

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This Question paper contains **38** questions. **All questions are compulsory.**
- (ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.
- (iii) In Section **A** – Question Number **1 to 18** are Multiple Choice Questions (MCQs) and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In Section **B** – Question Number **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In Section **C** – Question Number **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In Section **D** – Question Number **32 to 35** are Long Answer (LA) type questions, carrying **5** marks each.
- (vii) In Section **E** – Question Number **36 to 38** are case study based questions, carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in few questions in all the Sections except Section – **A**.
- (ix) Use of calculator is **NOT** allowed.

Section A (20 marks)

This section comprises 20 Multiple Choice Questions (MCQs) of 1 mark each.

1. Direction cosines of line $\frac{1-x}{0} = y = z$ are

[1 mark]

- (A) 1, 1, 1
- (B) $0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$
- (C) 1, 0, 0
- (D) 0, -1, -1

Correct Answer: (B) $0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

Solution:

Concept:

- Direction ratios (a, b, c) describe a direction vector of a line. A zero first ratio means the coordinate x remains constant.
- Direction cosines are normalized ratios: $(l, m, n) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$, and $l^2 + m^2 + n^2 = 1$.

Step 1: Read the direction of the line The symmetric equation denotes $x = 1$ with $y = z = t$. Thus a point on the line is $(1, t, t)$, and its direction ratios are $(0, 1, 1)$. Reversing the orientation gives the equally valid ratios $(0, -1, -1)$. The zero denominator is interpreted as a constant coordinate, not as ordinary division by zero.

Step 2: Normalize and check the options The magnitude of either direction vector is $\sqrt{2}$. In the orientation used by the options, the direction cosines are

$$\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Options (A) and (D) are unnormalized triples; their squared entries sum to 3 and 2 respectively. Option (C) points along the x -axis and contradicts constant x .

Step 3: Conclusion Hence, the correct option is (B): $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Quick Tip to Solve: First identify any constant coordinate; its direction cosine is zero. Normalize the other two ratios, allowing reversal of the whole direction.

2. In a linear programming problem, the linear function which has to be maximized or minimized is called [1 mark]

- (A) a feasible function
- (B) an objective function
- (C) an optimal function
- (D) a constraint

Correct Answer: (B) an objective function

Solution:

Concept:

- An objective function is the linear expression $Z = ax + by$ whose maximum or minimum value is sought.
- Constraints are the equations or inequalities restricting the decision variables; their common solution set is the feasible region.

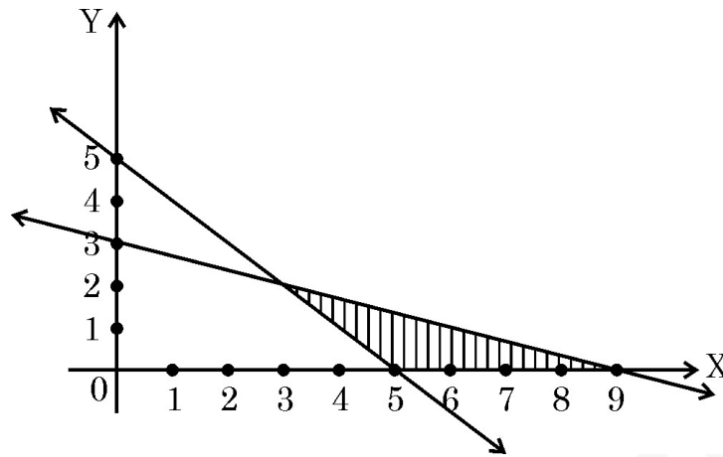
Step 1: Identify the role of the function The question asks for the name of the function being maximized or minimized. For example, if a problem seeks maximum profit, the profit expression is the objective function. The term describes the quantity being optimized rather than the restrictions imposed on the variables.

Step 2: Distinguish the other terms A feasible solution satisfies all constraints. An optimal solution is a feasible solution at which the objective attains its required extremum. Neither a feasible function nor an optimal function is the standard name for the expression being optimized. A constraint limits possible choices and is not the optimization objective.

Step 3: Conclusion The linear function to be maximized or minimized is an objective function. Hence, the correct option is (B).

Quick Tip to Solve: Ask whether an expression states the goal or a restriction: the goal is the objective function; restrictions are constraints.

3. For the feasible region shown below, the non-trivial constraints of the linear programming problem are [1 mark]



- (A) $x + y \leq 5$, $x + 3y \leq 9$
(B) $x + y \leq 5$, $x + 3y \geq 9$
(C) $x + y \geq 5$, $x + 3y \leq 9$
(D) $x + y \geq 5$, $3x + y \leq 9$

Correct Answer: (C) $x + y \geq 5$, $x + 3y \leq 9$

Solution:

Concept:

- A boundary line with intercepts a and b has equation $\frac{x}{a} + \frac{y}{b} = 1$.
- To choose the inequality sign, substitute a point inside the shaded feasible region. Non-trivial constraints exclude the usual non-negativity conditions.

Step 1: Find the boundary equations The steeper line cuts the axes at $(5, 0)$ and $(0, 5)$, hence its equation is $x + y = 5$. The shallower line cuts them at $(9, 0)$ and $(0, 3)$, hence $x + 3y = 9$. Their intersection is $(3, 2)$, in agreement with the graph.

Step 2: Choose the shaded sides The shaded triangle lies above the steeper line and below the shallower line. For instance, the interior point $(6, \frac{1}{2})$ gives $x + y = 6.5 > 5$ and $x + 3y = 7.5 < 9$. Thus $x + y \geq 5$ and $x + 3y \leq 9$. Options (A) and (B) choose the wrong side of the steeper line; option (D) uses a different second boundary with the intercepts exchanged.

Step 3: Conclusion The non-trivial constraints are $x + y \geq 5$ and $x + 3y \leq 9$. Hence, the correct option is (C).

Quick Tip to Solve: Read the intercepts before checking the shading; one interior point fixes both inequality signs without guessing from slopes.

4. For two events A and B such that $P(A) \neq 0$ and $P(B) \neq 1$, $P(A'/B') =$

[1 mark]

(A) $1 - P(A/B)$

(B) $1 - P(A'/B)$

(C) $\frac{1 - P(A \cap B)}{P(B')}$

(D) $\frac{1 - P(A \cup B)}{P(B')}$

Correct Answer: (D) $\frac{1 - P(A \cup B)}{P(B')}$

Solution:

Concept:

- Conditional probability is $P(E | F) = \frac{P(E \cap F)}{P(F)}$ for $P(F) > 0$.
- De Morgan's law gives $A' \cap B' = (A \cup B)'$; the probability of a complement is one minus the original probability.

Step 1: Apply conditional probability Here the conditioning event is B' , so its probability must be the denominator. Since $P(B) \neq 1$, we have $P(B') > 0$. Therefore

$$P(A'/B') = \frac{P(A' \cap B')}{P(B')}.$$

Step 2: Simplify the intersection of complements By De Morgan's law,

$$P(A' \cap B') = 1 - P(A \cup B).$$

The numerator in option (C) instead represents the complement of an intersection. Options (A) and (B) condition on B , whereas the required probability conditions on B' ; changing the conditioning event is not a valid complement rule.

Step 3: Conclusion $P(A'/B') = \frac{1 - P(A \cup B)}{P(B')}$. Hence, the correct option is (D).

Quick Tip to Solve: Keep the conditioning event fixed in the denominator, then apply De Morgan's law to the numerator.

5. A relation R on set $A = \{1, 2, 3\}$ is defined as $R = \{(1, 3), (3, 3), (1, 1), (2, 2), (3, 1)\}$ is

[1 mark]

- (A) only reflexive and symmetric
- (B) reflexive only
- (C) only reflexive and transitive
- (D) reflexive, symmetric and transitive

Correct Answer: (D) reflexive, symmetric and transitive

Solution:

Concept:

- A relation is reflexive when every (a, a) is present, symmetric when (a, b) implies (b, a) , and transitive when $(a, b), (b, c)$ imply (a, c) .
- Pairs of the form (a, a) act as loops; when composing them with another pair, the resulting pair is unchanged.

Step 1: Check reflexivity and symmetry The pairs $(1, 1), (2, 2), (3, 3)$ all belong to R , so it is reflexive on A . The only off-diagonal pairs are $(1, 3)$ and $(3, 1)$, each the reverse of the other. The diagonal pairs reverse to themselves, so the relation is also symmetric.

Step 2: Check all possible compositions Composing a loop with any pair reproduces a pair already in R . The only compositions using two off-diagonal pairs are $(1, 3), (3, 1)$, requiring $(1, 1)$, and $(3, 1), (1, 3)$, requiring $(3, 3)$; both are present. Element 2 is related only to itself. Thus every required transitive pair is present, so the options omitting transitivity or symmetry are incomplete.

Step 3: Conclusion The relation is reflexive, symmetric and transitive. Hence, the correct option is (D).

Quick Tip to Solve: Group the elements as $\{1, 3\}$ and $\{2\}$: every ordered pair within each group is present, which quickly verifies an equivalence relation.

6. If A and B are square matrices of same order, then which of the following statements is/are always true?

(i) $(A + B)(A - B) = A^2 - B^2$

(ii) $AB = BA$

(iii) $(A + B)^2 = A^2 + AB + BA + B^2$

(iv) $AB = 0 \Rightarrow A = 0$ or $B = 0$

[1 mark]

(A) Only (i) and (iii)

(B) Only (ii) and (iii)

(C) Only (iii)

(D) Only (iii) and (iv)

Correct Answer: (C) Only (iii)

Solution:

Concept:

- Matrix multiplication is distributive and associative but is generally not commutative: AB can differ from BA .
- Nonzero square matrices can have zero product, so scalar zero-product reasoning does not automatically apply to matrices.

Step 1: Expand the two products Without changing the order of factors,

$$\begin{aligned}(A + B)(A - B) &= A^2 - AB + BA - B^2, \\ (A + B)^2 &= A^2 + AB + BA + B^2.\end{aligned}$$

Thus (iii) always holds. Statement (i) requires cancellation of $-AB + BA$, which is not guaranteed; statement (ii) is the unsupported commutativity claim.

Step 2: Disprove the zero-product claim Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both are nonzero, but $AB = 0$, disproving (iv). To see that commutativity is not universal, take $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: then $AC = C$, whereas $CA = 0$. Therefore options containing (i), (ii), or (iv) cannot be always true.

Step 3: Conclusion Only statement (iii) is always true. Hence, the correct option is (C).

Quick Tip to Solve: Expand matrix products in their original factor order; never cancel mixed terms unless commutativity is explicitly given.

7. If $A = \begin{pmatrix} 1 & a & b \\ -1 & 2 & c \\ 0 & 5 & 3 \end{pmatrix}$ is a symmetric matrix, then the value of $3a + b + c$ is

[1 mark]

- (A) 2
- (B) 6
- (C) 4
- (D) 0

Correct Answer: (A) 2

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, equivalently $a_{ij} = a_{ji}$ for every pair of indices.
- The diagonal entries of a symmetric matrix need not be equal; symmetry compares entries across the main diagonal.

Step 1: Match the off-diagonal entries Compare the first-row, second-column entry with the second-row, first-column entry to get $a = -1$. Similarly, comparing positions $(1, 3)$ and $(3, 1)$ gives $b = 0$, and comparing $(2, 3)$ and $(3, 2)$ gives $c = 5$. The diagonal entries 1, 2 and 3 impose no additional restrictions.

Step 2: Evaluate the requested expression Substitute these values, retaining the negative sign of a :

$$3a + b + c = 3(-1) + 0 + 5 = -3 + 5 = 2.$$

The other printed numerical options 6, 4 and 0 disagree with this substitution. In particular, treating symmetry as equality of diagonal entries would be an incorrect use of the definition.

Step 3: Conclusion $3a + b + c = 2$. Hence, the correct option is (A).

Quick Tip to Solve: For symmetry, compare only the three pairs across the diagonal; then substitute into the expression with brackets around negative values.

8. If $A = \begin{pmatrix} \tan x & \cot x \\ -\cot x & \tan x \end{pmatrix}$ and $A + A' = 2I$, then value of $x \in \left[0, \frac{\pi}{2}\right]$ is

[1 mark]

(A) 0

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}$

Correct Answer: (C) $\frac{\pi}{4}$

Solution:

Concept:

- Transposition interchanges rows and columns. Thus corresponding off-diagonal entries switch positions in A^T .
- The identity matrix I has diagonal entries 1 and off-diagonal entries zero; equality of matrices means equality of corresponding entries.

Step 1: Compute the transpose and sum Here A' denotes the transpose. Therefore

$$A^T = \begin{pmatrix} \tan x & -\cot x \\ \cot x & \tan x \end{pmatrix}.$$

Adding it to A cancels the opposite off-diagonal terms, giving $A + A^T = 2 \tan x I$.

Step 2: Solve within the stated interval Comparing this sum with $2I$ gives $\tan x = 1$. In the interval $[0, \pi/2]$, the solution is $x = \pi/4$. The endpoint 0 makes $\cot x$ undefined, while $\pi/2$ makes $\tan x$ undefined, so neither can define the given matrix. At $\pi/3$, the tangent is $\sqrt{3}$, not 1.

Step 3: Conclusion $x = \frac{\pi}{4}$. Hence, the correct option is (C).

Quick Tip to Solve: For a matrix with opposite off-diagonal entries, adding its transpose leaves twice the diagonal; also check that the trigonometric entries are defined.

9. For a square matrix A , $(3A)^{-1}$ equals

[1 mark]

- (A) $3A^{-1}$
- (B) $9A^{-1}$
- (C) $\frac{1}{3}A^{-1}$
- (D) $\frac{1}{9}A^{-1}$

Correct Answer: (C) $\frac{1}{3}A^{-1}$

Solution:

Concept:

- For an invertible matrix A , its inverse obeys $AA^{-1} = A^{-1}A = I$.
- Multiplying a matrix by a nonzero scalar k multiplies its inverse by the reciprocal scalar: $(kA)^{-1} = k^{-1}A^{-1}$.

Step 1: Construct and verify the inverse The inverse notation presupposes that A is invertible. Test the candidate $\frac{1}{3}A^{-1}$ by multiplication:

$$(3A) \left(\frac{1}{3}A^{-1} \right) = AA^{-1} = I,$$
$$\left(\frac{1}{3}A^{-1} \right) (3A) = A^{-1}A = I.$$

Both defining products are the identity, so this candidate is the inverse.

Step 2: Reject the other scalar factors Multiplying $3A$ by $3A^{-1}$ gives $9I$; using $9A^{-1}$ gives $27I$; and using $\frac{1}{9}A^{-1}$ gives $\frac{1}{3}I$. None is the identity, so options (A), (B), and (D) fail the inverse test. The factor is reciprocal to 3, irrespective of the order of the square matrix.

Step 3: Conclusion $(3A)^{-1} = \frac{1}{3}A^{-1}$. Hence, the correct option is (C).

Quick Tip to Solve: Check an inverse candidate by multiplying it with the original matrix: the scalar coefficients must multiply to one.

10. If $\begin{vmatrix} -1 & -2 & 5 \\ -2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = -86$, then the sum of all possible values of a is

[1 mark]

- (A) 4
- (B) 5
- (C) -4
- (D) 9

Correct Answer: (C) -4

Solution:

Concept:

- Expand a determinant along a row, using alternating cofactor signs $+$, $-$, $+$.
- For $Aa^2 + Ba + C = 0$, the sum of its roots is $-B/A$.

Step 1: Expand along the first row Writing the determinant as D , its expansion is

$$D = (-1)(2a^2 + 4) - (-2)(-4a) + 5(-8) = -2a^2 - 8a - 44.$$

The middle minor is $(-2)(2a) - (-1)(0) = -4a$, so its contribution is $-8a$.

Step 2: Use the specified determinant value Setting $D = -86$, we obtain

$$a^2 + 4a - 21 = 0, \quad (a - 3)(a + 7) = 0.$$

The two possible values are 3 and -7 , whose sum is -4 . Thus 4 has the wrong sign, while 5 and 9 are not the sum of these roots.

Step 3: Conclusion The sum of all possible values is $\boxed{-4}$. Hence, the correct option is (C).

Quick Tip to Solve: For a requested sum, read $-B/A$ directly from the quadratic after checking the determinant signs.

11. If $x + y = xy$, then $\frac{dy}{dx}$ is

[1 mark]

(A) $\frac{y}{x-1}$

(B) $\frac{1}{x-1}$

(C) $\frac{y-1}{x-1}$

(D) $\frac{1-y}{x-1}$

Correct Answer: (D) $\frac{1-y}{x-1}$

Solution:

Concept:

- In implicit differentiation, treat y as a function of x .
- The product rule gives $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$.

Step 1: Differentiate both sides From $x + y = xy$, differentiation gives

$$1 + y' = y + xy'.$$

The derivative of the product contains two terms; omitting y would lead to a wrong numerator.

Step 2: Collect the derivative terms Rearrange to obtain

$$(1-x)y' = y-1, \quad y' = \frac{y-1}{1-x} = \frac{1-y}{x-1}.$$

Here $x = 1$ is impossible in the given relation, since it would imply $1 + y = y$. Options (A) and (B) have incorrect numerators, while (C) reverses only the denominator sign and hence gives the negative of the required derivative.

Step 3: Conclusion $\frac{dy}{dx} = \frac{1-y}{x-1}$. Hence, the correct option is (D).

Quick Tip to Solve: When reversing the sign of a denominator, reverse the numerator sign too to preserve the quotient.

12. $\int \frac{1}{\sec x + \tan x} dx$ is equal to

[1 mark]

- (A) $\log |\sec x + \tan x| + C$
- (B) $\log |\sec x - \tan x| + C$
- (C) $\log |1 + \cos x| + C$
- (D) $\log |1 + \sin x| + C$

Correct Answer: (D) $\log |1 + \sin x| + C$

Solution:

Concept:

- The identity $\sec^2 x - \tan^2 x = 1$ rationalises $\sec x + \tan x$.
- An antiderivative can be checked by differentiation on any interval where the integrand is defined.

Step 1: Simplify the integrand Multiply by the conjugate to get

$$\frac{1}{\sec x + \tan x} = \sec x - \tan x = \frac{1 - \sin x}{\cos x}.$$

Equivalently, multiplying numerator and denominator by $1 + \sin x$ gives $\cos x / (1 + \sin x)$.

Step 2: Integrate and test the options With $u = 1 + \sin x$, $du = \cos x dx$, so

$$\int \frac{\cos x}{1 + \sin x} dx = \log |1 + \sin x| + C.$$

The derivatives of options (A), (B) and (C) are respectively $\sec x$, $-\sec x$, and $-\sin x / (1 + \cos x)$; none equals the given integrand identically.

Step 3: Conclusion The integral is $\log |1 + \sin x| + C$. Hence, the correct option is (D).

Quick Tip to Solve: For a reciprocal involving secant plus tangent, use the conjugate identity before integrating.

13. For $f(x) = x + \frac{1}{x}$, ($x \neq 0$)

[1 mark]

- (A) local maximum value is 2
- (B) local minimum value is -2
- (C) local maximum value is -2
- (D) local minimum value $<$ local maximum value

Correct Answer: (C) local maximum value is -2

Solution:

Concept:

- A differentiable function can have a local extremum at a stationary point $f'(x) = 0$.
- At a stationary point, $f''(x) < 0$ gives a local maximum and $f''(x) > 0$ a local minimum.

Quick Tip to Solve: Classify each stationary point before substituting into the function; its x-coordinate is not its extremum value.

Step 1: Locate and classify stationary points For $f(x) = x + 1/x$,

$$f'(x) = 1 - \frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

Thus $f'(x) = 0$ at $x = \pm 1$. Since $f''(-1) = -2 < 0$, -1 is a local maximum point; since $f''(1) = 2 > 0$, 1 is a local minimum point.

Step 2: Evaluate the extremum values The local maximum value is $f(-1) = -2$, while the local minimum value is $f(1) = 2$. Options (A) and (B) interchange these values. Option (D) would require $2 < -2$, which is false. There is no contradiction in the local maximum being smaller than the local minimum because the two points lie on separate domain intervals.

Step 3: Conclusion The local maximum value is -2 . Hence, the correct option is (C).

Quick Tip to Solve: Classify each stationary point before substituting into the function; its x-coordinate is not its extremum value.

14. Which of the following expressions will give the area of region bounded by the curve $y = x^2$ and line $y = 16$? [1 mark]

- (A) $\int_0^4 x^2 dx$
(B) $2 \int_0^4 x^2 dx$
(C) $\int_0^{16} \sqrt{y} dy$
(D) $2 \int_0^{16} \sqrt{y} dy$

Correct Answer: (D) $2 \int_0^{16} \sqrt{y} dy$

Solution:

Concept:

- Using horizontal strips, area is $\int (x_{\text{right}} - x_{\text{left}}) dy$.
- The parabola $y = x^2$ has branches $x = \pm\sqrt{y}$ and is symmetric about the y-axis.

Step 1: Find the strip width and limits The line $y = 16$ meets the parabola at $x = \pm 4$. For a horizontal strip with $0 \leq y \leq 16$, the enclosed region extends from $x = -\sqrt{y}$ to $x = \sqrt{y}$, giving width $2\sqrt{y}$.

Step 2: Form the area integral Hence the area is

$$A = \int_0^{16} 2\sqrt{y} dy = 2 \int_0^{16} \sqrt{y} dy.$$

Option (C) counts only one symmetric half. Options (A) and (B) integrate the height x^2 above the x-axis rather than the gap $16 - x^2$ inside the required region. As a check, the full area is $2[\frac{2}{3}y^{3/2}]_0^{16} = 256/3$.

Quick Tip to Solve: Choose horizontal strips here: the right-minus-left width is immediately $2\sqrt{y}$.

Step 3: Conclusion The required expression is $2 \int_0^{16} \sqrt{y} dy$. Hence, the correct option is (D).

Quick Tip to Solve: Choose horizontal strips here: the right-minus-left width is immediately $2\sqrt{y}$.

15. The general solution of the differential equation $x^2 dy + y^2 dx = 0$ is

[1 mark]

(A) $x^3 + y^3 = k$

(B) $\frac{1}{y} - \frac{1}{x} = k$

(C) $\frac{1}{y} + \frac{1}{x} = k$

(D) $\log y^2 + \log x^2 = k$

Correct Answer: (C) $\frac{1}{y} + \frac{1}{x} = k$

Solution:

Concept:

- A separable equation can be integrated after placing functions of y with dy and functions of x with dx .
- The power rule gives $\int t^{-2} dt = -t^{-1} + C$.

Step 1: Separate the variables On a nonzero solution interval, dividing $x^2 dy + y^2 dx = 0$ by $x^2 y^2$ gives

$$\frac{dy}{y^2} + \frac{dx}{x^2} = 0.$$

This division produces inverse-square powers, rather than squares or logarithmic derivatives.

Step 2: Integrate and check the sign Integrating yields

$$-\frac{1}{y} - \frac{1}{x} = C, \quad \frac{1}{y} + \frac{1}{x} = k,$$

where $k = -C$ is arbitrary. Differentiating the last relation recovers $dy/y^2 + dx/x^2 = 0$. Option (B) instead gives a minus sign between these terms; (A) produces $x^2 dx + y^2 dy = 0$, and (D) produces logarithmic derivatives. The separate constant solution $y = 0$ is excluded by the division but does not change the listed general family.

Step 3: Conclusion The general solution is $\frac{1}{y} + \frac{1}{x} = k$. Hence, the correct option is (C).

Quick Tip to Solve: Keep both minus signs when integrating inverse squares; absorb their common sign into the constant.

16. The integrating factor of the differential equation $2x \frac{dy}{dx} - y = 3$ is

[1 mark]

- (A) $\sqrt{x}$
- (B) $\frac{1}{\sqrt{x}}$
- (C) e^x
- (D) e^{-x}

Correct Answer: (B) $\frac{1}{\sqrt{x}}$

Solution:

Concept:

- For $y' + P(x)y = Q(x)$, the integrating factor is $e^{\int P(x) dx}$.
- The coefficient of y' must first be made equal to one.

Step 1: Write the equation in standard form Divide $2xy' - y = 3$ by $2x$, for $x \neq 0$, to obtain

$$y' - \frac{1}{2x}y = \frac{3}{2x}.$$

Thus $P(x) = -1/(2x)$. The right-hand side does not enter the integrating-factor formula.

Step 2: Evaluate the integrating factor We have

$$\text{IF} = e^{-\frac{1}{2} \log |x|} = |x|^{-1/2}.$$

On the positive interval used by the real square-root options, this is $1/\sqrt{x}$. Option (A) would arise from a positive coefficient $1/(2x)$. Options (C) and (D) correspond to constant coefficients 1 and -1 , whereas the actual coefficient depends on x . On a negative interval the factor is $1/\sqrt{-x}$, with the same standard-form derivation.

Step 3: Conclusion The listed integrating factor is $\boxed{\frac{1}{\sqrt{x}}}$. Hence, the correct option is (B).

Quick Tip to Solve: Always divide by the coefficient of the derivative before reading $P(x)$ for the integrating factor.

17. If $|\vec{a}| = 5$ and $-2 \leq \lambda \leq 1$, then the sum of greatest and the smallest value of $|\lambda\vec{a}|$ is [1 mark]

- (A) -5
- (B) 5
- (C) 10
- (D) 15

Correct Answer: (C) 10

Solution:

Concept:

- Scalar multiplication changes magnitude by $|\lambda\vec{a}| = |\lambda| |\vec{a}|$.
- A magnitude is nonnegative; the minimum of $|\lambda|$ is zero when its interval contains zero.

Step 1: Find the smallest magnitude Here $|\lambda\vec{a}| = 5|\lambda|$. Since $-2 \leq \lambda \leq 1$ includes $\lambda = 0$, the smallest magnitude is 0, attained by the zero vector. A negative scalar reverses direction but never makes the magnitude negative.

Step 2: Find the greatest magnitude and add The largest value of $|\lambda|$ on this interval is 2, attained at $\lambda = -2$. Thus the greatest magnitude is 10, and the required sum is $10 + 0 = 10$. Option (A) is impossible for a sum of magnitudes. Option (B) overlooks the endpoint -2 , and option (D) incorrectly uses 5 as the minimum by comparing only the two endpoints.

Step 3: Conclusion The required sum is $\boxed{10}$. Hence, the correct option is (C).

Quick Tip to Solve: For a magnitude range, check zero as well as the endpoints before taking absolute scalar values.

18. Vector of magnitude 3 making equal angles with x and y axes and perpendicular to z axis is [1 mark]

- (A) $\hat{i} + 2\sqrt{2}\hat{j}$
- (B) $3\hat{k}$
- (C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$
- (D) $\sqrt{3}\hat{i} + \sqrt{3}\hat{j} + \sqrt{3}\hat{k}$

Correct Answer: (C) $\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}$

Solution:

Concept:

- For $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, its magnitude is $\sqrt{a^2 + b^2 + c^2}$.
- Equal angles with the positive x and y axes give equal direction cosines; perpendicularity to the z -axis gives $c = 0$.

Step 1: Apply the direction conditions Writing $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, perpendicularity gives $\vec{v} \cdot \hat{k} = c = 0$.

Quick Tip to Solve: Test all three requirements independently: magnitude 3, equal x and y components, and zero z -component.

Equal angles with x and y mean $a/3 = b/3$, hence $a = b$. With magnitude 3,

$$a^2 + b^2 = 9, \quad 2a^2 = 9.$$

The positive equal components among the options are $a = b = 3\sqrt{2}/2$.

Step 2: Check all listed vectors Option (C) has equal components, zero z-component and squared magnitude $9/2 + 9/2 = 9$. Option (A) has magnitude 3 but unequal x and y components. Option (B) lies along the z-axis. Option (D) has magnitude 3 and equal x and y components but a nonzero z-component. The negative of option (C) also satisfies the stated geometric conditions, but it is not among the listed choices.

Step 3: Conclusion The listed vector is $\boxed{\frac{3\sqrt{2}}{2}\hat{i} + \frac{3\sqrt{2}}{2}\hat{j}}$. Hence, the correct option is (C).

Quick Tip to Solve: Test all three requirements independently: magnitude 3, equal x and y components, and zero z-component.

19. For two vectors $\vec{a}$ and $\vec{b}$:

Assertion (A): $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

Reason (R): $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b}) \tan \theta, \left(\theta \neq \frac{\pi}{2}\right)$

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
 (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
 (C) Assertion (A) is true, but Reason (R) is false.
 (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- For vectors with angle θ , write $P = |\vec{a}||\vec{b}|$. Then $\vec{a} \cdot \vec{b} = P \cos \theta$ and $|\vec{a} \times \vec{b}| = P \sin \theta$.
- The identities $\sin^2 \theta + \cos^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$ connect the dot and cross products.

Step 1: Evaluate the assertion Substituting the two product formulae gives

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = P^2(\sin^2 \theta + \cos^2 \theta) = P^2.$$

Thus A is true, including the perpendicular case and the case of a zero vector.

Step 2: Evaluate the reason For the printed restriction $\theta \neq \pi/2$, $\cos \theta \neq 0$, so

$$(\vec{a} \cdot \vec{b}) \tan \theta = P \cos \theta \frac{\sin \theta}{\cos \theta} = P \sin \theta = |\vec{a} \times \vec{b}|.$$

Quick Tip to Solve: Cancel $\cos \theta$ only after checking the excluded perpendicular angle; verify that angle directly.

R is true; even for an obtuse angle the dot product and tangent are both negative, giving the required nonnegative result.

Step 3: Check the explanatory link Squaring R and adding the squared dot product gives

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = (P \cos \theta)^2(1 + \tan^2 \theta) = P^2.$$

This proves A under the stated restriction. For $\theta = \pi/2$, the dot product is zero and the cross-product magnitude is P , so the same identity holds directly. R therefore gives the correct explanation, rather than an unrelated fact. Options (B), (C) and (D) do not describe these truth values and this connection.

Step 4: Conclusion Both A and R are true, and R correctly explains A. Hence, the correct option is (A).

Quick Tip to Solve: Cancel $\cos \theta$ only after checking the excluded perpendicular angle; verify that angle directly.

20. Assertion (A): A line can have direction cosines $\langle 1, 1, 1 \rangle$.

Reason (R): $\cos \theta = 1$ is possible for $\theta = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- The direction cosines l, m, n of a line satisfy $l^2 + m^2 + n^2 = 1$. They are cosines of its angles with the three positive coordinate axes.
- A direction cosine may individually equal 1, because $\cos 0 = 1$, but the three values must also satisfy the common normalization condition.

Step 1: Evaluate the assertion For the proposed direction cosines $\langle 1, 1, 1 \rangle$,

$$l^2 + m^2 + n^2 = 1 + 1 + 1 = 3 \neq 1.$$

They cannot be direction cosines of a line. They can be direction ratios, whose corresponding direction cosines are $1/\sqrt{3}$ each. Thus A is false.

Step 2: Evaluate the reason The trigonometric value $\cos 0 = 1$ is correct. A line along the positive x -axis, for example, has direction cosines $\langle 1, 0, 0 \rangle$. Hence R is true.

Step 3: Check the relationship and options R concerns one angle only; it does not permit a single line to make zero angles with three mutually perpendicular axes simultaneously. Thus the assertion cannot follow from R. Options (A) and (B) wrongly declare A true, and option (C) wrongly declares R false.

Quick Tip to Solve: Always test $l^2 + m^2 + n^2 = 1$ before accepting a triple as direction cosines.

Step 4: Conclusion A is false and R is true. Hence, the correct option is (D).

Quick Tip to Solve: Always test $l^2 + m^2 + n^2 = 1$ before accepting a triple as direction cosines.

Section B (10 marks)

This section comprises Very Short Answer (VSA) type questions carrying 2 marks each.

21. Find the co-ordinates of the point on line $x = \frac{y-1}{2} = \frac{z-2}{3}$ whose y co-ordinate is 3 times the x co-ordinate. [2 marks]

Solution:

Concept:

- A point on a line in symmetric form is obtained by equating each expression to a common real parameter t .
- The coordinate condition that y is three times x means $y = 3x$; substitute the parameter expressions into this condition.

Step 1: Express the coordinates using a parameter For the printed line, put

$$x = \frac{y-1}{2} = \frac{z-2}{3} = t.$$

Each equality determines one coordinate:

$$x = t, \quad y = 2t + 1, \quad z = 3t + 2.$$

The constants in the numerators are shifts of the coordinates; they are retained when solving for y and z .

Step 2: Apply the given coordinate condition Substitute $x = t$ and $y = 2t + 1$ into $y = 3x$:

$$2t + 1 = 3t \implies t = 1.$$

This determines exactly one point, since the resulting equation has one solution.

Step 3: Find and verify the point At $t = 1$,

$$x = 1, \quad y = 2(1) + 1 = 3, \quad z = 3(1) + 2 = 5.$$

Check the three line expressions:

$$1 = \frac{3-1}{2} = \frac{5-2}{3}.$$

The additional condition also holds, because $3 = 3(1)$.

Step 4: Conclusion The required point has coordinates $(1, 3, 5)$.

Quick Tip to Solve: Use one common parameter for all three coordinates, then check both the line and $y = 3x$.

22(a). Check whether $f : \mathbb{R} - \{3\} \rightarrow \mathbb{R}$ defined as $f(x) = \frac{x-2}{x-3}$ is onto or not.

[2 marks]

Solution:

Concept:

- A function is onto its codomain if every element of that codomain is the image of at least one element of the domain.
- To test a rational function for surjectivity, set $y = f(x)$, solve for x , and check any excluded output values and the original domain restriction.

Step 1: Find whether the output one is attained Here the codomain is all of $\mathbb{R}$, so it contains 1. If $f(x) = 1$, then

$$\frac{x-2}{x-3} = 1 \implies x-2 = x-3.$$

Multiplication by $x-3$ is valid because $x \neq 3$ in the domain. The resulting equality $-2 = -3$ is impossible, so no domain element maps to 1.

Step 2: Determine the attainable outputs For completeness, let $y \neq 1$. Solving

$$y(x-3) = x-2$$

gives

$$(y-1)x = 3y-2, \quad x = \frac{3y-2}{y-1}.$$

This preimage is real and belongs to the domain, since

$$x-3 = \frac{1}{y-1} \neq 0.$$

Thus every real value other than 1 is attained, and the range is $\mathbb{R} \setminus \{1\}$.

Step 3: Conclusion The function is not onto the stated codomain $\mathbb{R}$, because the output 1 has no preimage.

Quick Tip to Solve: Rewrite $f(x) = 1 + 1/(x-3)$: the reciprocal term never vanishes, immediately excluding output 1.

OR

22(b). Check whether $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) defined as $f(x, y) = (2y, 3x)$ is injective or not.

[2 marks]

Solution:

Concept:

- A function is injective if equality of images forces equality of the original inputs.
- Equality of ordered pairs means equality of both corresponding components. Nonzero integer factors can be cancelled in such equations.

Quick Tip to Solve: For ordered-pair maps, equate corresponding output components and track which input variable each contains.

Step 1: Assume two ordered pairs have equal images Take arbitrary domain elements (x_1, y_1) and (x_2, y_2) in $\mathbb{Z} \times \mathbb{Z}$. Suppose

$$f(x_1, y_1) = f(x_2, y_2).$$

Using the definition of this function, this becomes

$$(2y_1, 3x_1) = (2y_2, 3x_2).$$

The first output component depends on the second input component, so the comparison must preserve this order.

Step 2: Compare and cancel the components Equality of ordered pairs gives the two equations

$$2y_1 = 2y_2, \quad 3x_1 = 3x_2.$$

Cancelling the nonzero factors 2 and 3 yields

$$y_1 = y_2, \quad x_1 = x_2.$$

Therefore $(x_1, y_1) = (x_2, y_2)$. This argument applies to every pair of integer inputs, establishing injectivity. It does not require every integer ordered pair in the codomain to be attained; that would be a separate test of surjectivity.

Step 3: Conclusion The given function $f(x, y) = (2y, 3x)$ is injective.

Quick Tip to Solve: For ordered-pair maps, equate corresponding output components and track which input variable each contains.

23. If $x = e^{\sin^{-1} t}$, $y = e^{\cos^{-1} t}$, find $\frac{dy}{dx}$ at $t = \frac{1}{\sqrt{2}}$.

[2 marks]

Solution:

Concept:

- For parametric functions $x(t), y(t)$, $dy/dx = (dy/dt)/(dx/dt)$, provided $dx/dt \neq 0$.
- For $-1 < t < 1$, $d(\sin^{-1} t)/dt = 1/\sqrt{1-t^2}$ and $d(\cos^{-1} t)/dt = -1/\sqrt{1-t^2}$. Here the exponent notation denotes inverse trigonometric functions.

Step 1: Differentiate with respect to the parameter Apply the chain rule separately to the two exponentials:

$$\frac{dx}{dt} = \frac{e^{\sin^{-1} t}}{\sqrt{1-t^2}}, \quad \frac{dy}{dt} = -\frac{e^{\cos^{-1} t}}{\sqrt{1-t^2}}.$$

The negative sign in the second derivative comes from the derivative of inverse cosine.

Step 2: Form the derivative ratio Divide the two expressions, cancelling the common denominator:

$$\frac{dy}{dx} = \frac{-e^{\cos^{-1} t}/\sqrt{1-t^2}}{e^{\sin^{-1} t}/\sqrt{1-t^2}} = -e^{\cos^{-1} t - \sin^{-1} t}.$$

Quick Tip to Solve: Cancel the common $\sqrt{1-t^2}$ factor before substituting; inverse sine and inverse cosine coincide at $1/\sqrt{2}$.

The ratio is valid at the requested parameter: $t = 1/\sqrt{2}$ lies inside $(-1, 1)$, and dx/dt is positive there.

Step 3: Substitute the requested value At $t = 1/\sqrt{2}$, both principal inverse trigonometric values equal $\pi/4$. Hence

$$\left. \frac{dy}{dx} \right|_{t=1/\sqrt{2}} = -e^{\pi/4 - \pi/4} = -1.$$

Equivalently, the derivative magnitudes are equal at this parameter, while their signs are opposite.

Step 4: Conclusion The required derivative is $\boxed{-1}$.

Quick Tip to Solve: Cancel the common $\sqrt{1-t^2}$ factor before substituting; inverse sine and inverse cosine coincide at $1/\sqrt{2}$.

24(a). Find the absolute maximum value of $f(x) = \cos x + \sin^2 x$, $x \in [0, \pi]$.

[2 marks]

Solution:

Concept:

- A continuous function on a closed interval attains its absolute maximum; compare its values at interior critical points and at both endpoints.
- Here $f'(x) = \sin x(2 \cos x - 1)$, using $\frac{d}{dx} \sin^2 x = 2 \sin x \cos x$.

Step 1: Locate the interior critical point Differentiating the given function gives

$$f'(x) = -\sin x + 2 \sin x \cos x = \sin x(2 \cos x - 1).$$

For $0 < x < \pi$, $\sin x > 0$. Hence $f'(x) = 0$ requires $\cos x = \frac{1}{2}$, giving the sole interior critical point $x = \pi/3$. The zeros $x = 0, \pi$ of $\sin x$ are endpoints and are checked separately.

Step 2: Compare the candidate values Since f is continuous on $[0, \pi]$, its absolute maximum occurs among the critical point and endpoints. Their values are

$$f(0) = 1, \quad f(\pi) = -1, \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{5}{4}.$$

This also agrees with the derivative being positive before $\pi/3$ and negative after it.

Step 3: Conclusion The absolute maximum value is $\boxed{\frac{5}{4}}$, attained at $\boxed{x = \pi/3}$.

Quick Tip to Solve: Write $f(x) = 1 + \cos x - \cos^2 x = \frac{5}{4} - (\cos x - \frac{1}{2})^2$; equality is attainable because $\pi/3$ lies in the interval.

24(b). If the volume of a solid hemisphere increases at a uniform rate, prove that its surface area varies inversely as its radius. [2 marks]

Solution:

Concept:

- For a solid hemisphere of radius $r > 0$, volume $V = \frac{2}{3}\pi r^3$ and total surface area $S = 3\pi r^2$.
- Related rates follow by differentiating both quantities with respect to time t ; a uniform volume increase means $dV/dt = k$, a positive constant.

Step 1: Express the rate of increase of radius Let r depend on time. Differentiating the volume formula and using the given uniform rate gives

$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt} = k, \quad \frac{dr}{dt} = \frac{k}{2\pi r^2}.$$

Thus the increasing radius has a rate that depends on its current value, even though the volume increases at a constant rate.

Step 2: Differentiate the surface area For the total surface area,

$$\frac{dS}{dt} = 6\pi r \frac{dr}{dt} = 6\pi r \frac{k}{2\pi r^2} = \frac{3k}{r}.$$

Therefore $r \, dS/dt = 3k$, a constant. If surface area denotes only the curved surface $S_c = 2\pi r^2$, the same calculation gives $dS_c/dt = 2k/r$, so the inverse dependence of the rate is unchanged. The area itself is proportional to r^2 ; the stated inverse variation is the related-rates result for its rate of increase.

Step 3: Conclusion The rate at which the hemisphere's surface area varies is inversely proportional to its radius: $\frac{dS}{dt} = \frac{3k}{r} \propto \frac{1}{r}$.

Quick Tip to Solve: Differentiate volume first to obtain dr/dt , then substitute it into the differentiated area formula; compare rates, since area itself grows as r^2 .

25. If $\vec{AB} = \hat{j} + \hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$ represent the two vectors along the sides AB and AC of $\triangle ABC$, prove that the median $\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2}$, where D is midpoint of BC . Hence, find the length of median AD . [2 marks]

Solution:

Concept:

- If D is the midpoint of BC , then $\vec{BD} = \frac{1}{2}\vec{BC}$. Vector addition along a path gives $\vec{AD} = \vec{AB} + \vec{BD}$.
- The magnitude of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$, since the unit coordinate vectors are mutually perpendicular.

Step 1: Prove the median vector formula Traversing B to A to C gives $\vec{BC} = \vec{AC} - \vec{AB}$. Since D

Quick Tip to Solve: Average the two vectors drawn from the same vertex to obtain its median vector, then take the magnitude after combining components.

bisects BC ,

$$\begin{aligned}\vec{AD} &= \vec{AB} + \frac{1}{2}\vec{BC} \\ &= \vec{AB} + \frac{1}{2}(\vec{AC} - \vec{AB}) \\ &= \frac{\vec{AB} + \vec{AC}}{2}.\end{aligned}$$

This proves the requested identity without any choice of origin.

Step 2: Substitute the given vectors and compute the magnitude Using the vectors printed in the question,

$$\vec{AD} = \frac{(\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k})}{2} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}.$$

The $\hat{j}$ components cancel. Thus

$$AD = |\vec{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}.$$

Step 3: Conclusion Hence $\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2}$, and the length of the median is $\frac{\sqrt{34}}{2}$ units.

Quick Tip to Solve: Average the two vectors drawn from the same vertex to obtain its median vector, then take the magnitude after combining components.

Section C (18 marks)

This section comprises Short Answer (SA) type questions of 3 marks each.

26(a). The probability of hitting the target by a trained sniper is three times the probability of not hitting the target on a stormy day due to high wind speed. The sniper fired two shots on the target on a stormy day when wind speed was very high. Find the probability that

(i) target is hit

(ii) atleast one shot misses the target.

[3 marks]

Solution:

Concept:

- A hit and a miss on one shot are complementary outcomes, so their probabilities sum to one.
- For independent repeated shots, the probability of a specified pair of outcomes is the product of the individual probabilities.
- The target is hit when at least one shot hits. Its complement is both shots missing; the complement of at least one miss is both shots hitting.

Step 1: Determine the probabilities for a single shot Let p be the probability of hitting the target and q the probability of missing it under the stated windy conditions. The given ratio yields $p = 3q$. Also,

Quick Tip to Solve: For two independent shots, calculate an “at least one” event by subtracting the probability of both opposite outcomes from one.

$p + q = 1$, because these two outcomes exhaust the possibilities. Therefore

$$3q + q = 1, \quad q = \frac{1}{4}, \quad p = \frac{3}{4}.$$

The same single-shot probabilities apply to each of the two shots, treated as independent trials.

Step 2: Find the probability that the target is hit Let H_1, H_2 denote hits on the first and second shots. The event that the target is hit is $H_1 \cup H_2$, including one hit or two hits. It fails to occur only if both shots miss. By independence,

$$P(H_1 \cup H_2) = 1 - P(H_1^c \cap H_2^c) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}.$$

Equivalently, adding the disjoint outcomes hit-hit, hit-miss and miss-hit gives $9/16 + 3/16 + 3/16$.

Step 3: Find the probability that at least one shot misses At least one miss includes the outcomes miss-hit, hit-miss and miss-miss. Its complementary event is hit-hit, whose probability is $p^2 = 9/16$. Consequently,

$$P(\text{at least one miss}) = 1 - P(H_1 \cap H_2) = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}.$$

Notice that these two requested events overlap when precisely one shot hits, so their probabilities need not add to one.

Step 4: Conclusion (i) The probability that the target is hit is $\boxed{15/16}$. (ii) The probability that at least one shot misses is $\boxed{7/16}$.

Quick Tip to Solve: For two independent shots, calculate an “at least one” event by subtracting the probability of both opposite outcomes from one.

OR

26(b). Mother, Father and Son line up at random for a family picture. Let events E: Son on one end and F: Father in the middle. Find $P(E/F)$. [3 marks]

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, provided $P(F) > 0$; conditioning restricts attention to the arrangements satisfying F .
- Three distinct people have $3! = 6$ equally likely linear arrangements when they line up at random.
- In a line of three people, the first and third positions are the ends; only the second position is the middle.

Step 1: List the equally likely arrangements Write M for Mother, F_a for Father and S for Son, distinguishing the person's symbol F_a from the event F . The sample space of left-to-right arrangements is

$$\Omega = \{MF_aS, MSF_a, F_aMS, F_aSM, SMF_a, SF_aM\}.$$

Quick Tip to Solve: Restrict the sample space to the two arrangements with Father in the middle; Son is on an end in every one of them.

Each of these six arrangements has probability $1/6$, because the line-up is random and all three people are distinct.

Step 2: Identify the conditioning event and its intersection Father is in the middle precisely in the arrangements

$$F = \{MF_aS, SF_aM\}, \quad P(F) = \frac{2}{6} = \frac{1}{3}.$$

Son is on an end in

$$E = \{MF_aS, F_aMS, SMF_a, SF_aM\}.$$

Both arrangements in F already put Son on an end. Hence $E \cap F = F$, and $P(E \cap F) = 2/6 = 1/3$. Once Father occupies the middle, Mother and Son must occupy the two ends.

Step 3: Apply the conditional probability formula Since $P(F) = 1/3 > 0$, the required conditional probability is defined. Substitution gives

$$P(E/F) = P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{1/3}{1/3} = 1.$$

The unconditional probability $P(E) = 4/6$ is not the requested quantity, since the information about Father's position changes the relevant sample space to the two arrangements in F .

Step 4: Conclusion The probability that Son is on one end, given that Father is in the middle, is $P(E/F) = 1$.

Quick Tip to Solve: Restrict the sample space to the two arrangements with Father in the middle; Son is on an end in every one of them.

27. Find: $\int \frac{3x-1}{\sqrt{x^2-4x}} dx$

[3 marks]

Solution:

Concept:

- First separate the numerator into a multiple of the derivative of the expression under the square root and a constant remainder.
- Completing the square gives $x^2 - 4x = (x - 2)^2 - 4$. Use $\int du/\sqrt{u^2 - a^2} = \log|u + \sqrt{u^2 - a^2}| + C$, where C is arbitrary.

Step 1: Split the numerator The derivative of $x^2 - 4x$ is $2x - 4$. Therefore,

$$3x - 1 = \frac{3}{2}(2x - 4) + 5.$$

This gives

$$I = \frac{3}{2} \int \frac{2x - 4}{\sqrt{x^2 - 4x}} dx + 5 \int \frac{dx}{\sqrt{x^2 - 4x}}.$$

The remainder must be 5: the derivative term produces $3x - 6$, and adding 5 restores the printed numerator.

Quick Tip to Solve: Match the numerator to $2x - 4$ first; the remainder is 5, and completing the square supplies the logarithm.

Step 2: Integrate both terms In the first integral set $t = x^2 - 4x$, so $dt = (2x - 4) dx$. Its contribution becomes

$$\frac{3}{2} \int t^{-1/2} dt = 3\sqrt{x^2 - 4x}.$$

For the remaining integral set $u = x - 2$. Then $du = dx$ and the radicand is $u^2 - 2^2$. Thus,

$$5 \int \frac{du}{\sqrt{u^2 - 4}} = 5 \log |x - 2 + \sqrt{x^2 - 4x}|.$$

The real integrand is defined on $x < 0$ and $x > 4$. The absolute value in the logarithm allows this expression on either interval. Differentiating these contributions gives $(3x - 6)/\sqrt{x^2 - 4x}$ and $5/\sqrt{x^2 - 4x}$, respectively, whose sum is the original integrand.

Step 3: Conclusion

$$I = 3\sqrt{x^2 - 4x} + 5 \log |x - 2 + \sqrt{x^2 - 4x}| + C.$$

Quick Tip to Solve: Match the numerator to $2x - 4$ first; the remainder is 5, and completing the square supplies the logarithm.

28(a). Evaluate: $\int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}$

[3 marks]

Solution:

Concept:

- Reflection of limits gives $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$. Here $a + b = \pi/2$.
- On the interval in this question, $\cot x > 0$, and $\cot(\pi/2 - x) = \tan x = 1/\cot x$. This makes the original and reflected integrands complementary.

Step 1: Reflect the variable about the midpoint of the interval Let

$$I = \int_{\pi/12}^{5\pi/12} \frac{dx}{1 + \sqrt{\cot x}}.$$

Substitute $x = \pi/2 - t$, so $dx = -dt$. The lower limit becomes $5\pi/12$ and the upper limit becomes $\pi/12$. Reversing the limits cancels the negative sign. Therefore,

$$I = \int_{\pi/12}^{5\pi/12} \frac{dt}{1 + \sqrt{\tan t}} = \int_{\pi/12}^{5\pi/12} \frac{\sqrt{\cot t}}{1 + \sqrt{\cot t}} dt.$$

The second equality follows by multiplying numerator and denominator by $\sqrt{\cot t}$, which is positive throughout the interval.

Step 2: Add the two equivalent integral forms The integration variable is a dummy variable, so

Quick Tip to Solve: When the limits add to $\pi/2$, pair x with $\pi/2 - x$; here the paired integrands sum to 1.

replace t by x and add the original and reflected expressions. Their denominators now coincide:

$$\begin{aligned} 2I &= \int_{\pi/12}^{5\pi/12} \frac{1 + \sqrt{\cot x}}{1 + \sqrt{\cot x}} dx \\ &= \int_{\pi/12}^{5\pi/12} 1 dx = \frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}. \end{aligned}$$

Thus the integral is half the interval length. No antiderivative of the original radical expression is needed: the symmetry directly removes it.

Step 3: Conclusion

$$I = \frac{\pi}{6}.$$

Quick Tip to Solve: When the limits add to $\pi/2$, pair x with $\pi/2 - x$; here the paired integrands sum to 1.

OR

28(b). Evaluate: $\int_{-\pi/6}^{\pi/2} (\sin |x| + \cos |x|) dx$

[3 marks]

Solution:

Concept:

- The definition of absolute value is $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$. Split the integral at $x = 0$, where this definition changes.
- Sine is odd and cosine is even: $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Consequently only the sine term changes sign on the negative interval.

Step 1: Remove the modulus on each interval Since 0 lies between the limits, write

$$I = \int_{-\pi/6}^0 (-\sin x + \cos x) dx + \int_0^{\pi/2} (\sin x + \cos x) dx.$$

On the first interval the sine argument is $-x$, but the cosine argument $-x$ gives the same cosine value. On the second interval both arguments are simply x . The bounds are not symmetric, so one cannot double the integral over the whole positive interval.

Step 2: Evaluate the negative and positive contributions An antiderivative of $-\sin x + \cos x$ is $\cos x + \sin x$. Thus,

$$\begin{aligned} I_1 &= [\cos x + \sin x]_{-\pi/6}^0 \\ &= 1 - \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) = \frac{3 - \sqrt{3}}{2}. \end{aligned}$$

For the positive part an antiderivative is $-\cos x + \sin x$, giving

$$I_2 = [-\cos x + \sin x]_0^{\pi/2} = 1 - (-1) = 2.$$

Quick Tip to Solve: Split at zero before integrating; on the negative side $\sin |x| = -\sin x$, whereas $\cos |x| = \cos x$.

Both contributions are positive, as expected because sine and cosine of $|x|$ are nonnegative over the respective angles. Adding the contributions produces the value of the entire integral.

Step 3: Conclusion

$$I = \frac{3 - \sqrt{3}}{2} + 2 = \frac{7 - \sqrt{3}}{2}.$$

Quick Tip to Solve: Split at zero before integrating; on the negative side $\sin |x| = -\sin x$, whereas $\cos |x| = \cos x$.

29. If $\frac{d}{dx}(F(x)) = \frac{1}{e^x + 1}$, then find $F(x)$ given that $F(0) = \log \frac{1}{2}$.

[3 marks]

Solution:

Concept:

- A function is recovered from its derivative by integration, and the given value at $x = 0$ fixes the arbitrary integration constant.
- Rewrite $1/(1 + e^x) = 1 - e^x/(1 + e^x)$. The second term is the derivative of $\log(1 + e^x)$, since the derivative of its argument is e^x .

Step 1: Integrate the given derivative The derivative may be expressed as

$$F'(x) = \frac{1}{e^x + 1} = 1 - \frac{e^x}{e^x + 1}.$$

Integrating gives

$$F(x) = x - \log(1 + e^x) + C.$$

To see the logarithmic term directly, put $u = 1 + e^x$, so $du = e^x dx$. Then $\int e^x/(1 + e^x) dx = \int du/u = \log u$. The argument $1 + e^x$ is strictly positive for every real x , so no absolute value is needed in this logarithm.

Step 2: Apply the value of the function at zero Substituting $x = 0$ in the integrated expression yields

$$F(0) = 0 - \log(1 + 1) + C = -\log 2 + C.$$

The prescribed value is $\log(1/2) = -\log 2$. Equating the values gives $C = 0$. As a check, differentiation of $x - \log(1 + e^x)$ gives

$$1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x},$$

and its value at zero is exactly $-\log 2$. Thus both the derivative and initial condition are satisfied, fixing the required function uniquely on the real line.

Step 3: Conclusion

$$F(x) = x - \log(1 + e^x) = \log\left(\frac{e^x}{1 + e^x}\right).$$

Quick Tip to Solve: Rewrite the integrand as $1 - e^x/(1 + e^x)$, integrate the logarithmic derivative, then substitute zero to determine C .

30(a). Solve the following differential equation: $x \frac{dy}{dx} = y - x \sin^2\left(\frac{y}{x}\right)$, given that $y(1) = \frac{\pi}{6}$. [3 marks]

Solution:

Concept:

- A homogeneous differential equation whose right-hand side depends on y/x is reduced by putting $v = y/x$, so $y = xv$ and $dy/dx = v + x dv/dx$.
- Separation of variables gives an equation with all functions of v on one side and functions of x on the other. Use $\int \csc^2 v dv = -\cot v$, then determine the integration constant from the given initial value.

Step 1: Substitute the ratio of the variables For $x \neq 0$, divide the differential equation by x :

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right).$$

Put $y = xv$. Differentiating this product and substituting gives

$$v + x \frac{dv}{dx} = v - \sin^2 v, \quad x \frac{dv}{dx} = -\sin^2 v.$$

Thus the terms v cancel, leaving a separable equation. The initial value corresponds to $x = 1$ and $v = \pi/6$, where $\sin v \neq 0$, so separation by $\sin^2 v$ is valid on the solution interval containing that point.

Step 2: Integrate and apply the initial condition Separate and integrate:

$$\csc^2 v dv = -\frac{dx}{x}, \quad -\cot v = -\log|x| + C_1.$$

Writing the arbitrary constant as $C = -C_1$, and restoring $v = y/x$, we obtain

$$\cot\left(\frac{y}{x}\right) = \log|x| + C.$$

Since the initial point has positive x , the interval containing it lies in $x > 0$, where $\log|x| = \log x$. At $x = 1$, use $y = \pi/6$:

$$C = \cot(\pi/6) - \log 1 = \sqrt{3}.$$

The branch of the implicit solution is chosen continuously through $y/x = \pi/6$; this preserves the given initial value.

Step 3: Conclusion The required solution through the given point is

$$\cot\left(\frac{y}{x}\right) = \log x + \sqrt{3}, \quad x > 0.$$

Quick Tip to Solve: Cancel the two copies of v after substituting $y=xv$; integrate the remaining negative sine-squared equation using the cotangent derivative.

30(b). Find the general solution of the differential equation: $y \log y \frac{dx}{dy} + x = \frac{2}{y}$.

[3 marks]

Solution:

Concept:

- Treat x as the dependent variable and y as the independent variable. A linear equation $\frac{dx}{dy} + P(y)x = Q(y)$ is solved using the integrating factor $IF = e^{\int P(y) dy}$.
- After multiplication by an integrating factor, the left side becomes the derivative of its product with x . Here $d(\log y)/dy = 1/y$, with $y > 0$ for the real logarithm.

Step 1: Put the equation in linear form The printed equation is

$$y \log y \frac{dx}{dy} + x = \frac{2}{y}.$$

On an interval with $y > 0$ and $y \neq 1$, divide by $y \log y$ to obtain

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{2}{y^2 \log y}.$$

Thus $P(y) = 1/(y \log y)$. Its antiderivative is $\log |\log y|$, so the integrating factor is $|\log y|$. Since the sign of $\log y$ is constant on either interval $(0, 1)$ or $(1, \infty)$, we may use the proportional integrating factor $\log y$.

Step 2: Integrate the product derivative Multiply the linear equation by $\log y$:

$$(\log y) \frac{dx}{dy} + \frac{x}{y} = \frac{2}{y^2}.$$

The product rule identifies the left side as $d(x \log y)/dy$. Therefore

$$\frac{d}{dy}(x \log y) = \frac{2}{y^2}, \quad x \log y = \int 2y^{-2} dy = -\frac{2}{y} + C.$$

As a direct check, differentiating this last relation gives $(\log y)x' + x/y = 2/y^2$; multiplying by y reproduces the original printed equation. The explicit formula below applies where $\log y \neq 0$; the implicit relation also describes any extension through $y = 1$ when such an extension exists.

Step 3: Conclusion The general solution is

$$x \log y + \frac{2}{y} = C,$$

or $x = \frac{C - 2/y}{\log y}$ on intervals with $y > 0$, $y \neq 1$.

Quick Tip to Solve: The dependent variable is x here: multiplying the linear form by $\log y$ immediately exposes $d(x \log y)/dy$.

31. Solve the following linear programming problem graphically:

Maximize $Z = 4500x + 5000y$

Subject to constraints

$$x + y \leq 250$$

$$25x + 40y \leq 7000$$

$$x \geq 0, y \geq 0$$

[3 marks]

Solution:

Concept:

- Each linear inequality determines a half-plane. The feasible region is their common intersection, restricted here to the first quadrant by $x \geq 0$ and $y \geq 0$.
- For a bounded nonempty feasible polygon, a linear objective function attains its maximum at a vertex. Plot the boundary lines, identify the feasible vertices, and evaluate $Z = 4500x + 5000y$ at each.

Step 1: Plot the constraints and identify the feasible region Draw $x + y = 250$ using its intercepts $(250, 0)$ and $(0, 250)$. Draw $25x + 40y = 7000$, equivalently $5x + 8y = 1400$, using its intercepts $(280, 0)$ and $(0, 175)$. The origin satisfies both inequalities, so choose the side containing the origin for each line, together with the first quadrant. The lines intersect where

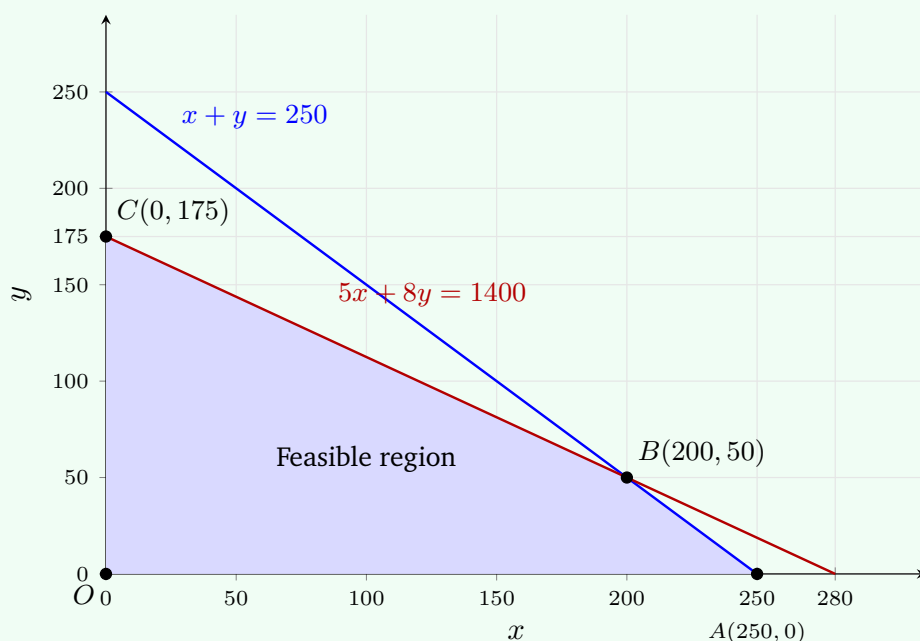
$$x + y = 250, \quad 5x + 8y = 1400.$$

Substitute $x = 250 - y$ in the second equation:

$$1250 - 5y + 8y = 1400, \quad 3y = 150, \quad y = 50, \quad x = 200.$$

Thus the shaded common feasible region is the bounded quadrilateral with vertices $O(0, 0)$, $A(250, 0)$, $B(200, 50)$ and $C(0, 175)$.

Quick Tip to Solve: Divide the second boundary by 5 before finding its intersection with $x+y=250$; then compare all four feasible vertices.



The shaded quadrilateral OABC is the feasible region.

Step 2: Compare the objective values at every corner Evaluate the objective function at the four vertices:

Vertex (x, y)	$Z = 4500x + 5000y$
$(0, 0)$	0
$(250, 0)$	11, 25, 000
$(200, 50)$	$9, 00, 000 + 2, 50, 000 = 11, 50, 000$
$(0, 175)$	8, 75, 000

The largest value occurs at $B(200, 50)$. At this point, $x + y = 250$ and $25x + 40y = 5000 + 2000 = 7000$, so both resource constraints are satisfied with equality, and both variables are nonnegative. Comparing all feasible corners ensures that the candidate is the global maximum on the shaded region.

Step 3: Conclusion The maximum value is $Z_{\max} = 11, 50, 000$, attained at $x = 200, y = 50$.

Quick Tip to Solve: Divide the second boundary by 5 before finding its intersection with $x + y = 250$; then compare all four feasible vertices.

Section D (20 marks)

This section comprises Long Answer (LA) type questions of 5 marks each.

32(a). Find the sub-interval of $\left(0, \frac{\pi}{2}\right)$ in which $f(x) = \log(\sin x + \cos x)$ is increasing and decreasing.

[5 marks]

Solution:

Concept:

- For a differentiable function, a positive derivative throughout an interval implies that the function is strictly increasing there; a negative derivative implies that it is strictly decreasing there.
- The logarithmic differentiation rule is $\frac{d}{dx} \log u(x) = \frac{u'(x)}{u(x)}$ for $u(x) > 0$. Thus the sign of the derivative depends on the numerator when the denominator is positive.
- On $0 < x < \pi/2$, both $\sin x$ and $\cos x$ are positive. Also, $\tan x$ is strictly increasing and takes the value 1 at $x = \pi/4$.

Step 1: Check the domain and differentiate The prescribed domain is the open interval $(0, \pi/2)$. On this interval $\sin x + \cos x > 0$, so the logarithm is defined and differentiable. Applying the chain rule gives

$$f'(x) = \frac{\cos x - \sin x}{\sin x + \cos x}.$$

Since the denominator is strictly positive throughout the prescribed interval, it never causes a change of sign or a discontinuity. We therefore only need to examine $\cos x - \sin x$.

Step 2: Locate the stationary point A stationary point satisfies

$$\begin{aligned} f'(x) = 0 &\iff \cos x - \sin x = 0 \\ &\iff \tan x = 1 \\ &\iff x = \frac{\pi}{4}. \end{aligned}$$

Division by $\cos x$ is valid because it is positive on the given interval. The monotonicity of tangent on this interval also shows that there is exactly one such point; no other critical point must be tested.

Step 3: Determine the sign on either side For $0 < x < \pi/4$, we have $\tan x < 1$, hence $\sin x < \cos x$. Consequently $f'(x) > 0$ throughout this sub-interval. For $\pi/4 < x < \pi/2$, we have $\tan x > 1$, hence $\sin x > \cos x$, so $f'(x) < 0$ throughout that sub-interval. At $x = \pi/4$ the derivative is zero, and its sign changes from positive to negative. This establishes the required increasing and decreasing behavior, rather than merely identifying a stationary point. The same sign change also verifies that the function attains its maximum at the midpoint of the given interval.

Step 4: Conclusion The function is strictly increasing on $\left(0, \pi/4\right)$ and strictly decreasing on $\left(\pi/4, \pi/2\right)$. The stationary point $x = \pi/4$ is a maximum; it may be included as the endpoint of either corresponding monotonic interval.

Quick Tip to Solve: For this logarithm, the denominator of the derivative is positive; compare sine and cosine at $\pi/4$ to determine both signs immediately.

32(b). A rectangle of perimeter 30 cm is revolved along one of its sides to sweep out a cylinder of maximum volume. Find the dimensions of the rectangle. [5 marks]

Solution:

Concept:

- Rotating a rectangle about one of its sides produces a cylinder. The side along the axis becomes its height h , and the perpendicular side becomes its radius r .
- The rectangle's perimeter is $2(h + r)$, while the volume swept out is $V = \pi r^2 h$. Use the perimeter constraint to express the volume in a single variable.
- An interior stationary point is a maximum if the derivative changes from positive to negative there. The second derivative and the behavior at the boundary provide independent checks of the extremum.

Step 1: Relate the cylinder dimensions to the rectangle Let the side about which the rectangle is revolved have length h cm. Let the other side have length r cm. Each point at distance r from the rotation axis sweeps a circle of radius r , so the generated cylinder has radius r and height h . From the given perimeter,

$$2(r + h) = 30 \implies h = 15 - r.$$

Both sides must be positive, so the admissible range is $0 < r < 15$. This is the geometric constraint to retain while optimizing; radius and height cannot be varied independently.

Step 2: Express and differentiate the volume Substituting the perimeter condition into the cylinder formula gives

$$\begin{aligned} V(r) &= \pi r^2(15 - r) \\ &= \pi(15r^2 - r^3), \\ V'(r) &= \pi(30r - 3r^2) = 3\pi r(10 - r). \end{aligned}$$

The volume has units of cubic centimetres. Solving $V'(r) = 0$ gives $r = 0$ or $r = 10$. The value $r = 0$ would give a degenerate rectangle and is excluded from the admissible interval. Thus $r = 10$ is the sole interior stationary point.

Step 3: Verify the maximum and find the remaining side When $0 < r < 10$, both r and $10 - r$ are positive, so $V'(r) > 0$. When $10 < r < 15$, the factor $10 - r$ is negative, so $V'(r) < 0$. Hence the volume increases up to $r = 10$ and decreases thereafter, proving that this point gives the maximum over the entire admissible interval. Also,

$$V''(r) = \pi(30 - 6r), \quad V''(10) = -30\pi < 0.$$

At either limiting boundary $r \rightarrow 0$ or $r \rightarrow 15$, the volume tends to zero, consistent with the global maximum just established. Substitution in the constraint now gives $h = 15 - 10 = 5$ cm. The perimeter check is $2(10 + 5) = 30$ cm, as required.

Step 4: Conclusion The rectangle must have dimensions $10 \text{ cm} \times 5 \text{ cm}$, and it must be revolved about its 5 cm side. The maximum swept volume is $\pi(10)^2(5) = 500\pi \text{ cm}^3$.

Quick Tip to Solve: Label the rotation-axis side as the height first: the perpendicular side is the radius and is squared in the volume formula, so the two sides play different roles.

33. Find the domain of $q(x) = \cos^{-1}(4x^2 - 3)$. Hence, find the value of x for which $q(x) = 0$. Also, write the range of $3q(x) - \pi$. [5 marks]

Solution:

Concept:

- The real principal inverse cosine $\cos^{-1} u$ is defined exactly for $-1 \leq u \leq 1$, and its principal values lie in $[0, \pi]$.
- To find the domain of a composite inverse trigonometric function, impose the inverse function's input restriction on its inner expression. Both endpoints are included when the inequalities are non-strict.
- A positive affine transformation $v = 3u - \pi$ preserves the order of the endpoints of an interval; its range follows by transforming the minimum and maximum attainable values of u .

Step 1: Impose the real inverse-cosine condition The expression $4x^2 - 3$ must be an allowed input of the inverse cosine. Therefore,

$$\begin{aligned} -1 \leq 4x^2 - 3 \leq 1 &\iff 2 \leq 4x^2 \leq 4 \\ &\iff \frac{1}{2} \leq x^2 \leq 1. \end{aligned}$$

Taking square roots of the bounds on x^2 gives $\frac{1}{\sqrt{2}} \leq |x| \leq 1$. The absolute value is necessary: both negative and positive values of x are possible. Thus the negative interval runs from -1 to $-\frac{1}{\sqrt{2}}$, and the positive interval runs from $\frac{1}{\sqrt{2}}$ to 1 . At each of their endpoints the inner expression equals either -1 or 1 , so all four endpoints are allowed.

Step 2: Solve the equation $q(x)$ equals zero On the principal branch, $\cos^{-1} u = 0$ holds if and only if $u = \cos 0 = 1$. Applying this to the given function,

$$\begin{aligned} 4x^2 - 3 &= 1, \\ x^2 &= 1, \\ x &= \pm 1. \end{aligned}$$

Both candidates lie in the domain just found. Direct substitution gives $q(1) = q(-1) = \cos^{-1}(1) = 0$, confirming that neither candidate is extraneous. No further values can satisfy the equation because the principal inverse cosine takes zero only at input 1 .

Step 3: Determine all attainable values of the transformed function As x varies over the positive domain interval $[\frac{1}{\sqrt{2}}, 1]$, the continuous expression $4x^2 - 3$ runs through every number from -1 to 1 . Consequently, $q(x)$ takes every principal inverse-cosine value from π down to 0 . The negative domain interval gives the same values because the inner expression depends on x^2 . Hence the entire interval $[0, \pi]$, rather than just its endpoints, is attained by q .

$$0 \leq q(x) \leq \pi \implies -\pi \leq 3q(x) - \pi \leq 2\pi.$$

The lower endpoint occurs at $x = \pm 1$, and the upper endpoint occurs at $x = \pm \frac{1}{\sqrt{2}}$. Continuity on either domain interval guarantees every intermediate value as well.

Quick Tip to Solve: First constrain the argument of inverse cosine to $[-1, 1]$; use an absolute value when passing from bounds on x squared to bounds on x , so the negative domain interval is retained.

Step 4: Conclusion The domain is $\left[-1, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, 1\right]$. The equation $q(x) = 0$ gives $x = \pm 1$, and the range of $3q(x) - \pi$ is $[-\pi, 2\pi]$.

Quick Tip to Solve: First constrain the argument of inverse cosine to $[-1, 1]$; use an absolute value when passing from bounds on x squared to bounds on x , so the negative domain interval is retained.

34. A line passing through the points $A(1, 2, 3)$ and $B(5, 8, 11)$ intersects the line $\vec{r} = 4\hat{i} + \hat{j} + \lambda(5\hat{i} + 2\hat{j} + \hat{k})$. Find the co-ordinates of the point of intersection. Hence, write the equation of a line passing through the point of intersection and perpendicular to both the lines. [5 marks]

Solution:

Concept:

- A line through two distinct points with position vectors $\vec{a}$ and $\vec{b}$ has equation $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is a real parameter.
- At an intersection, corresponding coordinates of the two line equations are equal. The parameters on the two lines are independent, so solve for both and check all three coordinates.
- A vector perpendicular to two non-parallel direction vectors $\vec{u}$ and $\vec{v}$ is their cross product $\vec{u} \times \vec{v}$. A line through the common point with this direction is perpendicular to both original lines.

Step 1: Write the line through A and B in parameter form The direction vector from $A(1, 2, 3)$ to $B(5, 8, 11)$ is

$$\vec{AB} = (5 - 1, 8 - 2, 11 - 3) = (4, 6, 8).$$

Dividing this vector by 2 gives the simpler parallel direction $\vec{u} = (2, 3, 4)$. Thus the line through A and B can be written as

$$(x, y, z) = (1, 2, 3) + t(2, 3, 4),$$

or $x = 1 + 2t$, $y = 2 + 3t$, $z = 3 + 4t$. The other line has base point $(4, 1, 0)$ and direction $\vec{v} = (5, 2, 1)$, so its coordinates are $x = 4 + 5\lambda$, $y = 1 + 2\lambda$, $z = \lambda$.

Step 2: Solve for the common point and check the remaining coordinate Equating the coordinates at an intersection gives

$$1 + 2t = 4 + 5\lambda,$$

$$2 + 3t = 1 + 2\lambda,$$

$$3 + 4t = \lambda.$$

Substitute $\lambda = 3 + 4t$ from the third equation into the first:

$$1 + 2t = 4 + 5(3 + 4t) = 19 + 20t,$$

$$-18t = 18,$$

$$t = -1, \quad \lambda = -1.$$

The second equation is also satisfied, because $2 + 3(-1) = -1 = 1 + 2(-1)$. Substituting $t = -1$ into the first line gives the common point $P = (-1, -1, -1)$. Substituting $\lambda = -1$ into the second

Quick Tip to Solve: Use the third coordinate to eliminate one line parameter, check the unused coordinate, and take the cross product of the two directions for the common perpendicular.

line independently gives $(4 - 5, 1 - 2, -1) = (-1, -1, -1)$, confirming the intersection in all three coordinates.

Step 3: Obtain a direction perpendicular to both original lines Compute the cross product of their direction vectors:

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} \\ &= (3 - 8)\hat{i} - (2 - 20)\hat{j} + (4 - 15)\hat{k} \\ &= -5\hat{i} + 18\hat{j} - 11\hat{k}.\end{aligned}$$

This is nonzero, so the original direction vectors are not parallel. Its dot products with the original directions are

$$\begin{aligned}(-5, 18, -11) \cdot (2, 3, 4) &= -10 + 54 - 44 = 0, \\ (-5, 18, -11) \cdot (5, 2, 1) &= -25 + 36 - 11 = 0.\end{aligned}$$

Hence it is perpendicular to both. Taking P as the base point yields the required line $\vec{r} = -\hat{i} - \hat{j} - \hat{k} + s(-5\hat{i} + 18\hat{j} - 11\hat{k})$, with $s \in \mathbb{R}$. It passes through the intersection when $s = 0$; multiplying its direction by any nonzero scalar describes the same line.

Step 4: Conclusion The point of intersection is $(-1, -1, -1)$. A line through it and perpendicular to both given lines is

$$\vec{r} = -\hat{i} - \hat{j} - \hat{k} + s(-5\hat{i} + 18\hat{j} - 11\hat{k}), \quad s \in \mathbb{R}.$$

Equivalently,
$$\frac{x+1}{-5} = \frac{y+1}{18} = \frac{z+1}{-11}.$$

Quick Tip to Solve: Use the third coordinate to eliminate one line parameter, check the unused coordinate, and take the cross product of the two directions for the common perpendicular.

35(a). If $P = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, find (QP) and hence solve the following system of equations using matrices:

$$x - y = 3, \quad 2x + 3y + 4z = 17, \quad y + 2z = 7$$

[5 marks]

Solution:

Concept:

- A system of three linear equations can be written as $PX = B$, where P is the coefficient matrix, X is the column of unknowns and B is the column of constants. The order of unknowns must be the same in every row.
- The entry in row i , column j of a product is the dot product of row i of the first matrix and column j of the second matrix. Matrix multiplication is performed in the stated order.
- If $QP = kI$ with $k \neq 0$, multiplying $PX = B$ on the left by Q gives $kX = QB$, and hence

Quick Tip to Solve: Once the product $QP = 6I$ is found, multiply the constant column by Q and divide each resulting component by 6; no separate inverse calculation is needed.

$$X = \frac{1}{k}QB. \text{ Equivalently, } P^{-1} = \frac{1}{k}Q.$$

Step 1: Calculate the product QP Using the given rows of Q and columns of P , we obtain

$$QP = \begin{pmatrix} 2(1) + 2(2) - 4(0) & 2(-1) + 2(3) - 4(1) & 2(0) + 2(4) - 4(2) \\ -4(1) + 2(2) - 4(0) & -4(-1) + 2(3) - 4(1) & -4(0) + 2(4) - 4(2) \\ 2(1) - 1(2) + 5(0) & 2(-1) - 1(3) + 5(1) & 2(0) - 1(4) + 5(2) \end{pmatrix}.$$

Simplifying the nine entries gives

$$QP = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I.$$

In particular, the non-zero scalar multiplying the identity allows us to isolate the unknown column after left multiplication by Q .

Step 2: Express the equations as a matrix equation Keeping the unknowns in the order x, y, z , define

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}.$$

The equations $x - y = 3$, $2x + 3y + 4z = 17$ and $y + 2z = 7$ have exactly the coefficient matrix P , so they are equivalent to $PX = B$. Multiplying both sides on the left by Q , and using associativity, gives

$$(QP)X = QB \implies 6X = QB.$$

Thus division by 6 is legitimate and gives the matrix method of solution directly.

Step 3: Evaluate the unknown column and check the equations Compute the right-hand column:

$$\begin{aligned} QB &= \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix}. \end{aligned}$$

Therefore $X = \frac{1}{6}(12, -6, 24)^T$. As a check, its components give

$$2 - (-1) = 3, \quad 2(2) + 3(-1) + 4(4) = 17, \quad -1 + 2(4) = 7.$$

All three original equations are satisfied. Since P has an inverse, this solution is unique.

Step 4: Conclusion $QP = 6I$, and the unique solution of the system is $x = 2, y = -1, z = 4$.

Quick Tip to Solve: Once the product $QP = 6I$ is found, multiply the constant column by Q and divide each resulting component by 6; no separate inverse calculation is needed.

35(b). Obtain the value of $\Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$ in terms of x, y and z . Further, if $\Delta = 0$ and x, y, z are non-zero real numbers, prove that $x^{-1} + y^{-1} + z^{-1} = -1$. [5 marks]

Solution:

Concept:

- For a 3×3 determinant, expansion along the first row uses the signs $+, -, +$. Each coefficient is multiplied by its corresponding 2×2 minor.
- The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ equals $ad - bc$. Expanding the minors explicitly prevents loss of terms in a determinant whose entries contain variables.
- Division by xyz is valid here because the second part specifies that all of x, y, z are non-zero. The identities $x^{-1} = 1/x$, $y^{-1} = 1/y$ and $z^{-1} = 1/z$ then convert products into reciprocals.

Step 1: Expand the determinant along its first row With the alternating cofactor signs included, the given determinant is

$$\begin{aligned} \Delta &= (1+x) \begin{vmatrix} 1+y & 1 \\ 1 & 1+z \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1+z \end{vmatrix} + \begin{vmatrix} 1 & 1+y \\ 1 & 1 \end{vmatrix} \\ &= (1+x)[(1+y)(1+z) - 1] \\ &\quad - [(1+z) - 1] + [1 - (1+y)]. \end{aligned}$$

The three minors simplify respectively to $y + z + yz$, z and $-y$. Notice that the middle minor is subtracted while the last minor is added; its value already contains a negative sign.

Step 2: Collect the terms in x, y, z Substituting the minors into the expansion gives

$$\begin{aligned} \Delta &= (1+x)(y + z + yz) - z - y \\ &= y + z + yz + xy + xz + xyz - z - y \\ &= xy + yz + zx + xyz. \end{aligned}$$

The terms y and z cancel. The expression is symmetric in the three variables, as expected because exchanging two variables corresponds to interchanging the corresponding rows and columns together, leaving the determinant unchanged. This calculation obtains the value for arbitrary real x, y, z , without requiring any division or non-zero assumptions.

Step 3: Apply the zero-determinant condition and divide For the second part, $\Delta = 0$ gives

$$xy + yz + zx + xyz = 0.$$

Since x, y and z are all non-zero, their product xyz is non-zero. Dividing the entire equation by that product yields

$$\frac{xy}{xyz} + \frac{yz}{xyz} + \frac{zx}{xyz} + 1 = 0.$$

Quick Tip to Solve: First collect the determinant as $xy + yz + zx + xyz$; then divide its zero equation by xyz to turn the three pairwise products into the required reciprocals.

Cancelling the common non-zero factors in each fraction, this becomes

$$\frac{1}{z} + \frac{1}{x} + \frac{1}{y} + 1 = 0.$$

Rearranging the three reciprocal terms and moving the constant to the other side proves the required identity. The non-zero condition is essential specifically for this division, rather than for evaluating the determinant itself.

Step 4: Conclusion $\Delta = xy + yz + zx + xyz$. Consequently, when $\Delta = 0$ and $xyz \neq 0$, $x^{-1} + y^{-1} + z^{-1} = -1$.

Quick Tip to Solve: First collect the determinant as $xy + yz + zx + xyz$; then divide its zero equation by xyz to turn the three pairwise products into the required reciprocals.

Section E (12 marks)

This section comprises of 3 case study based questions of 4 marks each.

36. Sports car racing is a form of motorsport which uses sports car prototypes. The competition is held on special tracks designed in various shapes. The equation of one such track is given as follows:

$$f(x) = \begin{cases} x^4 - 4x^2 + 4, & 0 \leq x < 3 \\ x^2 + 40, & x \geq 3 \end{cases}$$

Based on given information, answer the following questions:

[4 marks]

36(i). Find $f'(x)$ for $0 < x < 3$.

[1 mark]

Solution:

Concept:

- The power rule gives $\frac{d}{dx}(x^n) = nx^{n-1}$; the derivative of a constant is zero.
- For a piecewise function, use the formula valid on the interval containing the point. Every x with $0 < x < 3$ lies in the first branch.

Step 1: Select the polynomial branch On the requested open interval $0 < x < 3$, the track is represented by

$$f(x) = x^4 - 4x^2 + 4.$$

Each such point has a neighbourhood within this branch, so we can differentiate this polynomial directly. The formula $x^2 + 40$ applies only for $x \geq 3$ and must not be used here.

Step 2: Differentiate each term Applying the power rule and retaining the negative coefficient of the quadratic term gives

$$\begin{aligned} f'(x) &= 4x^3 - 4(2x) + 0 \\ &= 4x^3 - 8x. \end{aligned}$$

The constant 4 contributes zero, while the derivative of $-4x^2$ is $-8x$.

Step 3: Conclusion $f'(x) = 4x^3 - 8x$ for $0 < x < 3$.

Quick Tip to Solve: Choose the first branch for $0 < x < 3$, then reduce each polynomial exponent by one after multiplying by it.

36(ii). Find $f'(4)$.

[1 mark]

Solution:

Concept:

- The derivative at a point must come from the branch valid in a neighbourhood of that point.
- The power rule gives $\frac{d}{dx}(x^2) = 2x$, while the derivative of a constant is zero.

Step 1: Identify the branch containing four Since $4 > 3$, both 4 and all sufficiently nearby values belong to the second branch. Therefore, the relevant expression is

$$f(x) = x^2 + 40.$$

We differentiate this expression, rather than the quartic expression which is valid only for $0 \leq x < 3$. No junction calculation is needed because 4 is strictly inside the second interval.

Step 2: Differentiate and substitute Differentiating before inserting the point gives

$$f'(x) = 2x + 0 = 2x \quad (x > 3).$$

Hence the derivative at the requested point is obtained by the substitution

$$f'(4) = 2(4) = 8.$$

The added constant 40 changes the function's value but does not change its rate of change.

Step 3: Conclusion $f'(4) = 8$.

Quick Tip to Solve: At $x = 4$, use $x^2 + 40$ and differentiate before substituting; the constant contributes nothing to the slope.

36(iii)(a). Test for continuity of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- A function is continuous at $x = a$ when $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$.
- Each polynomial is continuous on its own interval. At the joining point of a piecewise function, evaluate the one-sided limits using their respective branches.

Step 1: Calculate the left-hand limit When x approaches 3 from below, the first branch applies. Direct substitution in this polynomial gives

$$\begin{aligned} \lim_{x \rightarrow 3^-} f(x) &= 3^4 - 4(3^2) + 4 \\ &= 81 - 36 + 4 = 49. \end{aligned}$$

Although the first branch excludes 3, its polynomial still determines the limit of values approaching

Quick Tip to Solve: At the junction, compare the left polynomial's value with the right polynomial's value, and use the branch including equality for $f(3)$.

the junction from the left.

Step 2: Calculate the right-hand limit and function value For values approaching 3 from above, use the second branch $x^2 + 40$. It also includes 3, since its domain is $x \geq 3$. Thus

$$\begin{aligned}\lim_{x \rightarrow 3^+} f(x) &= 3^2 + 40 = 49, \\ f(3) &= 3^2 + 40 = 49.\end{aligned}$$

The two one-sided limits agree, so the two-sided limit exists and is 49. It equals the actual value assigned by the function at the joining point, satisfying every part of the continuity criterion.

Step 3: Conclusion f is continuous at $x = 3$, since both one-sided limits and $f(3)$ equal 49.

Quick Tip to Solve: At the junction, compare the left polynomial's value with the right polynomial's value, and use the branch including equality for $f(3)$.

OR

36(iii)(b). Test for differentiability of $f(x)$ at $x = 3$.

[2 marks]

Solution:

Concept:

- Differentiability at a junction requires the left-hand derivative and right-hand derivative to exist and agree, with each difference quotient formed using the same value $f(3)$.
- Continuity is necessary for differentiability but is not sufficient. Even if the branch values agree, their limiting slopes can differ.

Step 1: Establish the common joining value The branch which includes $x = 3$ gives $f(3) = 3^2 + 40 = 49$. The polynomial to the left also tends to $3^4 - 4(3^2) + 4 = 49$, so the function is continuous at the joining point. We must still compare the one-sided difference quotients rather than infer differentiability from continuity alone.

Step 2: Evaluate both one-sided derivatives Writing the increment as h , expansion of the first polynomial gives

$$f(3 + h) = 49 + 84h + 50h^2 + 12h^3 + h^4 \quad (h < 0).$$

Therefore the left-hand derivative is

$$\lim_{h \rightarrow 0^-} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0^-} (84 + 50h + 12h^2 + h^3) = 84.$$

For $h > 0$, the second branch gives $f(3 + h) = 49 + 6h + h^2$, so the right-hand derivative is

$$\lim_{h \rightarrow 0^+} \frac{f(3 + h) - f(3)}{h} = \lim_{h \rightarrow 0^+} (6 + h) = 6.$$

Both slopes are finite, but $84 \neq 6$. Thus the two sides cannot produce a single limit of the difference quotient at this junction.

Quick Tip to Solve: After checking the branch values meet at 49, compare $4x^3 - 8x$ and $2x$ at $x = 3$; unequal slopes rule out differentiability.

Step 3: Conclusion f is not differentiable at $x = 3$, because the left-hand derivative is 84 and the right-hand derivative is 6.

Quick Tip to Solve: After checking the branch values meet at 49, compare $4x^3 - 8x$ and $2x$ at $x = 3$; unequal slopes rule out differentiability.

37. Smoking increases the risk of lung problems.

A study revealed that 170 in 1000 males who smoke develop lung complications, while 120 out of 1000 females who smoke develop lung related problems. In a colony, 50 people were found to be smokers of which 30 are males. A person is selected at random from these 50 people and tested for lung related problems.

Based on the given information, answer the following questions:

[4 marks]

37(i). What is the probability that selected person is a female?

[1 mark]

Solution:

Concept:

- For equally likely selections, $P(E) = \frac{\text{number of favourable persons}}{\text{total number of persons}}$.
- The male and female groups partition the given population of smokers.

Step 1: Count the female smokers There are 50 smokers in the colony, of whom 30 are males. Hence the number of females is $50 - 30 = 20$. The selection is from these 50 smokers, so the total population used in the denominator is 50.

Step 2: Form the selection probability Let F be the event that the selected person is female. Each of the 50 persons is equally likely to be selected. Therefore

$$P(F) = \frac{20}{50} = \frac{2}{5}.$$

The rates of lung complications do not enter this unconditional selection probability.

Step 3: Conclusion The probability that the selected person is female is $\frac{2}{5}$.

Quick Tip to Solve: First subtract the male count from the total; lung-problem rates are irrelevant to this part.

37(ii). If a male person is selected, what is the probability that he will not be suffering from lung problems?

[1 mark]

Solution:

Concept:

- For complementary events within a given group, $P(L^c | M) = 1 - P(L | M)$.
- Here M denotes a male smoker, L denotes lung problems, and L^c denotes their absence.

Quick Tip to Solve: When the selected gender is given, take the complement of that gender's complication rate directly.

Step 1: Use the given male complication rate The question already specifies that a male has been selected. Thus the relevant probability is conditional on being male, not the probability of first selecting a male. Among 1000 male smokers, 170 develop lung complications, giving $P(L | M) = 170/1000 = 0.17$.

Step 2: Take the complement within the male group The two possibilities, having and not having lung problems, exhaust the male group. Thus

$$P(L^c | M) = 1 - \frac{170}{1000} = \frac{830}{1000} = \frac{83}{100}.$$

Multiplication by 30/50 would instead give a joint probability and is not required here.

Step 3: Conclusion The required probability is $\boxed{\frac{83}{100}}$.

Quick Tip to Solve: When the selected gender is given, take the complement of that gender's complication rate directly.

37(iii)(a). A person selected at random is detected with lung complications. Find the probability that selected person is a female. [2 marks]

Solution:

Concept:

- Let M, F denote male and female smokers and L denote lung complications.
- Bayes' theorem gives $P(F | L) = \frac{P(F)P(L | F)}{P(M)P(L | M) + P(F)P(L | F)}$.

Step 1: Write the prior and conditional probabilities Of the 50 smokers, 30 are males and 20 are females. Thus $P(M) = 3/5$ and $P(F) = 2/5$. The study supplies $P(L | M) = 170/1000 = 17/100$ and $P(L | F) = 120/1000 = 3/25$. These complication probabilities are conditional on gender; they are not the posterior probability asked for.

Step 2: Find the total probability of lung complications The male and female groups are mutually exclusive and exhaustive. By the total probability theorem,

$$P(L) = \frac{3}{5} \frac{17}{100} + \frac{2}{5} \frac{3}{25} = \frac{51}{500} + \frac{24}{500} = \frac{75}{500} = \frac{3}{20}.$$

Step 3: Condition on the detected complication The probability of being female and having a complication is $P(F \cap L) = (2/5)(3/25) = 24/500$. Therefore

$$P(F | L) = \frac{24/500}{75/500} = \frac{24}{75} = \frac{8}{25}.$$

Conditioning changes the denominator to the probability of lung complications.

Step 4: Conclusion The probability that the detected person is female is $\boxed{\frac{8}{25}}$.

Quick Tip to Solve: Weight each complication rate by its gender proportion before applying Bayes' theorem.

37(iii)(b). A person selected at random is not having lung problems, find the probability that the person is a male. [2 marks]

Solution:

Concept:

- Let M, F denote male and female smokers and L^c the absence of lung problems.
- Use complements within each gender, followed by $P(M | L^c) = \frac{P(M)P(L^c | M)}{P(L^c)}$.

Step 1: Find the probabilities for healthy smokers The selection proportions are $P(M) = 30/50 = 3/5$ and $P(F) = 20/50 = 2/5$. The corresponding probabilities of not having lung problems are

$$P(L^c | M) = 1 - \frac{170}{1000} = \frac{83}{100}, \quad P(L^c | F) = 1 - \frac{120}{1000} = \frac{22}{25}.$$

The absence of problems is the condition on the selected person.

Step 2: Calculate the total probability of no lung problems Applying the total probability theorem to the two gender groups,

$$P(L^c) = \frac{3}{5} \frac{83}{100} + \frac{2}{5} \frac{22}{25} = \frac{249}{500} + \frac{176}{500} = \frac{425}{500} = \frac{17}{20}.$$

Both groups must be included in this denominator.

Step 3: Apply Bayes theorem The joint probability of being male and not having lung problems is $249/500$. Hence

$$P(M | L^c) = \frac{249/500}{425/500} = \frac{249}{425}.$$

This differs from the unconditional male selection probability $3/5$, because the person is already known not to have lung problems.

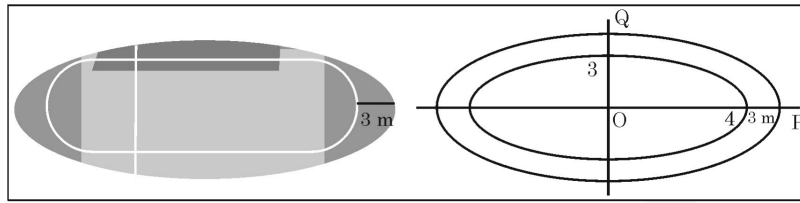
Step 4: Conclusion The required probability is $\boxed{\frac{249}{425}}$.

Quick Tip to Solve: Complement both gender-specific rates before weighting them; complementing the final posterior is incorrect.

38. A racing track is built around an elliptical ground whose equation is given by $9x^2 + 16y^2 = 144$. The width of the track is 3 m as shown below:

Based on given information, answer the following questions:

[4 marks]



38(i). Express y as a function of x from the given equation of ellipse.

[1 mark]

Solution:

Concept:

- An ellipse symmetric about the x -axis has an upper and a lower branch.
- For area calculations above the x -axis, choose the nonnegative square root.

Step 1: Isolate the square of y From the given equation $9x^2 + 16y^2 = 144$, move the x -term to the other side and divide by 16:

$$y^2 = \frac{144 - 9x^2}{16} = \frac{9}{16}(16 - x^2).$$

For real points on the ellipse, $16 - x^2 \geq 0$, so $-4 \leq x \leq 4$.

Step 2: Choose the branch needed for the ground area Taking square roots gives $y = \pm \frac{3}{4}\sqrt{16 - x^2}$. Both signs describe the complete ellipse. The upper boundary, used when finding the area by integrating above the x -axis, is $y = \frac{3}{4}\sqrt{16 - x^2}$; the lower boundary is its negative.

Step 3: Conclusion The upper branch is $y = \frac{3}{4}\sqrt{16 - x^2}$, $-4 \leq x \leq 4$; the complete ellipse has both signs.

Quick Tip to Solve: Factor 9 from the numerator before taking the square root, and state which branch you use.

38(ii). Integrate the function obtained in (i) with respect to x .

[1 mark]

Solution:

Concept:

- Use $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + C$.
- The upper branch found in part (i) is $y = \frac{3}{4}\sqrt{16 - x^2}$, with $a = 4$.

Step 1: Extract the constant factor For the upper branch of the ellipse,

$$\int y dx = \frac{3}{4} \int \sqrt{16 - x^2} dx.$$

Here $16 = 4^2$, so the standard square-root integral applies with $a = 4$. Keeping the factor $3/4$ outside avoids changing the coefficient of either term.

Quick Tip to Solve: Use $a = 4$ and multiply both terms of the standard antiderivative by $3/4$.

Step 2: Substitute into the standard antiderivative Using the formula,

$$\begin{aligned}\int y \, dx &= \frac{3}{4} \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right] + C \\ &= \frac{3x}{8} \sqrt{16 - x^2} + 6 \sin^{-1} \left(\frac{x}{4} \right) + C.\end{aligned}$$

The inverse sine is measured in radians, and the arbitrary constant is retained because this integral is indefinite.

Step 3: Conclusion $\int y \, dx = \frac{3x}{8} \sqrt{16 - x^2} + 6 \sin^{-1} \left(\frac{x}{4} \right) + C$ for the upper branch.

Quick Tip to Solve: Use $a = 4$ and multiply both terms of the standard antiderivative by $3/4$.

38(iii)(a). Find the area of the region enclosed within the elliptical ground excluding the track using integration. [2 marks]

Solution:

Concept:

- The ground is the inner ellipse $x^2/16 + y^2/9 = 1$, with intercepts 4 and 3.
- By symmetry its area is four times the first-quadrant area: $A = 4 \int_0^4 y \, dx$.

Step 1: Set up the area of the elliptical ground The equation describes the ground inside the track. Thus excluding the surrounding track means finding the area enclosed by this inner ellipse. Its upper boundary is $y = (3/4)\sqrt{16 - x^2}$. Symmetry about both axes gives

$$A = 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \, dx = 3 \int_0^4 \sqrt{16 - x^2} \, dx.$$

The track width is not subtracted from the given semi-axes.

Step 2: Evaluate the definite integral An antiderivative is

$$\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right).$$

At $x = 4$, the square-root term vanishes and $\sin^{-1}(1) = \pi/2$; at $x = 0$, both terms vanish. Hence

$$\begin{aligned}A &= 3 \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) \right]_0^4 \\ &= 3 \left(8 \frac{\pi}{2} - 0 \right) = 12\pi \, \text{m}^2.\end{aligned}$$

This is consistent with the ellipse area formula using semi-axes 4 m and 3 m, while the required calculation uses integration.

Step 3: Conclusion The area of the elliptical ground excluding the track is $12\pi \, \text{m}^2$.

Quick Tip to Solve: The given equation is for the inner ground: integrate its upper branch over 0 to 4 and multiply by four.

38(iii)(b). Write the co-ordinates of the points P and Q where the outer edge of the track cuts x axis and y axis in first quadrant and find the area of the triangle formed by points P , O , Q using integration. [2 marks]

Solution:

Concept:

- For $x^2/a^2 + y^2/b^2 = 1$, the positive axis intercepts are $(a, 0)$ and $(0, b)$.
- The area below a nonnegative function $y = f(x)$ is $\int_u^v f(x) dx$. For this triangle, integrate the straight line through its outer intercepts.

Step 1: Find the outer intercepts Dividing by 144 gives $x^2/16 + y^2/9 = 1$. The ground has positive intercepts $(4, 0)$ and $(0, 3)$. Adding the track width 3 m along each axis gives

$$P = (4 + 3, 0) = (7, 0), \quad Q = (0, 3 + 3) = (0, 6).$$

Step 2: Obtain the boundary of the triangle and integrate The straight line through $P = (7, 0)$ and $Q = (0, 6)$ has intercept form

$$\frac{x}{7} + \frac{y}{6} = 1, \quad y = 6 - \frac{6x}{7}.$$

For $0 \leq x \leq 7$, a vertical strip in the triangle has height $6 - 6x/7$. Hence

$$\begin{aligned} A &= \int_0^7 \left(6 - \frac{6x}{7} \right) dx \\ &= \left[6x - \frac{3x^2}{7} \right]_0^7 \\ &= 42 - 21 = 21 \text{ m}^2. \end{aligned}$$

The check $\frac{1}{2}(7)(6) = 21$ agrees with the integral.

Step 3: Conclusion The coordinates are $P = (7, 0)$, $Q = (0, 6)$, and the area of $\triangle POQ$, found using integration, is 21 m^2 .

Quick Tip to Solve: Add the track width to each positive ellipse intercept first; then integrate the straight line joining the outer intercepts, since the requested region is a triangle.