'/B') =$

[1 mark]

(A) $1 - P(A/B)$

(B) $1 - P(A'/B)$

(C) $\frac{1 - P(A \cap B)}{P(B')}$

(D) $\frac{1 - P(A \cup B)}{P(B')}$

Correct Answer: (D) $\frac{1 - P(A \cup B)}{P(B')}$

Solution:

Concept:

- For events E, F with $P(F) > 0$, conditional probability is $P(E | F) = P(E \cap F)/P(F)$.
- De Morgan's law gives $A' \cap B' = (A \cup B)'$, and the probability of a complement is one minus the original probability.

Step 1: Apply the conditional probability definition Since $P(B) \neq 1$, we have $P(B') = 1 - P(B) > 0$, so the requested conditional probability is defined. Keeping the conditioning event as B' ,

$$P(A' | B') = \frac{P(A' \cap B')}{P(B')}.$$

Step 2: Simplify the numerator The event that neither A nor B occurs is the complement of their union. Thus

$$P(A' \cap B') = 1 - P(A \cup B).$$

Options (A) and (B) condition on B , not B' , so they do not express the required quantity in general. Option (C) uses the complement of the intersection, which represents at least one event failing rather than both failing.

Step 3: Conclusion $P(A'/B') = \frac{1 - P(A \cup B)}{P(B')}$. Hence, the correct option is (D).

Quick Tip to Solve: Keep the denominator equal to the probability of the conditioning event; simplify the intersection of complements using De Morgan's law.

12. A relation R on set $A = \{1, 2, 3\}$ defined as $R = \{(1, 2), (2, 1), (2, 2)\}$ is

[1 mark]

- (A) Reflexive only
- (B) Reflexive and Transitive
- (C) Symmetric and Transitive
- (D) Symmetric only

Correct Answer: (D) Symmetric only

Solution:

Concept:

- A relation is reflexive when every (a, a) belongs to it, symmetric when (a, b) implies (b, a) , and transitive when $(a, b), (b, c)$ imply (a, c) .
- One counterexample is enough to disprove reflexivity or transitivity; symmetry requires checking each listed pair.

Step 1: Check reflexivity and symmetry Reflexivity on $A = \{1, 2, 3\}$ requires $(1, 1), (2, 2), (3, 3)$. The first and third are missing, so the relation is not reflexive. For symmetry, $(1, 2)$ has reverse $(2, 1)$, $(2, 1)$ has reverse $(1, 2)$, and $(2, 2)$ is its own reverse. Therefore all listed pairs satisfy symmetry.

Step 2: Test transitivity with a chain The pairs $(1, 2)$ and $(2, 1)$ both lie in R . Transitivity would consequently require $(1, 1)$, but that pair is absent. Hence R is not transitive. Options (A) and (B) incorrectly assert reflexivity; option (C) incorrectly asserts transitivity. The absence of pairs involving 3 does not invalidate symmetry, since symmetry only tests pairs already in the relation.

Step 3: Conclusion The relation is symmetric only. Hence, the correct option is (D).

Quick Tip to Solve: Use the chain $1 \rightarrow 2 \rightarrow 1$: its missing closing pair $(1, 1)$ immediately disproves transitivity.

13. If A and B are square matrices of same order, then which of the following statements is/are always true?

(i) $(A + B)(A - B) = A^2 - B^2$

(ii) $AB = BA$

(iii) $(A + B)^2 = A^2 + AB + BA + B^2$

(iv) $AB = 0 \Rightarrow A = 0$ or $B = 0$

[1 mark]

(A) Only (i) and (iii)

(B) Only (ii) and (iii)

(C) Only (iii)

(D) Only (iii) and (iv)

Correct Answer: (C) Only (iii)

Solution:

Concept:

- Matrix multiplication is distributive and associative, but square matrices need not commute: in general $AB \neq BA$.
- Nonzero matrices can have zero product, so the scalar zero-product rule cannot be used for matrices.

Step 1: Expand both expressions without changing order Distribution gives

$$\begin{aligned}(A + B)(A - B) &= A^2 - AB + BA - B^2, \\ (A + B)^2 &= A^2 + AB + BA + B^2.\end{aligned}$$

Thus (iii) is always true. Statement (i) needs $AB = BA$ to cancel its middle terms and is not always true.

Step 2: Disprove the remaining universal claims For (ii), take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$AB = B \neq 0$, whereas $BA = 0$, so commutativity fails. For (iv), take $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Both are nonzero, but $CD = 0$. Hence (iv) also fails. Options (A), (B) and (D) each include a statement that is not always valid.

Step 3: Conclusion Only statement (iii) is always true. Hence, the correct option is (C).

Quick Tip to Solve: Expand matrix products in the written order; never combine AB with BA unless commutativity is explicitly given.

14. If $A = \begin{pmatrix} 1 & a & b \\ -1 & 2 & c \\ 0 & 5 & 3 \end{pmatrix}$ is a symmetric matrix, then the value of $3a + b + c$ is

[1 mark]

- (A) 2
- (B) 6
- (C) 4
- (D) 0

Correct Answer: (A) 2

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, which means corresponding entries on opposite sides of the main diagonal are equal.
- The diagonal entries have no additional restriction from symmetry; determine the unknowns from off-diagonal pairs.

Step 1: Compare entries reflected across the diagonal The entries in positions (1, 2) and (2, 1) give $a = -1$. Similarly, positions (1, 3) and (3, 1) give $b = 0$, and positions (2, 3) and (3, 2) give $c = 5$. Each unknown is fixed directly by its reflected entry.

Step 2: Substitute into the requested expression Using these three values,

$$3a + b + c = 3(-1) + 0 + 5 = -3 + 5 = 2.$$

The main-diagonal entries 1, 2, 3 already equal their own transposes, so they do not enter the comparison. Options (B), (C) and (D) give 6, 4 and 0, respectively, none of which agrees with this substitution. In particular, the minus sign in $a = -1$ must be retained when multiplying by 3.

Step 3: Conclusion $3a + b + c = 2$. Hence, the correct option is (A).

Quick Tip to Solve: For symmetry, match entries across the main diagonal; preserve the negative entry when evaluating $3a$.

15. If $A = \begin{pmatrix} \frac{1}{2} \cos x & -\sin x \\ \sin x & \frac{1}{2} \cos x \end{pmatrix}$ and $A + A' = I$, then value of $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is

[1 mark]

- (A) $\frac{\pi}{2}$
(B) $\frac{\pi}{3}$
(C) 0
(D) $-\frac{\pi}{2}$

Correct Answer: (C) 0

Solution:

Concept:

- The transpose interchanges the rows and columns; the diagonal entries remain unchanged.
- In $A + A^T$, the opposite off-diagonal entries $-\sin x$ and $\sin x$ cancel, and each diagonal entry doubles.

Step 1: Add the matrix to its transpose The transpose is

$$A^T = \begin{pmatrix} \frac{1}{2} \cos x & \sin x \\ -\sin x & \frac{1}{2} \cos x \end{pmatrix}.$$

Consequently,

$$A + A^T = \begin{pmatrix} \cos x & 0 \\ 0 & \cos x \end{pmatrix} = (\cos x)I.$$

Equating this to I requires $\cos x = 1$.

Step 2: Use the specified interval and check options The general solution of $\cos x = 1$ is $x = 2k\pi$, where k is an integer. Within $[-\pi/2, \pi/2]$, only $k = 0$ is possible. The endpoint options $\pi/2$ and $-\pi/2$ have cosine zero, while $\pi/3$ has cosine $1/2$; none satisfies the matrix equation. For $x = 0$, the matrix is $\frac{1}{2}I$, whose sum with its transpose is indeed I .

Step 3: Conclusion The required value is $x = 0$. Hence, the correct option is (C).

Quick Tip to Solve: Cancel the sine entries in $A + A^T$ first, then compare a diagonal entry with the identity matrix.

16. For a square matrix A , $(3A)^{-1}$ equals

[1 mark]

- (A) $3A^{-1}$
- (B) $9A^{-1}$
- (C) $\frac{1}{3}A^{-1}$
- (D) $\frac{1}{9}A^{-1}$

Correct Answer: (C) $\frac{1}{3}A^{-1}$

Solution:

Concept:

- For a nonzero scalar k and an invertible matrix A , the inverse of kA is $k^{-1}A^{-1}$.
- An inverse must give the identity when multiplied on either side; this provides a direct check on a proposed scalar factor.

Step 1: Verify the inverse by multiplication The notation $(3A)^{-1}$ presupposes that A is invertible. Multiply the proposed matrix by $3A$:

$$(3A) \left(\frac{1}{3}A^{-1}\right) = AA^{-1} = I.$$

The reverse product also gives $\left(\frac{1}{3}A^{-1}\right)(3A) = A^{-1}A = I$. Thus the factor must compensate for the original scalar 3.

Step 2: Check the incorrect scalar factors Multiplying $3A$ by $3A^{-1}$ gives $9I$, and multiplying by $9A^{-1}$ gives $27I$. Multiplying by $\frac{1}{9}A^{-1}$ gives $\frac{1}{3}I$. These are not the identity, so options (A), (B) and (D) fail the defining property of an inverse. The factor is independent of the order of the square matrix.

Step 3: Conclusion $(3A)^{-1} = \frac{1}{3}A^{-1}$. Hence, the correct option is (C).

Quick Tip to Solve: Invert the scalar as well as the matrix: a factor 3 outside A becomes a factor $1/3$ outside A^{-1} .

17. If $\begin{vmatrix} -1 & -2 & 5 \\ -2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = -86$, then the sum of all possible values of a is

[1 mark]

- (A) 4
- (B) 5
- (C) -4
- (D) 9

Correct Answer: (C) -4

Solution:

Concept:

- Expansion along a row uses alternating cofactor signs $+, -, +$.

Quick Tip to Solve: After expansion, use the coefficient ratio $-q/p$ for the root sum; factoring is a useful sign check.

- For a quadratic $pa^2 + qa + r = 0$, the sum of its roots is $-q/p$; one may also factor it to verify all possible values.

Step 1: Expand the determinant along the first row Keeping the cofactor signs,

$$\begin{aligned} D &= (-1)(2a^2 + 4) - (-2)(-4a) + 5(-8) \\ &= -2a^2 - 4 - 8a - 40 \\ &= -2a^2 - 8a - 44. \end{aligned}$$

The middle minor is $(-2)(2a) - (-1)(0) = -4a$, and the last minor is $(-2)(4) - a(0) = -8$.

Step 2: Set the determinant equal to its given value The condition $D = -86$ gives

$$-2a^2 - 8a - 44 = -86 \implies a^2 + 4a - 21 = 0.$$

Factoring gives $(a + 7)(a - 3) = 0$, so the possible values are -7 and 3 . Their sum is -4 , also obtained from $-4/1$ by the root-sum formula. Options (A), (B) and (D) have positive sums and therefore do not match the quadratic obtained from the determinant.

Step 3: Conclusion The sum of all possible values of a is -4 . Hence, the correct option is (C).

Quick Tip to Solve: After expansion, use the coefficient ratio $-q/p$ for the root sum; factoring is a useful sign check.

18. If $e^{x+y} = 3x$, then $\frac{dy}{dx}$ equals

[1 mark]

- (A) $\frac{3}{e^{x+y}}$
 (B) $\frac{1}{e^{x+y}}$
 (C) $\frac{1 - e^{x+y}}{e^{x+y}}$
 (D) $\frac{3 - e^{x+y}}{e^{x+y}}$

Correct Answer: (D) $\frac{3 - e^{x+y}}{e^{x+y}}$

Solution:

Concept:

- When y depends on x , the chain rule gives $\frac{d}{dx}e^{x+y} = e^{x+y} \left(1 + \frac{dy}{dx}\right)$.
- Implicit differentiation requires differentiating both sides and then isolating dy/dx .

Step 1: Differentiate the equation implicitly From $e^{x+y} = 3x$, differentiation with respect to x gives

$$e^{x+y} \left(1 + \frac{dy}{dx}\right) = 3.$$

The factor $1 + dy/dx$ is the derivative of the entire exponent $x + y$; ignoring the derivative of x would

Quick Tip to Solve: Differentiate the full exponent $x + y$ before dividing by the exponential; the extra 1 is essential.

omit the 1.

Step 2: Isolate the derivative and compare the forms Since the exponential is positive, divide by it and rearrange:

$$\frac{dy}{dx} = \frac{3}{e^{x+y}} - 1 = \frac{3 - e^{x+y}}{e^{x+y}}.$$

Option (A) omits the subtraction of 1. Option (B) uses the wrong numerator and also omits that subtraction; option (C) uses 1 in place of the derivative 3 of $3x$. As a check, the original relation gives $y = \ln(3x) - x$ for $x > 0$, whose derivative $1/x - 1$ agrees with the result.

Step 3: Conclusion $\frac{dy}{dx} = \frac{3 - e^{x+y}}{e^{x+y}}$. Hence, the correct option is (D).

Quick Tip to Solve: Differentiate the full exponent $x + y$ before dividing by the exponential; the extra 1 is essential.

19. Assertion (A): A line can have direction cosines $\langle 1, 1, 1 \rangle$.

Reason (R): $\cos \theta = 1$ is possible for $\theta = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- The direction cosines l, m, n of a line are its unit direction vector components and satisfy $l^2 + m^2 + n^2 = 1$.
- For a real angle, $\cos 0 = 1$; each individual direction cosine can equal one, but the other two must then be zero.

Step 1: Judge the assertion For the proposed direction cosines $\langle 1, 1, 1 \rangle$, the sum of their squares is $1 + 1 + 1 = 3$, whereas a unit direction vector requires the sum to be 1. Thus the assertion is false. These numbers may be direction ratios, but cannot themselves be direction cosines.

Step 2: Judge the reason The identity $\cos 0 = 1$ is valid, so the reason is true. It concerns a single angle, and does not state that one line can simultaneously make angle zero with all three mutually perpendicular coordinate axes.

Step 3: Check the explanatory relation A true fact about one cosine cannot overcome the unit-vector condition violated by the assertion. Options (A) and (B) incorrectly require a true assertion, while (C) incorrectly declares the reason false.

Step 4: Conclusion Assertion is false and Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Before accepting direction cosines, square all three and check that their sum is exactly one.

20. For two vectors $\vec{a}$ and $\vec{b}$:

Assertion (A): $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

Reason (R): $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b}) \tan \theta, \left(\theta \neq \frac{\pi}{2}\right)$

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- If $\theta \in [0, \pi]$ is the angle between two nonzero vectors, then $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$ and $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$.
- The identities $\sin^2 \theta + \cos^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$ connect these products.

Step 1: Judge the assertion Using the magnitude formulae gives

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2.$$

Thus the assertion is true, including at $\theta = \pi/2$. If a vector is zero, both sides are zero.

Step 2: Judge the reason For the printed domain $\theta \neq \pi/2$, $\tan \theta$ is defined and

$$(\vec{a} \cdot \vec{b}) \tan \theta = |\vec{a}| |\vec{b}| \cos \theta \frac{\sin \theta}{\cos \theta} = |\vec{a}| |\vec{b}| \sin \theta = |\vec{a} \times \vec{b}|.$$

The reason is true. For an obtuse angle both the dot product and tangent are negative, so their product remains nonnegative.

Step 3: Check whether the reason explains the assertion Squaring the relation in the reason and adding $(\vec{a} \cdot \vec{b})^2$ gives $(\vec{a} \cdot \vec{b})^2 (1 + \tan^2 \theta) = |\vec{a}|^2 |\vec{b}|^2$. This is precisely the assertion wherever the stated tangent relation applies; the excluded right-angle case is verified directly in the first step. Hence the reason supplies its explanation. Options (B), (C) and (D) misclassify either that relation or a truth value.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Use the printed exclusion of the right angle before cancelling cosine in the tangent relation.

Section B (10 marks)

This section comprises Very Short Answer (VSA) type questions carrying 2 marks each.

21(a). Find the absolute maximum value of $f(x) = \cos x + \sin^2 x$, $x \in [0, \pi]$.

[2 marks]

Solution:

Concept:

- For a continuous function on a closed interval, the absolute maximum occurs at an endpoint or at an interior critical point.
- Differentiate using $\frac{d}{dx}(\sin^2 x) = 2 \sin x \cos x$.

Step 1: Find the critical point The given function is differentiable on $[0, \pi]$, and

$$f'(x) = -\sin x + 2 \sin x \cos x = \sin x(2 \cos x - 1).$$

In the interior $(0, \pi)$, $\sin x > 0$. Thus $f'(x) = 0$ requires $\cos x = 1/2$, giving the sole interior critical point $x = \pi/3$. The solutions of $\sin x = 0$ are the endpoints, which must also be checked.

Step 2: Compare the values at all candidates At the endpoints, $f(0) = 1$ and $f(\pi) = -1$. At the critical point,

$$f(\pi/3) = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}.$$

Since f is continuous on the closed interval, comparing these three values covers every possible location of an absolute maximum. The value $5/4$ exceeds both endpoint values.

Step 3: Conclusion The absolute maximum value is $\boxed{5/4}$, attained at $x = \pi/3$.

Quick Tip to Solve: Always compare the endpoint values with the stationary-point value when the domain is a closed interval.

OR

21(b). If the volume of a solid hemisphere increases at a uniform rate, prove that its surface area varies inversely as its radius.

[2 marks]

Solution:

Concept:

- For a solid hemisphere of radius r , volume is $V = \frac{2}{3}\pi r^3$, and total surface area is $S = 3\pi r^2$.
- With radius depending on time t , apply the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$ and similarly for surface area.

Step 1: Use the uniform increase of volume Let the constant positive rate of increase of volume be

Quick Tip to Solve: Differentiate both volume and area with respect to time; inverse proportionality here concerns the area growth rate.

$dV/dt = k$. Differentiating the volume formula with respect to time gives

$$k = 2\pi r^2 \frac{dr}{dt}, \quad \frac{dr}{dt} = \frac{k}{2\pi r^2}.$$

Thus the radius grows more slowly as the hemisphere becomes larger, even though the volume continues to increase uniformly.

Step 2: Find the rate of change of surface area Differentiate $S = 3\pi r^2$ with respect to time and substitute the preceding expression:

$$\frac{dS}{dt} = 6\pi r \frac{dr}{dt} = 6\pi r \frac{k}{2\pi r^2} = \frac{3k}{r}.$$

Therefore $r dS/dt = 3k$, a constant, proving the inverse dependence of the surface-area variation rate on radius. If surface area denotes only the curved area $2\pi r^2$, its rate is $2k/r$, with the same inverse dependence.

Step 3: State the precise meaning of the variation The varying quantity established by this related-rates calculation is the rate of increase of surface area. The area itself is $3\pi r^2$ and is proportional to the square of the radius, so a literal inverse-proportionality statement for the area itself would be false.

Step 4: Conclusion The surface-area rate satisfies $dS/dt = 3k/r \propto 1/r$. Thus its rate of variation is inversely proportional to the radius.

Quick Tip to Solve: Differentiate both volume and area with respect to time; inverse proportionality here concerns the area growth rate.

22. If $\overrightarrow{AB} = \hat{j} + \hat{k}$ and $\overrightarrow{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$ represent the two vectors along the sides AB and AC of $\triangle ABC$, prove that the median $\overrightarrow{AD} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}$, where D is midpoint of BC . Hence, find the length of median AD . [2 marks]

Solution:

Concept:

- A midpoint has position vector equal to the average of the position vectors of the two endpoints.
- The length of a vector $u\hat{i} + v\hat{j} + w\hat{k}$ is $\sqrt{u^2 + v^2 + w^2}$.

Step 1: Prove the median-vector relation Take any origin O , and let the position vectors of A, B, C be $\vec{a}, \vec{b}, \vec{c}$. Because D is the midpoint of BC , its position vector is $(\vec{b} + \vec{c})/2$. Consequently,

$$\overrightarrow{AD} = \frac{\vec{b} + \vec{c}}{2} - \vec{a} = \frac{(\vec{b} - \vec{a}) + (\vec{c} - \vec{a})}{2} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}.$$

This subtraction makes the result independent of the choice of origin.

Quick Tip to Solve: Use A as the temporary origin: the midpoint vector is half the sum of the two given side vectors.

Step 2: Substitute the given side vectors Adding the components in the printed vectors,

$$\overrightarrow{AB} + \overrightarrow{AC} = (\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k}) = 3\hat{i} + 5\hat{k}.$$

Hence $\overrightarrow{AD} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}$. Its $\hat{j}$ -component vanishes because the two side vectors have opposite $\hat{j}$ -components.

Step 3: Calculate the length The length is the magnitude of the median vector:

$$AD = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{9+25}{4}} = \frac{\sqrt{34}}{2}.$$

Averaging the vectors first, and then taking the magnitude, is essential; averaging the side lengths would not give the median length.

Step 4: Conclusion $\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{AC})/2$, and the median length is $\boxed{\sqrt{34}/2}$.

Quick Tip to Solve: Use A as the temporary origin: the midpoint vector is half the sum of the two given side vectors.

23. Find the co-ordinates of foot of perpendicular drawn from $(0, 0, 0)$ to line $\frac{x}{1} = \frac{y+1}{-1} = \frac{z-3}{-2}$. [2 marks]

Solution:

Concept:

- A point on a symmetric-form line is obtained by setting each ratio equal to one parameter t .
- At the perpendicular foot P , the vector from the origin to P has zero dot product with the line direction vector.

Step 1: Parametrise the printed line For the line $x/1 = (y+1)/(-1) = (z-3)/(-2)$, let the common ratio be t . Its points have coordinates

$$P(t) = (t, -1-t, 3-2t),$$

and its direction vector is $\vec{d} = (1, -1, -2)$. In particular, at $t = 0$ the line passes through $(0, -1, 3)$. The denominators, including their signs, determine the direction vector.

Step 2: Impose perpendicularity The vector from $O = (0, 0, 0)$ to $P(t)$ is $(t, -1-t, 3-2t)$. Therefore

$$\overrightarrow{OP} \cdot \vec{d} = t + (-1-t)(-1) + (3-2t)(-2) = 6t - 5.$$

The perpendicular condition gives $6t - 5 = 0$, so $t = 5/6$. Since $\vec{d} \neq \vec{0}$, this equation determines a unique foot on the line.

Step 3: Obtain and check the coordinates Substituting $t = 5/6$ yields

$$x = \frac{5}{6}, \quad y = -1 - \frac{5}{6} = -\frac{11}{6}, \quad z = 3 - \frac{10}{6} = \frac{4}{3}.$$

Quick Tip to Solve: Keep the signed denominators as direction ratios, then set the dot product of OP with that direction to zero.

The three line ratios are all $5/6$, and the dot product is $5/6 + 11/6 - 8/3 = 0$. Thus the computed point lies on the line and the segment from the origin is perpendicular to it.

Step 4: Conclusion The foot of perpendicular is $(5/6, -11/6, 4/3)$.

Quick Tip to Solve: Keep the signed denominators as direction ratios, then set the dot product of OP with that direction to zero.

24(a). Check whether $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x-2}{x-3}$ is onto or not.

[2 marks]

Solution:

Concept:

- A function is onto when every element of its stated codomain has at least one preimage in its domain.
- To test surjectivity, solve $f(x) = a$ for an arbitrary codomain value a , checking both exceptional values and the domain restriction $x \neq 3$.

Step 1: Test the codomain value 1 The codomain is $\mathbb{R}$, so it contains 1. If this value had a preimage, then

$$\frac{x-2}{x-3} = 1, \quad x \neq 3.$$

Multiplication by the nonzero denominator gives $x-2 = x-3$, or $-2 = -3$, a contradiction. Thus no allowed input maps to 1.

Step 2: Identify the other values in the range For $a \neq 1$, solving $a(x-3) = x-2$ gives

$$x = \frac{3a-2}{a-1}.$$

This input is allowed, since $x-3 = 1/(a-1) \neq 0$. Substitution gives $f(x) = a$. Hence the range is exactly $\mathbb{R} \setminus \{1\}$, which is smaller than the codomain.

Step 3: Conclusion The function is not onto $\mathbb{R}$, because the codomain element 1 has no preimage.

Quick Tip to Solve: Rewrite the function as $f(x) = 1 + 1/(x-3)$: the reciprocal term can never be zero, immediately revealing the missing value 1.

24(b). Check whether $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) defined as $f(x, y) = (2y, 3x)$ is injective or not. [2 marks]

Solution:

Concept:

- A function is injective if equality of the images of any two domain elements forces those domain elements to be equal.
- Two ordered pairs are equal exactly when their first coordinates are equal and their second coordinates are equal. Nonzero integer factors may be cancelled from such equalities.

Step 1: Equate the images of arbitrary input pairs Let (x_1, y_1) and (x_2, y_2) be arbitrary elements of $\mathbb{Z} \times \mathbb{Z}$, and suppose that their images are equal:

$$f(x_1, y_1) = f(x_2, y_2).$$

Using the given rule, this becomes

$$(2y_1, 3x_1) = (2y_2, 3x_2).$$

Coordinatewise equality therefore gives $2y_1 = 2y_2$ and $3x_1 = 3x_2$.

Step 2: Recover the original ordered pairs Cancelling 2 in the first coordinate equality gives $y_1 = y_2$, while cancelling 3 in the second gives $x_1 = x_2$. Consequently,

$$(x_1, y_1) = (x_2, y_2).$$

These deductions apply to any two integer input pairs, so distinct inputs cannot produce the same output. Also, $2y$ and $3x$ are integers for every integer input pair, as required by the codomain.

Step 3: Conclusion The function $f(x, y) = (2y, 3x)$ is injective on $\mathbb{Z} \times \mathbb{Z}$.

Quick Tip to Solve: Equate corresponding output coordinates; the swapped coordinate order does not affect injectivity because both nonzero factors can be cancelled.

25. If $x = \sin t - \cos t$, $y = \sin t \cos t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [2 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$ and $y = y(t)$, the derivative is $dy/dx = (dy/dt)/(dx/dt)$, provided $dx/dt \neq 0$ at the parameter value.
- The product rule gives $\frac{d}{dt}(\sin t \cos t) = \cos^2 t - \sin^2 t = \cos 2t$.

Quick Tip to Solve: Use $y = \frac{1}{2} \sin 2t$ to obtain $dy/dt = \cos 2t$ immediately, but still check that dx/dt is nonzero before dividing.

Step 1: Differentiate with respect to the parameter From $x = \sin t - \cos t$, differentiation gives

$$\frac{dx}{dt} = \cos t + \sin t.$$

Similarly, applying the product rule to $y = \sin t \cos t$ gives

$$\frac{dy}{dt} = \cos t \cos t + \sin t(-\sin t) = \cos^2 t - \sin^2 t.$$

Thus the required slope is the ratio of these derivatives.

Step 2: Evaluate at the given parameter value At $t = \pi/4$, both $\sin t$ and $\cos t$ equal $1/\sqrt{2}$. Therefore,

$$\left. \frac{dy}{dx} \right|_{t=\pi/4} = \frac{\frac{1}{2} - \frac{1}{2}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{0}{\sqrt{2}} = 0.$$

Since the denominator $dx/dt = \sqrt{2}$ is nonzero, this derivative exists. The curve has a horizontal tangent at the corresponding point $(0, 1/2)$.

Step 3: Conclusion The required value is $\left. \frac{dy}{dx} \right|_{t=\pi/4} = 0$.

Quick Tip to Solve: Use $y = \frac{1}{2} \sin 2t$ to obtain $dy/dt = \cos 2t$ immediately, but still check that dx/dt is nonzero before dividing.

Section C (18 marks)

This section comprises Short Answer (SA) type questions of 3 marks each.

26. If $\frac{d}{dx}(F(x)) = \frac{1}{e^x + 1}$, then find $F(x)$ given that $F(0) = \log \frac{1}{2}$.

[3 marks]

Solution:

Concept:

- An antiderivative $F(x)$ of a function $f(x)$ satisfies $F'(x) = f(x)$. Integrating determines F up to an additive constant C .
- The logarithmic rule is $\int g'(x)/g(x) dx = \log |g(x)| + C$. A given value of F determines the constant uniquely.

Step 1: Rewrite the integrand in a convenient form Direct integration becomes simpler after writing the numerator as $(e^x + 1) - e^x$. Thus

$$\frac{1}{e^x + 1} = 1 - \frac{e^x}{e^x + 1}.$$

The second fraction has the derivative of its denominator in the numerator, so the logarithmic integration rule applies. Also, $1 + e^x > 0$ for every real x , so no absolute-value sign is needed in this logarithm.

Quick Tip to Solve: For $1/(1 + e^x)$, subtract $e^x/(1 + e^x)$ from 1; the remaining fraction integrates immediately to a logarithm.

Step 2: Integrate and apply the initial condition Integrating both terms gives

$$F(x) = \int \left(1 - \frac{e^x}{1+e^x}\right) dx = x - \log(1+e^x) + C.$$

At $x = 0$, the given condition becomes

$$-\log 2 + C = \log \frac{1}{2} = -\log 2,$$

and hence $C = 0$. Here $\log$ denotes the natural logarithm, as usual in integration.

Step 3: Check the derivative and the given value Differentiating the resulting expression gives

$$\frac{d}{dx} [x - \log(1+e^x)] = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x}.$$

Substitution of $x = 0$ also gives $-\log 2 = \log(1/2)$, so both the differential condition and the specified value are satisfied.

Step 4: Conclusion The required function, valid for all real x , is

$$F(x) = x - \log(1+e^x).$$

Equivalently, $F(x) = \log\left(\frac{e^x}{1+e^x}\right)$.

Quick Tip to Solve: For $1/(1+e^x)$, subtract $e^x/(1+e^x)$ from 1; the remaining fraction integrates immediately to a logarithm.

27(a). Solve the following differential equation: $x \frac{dy}{dx} = y - x \sin^2\left(\frac{y}{x}\right)$, **given that** $y(1) = \frac{\pi}{6}$. [3 marks]

Solution:

Concept:

- A homogeneous differential equation has its right-hand side expressed in terms of y/x . Put $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$.
- A separable equation can be integrated after gathering functions of v with dv and functions of x with dx . Here $\int \csc^2 v dv = -\cot v$.

Step 1: Use the homogeneous substitution Since the initial point has $x = 1$, work on the positive x -interval containing it. Divide the given equation by x and set $v = y/x$:

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right), \quad v + x \frac{dv}{dx} = v - \sin^2 v.$$

Cancelling v from both sides gives $x dv/dx = -\sin^2 v$.

Step 2: Separate variables and integrate At the initial point, $v = \pi/6$, so $\sin v \neq 0$. Separating

Quick Tip to Solve: For $x dy/dx = y - x \sin^2(y/x)$, use $y = vx$: the two v terms cancel, leaving a directly separable equation.

variables for this solution branch gives

$$\csc^2 v \, dv = -\frac{dx}{x}.$$

On integration,

$$-\cot v = -\log x + C_1, \quad \cot v = \log x + C.$$

Constant-ratio solutions with $\sin v = 0$ do not satisfy the specified initial condition and therefore do not need to be included in this particular solution.

Step 3: Determine the constant from the initial value The condition $y(1) = \pi/6$ gives $v(1) = \pi/6$. Therefore

$$\cot \frac{\pi}{6} = \log 1 + C \implies C = \sqrt{3}.$$

Replacing v by y/x now yields the required relation. As a check, differentiating it gives

$$-\csc^2\left(\frac{y}{x}\right) \frac{x \, dy/dx - y}{x^2} = \frac{1}{x},$$

which rearranges to the original differential equation.

Step 4: Conclusion The required solution is $\cot\left(\frac{y}{x}\right) = \log x + \sqrt{3}$, $x > 0$, on the branch $0 < y/x < \pi$ passing through $(1, \pi/6)$.

Quick Tip to Solve: For $x \, dy/dx = y - x \sin^2(y/x)$, use $y = vx$: the two v terms cancel, leaving a directly separable equation.

OR

27(b). Find the general solution of the differential equation: $y \log y \frac{dx}{dy} + x = \frac{2}{y}$.

[3 marks]

Solution:

Concept:

- With y as the independent variable, a linear differential equation has the form $dx/dy + P(y)x = Q(y)$.
- An integrating factor is $I(y) = e^{\int P(y) \, dy}$. Multiplication by this factor turns the left side into $d(xI)/dy$, so $xI = \int QI \, dy + C$.

Step 1: Express the equation in linear form Treat x as a function of y . On an interval where $y > 0$ and $y \neq 1$, divide the given equation by $y \log y$:

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{2}{y^2 \log y}.$$

Thus $P(y) = 1/(y \log y)$ and $Q(y) = 2/(y^2 \log y)$. The factor y^2 on the right arises because the original right-hand side $2/y$ is also divided by $y \log y$.

Quick Tip to Solve: Notice that $(\log y) \, dx/dy + x/y$ is the derivative of $x \log y$; dividing the original equation by y exposes this product directly.

Step 2: Find and apply the integrating factor Using $u = \log y$, so $du = dy/y$, we obtain

$$\int \frac{dy}{y \log y} = \log |\log y|.$$

We may take the integrating factor as $\log y$ on either interval of constant sign, since a nonzero constant multiple is equally valid. Multiplying the linear equation by $\log y$ gives

$$(\log y) \frac{dx}{dy} + \frac{x}{y} = \frac{2}{y^2} \implies \frac{d}{dy}(x \log y) = \frac{2}{y^2}.$$

Step 3: Integrate and verify the resulting family Integration with respect to y yields

$$x \log y = \int 2y^{-2} dy = -\frac{2}{y} + C.$$

The negative sign follows from $\int y^{-2} dy = -y^{-1}$. Differentiating this relation and multiplying by y recovers

$$y \log y \frac{dx}{dy} + x = \frac{2}{y},$$

confirming that the family satisfies the given differential equation on its domain.

Step 4: Conclusion The general solution is $x \log y + \frac{2}{y} = C$, with arbitrary real constant C , on intervals in $y > 0$ away from $y = 1$. Equivalently, $x = (C - 2/y)/\log y$. The member $C = 2$ has a removable singularity at $y = 1$ and extends there with $x(1) = 2$.

Quick Tip to Solve: Notice that $(\log y) dx/dy + x/y$ is the derivative of $x \log y$; dividing the original equation by y exposes this product directly.

28. Solve the following linear programming problem graphically:

Maximize $Z = 8000x + 12000y$

Subject to constraints

$$3x + 4y \leq 60$$

$$x + 3y \leq 30$$

$$x \geq 0, y \geq 0$$

[3 marks]

Solution:

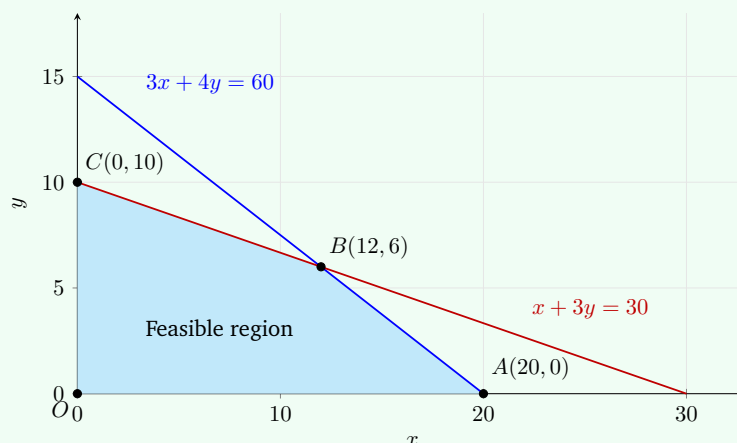
Concept:

- The feasible region is the intersection of the half-planes represented by the constraints.
- A linear objective function on a nonempty bounded feasible polygon attains its maximum at a corner point.

Step 1: Plot the boundary lines and shade their common region The line $3x + 4y = 60$ passes through $(20, 0)$ and $(0, 15)$, while $x + 3y = 30$ passes through $(30, 0)$ and $(0, 10)$. Both inequalities hold at the origin, so shade their common region on that side of each line, restricted to the first quadrant by

Quick Tip to Solve: Plot both intercept pairs, but test only corners of the common feasible region: here $(30, 0)$ and $(0, 15)$ are not feasible.

$x, y \geq 0$. Include the boundary segments.



The shaded feasible region is $OABC$.

Step 2: Find the intersection of the two oblique boundaries Solve the two boundary equations simultaneously. Multiplying the second equation by three and subtracting the first gives

$$\begin{aligned} 3x + 9y &= 90, & 3x + 4y &= 60, \\ 5y &= 30, & y &= 6, \\ x &= 30 - 3(6) = 12. \end{aligned}$$

Thus the feasible polygon has vertices $O(0,0)$, $A(20,0)$, $B(12,6)$ and $C(0,10)$. The intercepts $(30,0)$ and $(0,15)$ violate the other constraint.

Step 3: Evaluate the objective at every corner The region is closed and bounded, so the corner-point method applies directly. Substitution gives

Corner point (x, y)	$Z = 8000x + 12000y$
$(0, 0)$	0
$(20, 0)$	160000
$(12, 6)$	$96000 + 72000 = 168000$
$(0, 10)$	120000

The largest entry is the value at the intersection of the two lines.

Step 4: Conclusion The maximum value is $Z_{\max} = 168000$, attained at $x = 12, y = 6$.

Quick Tip to Solve: Plot both intercept pairs, but test only corners of the common feasible region: here $(30, 0)$ and $(0, 15)$ are not feasible.

29(a). The probability of hitting the target by a trained sniper is three times the probability of not hitting the target on a stormy day due to high wind speed. The sniper fired two shots on the target on a stormy day when wind speed was very high. Find the probability that

(i) target is hit

(ii) atleast one shot misses the target.

[3 marks]

Solution:

Concept:

- If p is the probability of hitting the target with one shot and q that of missing, then $p + q = 1$. Hitting and missing are complementary outcomes.
- For independent shots with the same hit probability, multiply probabilities to obtain a specified sequence of outcomes. The complement rule is $P(A) = 1 - P(A')$.

Step 1: Determine the probability of each single-shot outcome The given relationship is $p = 3q$. Combining this with $p + q = 1$, we obtain

$$3q + q = 1 \Rightarrow q = \frac{1}{4}, \quad p = \frac{3}{4}.$$

Use these probabilities for each of the two independent shots. Let H denote a hit and M a miss, with the order indicating the first and second shots.

Step 2: Find the probability that the target is hit (i) The target is hit if at least one of the two shots hits it. The only outcome failing this condition is MM . Therefore

$$P(\text{target is hit}) = 1 - P(MM) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}.$$

This event includes HH , HM and MH : both hits as well as exactly one hit count. Their respective probabilities are $9/16$, $3/16$ and $3/16$, confirming the sum.

Step 3: Find the probability of at least one miss (ii) At least one miss means HM , MH or MM . Its complement is that both shots hit, namely HH . Hence

$$P(\text{at least one miss}) = 1 - P(HH) = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}.$$

Notice that the two requested events are not complements of one another: HM and MH belong to both. Their probabilities therefore need not add to one.

Step 4: Conclusion (i) The probability that the target is hit is $\boxed{15/16}$. (ii) The probability that at least one shot misses the target is $\boxed{7/16}$.

Quick Tip to Solve: For two shots, compute at least one hit as one minus both misses, and at least one miss as one minus both hits.

29(b). Mother, Father and Son line up at random for a family picture. Let events E : Son on one end and F : Father in the middle. Find $P(E/F)$. [3 marks]

Solution:

Concept:

- For equally likely arrangements, an event's probability is its number of favourable arrangements divided by the total number of arrangements.
- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, provided $P(F) > 0$. Conditioning on F restricts consideration to arrangements in which F occurs.

Step 1: List the equally likely arrangements Write M , F_a and S for Mother, Father and Son respectively; F_a denotes the person, whereas F below denotes the event. Three distinct people can stand in a line in $3! = 6$ equally likely orders. From left to right these are

$$MF_aS, \quad MSF_a, \quad F_aMS, \quad F_aSM, \quad SMF_a, \quad SF_aM.$$

The two ends are the leftmost and rightmost positions. Event E consists of orders with Son occupying either of these positions.

Step 2: Identify the conditioning event and intersection Father is in the middle only in the arrangements MF_aS and SF_aM . Thus event F contains exactly two outcomes, giving

$$P(F) = \frac{2}{6} = \frac{1}{3}.$$

In both these arrangements Son is on an end. Therefore every outcome of F is also an outcome of E ; that is, $F \subseteq E$. Consequently $E \cap F = F$, and

$$P(E \cap F) = \frac{2}{6} = \frac{1}{3}.$$

Step 3: Apply the conditional probability formula Restricting the sample space to the two orders in which Father occupies the middle leaves two favourable orders out of two. Equivalently, the formula gives

$$P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{1/3}{1/3} = 1.$$

The unconditional chance of Son being on an end is $2/3$, but it is not the requested probability: the information about Father's position makes Son's end position certain.

Step 4: Conclusion The required conditional probability is $P(E/F) = 1$, since Father in the middle necessarily leaves Son at one of the ends.

Quick Tip to Solve: Before counting all arrangements, check whether the conditioning event forces the desired event. Here Father in the middle forces Son to an end.

30. Find: $\int \frac{2x+1}{\sqrt{6x+x^2}} dx$

[3 marks]

Solution:

Concept:

- Split the numerator into a multiple of the derivative of the expression under the square root and a constant. Here, $\frac{d}{dx}(x^2 + 6x) = 2x + 6$.
- Complete the square and use $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln |u + \sqrt{u^2 - a^2}| + C$, where a is constant.

Step 1: Split the numerator Write $2x + 1 = (2x + 6) - 5$. This separates the integral into a direct substitution term and a standard logarithmic term. If I denotes the required integral, then

$$I = \int \frac{2x+6}{\sqrt{x^2+6x}} dx - 5 \int \frac{dx}{\sqrt{x^2+6x}}.$$

The real integrand is defined on intervals where $x^2 + 6x > 0$, namely $x < -6$ or $x > 0$.

Step 2: Integrate the derivative term Put $t = x^2 + 6x$, so $dt = (2x + 6) dx$. Therefore,

$$\int \frac{2x+6}{\sqrt{x^2+6x}} dx = \int t^{-1/2} dt = 2\sqrt{x^2+6x}.$$

A single arbitrary constant will be added after combining the terms.

Step 3: Evaluate the remaining standard integral Completing the square gives $x^2 + 6x = (x + 3)^2 - 9$. With $u = x + 3$ and $du = dx$, we get

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2+6x}} &= \int \frac{du}{\sqrt{u^2-3^2}} \\ &= \ln |x+3+\sqrt{x^2+6x}|. \end{aligned}$$

The absolute value keeps the logarithmic antiderivative valid on either real interval. Its derivative is $1/\sqrt{x^2+6x}$, so combining the derivatives yields the required numerator $(2x+6) - 5 = 2x+1$.

Step 4: Conclusion Hence,

$$\boxed{2\sqrt{x^2+6x} - 5 \ln |x+3+\sqrt{x^2+6x}| + C}.$$

Quick Tip to Solve: Compare the numerator with $2x + 6$ first: the missing constant is -5 , and completing the square turns its integral into the logarithmic standard form.

31(a). Evaluate: $\int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}$

[3 marks]

Solution:

Concept:

- Reflection of the integration variable gives $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$. The sum of these limits is $\pi/2$.
- On $0 < x < \pi/2$, $\cot(\pi/2 - x) = \tan x$ and $\sqrt{\tan x} = 1/\sqrt{\cot x}$; positivity allows these reciprocal square roots.

Step 1: Reflect the variable about the midpoint Denote the given integral by

$$I = \int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}.$$

Set $x = \pi/2 - t$. The lower and upper limits become $5\pi/12$ and $\pi/12$, respectively. Since $dx = -dt$, reversing the limits cancels the minus sign. Thus, after renaming t as x ,

$$I = \int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\tan x}}.$$

Step 2: Express the reflected integrand using cotangent All points of the interval lie in the first quadrant, so $\sqrt{\cot x}$ is positive and nonzero. Multiply the numerator and denominator of the reflected integrand by this quantity:

$$\frac{1}{1 + \sqrt{\tan x}} = \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}}.$$

This gives a second expression for the same integral, with exactly the denominator needed to add it to the original expression.

Step 3: Add the two expressions and integrate Adding the original and reflected forms gives

$$\begin{aligned} 2I &= \int_{\pi/12}^{5\pi/12} \left(\frac{1}{1 + \sqrt{\cot x}} + \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}} \right) dx \\ &= \int_{\pi/12}^{5\pi/12} 1 dx = \frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}. \end{aligned}$$

The reflection therefore makes the mean value of the integrand over this interval equal to one half.

Step 4: Conclusion Dividing by two, the required definite integral is $\boxed{\pi/6}$.

Quick Tip to Solve: When the limits add to $\pi/2$, try $x \mapsto \pi/2 - x$. Here it exchanges cotangent and tangent, so the two integrands add to 1.

31(b). Evaluate: $\int_{-\pi/6}^{\pi/2} (\sin |x| + \cos |x|) dx$

[3 marks]

Solution:

Concept:

- The definition $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$ requires splitting an integral at zero when its limits have opposite signs.
- Use $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Also, $\int \sin x dx = -\cos x$ and $\int \cos x dx = \sin x$.

Step 1: Remove the modulus on each interval Let I denote the required integral. For $-\pi/6 \leq x < 0$, we have $\sin |x| = -\sin x$ and $\cos |x| = \cos x$. For $0 \leq x \leq \pi/2$, the integrand is $\sin x + \cos x$. Therefore,

$$I = \int_{-\pi/6}^0 (-\sin x + \cos x) dx + \int_0^{\pi/2} (\sin x + \cos x) dx.$$

The sine term changes sign on the negative interval; the cosine term does not, since cosine is an even function.

Step 2: Evaluate the contribution from negative inputs An antiderivative of $-\sin x + \cos x$ is $\cos x + \sin x$. Applying the lower limit carefully gives

$$\begin{aligned} I_1 &= [\cos x + \sin x]_{-\pi/6}^0 \\ &= 1 - \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \\ &= \frac{3 - \sqrt{3}}{2}. \end{aligned}$$

Here $\cos(-\pi/6) = \sqrt{3}/2$ while $\sin(-\pi/6) = -1/2$, so the subtraction of the lower-limit value includes a positive half.

Step 3: Evaluate the contribution from nonnegative inputs On the positive interval, integrate the original sine and cosine terms directly:

$$\begin{aligned} I_2 &= [-\cos x + \sin x]_0^{\pi/2} \\ &= (0 + 1) - (-1 + 0) = 2. \end{aligned}$$

Both contributions are positive, as expected because $|x|$ stays between 0 and $\pi/2$, where sine and cosine are nonnegative.

Step 4: Conclusion Adding the two contributions gives

$$I = \frac{3 - \sqrt{3}}{2} + 2 = \frac{7 - \sqrt{3}}{2}.$$

Quick Tip to Solve: Split at $x = 0$ before integrating: only the sine term changes sign on the negative interval. Keep the entire lower-limit value inside parentheses when subtracting it.

Section D (20 marks)

This section comprises Long Answer (LA) type questions of 5 marks each.

32. Find the domain of $p(x) = \sin^{-1}(1 - 2x^2)$. Hence, find the value of x for which $p(x) = \frac{\pi}{6}$. Also, write the range of $2p(x) + \frac{\pi}{2}$. [5 marks]

Solution:

Concept:

- The real principal inverse sine $\sin^{-1} u$ is defined precisely when $-1 \leq u \leq 1$, and its values lie in $[-\pi/2, \pi/2]$.
- For an angle θ in the principal interval, $\sin^{-1} u = \theta$ is equivalent to $u = \sin \theta$. Both signs must be retained when solving an equation in x^2 .
- To transform the range of a function, apply the stated linear transformation to its endpoint values. Multiplication by a positive number preserves the order of the bounds.

Step 1: Impose the inverse-sine domain condition For $p(x) = \sin^{-1}(1 - 2x^2)$ to be real, its entire argument must lie between -1 and 1 . Therefore,

$$-1 \leq 1 - 2x^2 \leq 1.$$

The right-hand inequality gives $-2x^2 \leq 0$, which holds for every real x . The left-hand inequality gives

$$-1 \leq 1 - 2x^2 \implies 2x^2 \leq 2 \implies x^2 \leq 1.$$

Thus the permitted inputs satisfy $-1 \leq x \leq 1$. Both endpoints are included because their inverse-sine argument equals -1 , an allowed value.

Step 2: Solve the specified inverse-trigonometric equation The requested angle $\pi/6$ belongs to the principal interval, so taking sine introduces no ambiguity in the inverse function:

$$\begin{aligned}\sin^{-1}(1 - 2x^2) &= \frac{\pi}{6}, \\ 1 - 2x^2 &= \sin \frac{\pi}{6} = \frac{1}{2}, \\ 2x^2 &= \frac{1}{2}, \quad x^2 = \frac{1}{4}.\end{aligned}$$

Hence the two candidate inputs are $x = 1/2$ and $x = -1/2$. Both lie in the domain, and each gives $1 - 2x^2 = 1/2$, whose principal inverse sine is exactly $\pi/6$.

Step 3: Determine the full range and transform it As x varies over $[-1, 1]$, its square attains every value in $[0, 1]$. Consequently $1 - 2x^2$ attains every value in $[-1, 1]$, so the full principal range of inverse sine is attained:

$$-\frac{\pi}{2} \leq p(x) \leq \frac{\pi}{2}.$$

Quick Tip to Solve: First bound the whole inverse-sine argument. For the range here, use $x^2 \in [0, 1]$; for the equation, retain both square roots and check them in the domain.

Multiply these bounds by 2 and then add $\pi/2$:

$$-\pi \leq 2p(x) \leq \pi \implies -\frac{\pi}{2} \leq 2p(x) + \frac{\pi}{2} \leq \frac{3\pi}{2}.$$

The lower bound is attained at $x = \pm 1$, and the upper bound at $x = 0$. Continuity ensures that every intermediate value is attained as well.

Step 4: Conclusion The domain is $[-1, 1]$. The equation $p(x) = \pi/6$ holds for $x = \pm \frac{1}{2}$. The range of $2p(x) + \pi/2$ is $[-\pi/2, 3\pi/2]$.

Quick Tip to Solve: First bound the whole inverse-sine argument. For the range here, use $x^2 \in [0, 1]$; for the equation, retain both square roots and check them in the domain.

33. A line passing through the points $A(1, 2, 3)$ and $B(5, 8, 11)$ intersects the line $\vec{r} = 4\hat{i} + \hat{j} + \lambda(5\hat{i} + 2\hat{j} + \hat{k})$. Find the co-ordinates of the point of intersection. Hence, write the equation of a line passing through the point of intersection and perpendicular to both the lines. [5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$ and nonzero direction vector $\vec{d}$ has equation $\vec{r} = \vec{a} + t\vec{d}$, where t is a real parameter.
- At an intersection, the corresponding coordinates of the two lines are equal. Use independent parameters for the lines and verify all three coordinate equations.
- For two nonparallel direction vectors $\vec{d}_1$ and $\vec{d}_2$, their cross product is perpendicular to both. A required common perpendicular through their intersection uses this cross product as its direction vector.

Step 1: Write the line through the two given points The direction vector from $A(1, 2, 3)$ to $B(5, 8, 11)$ is

$$\overrightarrow{AB} = (5 - 1, 8 - 2, 11 - 3) = (4, 6, 8) = 2(2, 3, 4).$$

Dividing a direction vector by a nonzero scalar does not change its line. Thus, with parameter t , the first line can be written as

$$(x, y, z) = (1 + 2t, 2 + 3t, 3 + 4t).$$

The given second line, with its own parameter λ , is

$$(x, y, z) = (4 + 5\lambda, 1 + 2\lambda, \lambda).$$

Step 2: Equate coordinates and verify the intersection Equality of the three coordinates at a common point requires

$$1 + 2t = 4 + 5\lambda,$$

$$2 + 3t = 1 + 2\lambda,$$

$$3 + 4t = \lambda.$$

Quick Tip to Solve: Here the second line has $z = \lambda$, so use the z -equation to eliminate λ first. Check the remaining coordinate, then cross the two direction vectors.

Substitute $\lambda = 3 + 4t$ into the first equation:

$$1 + 2t = 4 + 5(3 + 4t) = 19 + 20t \implies t = -1.$$

This gives $\lambda = 3 + 4(-1) = -1$. The unused second equation is also satisfied, since $2 + 3(-1) = -1 = 1 + 2(-1)$. Substitution in either line therefore gives the actual common point $P = (-1, -1, -1)$, rather than merely a solution of two coordinate equations.

Step 3: Find a direction perpendicular to both lines Choose $\vec{d}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{d}_2 = 5\hat{i} + 2\hat{j} + \hat{k}$. Their cross product is

$$\begin{aligned}\vec{d}_1 \times \vec{d}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} \\ &= (3 - 8)\hat{i} - (2 - 20)\hat{j} + (4 - 15)\hat{k} \\ &= -5\hat{i} + 18\hat{j} - 11\hat{k}.\end{aligned}$$

It is nonzero. Direct dot-product checks confirm the two perpendicularities:

$$(-5, 18, -11) \cdot (2, 3, 4) = -10 + 54 - 44 = 0,$$

$$(-5, 18, -11) \cdot (5, 2, 1) = -25 + 36 - 11 = 0.$$

Use this direction with base point P to construct the required line; because both original lines also contain P , it meets each of them there at a right angle.

Step 4: Conclusion The intersection is $(-1, -1, -1)$. A line through this point and perpendicular to both given lines is

$$\vec{r} = -\hat{i} - \hat{j} - \hat{k} + \mu(-5\hat{i} + 18\hat{j} - 11\hat{k}), \quad \mu \in \mathbb{R}.$$

Equivalently, $\frac{x+1}{-5} = \frac{y+1}{18} = \frac{z+1}{-11}.$

Quick Tip to Solve: Here the second line has $z = \lambda$, so use the z -equation to eliminate λ first. Check the remaining coordinate, then cross the two direction vectors.

34(a). If $P = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, find (QP) and hence solve the following system of equations using matrices:

$$x - y = 3, \quad 2x + 3y + 4z = 17, \quad y + 2z = 7$$

[5 marks]

Solution:

Concept:

- For matrix multiplication, the entry in row i , column j of QP is the dot product of row i of Q with column j of P . The order of the two matrices matters.
- Write a linear system as $PX = B$, where P contains its coefficients, $X = (x, y, z)^T$ contains its

Quick Tip to Solve: Once $QP = 6I$ is known, premultiply $PX = B$ by Q and divide QB by six; do not compute the inverse by cofactors again.

unknowns, and B contains its constants in the same equation order.

- If $QP = kI$ for a non-zero scalar k , multiplying $PX = B$ on the left by Q gives $kX = QB$. Here I is the identity matrix of order three.

Step 1: Multiply the matrices in the stated order Using the rows of Q and the columns of P , calculate all nine entries. In particular, the columns of P are $(1, 2, 0)^T$, $(-1, 3, 1)^T$ and $(0, 4, 2)^T$.

$$QP = \begin{pmatrix} 2+4+0 & -2+6-4 & 0+8-8 \\ -4+4+0 & 4+6-4 & 0+8-8 \\ 2-2+0 & -2-3+5 & 0-4+10 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I.$$

Thus the product is a non-zero scalar multiple of the identity. This also shows that P is invertible, with $P^{-1} = \frac{1}{6}Q$, so the given system has a unique solution.

Step 2: Express the equations as one matrix equation Preserve the equation order so that every row has its corresponding constant. The system becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}, \quad PX = B.$$

The zero in the first row represents the absence of z , and the zero in the third row represents the absence of x .

Step 3: Use the product to determine the unknown column Premultiply both sides by Q . By associativity, $Q(PX) = (QP)X = 6X$, and therefore $X = \frac{1}{6}QB$. Compute the right-hand side using the same row-by-column rule:

$$\begin{aligned} QB &= \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix}, \\ X &= \frac{1}{6} \begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}. \end{aligned}$$

Step 4: Verify all three equations Substitute the obtained values into each original equation, rather than checking only the matrix product:

$$\begin{aligned} x - y &= 2 - (-1) = 3, \\ 2x + 3y + 4z &= 4 - 3 + 16 = 17, \\ y + 2z &= -1 + 8 = 7. \end{aligned}$$

Every equation is satisfied, confirming both the signs and the placement of the constants in B .

Quick Tip to Solve: Once $QP = 6I$ is known, premultiply $PX = B$ by Q and divide QB by six; do not compute the inverse by cofactors again.

Step 5: Conclusion $QP = 6I$. The unique solution of the given system is $x = 2, y = -1, z = 4$.

Quick Tip to Solve: Once $QP = 6I$ is known, premultiply $PX = B$ by Q and divide QB by six; do not compute the inverse by cofactors again.

OR

34(b). Obtain the value of $\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$ in terms of x, y and z . Further, if $\Delta = 0$ and x, y, z are non-zero real numbers, prove that $x^{-1} + y^{-1} + z^{-1} = -1$. [5 marks]

Solution:

Concept:

- Adding a multiple of one row to another row leaves the determinant unchanged. Thus operations such as $R_1 \rightarrow R_1 - R_3$ simplify entries without changing its value.
- Expansion along the first row of a determinant of order three uses the signs $+, -, +$. Each term is its entry multiplied by the determinant of the remaining two rows and columns.
- For a determinant of order two, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Division by xyz is valid here only after using the given condition that x, y, z are all non-zero.

Step 1: Simplify two rows while retaining the determinant value Starting with the printed determinant, subtract the unchanged third row from each of the first two rows. These operations do not alter Δ :

$$\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix},$$
$$R_1 \rightarrow R_1 - R_3, \quad R_2 \rightarrow R_2 - R_3,$$
$$\Delta = \begin{vmatrix} x & 0 & -z \\ 0 & y & -z \\ 1 & 1 & 1+z \end{vmatrix}.$$

For example, the last entry of the first new row is $1 - (1 + z) = -z$. Keeping this sign is essential when forming the cofactor term.

Step 2: Expand along the simplified first row The middle entry of this row is zero, so its cofactor

Quick Tip to Solve: Subtract the third row from the first two rows to create zeros, then expand. Divide the resulting zero equation by xyz only after checking that all three variables are non-zero.

term vanishes. The third entry carries a positive cofactor sign because its position is $(1, 3)$. Therefore

$$\begin{aligned}\Delta &= x \begin{vmatrix} y & -z \\ 1 & 1+z \end{vmatrix} + (-z) \begin{vmatrix} 0 & y \\ 1 & 1 \end{vmatrix} \\ &= x[y(1+z) - (-z)] + (-z)(0-y) \\ &= x(y+yz+z) + yz \\ &= xy + xyz + xz + yz.\end{aligned}$$

This polynomial gives the value in terms of x, y, z for all their values; no division or non-zero assumption was needed in evaluating the determinant.

Step 3: Apply the zero-determinant condition Now use the additional hypotheses $\Delta = 0$ and $xyz \neq 0$. The evaluated determinant gives

$$xyz + xy + xz + yz = 0.$$

Divide every term by the same non-zero quantity xyz , obtaining

$$\begin{aligned}1 + \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} &= 0, \\ 1 + \frac{1}{z} + \frac{1}{y} + \frac{1}{x} &= 0.\end{aligned}$$

Each cancellation is valid because none of x, y, z is zero. Reorder the reciprocal terms and transpose the constant to the right-hand side to obtain the required identity. The non-zero hypothesis is thus used precisely at the division stage, not during the determinant expansion.

Step 4: Conclusion $\Delta = xyz + xy + yz + zx$. Consequently, if $\Delta = 0$ and $xyz \neq 0$, then $x^{-1} + y^{-1} + z^{-1} = -1$.

Quick Tip to Solve: Subtract the third row from the first two rows to create zeros, then expand. Divide the resulting zero equation by xyz only after checking that all three variables are non-zero.

35(a). Find the sub-interval of $(0, \pi)$ in which $f(x) = \tan^{-1}(\sin x - \cos x)$ is increasing and decreasing.

[5 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where its derivative is positive and decreasing where its derivative is negative. To find the required intervals, locate all zeros of the derivative inside the given domain and check its sign on each resulting interval.
- For a differentiable function $u(x)$, the chain rule gives $\frac{d}{dx} \tan^{-1} u(x) = \frac{u'(x)}{1+u(x)^2}$. Its denominator is strictly positive for every real value of $u(x)$.
- The identity $\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ makes the sign of the numerator easy to determine on $(0, \pi)$.

Quick Tip to Solve: Ignore the strictly positive inverse-tangent denominator and test the sign of $\sqrt{2} \sin(x + \pi/4)$; its only zero inside the domain is $3\pi/4$.

Step 1: Differentiate using the chain rule Put $u(x) = \sin x - \cos x$. Then $u'(x) = \cos x + \sin x$, so

$$f'(x) = \frac{\sin x + \cos x}{1 + (\sin x - \cos x)^2}.$$

Since the denominator is at least 1, it never vanishes and cannot change the sign of $f'(x)$. Thus only the numerator needs a sign analysis; there are no additional points at which the derivative is undefined.

Step 2: Locate the stationary point inside the domain The equation $f'(x) = 0$ becomes

$$\sqrt{2} \sin \left(x + \frac{\pi}{4} \right) = 0.$$

For $0 < x < \pi$, the shifted angle lies between $\pi/4$ and $5\pi/4$. The only zero of sine in this range is π . Hence the only stationary point is $x = 3\pi/4$, which divides the domain into two intervals.

Step 3: Check the sign on both sub-intervals If $0 < x < 3\pi/4$, then $\pi/4 < x + \pi/4 < \pi$, where sine is positive. Therefore $f'(x) > 0$ throughout this interval.

If $3\pi/4 < x < \pi$, then $\pi < x + \pi/4 < 5\pi/4$, where sine is negative. Therefore $f'(x) < 0$ throughout this interval. At the separating point $f'(3\pi/4) = 0$; the derivative changes from positive to negative there, so the increasing and decreasing portions have been identified consistently.

Step 4: Conclusion The function is increasing on $(0, 3\pi/4)$ and decreasing on $(3\pi/4, \pi)$.

Quick Tip to Solve: Ignore the strictly positive inverse-tangent denominator and test the sign of $\sqrt{2} \sin(x + \pi/4)$; its only zero inside the domain is $3\pi/4$.

OR

35(b). A rectangle of perimeter 24 cm is revolved along one of its sides to sweep out a cylinder of maximum volume. Find the dimensions of the rectangle. [5 marks]

Solution:

Concept:

- When a rectangle revolves about one of its sides, that side becomes the cylinder's height h , while the perpendicular side becomes its radius r . The swept solid has volume $V = \pi r^2 h$.
- The rectangle's perimeter fixes the sum of its adjacent sides: $2(r + h) = 24$. Hence $h = 12 - r$, with $0 < r < 12$ for a non-degenerate rectangle.
- To maximize the volume under this constraint, express it as a function of one variable, solve $V'(r) = 0$, and verify that the stationary point gives a maximum over the full admissible interval.

Step 1: Identify the radius and the axis of revolution Let the side about which the rectangle revolves have length h cm, and let its perpendicular side have length r cm. Every point on the opposite side moves in a circle of radius r , giving a cylinder with height h . The perimeter condition gives

$$2(h + r) = 24, \quad h + r = 12, \quad h = 12 - r.$$

Quick Tip to Solve: Assign the axis side to height before differentiating: maximize $r^2(12 - r)$, then use the perimeter to recover the axis side.

Both sides must be positive, so the allowable radius is $0 < r < 12$.

Step 2: Form and differentiate the volume function Substituting the constraint into the cylinder's volume,

$$V(r) = \pi r^2(12 - r) = \pi(12r^2 - r^3),$$

$$V'(r) = \pi(24r - 3r^2) = 3\pi r(8 - r).$$

A stationary point satisfies $3\pi r(8 - r) = 0$. The value $r = 0$ is outside the allowed interval and corresponds to a collapsed rectangle. The only admissible stationary value is therefore $r = 8$.

Step 3: Verify the maximum and recover the other side For $0 < r < 8$, both r and $8 - r$ are positive, so $V'(r) > 0$. For $8 < r < 12$, the second factor is negative, so $V'(r) < 0$. The volume therefore increases up to $r = 8$ and decreases afterwards, proving this is the global maximum on the admissible interval. As an additional check, $V''(r) = \pi(24 - 6r)$, giving $V''(8) = -24\pi < 0$. The height is $h = 12 - 8 = 4$ cm. These lengths satisfy the original perimeter condition, since $2(8 + 4) = 24$ cm. The associated maximum volume is $\pi(8)^2(4) = 256\pi \text{ cm}^3$.

Step 4: Conclusion The required rectangle has dimensions $8 \text{ cm} \times 4 \text{ cm}$. It must revolve about its 4 cm side, giving radius 8 cm and height 4 cm.

Quick Tip to Solve: Assign the axis side to height before differentiating: maximize $r^2(12 - r)$, then use the perimeter to recover the axis side.

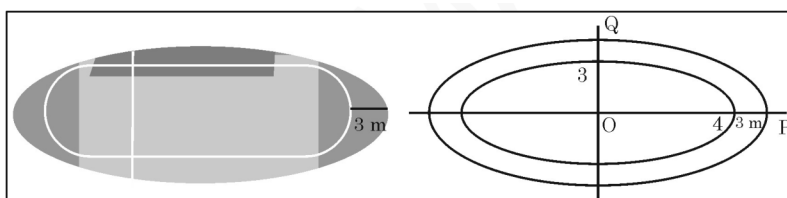
Section E (12 marks)

This section comprises of 3 case study based questions of 4 marks each.

36. A racing track is built around an elliptical ground whose equation is given by $9x^2 + 16y^2 = 144$. The width of the track is 3 m as shown below:

Based on given information, answer the following questions:

[4 marks]



36(i). Express y as a function of x from the given equation of ellipse.

[1 mark]

Solution:

Concept:

- To express the ellipse in terms of y , isolate y^2 and then take square roots.
- The upper and lower halves of an ellipse are separate functions of x ; the upper half has $y \geq 0$.

Step 1: Isolate the square of the ordinate Starting from the given ellipse, subtract $9x^2$ and divide by 16:

$$16y^2 = 144 - 9x^2 = 9(16 - x^2), \quad y^2 = \frac{9}{16}(16 - x^2).$$

Quick Tip to Solve: Factor out 9 before taking the square root, and use the positive branch when integrating the first-quadrant area.

Step 2: Choose the branch for the upper half Taking square roots gives $y = \pm \frac{3}{4}\sqrt{16 - x^2}$. The radical is real for $-4 \leq x \leq 4$. For the upper half, used to calculate the first-quadrant area, choose the positive sign; the negative sign describes the lower half.

Step 3: Conclusion The upper-half function is $y = \frac{3}{4}\sqrt{16 - x^2}$, $-4 \leq x \leq 4$. The complete ellipse has the two branches $y = \pm \frac{3}{4}\sqrt{16 - x^2}$.

Quick Tip to Solve: Factor out 9 before taking the square root, and use the positive branch when integrating the first-quadrant area.

36(ii). Integrate the function obtained in (i) with respect to x .

[1 mark]

Solution:

Concept:

- A constant factor can be taken outside an integral: $\int kg(x) dx = k \int g(x) dx$.
- For $a > 0$, $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$.

Step 1: Substitute the upper-half function Part (i) gives $y = \frac{3}{4}\sqrt{16 - x^2}$ for the upper half of the ellipse. Therefore the required antiderivative is obtained from

$$\int y dx = \frac{3}{4} \int \sqrt{4^2 - x^2} dx.$$

Step 2: Apply the standard integral Set $a = 4$ in the square-root formula and distribute the factor $3/4$:

$$\begin{aligned} \int y dx &= \frac{3}{4} \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right] + C \\ &= \frac{3x}{8} \sqrt{16 - x^2} + 6 \sin^{-1} \left(\frac{x}{4} \right) + C. \end{aligned}$$

Here C is an arbitrary constant; $\sin^{-1}$ denotes the inverse sine function.

Step 3: Conclusion $\int y dx = \frac{3x}{8} \sqrt{16 - x^2} + 6 \sin^{-1}(x/4) + C$ for the upper branch. For the lower branch, negate both nonconstant terms.

Quick Tip to Solve: Read $a^2 = 16$ as $a = 4$; the inverse-sine argument is $x/4$, not $x/16$.

36(iii)(a). Find the area of the region enclosed within the elliptical ground excluding the track using integration.

[2 marks]

Solution:

Concept:

- The area below a nonnegative curve $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x) dx$.

Quick Tip to Solve: Integrate only from 0 to the ground's intercept 4, then multiply by four; the outer track intercept is irrelevant here.

- An ellipse centred at the origin is symmetric about both coordinate axes, so its area is four times its first-quadrant area.

Step 1: Set up the ground area integral Dividing the ellipse equation by 144 gives $x^2/16 + y^2/9 = 1$. The ground therefore meets the positive axes at (4, 0) and (0, 3). Its upper boundary is $y = \frac{3}{4}\sqrt{16 - x^2}$. The track lies outside this ellipse, so the requested ground area is exactly the area within the given curve:

$$A = 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} dx = 3 \int_0^4 \sqrt{16 - x^2} dx.$$

Step 2: Evaluate the definite integral Using the square-root antiderivative and the values $\sin^{-1}(1) = \pi/2$, $\sin^{-1}(0) = 0$,

$$\begin{aligned} A &= 3 \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right]_0^4 \\ &= 3 \left[0 + 8 \left(\frac{\pi}{2} \right) - 0 \right] = 12\pi. \end{aligned}$$

The radical term vanishes at both limits. No track area is included in these integration bounds.

Step 3: Conclusion The area enclosed within the elliptical ground, excluding the track, is $12\pi \text{ m}^2$.

Quick Tip to Solve: Integrate only from 0 to the ground's intercept 4, then multiply by four; the outer track intercept is irrelevant here.

OR

36(iii)(b). Write the co-ordinates of the points P and Q where the outer edge of the track cuts x axis and y axis in first quadrant and find the area of the triangle formed by points P , O , Q using integration. [2 marks]

Solution:

Concept:

- The positive intercepts of $x^2/a^2 + y^2/b^2 = 1$ are $(a, 0)$ and $(0, b)$.
- A straight line joining intercepts $(p, 0)$ and $(0, q)$ has equation $x/p + y/q = 1$. The area beneath this line from 0 to p is $\int_0^p y dx$.

Step 1: Locate the outer intercepts The given ground is $x^2/16 + y^2/9 = 1$, so its positive x - and y -intercepts are 4 m and 3 m. The track extends another 3 m outwards at each axis. Hence the outer edge meets the positive axes at

$$P = (4 + 3, 0) = (7, 0), \quad Q = (0, 3 + 3) = (0, 6).$$

The third vertex is the origin $O = (0, 0)$.

Step 2: Integrate under the side joining the intercepts The line PQ is $x/7 + y/6 = 1$, or $y = 6 - 6x/7$. Each vertical strip of triangle POQ , for $0 \leq x \leq 7$, runs from the x -axis up to this line. Its area is

Quick Tip to Solve: Add the track width to both positive intercepts, then integrate the line joining them; do not integrate the outer curve for a triangle.

therefore

$$\begin{aligned} A &= \int_0^7 \left(6 - \frac{6x}{7}\right) dx \\ &= \left[6x - \frac{3x^2}{7}\right]_0^7 = 42 - 21 = 21. \end{aligned}$$

These bounds cover precisely the triangle, irrespective of the curved outer boundary of the track.

Step 3: Conclusion $P = (7, 0)$, $Q = (0, 6)$, and the area of triangle POQ , found by integration, is 21 m^2 .

Quick Tip to Solve: Add the track width to both positive intercepts, then integrate the line joining them; do not integrate the outer curve for a triangle.

37. Sports car racing is a form of motorsport which uses sports car prototypes. The competition is held on special tracks designed in various shapes. The equation of one such track is given as follows:

$$f(x) = \begin{cases} x^4 - 4x^2 + 4, & 0 \leq x < 3 \\ x^2 + 40, & x \geq 3 \end{cases}$$

Based on given information, answer the following questions:

[4 marks]

37(i). Find $f'(x)$ for $0 < x < 3$.

[1 mark]

Solution:

Concept:

- To differentiate a piecewise function inside one of its intervals, use the formula valid throughout a neighbourhood of that point.
- The power rule is $\frac{d}{dx}x^n = nx^{n-1}$; differentiation is linear and the derivative of a constant is zero.

Step 1: Select the applicable branch Every point in the requested interval $0 < x < 3$ lies in the first branch, where $f(x) = x^4 - 4x^2 + 4$. The formula $x^2 + 40$ applies only for $x \geq 3$, so it is not used here.

Step 2: Differentiate each term Apply the power rule separately to the quartic and quadratic terms; the constant contributes zero:

$$f'(x) = 4x^3 - 4(2x) + 0 = 4x^3 - 8x.$$

This calculation is valid at every interior point of the given interval and does not require a check at the branch boundary.

Step 3: Conclusion $f'(x) = 4x^3 - 8x$ for $0 < x < 3$.

Quick Tip to Solve: Choose the branch from the stated interval before differentiating, and remember that the constant 4 contributes zero.

37(ii). Find $f'(4)$.

[1 mark]

Solution:

Concept:

- At an interior point of a piecewise branch, the derivative is calculated using that branch's formula in a neighbourhood of the point.
- The derivative of x^2 is $2x$, while the derivative of a constant is zero.

Step 1: Identify the formula around four Since $4 > 3$, both sufficiently close left and right neighbours of 4 lie in the interval $x \geq 3$. Thus the relevant expression is $f(x) = x^2 + 40$, rather than the first branch $x^4 - 4x^2 + 4$.

Step 2: Differentiate before substituting Apply the power rule and then put $x = 4$:

$$f'(x) = \frac{d}{dx}(x^2 + 40) = 2x, \quad f'(4) = 2(4) = 8.$$

The additive constant 40 changes function values but has no effect on the rate of change. Evaluating $f(4)$ itself would therefore answer a different question.

Step 3: Conclusion The required derivative is $f'(4) = 8$.

Quick Tip to Solve: Locate 4 in the piecewise definition, differentiate $x^2 + 40$, and only then substitute $x = 4$.

37(iii)(a). Test for continuity of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- A function is continuous at an interior point $x = a$ when its left-hand limit and right-hand limit both equal its defined value $f(a)$.
- Polynomial expressions are continuous, so their one-sided limits are found by direct substitution into the formula valid on the corresponding side.

Step 1: Calculate the left-hand limit For $x < 3$ close to 3, the first branch $x^4 - 4x^2 + 4$ applies. Substituting $x = 3$ into this polynomial gives

$$\lim_{x \rightarrow 3^-} f(x) = 3^4 - 4(3^2) + 4 = 81 - 36 + 4 = 49.$$

Although this branch does not define the function at 3, it determines the approach from the left.

Step 2: Calculate the right-hand limit and function value For $x > 3$, the second branch is $x^2 + 40$. The same branch includes equality at $x = 3$, so it also supplies the actual function value:

$$\lim_{x \rightarrow 3^+} f(x) = 3^2 + 40 = 49, \quad f(3) = 3^2 + 40 = 49.$$

Hence the two-sided limit exists and agrees with the function value. Merely comparing the two limits would not suffice unless this value were checked as well.

Quick Tip to Solve: At the join, compute three quantities: the left limit, the right limit and the value from the branch containing the equality sign.

Step 3: Conclusion The one-sided limits and function value agree:

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) = 49.$$

Therefore, f is continuous at $x = 3$.

Quick Tip to Solve: At the join, compute three quantities: the left limit, the right limit and the value from the branch containing the equality sign.

OR

37(iii)(b). Test for differentiability of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- Differentiability at a joining point requires equal finite left-hand and right-hand derivatives, defined using the actual value of the function at that point.
- Continuity is necessary for differentiability; matching function values alone does not guarantee matching slopes.

Step 1: Check the value at the joining point At $x = 3$, the second branch applies, giving $f(3) = 3^2 + 40 = 49$. The first branch tends to $3^4 - 4(3^2) + 4 = 49$, so the two branches meet continuously. We must now compare their one-sided slopes.

Step 2: Calculate both one-sided derivatives Using h as the increment from 3, the left-hand difference quotient is

$$\begin{aligned} f'_-(3) &= \lim_{h \rightarrow 0^-} \frac{(3+h)^4 - 4(3+h)^2 + 4 - 49}{h} \\ &= \lim_{h \rightarrow 0^-} (84 + 50h + 12h^2 + h^3) = 84. \end{aligned}$$

For positive increments the second branch gives

$$f'_+(3) = \lim_{h \rightarrow 0^+} \frac{(3+h)^2 + 40 - 49}{h} = \lim_{h \rightarrow 0^+} (6 + h) = 6.$$

Each one-sided derivative exists, but their values are unequal.

Step 3: Conclusion Since $f'_-(3) = 84 \neq 6 = f'_+(3)$, f is not differentiable at $x = 3$, although it is continuous there.

Quick Tip to Solve: At a piecewise join, check continuity first, then compare the slopes from the two branches; equal heights need not mean equal slopes.

38. Smoking increases the risk of lung problems.

A study revealed that 170 in 1000 males who smoke develop lung complications, while 120 out of 1000 females who smoke develop lung related problems. In a colony, 50 people were found to be smokers of which 30 are males. A person is selected at random from these 50 people and tested for lung related problems.

Based on the given information, answer the following questions:

[4 marks]

38(i). What is the probability that selected person is a female?

[1 mark]

Solution:

Concept:

- For equally likely selections, the probability of an event is its favourable number of outcomes divided by the total number of outcomes.
- Male and female smokers form the two groups in this colony, so the female count is the total minus the male count.

Step 1: Count the female smokers The selection is from the 50 smokers in the colony. Of these, 30 are males, so the number of females is $50 - 30 = 20$.

Step 2: Form the favourable-to-total ratio Let F be the event that the selected person is female. Every one of the 50 people is equally likely to be selected. Therefore,

$$P(F) = \frac{20}{50} = \frac{2}{5}.$$

The lung-complication rates do not enter this calculation because selection has no stated condition about health.

Step 3: Conclusion The probability that the selected person is female is $\boxed{\frac{2}{5}}$.

Quick Tip to Solve: Use the colony counts for the sex of a random smoker; use the study rates only when lung problems are involved.

38(ii). If a male person is selected, what is the probability that he will not be suffering from lung problems?

[1 mark]

Solution:

Concept:

- If M denotes the event of selecting a male and L denotes lung problems, the given male rate is the conditional probability $P(L | M)$.
- Complementary conditional events add to one: $P(L^c | M) = 1 - P(L | M)$, where L^c means no lung problems.

Step 1: Identify the male complication rate The question already specifies that a male has been selected. Among males who smoke, 170 out of 1000 develop lung complications. Thus $P(L | M) = 170/1000 = 17/100$.

Quick Tip to Solve: When the selected person's group is already specified, take the complement of the rate within that group.

Step 2: Take the complement within the male group A male either suffers from the stated problems or does not. Hence,

$$P(L^c | M) = 1 - \frac{17}{100} = \frac{83}{100}.$$

Multiplication by the chance of selecting a male is unnecessary because the selection is already known to be male.

Step 3: Conclusion The required probability is $\boxed{\frac{83}{100}}$, or 0.83.

Quick Tip to Solve: When the selected person's group is already specified, take the complement of the rate within that group.

38(iii)(a). A person selected at random is detected with lung complications. Find the probability that selected person is a female. [2 marks]

Solution:

Concept:

- Let M , F , and L denote selecting a male, selecting a female, and detecting lung complications, respectively. The male and female groups partition the sample space.
- Total probability gives $P(L) = P(M)P(L | M) + P(F)P(L | F)$.
- Bayes' theorem reverses the conditioning: $P(F | L) = P(F)P(L | F)/P(L)$.

Step 1: Record the group probabilities and complication rates The colony has 30 male smokers and $50 - 30 = 20$ female smokers. Hence $P(M) = 3/5$ and $P(F) = 2/5$. The study supplies $P(L | M) = 17/100$ and $P(L | F) = 12/100$. These are rates within each group, not probabilities of the person's sex after detection.

Step 2: Find the total probability of lung complications A detected person can be male or female, so both disjoint contributions must be included:

$$\begin{aligned} P(L) &= \frac{3}{5} \cdot \frac{17}{100} + \frac{2}{5} \cdot \frac{12}{100} \\ &= \frac{51}{500} + \frac{24}{500} = \frac{75}{500}. \end{aligned}$$

Step 3: Condition on the positive detection The female contribution to all detections is $P(F \cap L) = 24/500$. Dividing it by the total detection probability gives

$$P(F | L) = \frac{24/500}{75/500} = \frac{24}{75} = \frac{8}{25}.$$

Step 4: Conclusion Given that lung complications are detected, the probability that the person is female is $\boxed{\frac{8}{25}}$, or 0.32.

Quick Tip to Solve: Weight each complication rate by its group proportion before applying Bayes' theorem; the female rate alone is not the requested posterior probability.

38(iii)(b). A person selected at random is not having lung problems, find the probability that the person is a male. [2 marks]

Solution:

Concept:

- Let M and F denote the male and female groups, and let N denote having no lung problems. The complementary rates give $P(N | M) = 1 - P(L | M)$, where L denotes lung problems.
- Total probability gives $P(N) = P(M)P(N | M) + P(F)P(N | F)$.
- Bayes' theorem gives $P(M | N) = P(M)P(N | M)/P(N)$: the male contribution divided by all people without problems in probability terms.

Step 1: Find the probabilities of no lung problems The group proportions are $P(M) = 30/50 = 3/5$ and $P(F) = 20/50 = 2/5$. Taking complements of the stated complication rates within each group gives

$$P(N | M) = 1 - \frac{170}{1000} = \frac{83}{100}, \quad P(N | F) = 1 - \frac{120}{1000} = \frac{88}{100}.$$

Step 2: Combine the two group contributions The selected person without problems may be from either group. The total probability is therefore

$$\begin{aligned} P(N) &= \frac{3}{5} \cdot \frac{83}{100} + \frac{2}{5} \cdot \frac{88}{100} \\ &= \frac{249}{500} + \frac{176}{500} = \frac{425}{500}. \end{aligned}$$

Step 3: Apply the given health condition The joint probability of being male and having no problems is $249/500$. Restricting the sample space to those without problems gives

$$P(M | N) = \frac{249/500}{425/500} = \frac{249}{425}.$$

Step 4: Conclusion Given that the selected person has no lung problems, the probability that the person is male is $\frac{249}{425}$.

Quick Tip to Solve: Complement both sex-specific complication rates first, then use Bayes' theorem with the male contribution in the numerator.