

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-3-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 3 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 198$ and $|\vec{a}| = 10|\vec{b}|$, then :

[1 mark]

- (A) $|\vec{a}| = \sqrt{2}$
- (B) $|\vec{b}| = \sqrt{2}$
- (C) $|\vec{b}| = 10\sqrt{2}$
- (D) $|\vec{a}| = \frac{10}{\sqrt{2}}$

Correct Answer: (B) $|\vec{b}| = \sqrt{2}$

Solution:

Concept:

- The scalar product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- The square of a vector's magnitude is $\vec{a} \cdot \vec{a} = |\vec{a}|^2$; magnitudes are nonnegative.

Step 1: Expand the scalar product The mixed products cancel because the scalar product is commutative. Therefore,

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = 198.$$

Step 2: Substitute the magnitude relation Put $|\vec{b}| = t$, so $|\vec{a}| = 10t$. Then

$$100t^2 - t^2 = 198 \Rightarrow t^2 = 2 \Rightarrow t = \sqrt{2}.$$

Thus $|\vec{a}| = 10\sqrt{2}$. Option (A) wrongly assigns the smaller magnitude to $\vec{a}$, and (C) assigns the larger one to $\vec{b}$. Option (D) gives $5\sqrt{2}$, which is not the magnitude of $\vec{a}$.

Step 3: Conclusion $|\vec{b}| = \sqrt{2}$. Hence, the correct option is (B).

Quick Tip to Solve: Cancel the mixed dot products first, then square the given magnitude ratio before substituting.

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of lines L_1 and L_2 respectively and θ is the acute angle between them, then :

[1 mark]

(A) $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(B) $\sin \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(C) $\tan \theta = \frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2}$

(D) $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

Correct Answer: (D) $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

Solution:

Concept:

- Direction cosines (l, m, n) form a unit direction vector because $l^2 + m^2 + n^2 = 1$.
- The dot product of unit vectors equals the cosine of their angle; the acute angle between unoriented lines uses its absolute value.

Step 1: Form the dot product Let the unit direction vectors of the two lines be $\vec{u} = (l_1, m_1, n_1)$ and $\vec{v} = (l_2, m_2, n_2)$. Their dot product is

$$s = \vec{u} \cdot \vec{v} = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

This gives the cosine of the angle between the chosen vector directions.

Step 2: Select the acute line angle Reversing one direction vector represents the same line but changes the sign of s . The acute angle between the lines must have a positive cosine, so $\cos \theta = |s|$. Option (A) fails when the chosen directions have a negative dot product; (B) incorrectly uses sine, and (C) is not the dot-product angle formula and can even divide by zero.

Step 3: Conclusion $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$. Hence, the correct option is (D).

Quick Tip to Solve: When the question asks for the acute angle between lines, take the absolute value of their unit-vector dot product.

3. Direction ratios of lines l_1 and l_2 respectively are $\langle 1, 0, 0 \rangle$ and $\langle 0, -1, 0 \rangle$. The direction ratios of the line perpendicular to both l_1 and l_2 are :

[1 mark]

(A) $\langle 1, 1, 0 \rangle$

(B) $\langle 0, 0, -1 \rangle$

(C) $\langle 1, 1, 1 \rangle$

(D) $\langle 1, 0, -1 \rangle$

Correct Answer: (B) $\langle 0, 0, -1 \rangle$

Solution:

Concept:

- Two nonzero direction vectors are perpendicular exactly when their scalar product is zero.
- Direction ratios need not be unit vectors: any nonzero scalar multiple describes the same line

Quick Tip to Solve: A line perpendicular to both the x-axis and y-axis must have only its z direction ratio nonzero.

direction.

Step 1: Apply both perpendicularity conditions Let the required direction ratios be (a, b, c) . Taking dot products with the two given directions gives

$$(a, b, c) \cdot (1, 0, 0) = a = 0, \quad (a, b, c) \cdot (0, -1, 0) = -b = 0.$$

Thus the first two direction ratios must vanish, whereas $c \neq 0$ is needed for a valid line direction.

Step 2: Test the listed triples The triple $(0, 0, -1)$ has zero scalar product with both given directions. Options (A), (C) and (D) each have first component 1, giving dot product 1 with $(1, 0, 0)$; hence none is perpendicular to the first line. The negative third component in (B) merely reverses the direction and is permissible.

Step 3: Conclusion The required direction ratios are $\langle 0, 0, -1 \rangle$. Hence, the correct option is (B).

Quick Tip to Solve: A line perpendicular to both the x-axis and y-axis must have only its z direction ratio nonzero.

4. If E and F are two independent events such that $P(E) = \frac{3}{10}$, $P(E \cup F) = \frac{1}{2}$, then $P(E|F) - P(F|E)$ is equal to : [1 mark]

- (A) $\frac{2}{7}$
- (B) $\frac{3}{35}$
- (C) $\frac{1}{70}$
- (D) $\frac{1}{7}$

Correct Answer: (C) $\frac{1}{70}$

Solution:

Concept:

- For independent events, $P(E \cap F) = P(E)P(F)$, and conditioning on an event of positive probability does not change the other probability.
- The addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Step 1: Determine the probability of F Put $P(F) = p$. Using independence in the addition rule,

$$\frac{1}{2} = \frac{3}{10} + p - \frac{3}{10}p \Rightarrow \frac{1}{5} = \frac{7}{10}p \Rightarrow p = \frac{2}{7}.$$

Both conditioning events have positive probabilities, so the conditional probabilities are defined.

Step 2: Subtract the conditional probabilities Independence gives $P(E | F) = 3/10$ and $P(F | E) = 2/7$. Consequently,

$$P(E | F) - P(F | E) = \frac{3}{10} - \frac{2}{7} = \frac{21 - 20}{70} = \frac{1}{70}.$$

Quick Tip to Solve: For independent events, first replace the two conditional probabilities by their unconditional values, then use the union to find the missing one.

Option (A) is only $P(F)$; (B) equals the intersection probability $(3/10)(2/7)$, and (D) is not the computed difference.

Step 3: Conclusion The required difference is $\frac{1}{70}$. Hence, the correct option is (C).

Quick Tip to Solve: For independent events, first replace the two conditional probabilities by their unconditional values, then use the union to find the missing one.

5. The domain of $\cos^{-1}(4x + 1)$ is :

[1 mark]

(A) $[-1, 1]$

(B) $[-3, 5]$

(C) $[-4, 4]$

(D) $\left[-\frac{1}{2}, 0\right]$

Correct Answer: (D) $\left[-\frac{1}{2}, 0\right]$

Solution:

Concept:

- The real-valued inverse cosine $\cos^{-1} u$ is defined exactly when $-1 \leq u \leq 1$.
- To find the domain of a composite function, impose the outer function's input restriction on the entire inner expression.

Step 1: Restrict the argument Here the input to inverse cosine is $4x + 1$, so it must satisfy

$$-1 \leq 4x + 1 \leq 1.$$

Restricting x directly to $[-1, 1]$ would ignore the transformation inside the inverse function.

Step 2: Solve the double inequality Subtract 1 from every member and then divide by the positive number 4, preserving both inequality directions:

$$-2 \leq 4x \leq 0 \Rightarrow -\frac{1}{2} \leq x \leq 0.$$

Both endpoints are included because they give arguments -1 and 1 , where inverse cosine is defined. Options (A), (B) and (C) all contain $x = 1$, whose argument is 5 , so none can be the required domain.

Step 3: Conclusion The domain is $\left[-\frac{1}{2}, 0\right]$. Hence, the correct option is (D).

Quick Tip to Solve: Place the entire expression inside inverse cosine between -1 and 1 , then solve the two inequalities together.

6. If $A^2 = 4A + 3I$ and $A^{-1} = xA + yI$, then the value of $(x + y)$ is :

[1 mark]

- (A) -1
- (B) 1
- (C) $\frac{5}{3}$
- (D) 7

Correct Answer: (A) -1

Solution:

Concept:

- Multiplying a matrix identity by A^{-1} reduces A^2 to A and A to the identity matrix I .
- A polynomial relation with a nonzero constant term can be rearranged to express the inverse as a polynomial in the matrix.

Step 1: Isolate the identity matrix Rearrange the given relation to obtain

$$A^2 - 4A = 3I \Rightarrow A(A - 4I) = 3I.$$

Thus $\frac{1}{3}(A - 4I)$ is an inverse of A . Equivalently, multiply the original relation by A^{-1} to get $A = 4I + 3A^{-1}$.

Step 2: Read the coefficients of the inverse expression Solving the last identity for the inverse gives

$$A^{-1} = \frac{1}{3}A - \frac{4}{3}I.$$

In the required form $xA + yI$, take $x = 1/3$ and $y = -4/3$, giving $x + y = (1 - 4)/3$. Option (C) would result from losing the minus sign of the identity term; (B) reverses the final sign, while (D) does not follow from dividing by 3.

Step 3: Conclusion $x + y = -1$. Hence, the correct option is (A).

Quick Tip to Solve: Move the term in A to the left before dividing by the constant term; this preserves the negative coefficient of I in the inverse.

7. If A and B are skew-symmetric matrices of same order, then $AB' + BA'$ is a/an :

[1 mark]

- (A) symmetric matrix
- (B) skew-symmetric matrix
- (C) null matrix
- (D) identity matrix

Correct Answer: (A) symmetric matrix

Solution:

Concept:

- A matrix S is symmetric if $S' = S$, while a skew-symmetric matrix satisfies $S' = -S$; the prime denotes transpose.
- Transposition reverses the order in a product: $(AB)' = B'A'$, and transposition twice returns the original matrix.

Step 1: Transpose the given sum Set $M = AB' + BA'$. Since the matrices have the same order, both products and their sum are defined. Using the product rule for transpose,

$$M' = (AB')' + (BA')' = BA' + AB'.$$

The order of factors reverses separately in each product; it is not necessary to assume $AB = BA$.

Step 2: Compare with the original matrix Addition of matrices is commutative, so $BA' + AB' = AB' + BA' = M$. Hence the sum is symmetric. It is not necessarily null, identity or skew-symmetric: for example, choosing

$$A = B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

gives $AB' + BA' = 2I$, which has none of those three properties. This also shows that the symmetry conclusion holds without needing any extra relation between the two matrices.

Step 3: Conclusion $AB' + BA'$ is a symmetric matrix. Hence, the correct option is (A).

Quick Tip to Solve: For a sum of the form $AB' + BA'$, transpose each product and observe that the two terms simply exchange places.

8. If a matrix X is such that $\begin{bmatrix} 2 & 1 \end{bmatrix} X = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$, then the order of matrix X is :

[1 mark]

- (A) 3×1
- (B) 2×3
- (C) 1×2
- (D) 1×3

Correct Answer: (B) 2×3

Solution:

Concept:

- A matrix of order $m \times n$ has m rows and n columns.
- The product of an $m \times n$ matrix and an $n \times p$ matrix is defined and has order $m \times p$.

Step 1: Match the inner dimensions The left row matrix $\begin{pmatrix} 2 & 1 \end{pmatrix}$ has order 1×2 . If X has order $r \times c$, the product is defined only if $r = 2$, because the number of columns in the left factor must equal the number of rows in X .

Step 2: Match the output dimensions The resulting product has order $1 \times c$. The right row matrix $\begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$ has order 1×3 , so $c = 3$. The dimensions therefore obey

$$(1 \times 2)(2 \times 3) = 1 \times 3.$$

Option (A) has three rows rather than the required two; (C) and (D) each have only one row. Their inner dimensions do not match the given left factor, so those products cannot be formed.

Step 3: Conclusion The order of X is 2×3 . Hence, the correct option is (B).

Quick Tip to Solve: Read the unknown matrix's rows from the left factor's columns and its columns from the product's columns.

9. If a square matrix A is such that $A^2 = A$ and $(I - A)^3 = xA + I$, then value of x must be : [1 mark]

- (A) 7
- (B) 5
- (C) -7
- (D) -1

Correct Answer: (D) -1

Solution:

Concept:

- An idempotent matrix satisfies $A^2 = A$; hence every positive integral power of A equals A .
- The identity matrix I commutes with A , so the ordinary binomial expansion applies to $(I - A)^3$.

Step 1: Reduce the matrix powers Use the given relation before comparing coefficients. Since

Quick Tip to Solve: Replace every positive power of an idempotent matrix by the matrix itself before combining coefficients.

$A^3 = A^2 A = A^2 = A$, the expansion becomes

$$(I - A)^3 = I - 3A + 3A^2 - A^3 = I - 3A + 3A - A = I - A.$$

Step 2: Compare the coefficient of A The reduced identity has coefficient -1 multiplying A , so it agrees with $I + xA$ for $x = -1$. The values $7, 5, -7$ do not produce this identity for a general idempotent matrix; for example, take $A = I$. If A happens to be the zero matrix alone, every x satisfies the equation, but -1 is the listed value that works for every idempotent matrix.

Step 3: Conclusion The required universal coefficient is $x = -1$. Hence, the correct option is (D).

Quick Tip to Solve: Replace every positive power of an idempotent matrix by the matrix itself before combining coefficients.

10. If $A(\text{adj } A) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then the value of $|2A|$ is :

[1 mark]

- (A) 6
- (B) 54
- (C) 12
- (D) 24

Correct Answer: (D) 24

Solution:

Concept:

- For any square matrix A , $A \text{ adj } A = |A|I$, where $|A|$ is its determinant.
- For a matrix of order n , multiplying every entry by k multiplies the determinant by k^n .

Step 1: Identify the determinant and order The displayed product is $3I_3$. Comparing this with $A \text{ adj } A = |A|I_3$ gives $|A| = 3$, and the order of A is 3. We read the common diagonal entry, rather than the determinant 27 of the displayed product.

Step 2: Scale all three rows In $2A$, each of the three rows has a factor of 2. Thus

$$|2A| = 2^3|A| = 8 \times 3.$$

Option (A), 6, uses only one factor of 2, and option (C), 12, uses only two. Option (B), 54, incorrectly doubles the determinant of the displayed product.

Step 3: Conclusion $|2A| = 24$. Hence, the correct option is (D).

Quick Tip to Solve: First read $|A|$ from $A \text{ adj } A$, then raise the scalar multiplier to the matrix order.

11. The value of k for which the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ k(x+1), & x = 0 \end{cases}$ is a continuous function, is :

[1 mark]

- (A) $\frac{1}{4}$
 (B) 2
 (C) $\frac{1}{2}$
 (D) 0

Correct Answer: (D) 0

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0)$.
- The bound $|\sin t| \leq 1$ allows the squeeze theorem to handle an oscillating trigonometric factor.

Step 1: Evaluate the limit at zero For $x \neq 0$,

$$-x^2 \leq x^2 \sin(1/x) \leq x^2.$$

Both bounds tend to zero as $x \rightarrow 0$, from either side. Consequently $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0$, although $\sin(1/x)$ itself has no limit.

Step 2: Match the defined value The second branch gives $f(0) = k(0+1) = k$, so continuity requires $k = 0$. At every nonzero real number the first branch is a composition and product of continuous functions. The other listed values, $1/4, 2, 1/2$, make the value at zero unequal to its limiting value.

Step 3: Conclusion $k = 0$ makes the function continuous. Hence, the correct option is (D).

Quick Tip to Solve: For a bounded oscillating factor multiplied by x^2 , squeeze the product between $-x^2$ and x^2 .

12. If $2 \cos^{-1} x = y$, then $\frac{dy}{dx}$ is :

[1 mark]

- (A) $-2 \sin^{-1} x$
 (B) $-\frac{1}{2} \sin \frac{y}{2}$
 (C) $\frac{-1}{\sqrt{1-2x^2}}$
 (D) $-2 \operatorname{cosec} \frac{y}{2}$

Correct Answer: (D) $-2 \operatorname{cosec} \frac{y}{2}$

Solution:

Concept:

- The derivative of $\cos^{-1} x$ is $-1/\sqrt{1-x^2}$ for $-1 < x < 1$.

Quick Tip to Solve: When inverse-trigonometric answers are expressed using y , rewrite the relation as $x = \cos(y/2)$ first.

- From $y = 2 \cos^{-1} x$, one obtains $x = \cos(y/2)$; on the principal branch, $\sin(y/2) = \sqrt{1 - x^2}$.

Step 1: Differentiate the inverse relation Apply the chain rule to $x = \cos(y/2)$:

$$1 = -\sin(y/2) \frac{1}{2} \frac{dy}{dx}.$$

The factor $1/2$ comes from differentiating the argument $y/2$; it must be moved to the other side when solving for the derivative.

Step 2: Express the derivative in the given form Rearrangement gives

$$\frac{dy}{dx} = -\frac{2}{\sin(y/2)} = -2 \operatorname{cosec}(y/2).$$

Option (B) incorrectly multiplies by $\sin(y/2)$. Option (A) involves an inverse sine instead of a derivative. The printed option (C), $-1/\sqrt{1 - 2x^2}$, has both the wrong coefficient and the wrong radicand.

Step 3: Conclusion $\frac{dy}{dx} = -2 \operatorname{cosec}(y/2)$, for $-1 < x < 1$. Hence, the correct option is (D).

Quick Tip to Solve: When inverse-trigonometric answers are expressed using y , rewrite the relation as $x = \cos(y/2)$ first.

13. The rate of change of volume of a sphere with respect to its diameter, when its radius is 5 cm, is : [1 mark]

- (A) $400\pi \text{ cm}^3/\text{cm}$
- (B) $100\pi \text{ cm}^3/\text{cm}$
- (C) $50\pi \text{ cm}^3/\text{cm}$
- (D) $25\pi \text{ cm}^3/\text{cm}$

Correct Answer: (C) $50\pi \text{ cm}^3/\text{cm}$

Solution:

Concept:

- The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.
- If $d = 2r$ is the diameter, the chain rule gives $dV/dd = (dV/dr)(dr/dd)$, with $dr/dd = 1/2$.

Step 1: Differentiate with respect to the diameter The requested variable is the diameter, so the radius derivative alone is insufficient. Using $r = d/2$,

$$\frac{dV}{dd} = 4\pi r^2 \times \frac{1}{2} = 2\pi r^2.$$

Equivalently, $V = \pi d^3/6$, whose derivative is $\pi d^2/2$.

Quick Tip to Solve: For any quantity depending on sphere radius, its rate with respect to diameter is half its rate with respect to radius.

Step 2: Substitute the given radius At $r = 5$ cm, the diameter is 10 cm, and

$$\frac{dV}{dd} = 2\pi(5)^2 = 50\pi \text{ cm}^3/\text{cm}.$$

The value 100π is the rate with respect to radius. The value 400π results from putting the diameter into $4\pi r^2$, while 25π is merely πr^2 .

Step 3: Conclusion The required rate is $50\pi \text{ cm}^3/\text{cm}$. Hence, the correct option is (C).

Quick Tip to Solve: For any quantity depending on sphere radius, its rate with respect to diameter is half its rate with respect to radius.

14. $\int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx$ is equal to :

[1 mark]

- (A) $\tan^{-1}(\sin x) + C$
(B) $\sin^{-1}(\sin x) + C$
(C) $\log|\sin x + \sqrt{1 + \sin^2 x}| + C$
(D) $\log|\sin x - \sqrt{\sin^2 x + 1}| + C$

Correct Answer: (C) $\log|\sin x + \sqrt{1 + \sin^2 x}| + C$

Solution:

Concept:

- Use substitution when a composite expression and its derivative occur together: here $u = \sin x$ gives $du = \cos x dx$.
- The standard primitive is $\int du/\sqrt{1+u^2} = \log|u + \sqrt{1+u^2}| + C$.

Step 1: Reduce to a standard integral Set $u = \sin x$. The numerator becomes exactly du , giving

$$\int \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx = \int \frac{du}{\sqrt{1 + u^2}}.$$

The plus sign inside the square root identifies the logarithmic standard form.

Step 2: Check the alternative primitives The standard formula yields $\log|u + \sqrt{1+u^2}| + C$. Option (A) differentiates to $\cos x/(1 + \sin^2 x)$, without the square root, and option (B) instead introduces $\sqrt{1 - \sin^2 x}$. For option (D), $|u - \sqrt{1+u^2}| = 1/(u + \sqrt{1+u^2})$, so its logarithm is the negative of the required primitive, not the same primitive plus a constant.

Step 3: Conclusion The integral is $\log|\sin x + \sqrt{1 + \sin^2 x}| + C$. Hence, the correct option is (C).

Quick Tip to Solve: After substituting $u = \sin x$, distinguish $1 + u^2$, $\sqrt{1 - u^2}$ and $\sqrt{1 + u^2}$ before choosing a formula.

15. $\int_{-1}^1 (1 - |x|) dx$ is equal to :

[1 mark]

(A) $2 \int_0^1 (1 + x) dx$

(B) $2 \int_{-1}^0 (1 + x) dx$

(C) 0

(D) $2 \int_{-1}^0 (1 - x) dx$

Correct Answer: (B) $2 \int_{-1}^0 (1 + x) dx$

Solution:

Concept:

- The modulus is $|x| = -x$ on $[-1, 0]$ and $|x| = x$ on $[0, 1]$.
- An even integrand has equal integrals over the two halves of a symmetric interval; $1 - |x|$ is even.

Step 1: Split at the change of modulus Let the required integral be I . Using the correct sign of the modulus on each side,

$$I = \int_{-1}^0 (1 + x) dx + \int_0^1 (1 - x) dx.$$

The two integrals are equal by symmetry, and each equals $1/2$.

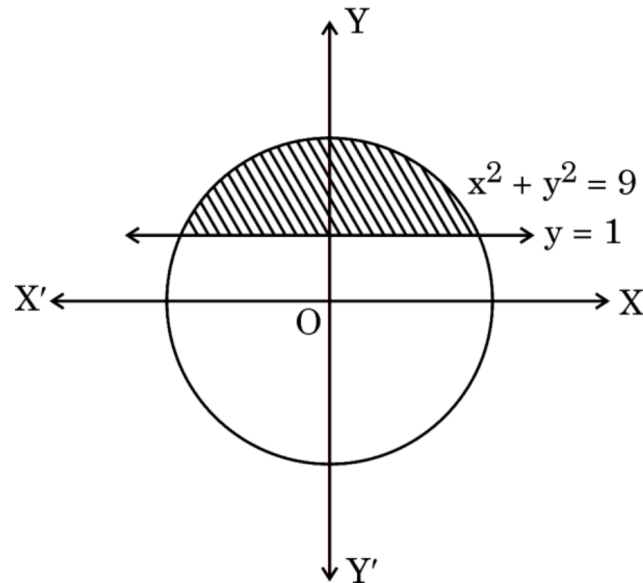
Step 2: Compare the printed alternatives Thus $I = 2 \int_{-1}^0 (1 + x) dx = 1$, which is option (B). Directly evaluating option (A) gives $2 \int_0^1 (1 + x) dx = 3$, and option (D) gives $2 \int_{-1}^0 (1 - x) dx = 3$. Option (C) is zero; symmetry does not make this integral zero because the integrand is even and nonnegative, not odd.

Step 3: Conclusion $I = 2 \int_{-1}^0 (1 + x) dx = 1$. Hence, the correct option is (B).

Quick Tip to Solve: When doubling the negative half of this integral, replace $|x|$ by $-x$, so the integrand becomes $1 + x$.

16. The area of the shaded region of the circle given below is equal to :

[1 mark]



- (A) $\int_1^3 \sqrt{9 - y^2} dy$
(B) $2 \int_1^3 \sqrt{9 - y^2} dy$
(C) $\int_0^3 \sqrt{9 - x^2} dx$
(D) $2 \int_0^3 \sqrt{9 - x^2} dx$

Correct Answer: (B) $2 \int_1^3 \sqrt{9 - y^2} dy$

Solution:

Concept:

- The circle $x^2 + y^2 = 9$ has radius 3; the shaded cap lies above $y = 1$.
- Integrate the horizontal strip width, $x_{\text{right}} - x_{\text{left}}$, with respect to y .

Step 1: Find the strip width The circle gives $x = \pm\sqrt{9 - y^2}$. The strip crosses both halves of the circle, so its width is $2\sqrt{9 - y^2}$. The shaded region extends from the chord $y = 1$ to the top $y = 3$.

Step 2: Compare the alternatives Integrating this width between these heights gives option (B). Option (A) counts only half the cap. Options (C) and (D) give a quarter-circle and a semicircle respectively; neither uses the chord height as a boundary.

Step 3: Conclusion The shaded area is $\boxed{2 \int_1^3 \sqrt{9 - y^2} dy}$. Hence, the correct option is (B).

Quick Tip to Solve: A horizontal chord suggests integrating horizontal widths in y , starting at the chord height.

17. $\frac{dy}{dx} = F(x, y)$ will be a homogeneous differential equation for which of the following functions ?

- (i) $F(x, y) = 3x + 2y$
(ii) $F(x, y) = \sin \frac{y}{x} + \log y - \log x$
(iii) $F(x, y) = e^{y/x} + 1$
(iv) $F(x, y) = \sqrt{x^2 + y^2} - y$

[1 mark]

- (A) (i) and (ii)
(B) (i), (ii) and (iii)
(C) (ii), (iii) and (iv)
(D) (ii) and (iii)

Correct Answer: (D) (ii) and (iii)

Solution:

Concept:

- The equation $dy/dx = F(x, y)$ is homogeneous when F is a function of y/x alone, on its domain.
- Equivalently, the right side has degree zero: $F(kx, ky) = F(x, y)$ for admissible positive k .

Step 1: Test the ratio-dependent expressions In (ii), the logarithms require $x, y > 0$, and $\log y - \log x = \log(y/x)$. Thus the entire expression is $\sin(y/x) + \log(y/x)$, a function of the ratio only. Expression (iii), $e^{y/x} + 1$, also depends only on this ratio. Both satisfy the required condition.

Step 2: Exclude expressions of degree one Expression (i) becomes $k(3x + 2y)$ when both variables are multiplied by k . For (iv), positive scaling gives $k(\sqrt{x^2 + y^2} - y)$. These expressions have degree one, not zero, so neither yields the required homogeneous differential equation. Options (A), (B) and (C) each include at least one of these inadmissible expressions.

Step 3: Conclusion Only (ii) and (iii) give homogeneous differential equations. Hence, the correct option is (D).

Quick Tip to Solve: First combine the logarithms into $\log(y/x)$; then test whether any factor of x remains outside the ratio.

18. For any two vectors $\vec{a}$ and $\vec{b}$, which of the following statements is always true ?

[1 mark]

- (A) $\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}|$
(B) $|\vec{a} + \vec{b}| \geq |\vec{a}| + |\vec{b}|$
(C) $|\vec{a} - \vec{b}| = |\vec{a}| - |\vec{b}|$
(D) $|\vec{a} \times \vec{b}| \geq |\vec{a}||\vec{b}|$

Correct Answer: (A) $\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}|$

Solution:

Concept:

- For nonzero vectors making angle θ , $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$, with $-1 \leq \cos \theta \leq 1$.
- The triangle inequality gives $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$, while $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$.

Quick Tip to Solve: To disprove an alleged universal vector inequality, try equal, opposite or perpendicular unit vectors.

Step 1: Apply the bound on cosine Since $|\vec{a}||\vec{b}|$ is nonnegative and $\cos \theta \leq 1$, multiplying preserves the inequality:

$$\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}|.$$

If either vector is zero, both sides are zero, so the statement remains true in that case also.

Step 2: Test the remaining options Option (B) reverses the triangle inequality: for $\vec{a} = \hat{i}$ and $\vec{b} = -\hat{i}$, it would assert $0 \geq 2$. For (C), take $\vec{a} = \hat{i}$, $\vec{b} = \hat{j}$; the left side is $\sqrt{2}$, but the right side is zero. For (D), parallel unit vectors have zero cross product but a product of magnitudes equal to one. Thus none of these three statements always holds.

Step 3: Conclusion The universally valid inequality is $\vec{a} \cdot \vec{b} \leq |\vec{a}||\vec{b}|$. Hence, the correct option is (A).

Quick Tip to Solve: To disprove an alleged universal vector inequality, try equal, opposite or perpendicular unit vectors.

19. Assertion (A): One of the particular solutions of the differential equation $\frac{dy}{dx} = e^{x+y}$ can be $e^x + e^{-y} = -2$.

Reason (R): $e^x + e^{-y} = C$ is the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
 (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
 (C) Assertion (A) is true, but Reason (R) is false.
 (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A real exponential is strictly positive: $e^u > 0$ for every real u .
- Separation of variables transforms $dy/dx = e^{x+y}$ into $e^{-y} dy = e^x dx$; integration introduces an arbitrary constant.

Step 1: Evaluate the assertion For real x, y , both e^x and e^{-y} are positive, so their sum cannot equal -2 . The stated relation has no real solution points and therefore cannot describe a particular real solution of the differential equation. Assertion (A) is false.

Step 2: Evaluate the reason Integrating the separated equation gives

$$-e^{-y} = e^x + C_1, \quad e^x + e^{-y} = C.$$

This is the general solution, restricted to intervals where $C - e^x > 0$, so that $y = -\log(C - e^x)$ is real. Reason (R) is true.

Step 3: Assess the asserted explanation An arbitrary constant in an implicit general solution must

Quick Tip to Solve: Before choosing a constant in an exponential solution, check that the resulting equation has real points.

still permit a real solution. Taking $C = -2$ violates positivity, so the true general solution does not establish the assertion. Options (A), (B) and (C) all require a true assertion and are excluded.

Step 4: Conclusion The assertion is false, but the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Before choosing a constant in an exponential solution, check that the resulting equation has real points.

20. Assertion (A): The vectors $\vec{a}$ and $(-2\vec{a})$, where $\vec{a} \neq \vec{0}$, are collinear vectors.

Reason (R): $\vec{a} \cdot (-2\vec{a}) = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- Nonzero vectors are collinear when one is a scalar multiple of the other. A negative multiplier reverses direction.
- The dot product of a vector with itself is its squared magnitude: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

Step 1: Evaluate the assertion The second vector is obtained by multiplying the nonzero vector $\vec{a}$ by the scalar -2 . It is parallel to the same line, has twice the magnitude and points in the opposite direction. Thus the vectors are collinear and the assertion is true.

Step 2: Evaluate the reason Using the scalar multiplication property,

$$\vec{a} \cdot (-2\vec{a}) = -2(\vec{a} \cdot \vec{a}) = -2|\vec{a}|^2.$$

Since $\vec{a} \neq \vec{0}$, its squared magnitude is positive, so this dot product is strictly negative, not zero. Therefore the reason is false.

Step 3: Check the explanation and select the case The scalar-multiple relation explains collinearity. A zero dot product would instead express perpendicularity for two nonzero vectors. Consequently the stated reason cannot explain the assertion; the options claiming a true reason or a false assertion are eliminated.

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: A negative scalar multiple is antiparallel, not perpendicular; its dot product with the original nonzero vector is negative.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). If the lines $\frac{x-3}{1} = \frac{1-y}{1} = \frac{z+2}{p}$ and $\frac{2-x}{3} = \frac{y+1}{5} = \frac{z+56}{2p}$ are perpendicular to each other, then find the value(s) of p . [2 marks]

Solution:

Concept:

- In the symmetric equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the direction ratios of the line are (a, b, c) .
- Two lines have perpendicular directions when the dot product of their direction vectors is zero:
 $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Step 1: Read the signed direction ratios The reversed numerators must first be put into the usual form. In the first line,

$$\frac{1-y}{1} = \frac{y-1}{-1},$$

so its direction vector is $\vec{d}_1 = (1, -1, p)$. In the second line,

$$\frac{2-x}{3} = \frac{x-2}{-3},$$

so its direction vector is $\vec{d}_2 = (-3, 5, 2p)$. The constants locating points on the lines do not affect their directions.

Step 2: Apply the perpendicularity condition Equate the dot product to zero and simplify:

$$\begin{aligned}(1)(-3) + (-1)(5) + (p)(2p) &= 0, \\ -8 + 2p^2 &= 0, \\ p^2 &= 4.\end{aligned}$$

Both roots are admissible: each makes the dot product zero, and neither direction vector becomes a zero vector. The sign of p cannot be discarded because the condition contains its square.

Step 3: Conclusion The required values are $p = 2$ or $p = -2$.

Quick Tip to Solve: Change $1 - y$ and $2 - x$ into standard coordinate differences before using the denominators as direction ratios.

21(b). Find the vector equation of a line passing through the origin and perpendicular to both the lines $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and $\vec{r} = \mu(\hat{i} - \hat{j} + \hat{k})$. [2 marks]

Solution:

Concept:

- A vector perpendicular to two nonparallel vectors $\vec{u}$ and $\vec{v}$ is their cross product $\vec{u} \times \vec{v}$.
- A line through position vector $\vec{a}$, with direction vector $\vec{d} \neq \vec{0}$, has vector equation $\vec{r} = \vec{a} + t\vec{d}$, where t is real.

Step 1: Extract the two directions The coefficients of the parameters in the given equations supply the direction vectors:

$$\vec{u} = 3\hat{i} + 4\hat{j} + 2\hat{k}, \quad \vec{v} = \hat{i} - \hat{j} + \hat{k}.$$

The position vector $2\hat{i} - \hat{j} + 2\hat{k}$ fixes a point on the first given line; it is not its direction vector. The required direction must be perpendicular to both $\vec{u}$ and $\vec{v}$.

Step 2: Compute and verify a common normal direction Expanding the cross product gives

$$\begin{aligned} \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 1 & -1 & 1 \end{vmatrix} \\ &= (4 + 2)\hat{i} - (3 - 2)\hat{j} + (-3 - 4)\hat{k} \\ &= 6\hat{i} - \hat{j} - 7\hat{k}. \end{aligned}$$

The checks $\vec{d} \cdot \vec{u} = 18 - 4 - 14 = 0$ and $\vec{d} \cdot \vec{v} = 6 + 1 - 7 = 0$ confirm both perpendicular directions. Since the required line passes through the origin, its fixed position vector is zero.

Step 3: Conclusion A required vector equation is $\boxed{\vec{r} = t(6\hat{i} - \hat{j} - 7\hat{k}), t \in \mathbb{R}}$.

Quick Tip to Solve: Cross the parameter coefficients, then use the origin as the position vector; any nonzero multiple of the cross product gives the same line.

22. If $x = e^{\sin^{-1} t}$ and $y = e^{\cos^{-1} t}$, then find $\frac{dy}{dx}$ at $t = \frac{1}{\sqrt{2}}$. [2 marks]

Solution:

Concept:

- For differentiable parametric functions $x = x(t)$, $y = y(t)$, $dy/dx = (dy/dt)/(dx/dt)$ whenever $dx/dt \neq 0$.
- The chain rule gives $d(e^u)/dt = e^u u'$, while $d(\sin^{-1} t)/dt = 1/\sqrt{1-t^2}$ and $d(\cos^{-1} t)/dt = -1/\sqrt{1-t^2}$ for $-1 < t < 1$.

Quick Tip to Solve: Cancel the common factor $1/\sqrt{1-t^2}$ before substituting; at $t = 1/\sqrt{2}$, both inverse angles equal $\pi/4$.

Step 1: Differentiate both coordinates Apply the chain rule separately to the two exponentials:

$$\frac{dx}{dt} = \frac{e^{\sin^{-1} t}}{\sqrt{1-t^2}}, \quad \frac{dy}{dt} = -\frac{e^{\cos^{-1} t}}{\sqrt{1-t^2}}.$$

At the specified value of t , the square-root denominator is positive and $dx/dt \neq 0$, so the parametric derivative formula applies. The minus sign arises from differentiating inverse cosine.

Step 2: Form the ratio and substitute the parameter Dividing cancels the common square-root factor:

$$\frac{dy}{dx} = -e^{\cos^{-1} t - \sin^{-1} t}.$$

For $t = 1/\sqrt{2}$, the principal values of both inverse trigonometric functions are $\pi/4$. Hence the exponent becomes $\pi/4 - \pi/4 = 0$, and the derivative is $-e^0$. The increasing $x(t)$ and decreasing $y(t)$ also confirm its negative sign.

Step 3: Conclusion At $t = 1/\sqrt{2}$, $\boxed{dy/dx = -1}$.

Quick Tip to Solve: Cancel the common factor $1/\sqrt{1-t^2}$ before substituting; at $t = 1/\sqrt{2}$, both inverse angles equal $\pi/4$.

23. Find the sub-interval of $(0, \infty)$ in which $f(x) = x \log x$ is increasing.

[2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where its derivative is positive; the sign of the derivative must be tested within the stated domain.
- For the natural logarithm, $\frac{d}{dx} \log x = \frac{1}{x}$ for $x > 0$. The product rule is $(uv)' = u'v + uv'$.

Step 1: Differentiate the product Both factors x and $\log x$ are differentiable throughout the given domain $(0, \infty)$. Applying the product rule gives

$$f'(x) = 1 \cdot \log x + x \cdot \frac{1}{x} = \log x + 1.$$

Step 2: Determine the positive-derivative interval Solve the strict inequality for an increasing function:

$$f'(x) > 0 \iff \log x > -1 \iff x > e^{-1}.$$

The exponential function preserves the inequality because it is increasing. Also, $f'(x) < 0$ for $0 < x < e^{-1}$, and $f'(e^{-1}) = 0$. Thus the derivative changes from negative to positive at the boundary.

Step 3: Conclusion The required increasing sub-interval is $\boxed{(1/e, \infty)}$. The stationary endpoint may also be included: f is strictly increasing on $[1/e, \infty)$.

Quick Tip to Solve: For a product containing $\log x$, differentiate both factors; the extra $+1$ shifts the boundary from 1 to $1/e$.

24(a). Find the value of $\sin[\cot^{-1} \sqrt{2} (\cos(\tan^{-1} 1))]$.

[2 marks]

Solution:

Concept:

- Principal values satisfy $\tan^{-1} t \in (-\pi/2, \pi/2)$ and $\cot^{-1} t \in (0, \pi)$. These ranges determine which angle to use.
- Evaluate a composition from the innermost function outward: first the inverse tangent, then its cosine, then the argument of the inverse cotangent, and finally sine.

Step 1: Evaluate the inner cosine Since $\tan(\pi/4) = 1$ and $\pi/4$ lies in the principal range of inverse tangent,

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \cos(\tan^{-1} 1) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Here the inverse tangent gives an angle, whereas its cosine gives the number multiplied by $\sqrt{2}$.

Step 2: Simplify the inverse-cotangent argument The argument of $\cot^{-1}$ is the product $\sqrt{2} \cos(\tan^{-1} 1)$. Substitution gives

$$\sqrt{2} \cos(\tan^{-1} 1) = \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1, \quad \cot^{-1} 1 = \frac{\pi}{4}.$$

The latter is the principal value because $\cot(\pi/4) = 1$ and $\pi/4 \in (0, \pi)$.

Step 3: Conclusion The given value is $\sin(\pi/4) = \boxed{\frac{1}{\sqrt{2}}}$.

Quick Tip to Solve: Simplify $\sqrt{2} \cos(\tan^{-1} 1)$ first; its value is 1, so the outer inverse cotangent becomes immediate.

OR

24(b). A relation R on $A = \{1, 2, 3\}$ is defined as $R = \{(1, 1), (3, 3), (1, 2)\}$. Is R a symmetric relation? Justify. Write the smallest relation set R_1 such that $R \cup R_1$ becomes an equivalence relation on the set $\{1, 2, 3\}$.

[2 marks]

Solution:

Concept:

- A relation is symmetric when $(a, b) \in R$ implies $(b, a) \in R$ for all elements of its underlying set.
- An equivalence relation is reflexive, symmetric and transitive. Reflexivity requires every pair (a, a) ; transitivity requires (a, c) whenever (a, b) and (b, c) occur.

Step 1: Test the symmetry of the given relation In $R = \{(1, 1), (3, 3), (1, 2)\}$, the pair $(1, 2)$ is present but its reverse $(2, 1)$ is absent. This single counterexample disproves symmetry. The self-pairs $(1, 1)$ and $(3, 3)$ do not compensate for the missing reverse pair.

Step 2: Add the necessary pairs and check equivalence Reflexivity on $A = \{1, 2, 3\}$ forces the missing

Quick Tip to Solve: Add missing diagonal and reverse pairs first; then identify the equivalence classes to check whether transitivity forces any further pairs.

pair (2, 2), and symmetry forces (2, 1). With these additions,

$$R \cup \{(2, 2), (2, 1)\} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}.$$

This is precisely the relation whose equivalence classes are $\{1, 2\}$ and $\{3\}$: all ordered pairs within each class occur. It is therefore reflexive, symmetric and transitive. Neither added pair can be omitted, and no pair joining 3 to 1 or 2 is necessary.

Step 3: Conclusion R is not symmetric. The smallest required relation is $R_1 = \{(2, 2), (2, 1)\}$.

Quick Tip to Solve: Add missing diagonal and reverse pairs first; then identify the equivalence classes to check whether transitivity forces any further pairs.

25. If for two unit vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + 2\vec{b}| = |2\vec{a} - \vec{b}|$, then find the angle between $\vec{a}$ and $\vec{b}$. [2 marks]

Solution:

Concept:

- For any vector $\vec{u}$, its squared magnitude is $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$. Squaring a sum or difference therefore introduces the dot-product cross term.
- If $\theta \in [0, \pi]$ is the angle between unit vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}| = |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = \cos \theta$.

Step 1: Square and expand both magnitudes The given magnitudes are nonnegative, so their equality can be squared. Expanding with the distributive law for dot products,

$$\begin{aligned} |\vec{a} + 2\vec{b}|^2 &= |\vec{a}|^2 + 4|\vec{b}|^2 + 4\vec{a} \cdot \vec{b}, \\ |2\vec{a} - \vec{b}|^2 &= 4|\vec{a}|^2 + |\vec{b}|^2 - 4\vec{a} \cdot \vec{b}. \end{aligned}$$

The opposite signs of the cross terms come from the plus and minus signs in the respective vector expressions.

Step 2: Use the unit-vector condition Substituting $|\vec{a}|^2 = |\vec{b}|^2 = 1$ into the equality gives

$$5 + 4\vec{a} \cdot \vec{b} = 5 - 4\vec{a} \cdot \vec{b} \Rightarrow 8\vec{a} \cdot \vec{b} = 0.$$

Thus $\cos \theta = 0$. There is only one angle with this cosine in the standard interval $[0, \pi]$ for angles between nonzero vectors.

Step 3: Conclusion The vectors are perpendicular, and their angle is $\theta = \pi/2 = 90^\circ$.

Quick Tip to Solve: Square the magnitudes before comparing vectors: the equal constant terms cancel and immediately isolate the dot product.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. A survey was conducted on the patients who have undergone knee replacement surgeries. It was found that Robotic Knee replacement surgeries have 90% success rate.

On a particular day, robotic surgery was performed on three patients, A , B and C , one after the other. Assuming that the success and failure of each surgery is independent of each other, find the probability that :

- (i) exactly one surgery is successful,
- (ii) at most two surgeries are successful.

[3 marks]

Solution:

Concept:

- Independent trials have probabilities that multiply: the probability of a specified pattern of successes and failures is the product of the individual probabilities.
- If X counts successes in n independent trials, each with success probability p , then $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$. Here $n = 3$, $p = 0.9$ and $1 - p = 0.1$.

Step 1: Identify the success and failure probabilities The success rate of 90% means that the probability of a successful surgery is 0.9. Each surgery has only the two stated outcomes, so its failure probability is $1 - 0.9 = 0.1$. Because the three outcomes are independent and have the same success probability, their total number of successes X follows a binomial distribution with $n = 3$.

Step 2: Calculate the probability of exactly one success Exactly one success can occur as success-failure-failure, failure-success-failure, or failure-failure-success for patients A, B, C respectively. These three disjoint patterns each have probability $0.9(0.1)^2$. Thus

$$P(X = 1) = \binom{3}{1} (0.9)(0.1)^2 = 3(0.9)(0.01) = 0.027.$$

The factor 3 counts which of the three patients has the successful surgery.

Step 3: Use the complement for at most two successes At most two successes includes zero, one and two successes. Its only excluded outcome is that all three surgeries succeed. Independence gives

$$P(X = 3) = (0.9)^3 = 0.729.$$

Therefore

$$P(X \leq 2) = 1 - P(X = 3) = 1 - 0.729 = 0.271.$$

As a check, summing the separate probabilities gives $0.001 + 0.027 + 0.243 = 0.271$, the same result.

Step 4: Conclusion (i) The probability of exactly one successful surgery is $\boxed{0.027}$. (ii) The probability of at most two successful surgeries is $\boxed{0.271}$.

Quick Tip to Solve: Count the three placements of a single success; for at most two successes, subtract the all-success probability from 1.

27(a). Find : $\int \frac{dx}{x^{1/2} + x^{1/3}}$

[3 marks]

Solution:

Concept:

- To remove fractional powers, substitute $x = t^6$, where 6 is the least common multiple of the denominators 2 and 3. For the real integrand, $x > 0$, so $t = x^{1/6} > 0$.
- Simplify an improper rational function by polynomial division. Then apply $\int t^n dt = t^{n+1}/(n+1)$ and $\int dt/(t+1) = \ln|t+1|$.

Step 1: Remove the fractional powers Put $t = x^{1/6}$. Then $x = t^6$, $dx = 6t^5 dt$, $x^{1/2} = t^3$ and $x^{1/3} = t^2$. Hence the integral becomes

$$I = \int \frac{6t^5}{t^3 + t^2} dt = 6 \int \frac{t^3}{t+1} dt.$$

The factor t^2 can be cancelled because $x > 0$ and therefore $t \neq 0$. This substitution turns both fractional powers into integer powers simultaneously.

Step 2: Divide the polynomial and integrate Polynomial division gives

$$t^3 = (t+1)(t^2 - t + 1) - 1, \quad \frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1}.$$

Thus each term has an elementary antiderivative:

$$\begin{aligned} I &= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \\ &= 2t^3 - 3t^2 + 6t - 6 \ln|t+1| + C. \end{aligned}$$

Since $t > 0$, the logarithm has positive argument. Replace t , t^2 and t^3 by the corresponding powers of x .

Step 3: Conclusion

$$\boxed{2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6 \ln(1 + x^{1/6}) + C}, \quad x > 0.$$

Quick Tip to Solve: For powers $1/2$ and $1/3$, use $x = t^6$; cancel the common t^2 before dividing t^3 by $t+1$.

OR

27(b). Find : $\int \tan^{-1} \left(\frac{1-x}{1+x} \right) dx$

[3 marks]

Solution:

Concept:

- Integration by parts is $\int u dv = uv - \int v du$. Taking the inverse tangent as u simplifies its derivative.

Quick Tip to Solve: Differentiate the inverse tangent directly: its derivative simplifies to $-1/(1+x^2)$, so integration by parts leaves a logarithm.

- The chain rule gives $\frac{d}{dx} \tan^{-1} z = \frac{z'}{1+z^2}$. Also $\int x/(1+x^2) dx = \frac{1}{2} \ln(1+x^2) + C$.

Step 1: Differentiate the inverse tangent Set $u = \tan^{-1}\left(\frac{1-x}{1+x}\right)$ and $dv = dx$, so $v = x$. For $x \neq -1$, first differentiate the inner quotient:

$$\frac{d}{dx} \left(\frac{1-x}{1+x} \right) = \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}.$$

The denominator in the inverse-tangent derivative simplifies as

$$1 + \left(\frac{1-x}{1+x} \right)^2 = \frac{(1+x)^2 + (1-x)^2}{(1+x)^2} = \frac{2(1+x^2)}{(1+x)^2}.$$

Dividing these expressions gives $du = -dx/(1+x^2)$.

Step 2: Apply integration by parts Substitute u , v and du into the formula. The negative derivative makes the remaining integral positive:

$$I = x \tan^{-1}\left(\frac{1-x}{1+x}\right) + \int \frac{x}{1+x^2} dx.$$

For the remaining integral put $s = 1+x^2$, so $ds = 2x dx$. Hence it is $\frac{1}{2} \ln(1+x^2)$. Keeping the original inverse tangent avoids introducing different inverse-tangent branches on the two sides of $x = -1$.

Step 3: Conclusion

$$x \tan^{-1}\left(\frac{1-x}{1+x}\right) + \frac{1}{2} \ln(1+x^2) + C.$$

This formula holds on each interval of the domain $x \neq -1$, with an arbitrary constant on each interval.

Quick Tip to Solve: Differentiate the inverse tangent directly: its derivative simplifies to $-1/(1+x^2)$, so integration by parts leaves a logarithm.

28. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin 2x} dx$

[3 marks]

Solution:

Concept:

- Reflection of a definite integral gives $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. Here reflection about $\pi/4$ interchanges $\sin x$ and $\cos x$.
- Use $1 + \sin 2x = (\sin x + \cos x)^2 = 2 \cos^2(x - \pi/4)$, together with $\int \sec^2 u du = \tan u$.

Step 1: Reflect the integral and add Denote the required integral by I . Replace x by $\pi/2 - x$. Since $\sin(\pi/2 - x) = \cos x$ and $\sin(\pi - 2x) = \sin 2x$, the same integral is

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin 2x} dx.$$

Quick Tip to Solve: Reflect x to $\pi/2 - x$ first: the two numerators add to 1, while $1 + \sin 2x$ stays unchanged.

Add this expression to the given one and use $\sin^2 x + \cos^2 x = 1$:

$$2I = \int_0^{\pi/2} \frac{1}{1 + \sin 2x} dx.$$

The denominator is positive throughout the interval, so these manipulations do not introduce any singularity.

Step 2: Integrate the resulting squared secant The sum-of-squares identity rewrites the denominator as $2 \cos^2(x - \pi/4)$. Therefore

$$\begin{aligned} 2I &= \frac{1}{2} \int_0^{\pi/2} \sec^2(x - \pi/4) dx \\ &= \frac{1}{2} [\tan(x - \pi/4)]_0^{\pi/2} \\ &= \frac{1}{2} \left(\tan \frac{\pi}{4} - \tan \frac{-\pi}{4} \right) \\ &= \frac{1}{2} (1 - (-1)) = 1. \end{aligned}$$

Here the shifted argument ranges from $-\pi/4$ to $\pi/4$, where the tangent antiderivative is continuous. Finally divide by the factor 2 introduced by adding the two equal integrals.

Step 3: Conclusion

$$\boxed{\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin 2x} dx = \frac{1}{2}}.$$

Quick Tip to Solve: Reflect x to $\pi/2 - x$ first: the two numerators add to 1, while $1 + \sin 2x$ stays unchanged.

29(a). Find : $\int \frac{\cos x}{(2 + \sin x)(4 + \sin x)} dx$

[3 marks]

Solution:

Concept:

- Use substitution when a function and its derivative both occur: $t = \sin x$ gives $dt = \cos x dx$, reducing this trigonometric integral to a rational integral.
- Resolve a rational function with distinct linear factors into partial fractions. Each term then uses $\int dt/(t + a) = \ln |t + a| + C$, where a is a constant.

Step 1: Substitute and resolve into partial fractions Let I denote the required integral. Set $t = \sin x$, so the numerator $\cos x dx$ is exactly dt . Hence

$$I = \int \frac{dt}{(t + 2)(t + 4)}.$$

Write

$$\frac{1}{(t + 2)(t + 4)} = \frac{A}{t + 2} + \frac{B}{t + 4}.$$

Quick Tip to Solve: The linear factors differ by 2, so their reciprocal difference equals $2/((t + 2)(t + 4))$; multiply by $1/2$.

Clearing denominators gives $1 = A(t + 4) + B(t + 2)$. Comparing coefficients of t and the constant term yields $A + B = 0$ and $4A + 2B = 1$. Thus $A = 1/2$ and $B = -1/2$.

Step 2: Integrate and restore the original variable Substitute these constants and integrate the two linear-denominator terms separately:

$$\begin{aligned} I &= \frac{1}{2} \int \left(\frac{1}{t+2} - \frac{1}{t+4} \right) dt \\ &= \frac{1}{2} \ln |t+2| - \frac{1}{2} \ln |t+4| + C \\ &= \frac{1}{2} \ln \left| \frac{t+2}{t+4} \right| + C. \end{aligned}$$

Now replace t by $\sin x$. Since $-1 \leq \sin x \leq 1$, both $2 + \sin x$ and $4 + \sin x$ are positive for every real x , so the absolute-value signs may be omitted. Differentiating the difference of logarithms reproduces the two partial fractions multiplied by $\cos x$.

Step 3: Conclusion

$$\boxed{\frac{1}{2} \ln \left(\frac{2 + \sin x}{4 + \sin x} \right) + C}.$$

Quick Tip to Solve: The linear factors differ by 2, so their reciprocal difference equals $2/((t+2)(t+4))$; multiply by $1/2$.

OR

29(b). Find : $\int \frac{x+3}{x^2+4x+5} dx$

[3 marks]

Solution:

Concept:

- For a differentiable function $f(x)$, the logarithmic pattern is $\int f'(x)/f(x) dx = \ln |f(x)| + C$, where C is an arbitrary constant.
- Completing the square reduces an irreducible quadratic to the inverse-tangent form: $\int du/(1+u^2) = \tan^{-1} u + C$.

Step 1: Split the numerator using the denominator's derivative Let I denote the required integral. The denominator has derivative $2x + 4$, so write the numerator as

$$x + 3 = \frac{1}{2}(2x + 4) + 1.$$

This separates the integral into one logarithmic term and one term involving a quadratic:

$$I = \frac{1}{2} \int \frac{2x+4}{x^2+4x+5} dx + \int \frac{dx}{x^2+4x+5}.$$

The extra constant is 1, since half the derivative contributes $x + 2$, not $x + 3$.

Quick Tip to Solve: For this numerator, first remove half the derivative of the denominator; the remainder is exactly 1, producing the inverse-tangent term after completing the square.

Step 2: Complete the square and integrate each term The identity $x^2 + 4x + 5 = (x + 2)^2 + 1$ shows that the denominator is positive for every real x . Thus the first integral is $\frac{1}{2} \ln(x^2 + 4x + 5)$; absolute-value signs are unnecessary here. For the second integral, set $u = x + 2$, giving $du = dx$. Hence

$$\int \frac{dx}{(x + 2)^2 + 1} = \int \frac{du}{u^2 + 1} = \tan^{-1}(x + 2).$$

As a check, the derivatives of these two terms combine to

$$\frac{x + 2}{x^2 + 4x + 5} + \frac{1}{(x + 2)^2 + 1} = \frac{x + 3}{x^2 + 4x + 5},$$

which reproduces the original integrand.

Step 3: Conclusion Therefore,

$$\int \frac{x + 3}{x^2 + 4x + 5} dx = \frac{1}{2} \ln(x^2 + 4x + 5) + \tan^{-1}(x + 2) + C.$$

Quick Tip to Solve: For this numerator, first remove half the derivative of the denominator; the remainder is exactly 1, producing the inverse-tangent term after completing the square.

30(a). Find the general solution of the differential equation $(x^2 - y^2) dx + 2xy dy = 0$. [3 marks]

Solution:

Concept:

- A first-order differential equation is homogeneous when its derivative form depends only on y/x . Use $y = vx$, where v is a function of x , so $dy/dx = v + x dv/dx$.
- After separating the variables, integrate each side and absorb nonzero constant factors into an arbitrary integration constant. An implicit final relation can be checked by differentiation.

Step 1: Identify the homogeneous form and substitute The printed equation gives, wherever the derivative form is defined,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{(y/x)^2 - 1}{2(y/x)}.$$

Both coefficients of the original differential equation have degree two. Set $y = vx$, with $x \neq 0$. Substitution gives

$$v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v},$$

$$2vx \frac{dv}{dx} = -(v^2 + 1).$$

The minus sign appears because the term $2v^2$ moves from the left to the right.

Step 2: Separate the variables and integrate Since $v^2 + 1 > 0$ for real v , separation introduces no lost

Quick Tip to Solve: After substituting $y = vx$, simplify the numerator before separating: it becomes $-(1 + v^2)$. This prevents the common sign error that would incorrectly produce $x^2 - y^2$.

constant real value of v :

$$\frac{2v}{v^2 + 1} dv = -\frac{dx}{x}.$$

Integrating yields

$$\ln(v^2 + 1) = -\ln|x| + C_1.$$

Combining the logarithms gives $|x|(v^2 + 1) = K$. On an interval where x has fixed sign, absorb that sign into a constant C , so that $x(v^2 + 1) = C$. Substituting $v = y/x$ then gives $x + y^2/x = C$.

Step 3: Verify the implicit family Multiplying the last relation by x gives the candidate implicit family. Differentiating $x^2 + y^2 = Cx$ gives

$$(2x - C) dx + 2y dy = 0.$$

Multiply by x and substitute $Cx = x^2 + y^2$. The result is exactly

$$(x^2 - y^2) dx + 2xy dy = 0,$$

confirming the signs and the constant's placement.

Step 4: Conclusion The general solution is

$$x^2 + y^2 = Cx,$$

where C is an arbitrary real constant on a solution interval.

Quick Tip to Solve: After substituting $y = vx$, simplify the numerator before separating: it becomes $-(1 + v^2)$. This prevents the common sign error that would incorrectly produce $x^2 - y^2$.

OR

30(b). Solve the differential equation $\sin x \cos y dx + \cos x \sin y dy = 0$, **given that** $y = \frac{\pi}{4}$ **when** $x = 0$.
[3 marks]

Solution:

Concept:

- Separate a differential equation by bringing all terms involving x with dx , and all terms involving y with dy .
- The standard integral is $\int \tan t dt = -\ln|\cos t| + C$. An initial condition determines the arbitrary constant and selects the local branch of the solution.

Step 1: Separate the variables near the initial point At $x = 0$, $y = \pi/4$, both $\cos x$ and $\cos y$ are nonzero. We may therefore divide the equation by $\cos x \cos y$ on an interval containing the initial point:

$$\frac{\sin x}{\cos x} dx + \frac{\sin y}{\cos y} dy = 0.$$

Quick Tip to Solve: Recognize the left side as $-d(\cos x \cos y)$. The product is constant, and substituting the initial point immediately determines that constant.

Thus the separated equation is

$$\tan x \, dx + \tan y \, dy = 0.$$

This form is convenient because each tangent has a standard logarithmic integral.

Step 2: Integrate and apply the initial condition Integrating each term gives

$$-\ln |\cos x| - \ln |\cos y| = C_1.$$

Combining logarithms and absorbing the constant yields $\cos x \cos y = C$ on the interval under consideration. The product is positive at the given initial point, so this step agrees with the absolute values in the integrated expression. Now use $x = 0$ and $y = \pi/4$:

$$C = \cos 0 \cos \frac{\pi}{4} = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Step 3: Check the relation in the original equation Differentiating the constant-product relation gives

$$-\sin x \cos y \, dx - \cos x \sin y \, dy = 0.$$

Multiplication by -1 recovers the printed differential equation. The relation also gives $\cos y = 1/\sqrt{2}$ at $x = 0$, and the specified value $y = \pi/4$ chooses the required local branch.

Step 4: Conclusion The required solution satisfying the initial condition is

$$\cos x \cos y = \frac{1}{\sqrt{2}}.$$

Equivalently, its branch through the given point is $y = \cos^{-1}(1/(\sqrt{2} \cos x))$ near $x = 0$.

Quick Tip to Solve: Recognize the left side as $-d(\cos x \cos y)$. The product is constant, and substituting the initial point immediately determines that constant.

31. If $(\sin x)^y = y^{\cos x}$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- Logarithmic differentiation brings a variable exponent down as a multiplier: $\ln(a^b) = b \ln a$ for a positive base a .
- In implicit differentiation, $y = y(x)$. Thus $d(\ln y)/dx = y'/y$, while $d(\ln(\sin x))/dx = \cot x$, where $y' = dy/dx$.

Step 1: Take logarithms of the given relation On the real logarithmic domain $\sin x > 0$, $y > 0$, take natural logarithms of both sides of $(\sin x)^y = y^{\cos x}$. This converts both variable powers into products:

$$y \ln(\sin x) = \cos x \ln y.$$

Quick Tip to Solve: Take logarithms before differentiating either variable power. Then collect every y' term first and multiply by y only at the end to simplify the fraction.

Both y and $\sin x$ depend on x , so neither factor on the left can be treated as constant. Similarly, the right side requires the product rule.

Step 2: Differentiate each product with respect to x Applying the product rule and the chain rule gives

$$y' \ln(\sin x) + y \frac{\cos x}{\sin x} = -\sin x \ln y + \frac{\cos x}{y} y'.$$

Here $-\sin x \ln y$ comes from differentiating $\cos x$, and the other term on the right contains y' because the logarithm has the dependent variable y as its argument.

Step 3: Collect the derivative terms and simplify Bring both terms involving y' to the same side:

$$y' \left(\ln(\sin x) - \frac{\cos x}{y} \right) = -\sin x \ln y - y \cot x.$$

Multiply both sides by $-y$ to eliminate the small denominator without changing the equality:

$$y' (\cos x - y \ln(\sin x)) = y \sin x \ln y + y^2 \cot x.$$

Dividing by the coefficient of y' gives the derivative wherever that coefficient is nonzero.

Step 4: Conclusion Hence,

$$\frac{dy}{dx} = \frac{y^2 \cot x + y \sin x \ln y}{\cos x - y \ln(\sin x)}.$$

This formula applies on a differentiable real branch where the logarithms are defined and the denominator is nonzero.

Quick Tip to Solve: Take logarithms before differentiating either variable power. Then collect every y' term first and multiply by y only at the end to simplify the fraction.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Represent the equations of lines l_1 and l_2 in vector form and check whether they are intersecting or not.

$$l_1 : \frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$$

$$l_2 : \frac{x+1}{-1} = \frac{2-y}{-2} = \frac{z-5}{5}$$

[5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has vector equation $\vec{r} = \vec{a} + t\vec{b}$, where t is real.
- Two lines intersect only if their three coordinate equations admit the same pair of parameter values. Agreement in two coordinates alone does not establish intersection in space.

Quick Tip to Solve: First rewrite $(2-y)/(-2) = (y-2)/2$. Here each line's position vector equals its direction vector, so setting its parameter to -1 immediately locates the origin.

Step 1: Convert the first line to parametric form Put each ratio in the first equation equal to λ :

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} = \lambda.$$

Rearranging each coordinate gives

$$x = -3 - 3\lambda, \quad y = 1 + \lambda, \quad z = 5 + 5\lambda.$$

Thus the line passes through $(-3, 1, 5)$ and has direction ratios $(-3, 1, 5)$. Its vector representation is

$$l_1: \quad \vec{r} = -3\hat{i} + \hat{j} + 5\hat{k} + \lambda(-3\hat{i} + \hat{j} + 5\hat{k}).$$

Step 2: Convert the second line with care over the sign Use an independent real parameter μ for the second line:

$$\frac{x+1}{-1} = \frac{2-y}{-2} = \frac{z-5}{5} = \mu.$$

In particular, $2 - y = -2\mu$ gives $y = 2 + 2\mu$, so the second direction ratio is positive. Consequently,

$$x = -1 - \mu, \quad y = 2 + 2\mu, \quad z = 5 + 5\mu,$$

and

$$l_2: \quad \vec{r} = -\hat{i} + 2\hat{j} + 5\hat{k} + \mu(-\hat{i} + 2\hat{j} + 5\hat{k}).$$

Step 3: Solve the conditions for a common point At an intersection all three coordinates must coincide, so

$$-3 - 3\lambda = -1 - \mu,$$

$$1 + \lambda = 2 + 2\mu,$$

$$5 + 5\lambda = 5 + 5\mu.$$

The last equation gives $\lambda = \mu$. Substituting in the first gives

$$-3 - 3\lambda = -1 - \lambda \Rightarrow -2\lambda = 2 \Rightarrow \lambda = \mu = -1.$$

Now verify the remaining equation: both $1 + \lambda$ and $2 + 2\mu$ are zero. Thus this pair satisfies every coordinate condition. Substitution into either line gives $x = 0, y = 0, z = 0$. The direction vectors are not proportional, so the lines are distinct and this common point is their unique intersection.

Step 4: Conclusion The vector forms are

$$l_1: \quad \vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(-3\hat{i} + \hat{j} + 5\hat{k}),$$

$$l_2: \quad \vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu(-\hat{i} + 2\hat{j} + 5\hat{k}).$$

They intersect at the origin $(0, 0, 0)$, corresponding to $\lambda = \mu = -1$.

Quick Tip to Solve: First rewrite $(2 - y)/(-2) = (y - 2)/2$. Here each line's position vector equals its direction vector, so setting its parameter to -1 immediately locates the origin.

32(b). Opposite sides of a square are along the lines :

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Find the area of the square if direction ratios of other pair of opposite sides of the square are given by $\langle -3, 6, p \rangle$. Also, find the value of p . [5 marks]

Solution:

Concept:

- For parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$, their perpendicular distance is $s = |(\vec{a}_2 - \vec{a}_1) \times \vec{b}|/|\vec{b}|$.
- The distance between opposite sides of a square equals its side length, so its area is s^2 . Adjacent side directions are perpendicular and hence have zero scalar product.
- In space, perpendicularity alone is not sufficient: an adjacent side must also lie in the plane of the given parallel lines and connect them.

Step 1: Identify the parallel lines and their separation vector The given lines have the same direction vector

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}, \quad |\vec{b}| = \sqrt{4 + 9 + 36} = 7.$$

Take points $A = (1, 2, -4)$ and $B = (3, 3, -5)$ on the first and second lines respectively. Their displacement is

$$\vec{d} = \overrightarrow{AB} = 2\hat{i} + \hat{j} - \hat{k}.$$

This displacement is not itself perpendicular to the lines because $\vec{d} \cdot \vec{b} = 4 + 3 - 6 = 1$. Therefore its length cannot be used directly as the side of a square.

Step 2: Calculate the area determined by the two lines Use the cross product to remove the component along the common direction:

$$\begin{aligned} \vec{d} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} \\ &= 9\hat{i} - 14\hat{j} + 4\hat{k}. \end{aligned}$$

Hence

$$|\vec{d} \times \vec{b}|^2 = 81 + 196 + 16 = 293, \quad s = \frac{\sqrt{293}}{7}.$$

A square whose opposite sides lie on these two lines would therefore have area $s^2 = 293/49$ square units. This also follows from $s^2 = |\vec{d}|^2 - (\vec{d} \cdot \vec{b})^2/|\vec{b}|^2 = 6 - 1/49$.

Step 3: Apply the perpendicularity condition to find the candidate value Write the specified direction of the other sides as $\vec{c} = -3\hat{i} + 6\hat{j} + p\hat{k}$. Adjacent sides must be perpendicular, so

$$\begin{aligned} \vec{b} \cdot \vec{c} &= 0, \\ 2(-3) + 3(6) + 6p &= 0, \\ 12 + 6p &= 0. \end{aligned}$$

Quick Tip to Solve: Square the distance formula directly to obtain the area without retaining radicals. In three dimensions also check that the proposed adjacent-side direction is parallel to the perpendicular separation vector.

This necessary condition gives the candidate $p = -2$.

Step 4: Check that all the printed conditions are compatible The unique perpendicular displacement between the parallel lines is

$$\begin{aligned}\vec{d}_{\perp} &= \vec{d} - \frac{\vec{d} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} \\ &= \frac{1}{49}(96\hat{i} + 46\hat{j} - 55\hat{k}).\end{aligned}$$

Any side joining these two lines at right angles must be parallel to this vector. At $p = -2$, however, the stated direction is $(-3, 6, -2)$, which is not proportional to $(96, 46, -55)$. Thus the extra direction condition is inconsistent with the two specified opposite-side lines. The distance and dot-product computations give the usual formal values, but do not establish the existence of a square satisfying every printed condition.

Step 5: Conclusion The two-line distance calculation gives $\boxed{293/49 \text{ square units}}$, and perpendicularity gives $\boxed{p = -2}$. These are the formal requested values. Taken literally together, the printed data admit no such square, because the prescribed adjacent-side direction does not join the two lines perpendicularly.

Quick Tip to Solve: Square the distance formula directly to obtain the area without retaining radicals. In three dimensions also check that the proposed adjacent-side direction is parallel to the perpendicular separation vector.

33. Show that a function $f : \mathbb{R}_+ \rightarrow A \subset \mathbb{N}$, defined as $f(x) = 4x^2 + 12x + 15$ is one-one. Find set A so that f is onto where $\mathbb{R}_+ = [0, \infty)$. Also, find if there exists $a \in \mathbb{R}_+$ such that $f(a) = 7$. Justify. [5 marks]

Solution:

Concept:

- A real-valued rule is one-one if $f(u) = f(v)$ for domain elements u, v implies $u = v$.
- A function is onto precisely when its codomain equals its range. Every output must first belong to the declared codomain for the function to be well-defined.
- Completing the square gives $4x^2 + 12x + 15 = (2x + 3)^2 + 6$, which determines the range when $x \geq 0$.

Step 1: Check the printed codomain condition The printed question specifies $A \subset \mathbb{N}$, while the domain contains every nonnegative real number. This creates a mathematical inconsistency: for example, $x = \frac{1}{4}$ belongs to the domain but

$$f\left(\frac{1}{4}\right) = 4\left(\frac{1}{16}\right) + 12\left(\frac{1}{4}\right) + 15 = \frac{73}{4} \notin \mathbb{N}.$$

Thus no subset of $\mathbb{N}$ can be a codomain for this rule on the entire stated domain. We establish the injectivity and range of the real-valued polynomial rule, and then distinguish this result from the impossible printed restriction.

Step 2: Prove that the polynomial rule is one-one Take arbitrary $u, v \in [0, \infty)$ and suppose $f(u) = f(v)$.

Quick Tip to Solve: Check the codomain before proving onto: a noninteger output immediately rules out a natural-number codomain. Then use $(2x + 3)^2 + 6$ to obtain the range.

Cancelling the constant terms and factoring gives

$$\begin{aligned}4u^2 + 12u + 15 &= 4v^2 + 12v + 15, \\4(u^2 - v^2) + 12(u - v) &= 0, \\4(u - v)(u + v + 3) &= 0.\end{aligned}$$

Since $u, v \geq 0$, the factor $u + v + 3 \geq 3 > 0$. Therefore $u - v = 0$, so equal outputs force equal inputs. This proves injectivity of the rule considered as a real-valued function on the stated domain.

Step 3: Determine the exact range and the onto codomain For every $x \geq 0$, we have $2x + 3 \geq 3$, and hence

$$f(x) = (2x + 3)^2 + 6 \geq 9 + 6 = 15.$$

The endpoint is attained at $x = 0$. Conversely, for any real $y \geq 15$, define

$$x = \frac{\sqrt{y - 6} - 3}{2}.$$

Then $\sqrt{y - 6} \geq 3$, so $x \geq 0$; substituting yields $f(x) = y$. Therefore the range is exactly $[15, \infty)$. If the codomain is allowed to be a subset of $\mathbb{R}$, the unique choice making the function onto is $A = [15, \infty)$. This interval is not a subset of $\mathbb{N}$, so there is no such $A \subset \mathbb{N}$ as printed.

Step 4: Test whether 7 has a pre-image Since every value on the domain is at least 15, the value 7 is outside the range. Algebraically,

$$f(a) = 7 \implies (2a + 3)^2 = 1 \implies a = -1 \text{ or } a = -2.$$

Both possible real roots are negative and therefore excluded by $a \in [0, \infty)$.

Step 5: Conclusion The polynomial rule is one-one on $[0, \infty)$, with range $[15, \infty)$, and 7 has no pre-image in this domain. As printed, no $A \subset \mathbb{N}$ makes the stated function well-defined or onto. With a real codomain, the onto choice is $A = [15, \infty)$.

Quick Tip to Solve: Check the codomain before proving onto: a noninteger output immediately rules out a natural-number codomain. Then use $(2x + 3)^2 + 6$ to obtain the range.

34. Solve the following Linear Programming Problem graphically :

Maximize $Z = 8x + 8y$

subject to the constraints

$$x - 20 \leq 0$$

$$2x + 3y \leq 120$$

$$2x + y \leq 60$$

$$x \geq 0, y \geq 0$$

[5 marks]

Solution:

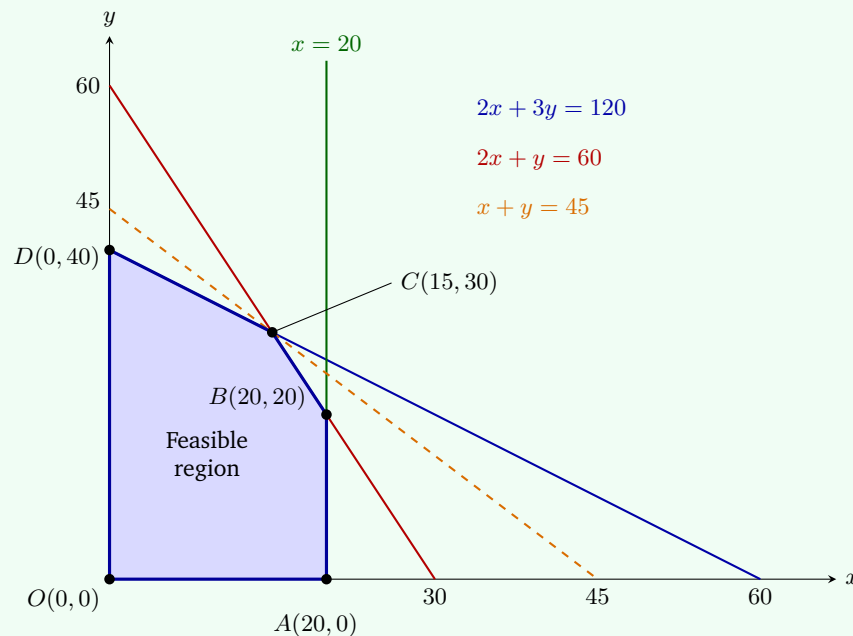
Concept:

- The feasible region is the intersection of all constraint half-planes, including $x \geq 0$ and $y \geq 0$.
- A linear objective on a nonempty bounded polygon attains its maximum at a corner point. Here $Z = 8(x + y)$, so objective lines have slope -1 .

Step 1: Draw the boundary lines and select the half-planes Rewrite the first constraint as $x \leq 20$. Its boundary is the vertical line $x = 20$, and its allowed half-plane is to the left. For the remaining boundaries, use their axis intercepts:

Boundary	x -intercept	y -intercept
$2x + 3y = 120$	$(60, 0)$	$(0, 40)$
$2x + y = 60$	$(30, 0)$	$(0, 60)$

The origin satisfies both inequalities, so shade below each sloping line, retaining their common portion with $0 \leq x \leq 20$ and $y \geq 0$. The resulting feasible polygon is $OABCD$ in the graph. Solid boundary segments are included because all inequalities are non-strict.



Shaded feasible region $OABCD$; the dashed objective line touches it at $C(15, 30)$.

Step 2: Compute and verify every corner On the x -axis, the strongest bound is $x \leq 20$, giving

Quick Tip to Solve: Since the objective is $8(x + y)$, compare $x + y$ at the five feasible corners and multiply the largest value by 8; exclude boundary intersections violating any other constraint.

$O = (0, 0)$ and $A = (20, 0)$. At $x = 20$, the bounds on y are $y \leq 80/3$ and $y \leq 20$; therefore $B = (20, 20)$. The two sloping boundaries meet where

$$2x + 3y = 120,$$

$$2x + y = 60.$$

Subtraction gives $2y = 60$, so $y = 30$ and $2x = 30$, giving $C = (15, 30)$. This point satisfies $x \leq 20$. On the y -axis the stronger bound is $y \leq 40$, giving $D = (0, 40)$. The intersection $(20, 80/3)$ is excluded because it violates $2x + y \leq 60$. The intercepts $(30, 0)$ and $(60, 0)$ violate $x \leq 20$, and $(0, 60)$ violates $2x + 3y \leq 120$. Thus none is an additional feasible corner.

Step 3: Evaluate the objective at the feasible corners The shaded region is nonempty, closed and bounded, so the corner-point method applies. Substituting each vertex into $Z = 8x + 8y$ gives

Corner	$Z = 8x + 8y$
$O(0, 0)$	0
$A(20, 0)$	160
$B(20, 20)$	320
$C(15, 30)$	360
$D(0, 40)$	320

The objective line $x + y = 45$, shown dashed, passes through C and supports the feasible region there. Shifting it further away from the origin would give no feasible point. Its slope -1 differs from the slopes of both edges meeting at C , so the maximizing point is unique.

Step 4: Conclusion The maximum value is $Z_{\max} = 360$, attained uniquely at $x = 15, y = 30$.

Quick Tip to Solve: Since the objective is $8(x + y)$, compare $x + y$ at the five feasible corners and multiply the largest value by 8; exclude boundary intersections violating any other constraint.

35(a). On the inauguration day of a new showroom, a lucky draw was organized and some vouchers of Rs 1,000 and Rs 500 were given to the lucky draw winners.

A total of 60 vouchers were given on the day. The number of Rs 1,000 vouchers added to 3 times the number of Rs 500 vouchers, gives 100. Express the given information as a system of linear equations in two variables. Hence, find the number of vouchers of each type by matrix method. [5 marks]

Solution:

Concept:

- Introduce one variable for the number of vouchers of each denomination. A statement about their total number gives a linear equation in these variables.
- A linear system can be written as $AX = B$, where A contains the coefficients, X is the column of unknowns and B is the column of constants.
- If $|A| \neq 0$, the unique solution is $X = A^{-1}B$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Quick Tip to Solve: Define the variables as voucher counts, not rupee totals. In the inverse calculation, keep the variable order and the row order unchanged.

Step 1: Define the variables and form the equations Let x be the number of Rs 1,000 vouchers and y the number of Rs 500 vouchers. Since 60 vouchers were distributed in all,

$$x + y = 60.$$

The other condition concerns the numbers of vouchers, rather than their monetary values: the first number plus three times the second number is 100. Thus,

$$x + 3y = 100.$$

These are the required equations in two variables.

Step 2: Write the system in matrix form Keep the same order x, y in both equations and place their constants in the corresponding rows:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 60 \\ 100 \end{pmatrix}}_B.$$

To use the inverse method, first check that the coefficient matrix is nonsingular:

$$|A| = (1)(3) - (1)(1) = 2 \neq 0.$$

Therefore the inverse exists and determines exactly one pair of voucher counts.

Step 3: Obtain the inverse and multiply by the constant column Interchange the diagonal entries of A , change the signs of the other two entries, and divide by its determinant:

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}.$$

Multiplying $AX = B$ on the left by A^{-1} gives

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 60 \\ 100 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 180 - 100 \\ -60 + 100 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \end{pmatrix}. \end{aligned}$$

Step 4: Check the counts against both statements Both entries obtained are nonnegative integers, as voucher counts must be. Substitution into the original conditions gives

$$40 + 20 = 60, \quad 40 + 3(20) = 100.$$

Thus the computed counts satisfy both the total-number condition and the weighted-number condition.

Step 5: Conclusion The required system is $x + y = 60$, $x + 3y = 100$. There were 40 vouchers of Rs

Quick Tip to Solve: Define the variables as voucher counts, not rupee totals. In the inverse calculation, keep the variable order and the row order unchanged.

1,000 and 20 vouchers of Rs 500.

Quick Tip to Solve: Define the variables as voucher counts, not rupee totals. In the inverse calculation, keep the variable order and the row order unchanged.

OR

35(b). Given that $P = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $Q = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$ and $R = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, find a matrix S such that $PQ - RS$ is a null matrix. [5 marks]

Solution:

Concept:

- Matrix multiplication uses the row-by-column rule: the entry in row i , column j of a product is the sum of products of corresponding entries of row i and column j .
- The equation $PQ - RS = O$, where O is the null matrix, gives $RS = PQ$. If R is invertible, left multiplication by R^{-1} gives $S = R^{-1}PQ$.
- A square matrix is invertible if its determinant is nonzero. For $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $R^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Step 1: Calculate the product of the first two matrices Use each row of P with each column of Q , taking care of the negative entry in the first row of P :

$$\begin{aligned} PQ &= \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2(5) - 1(7) & 2(2) - 1(4) \\ 3(5) + 4(7) & 3(2) + 4(4) \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}. \end{aligned}$$

The required relation is therefore $RS = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}$.

Step 2: Find the inverse of the matrix multiplying the unknown For the given matrix R , calculate

$$|R| = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2(8) - 5(3) = 1.$$

Since this determinant is nonzero, R has an inverse. The formula for a two-by-two matrix yields

$$R^{-1} = \frac{1}{1} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}.$$

Quick Tip to Solve: In $RS = PQ$, cancel R by multiplying on the left, giving $S = R^{-1}PQ$. Multiplying on the right changes the order and does not isolate S .

Step 3: Isolate the unknown matrix by left multiplication Multiply $RS = PQ$ on the left by R^{-1} . This places R^{-1} next to R , so their product is the identity matrix:

$$R^{-1}RS = R^{-1}PQ \implies S = R^{-1}(PQ).$$

Consequently,

$$\begin{aligned} S &= \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 24 - 215 & 0 - 110 \\ -9 + 86 & 0 + 44 \end{pmatrix} \\ &= \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}. \end{aligned}$$

The multiplication order matters because matrix products do not in general commute.

Step 4: Verify the original null-matrix condition Substitution provides an independent check on all four entries:

$$\begin{aligned} RS &= \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix} \\ &= \begin{pmatrix} -382 + 385 & -220 + 220 \\ -573 + 616 & -330 + 352 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} = PQ. \end{aligned}$$

Hence subtracting these equal products indeed gives the two-by-two null matrix.

Step 5: Conclusion The unique required matrix is $S = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}$, for which $PQ - RS = O$.

Quick Tip to Solve: In $RS = PQ$, cancel R by multiplying on the left, giving $S = R^{-1}PQ$. Multiplying on the right changes the order and does not isolate S .

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

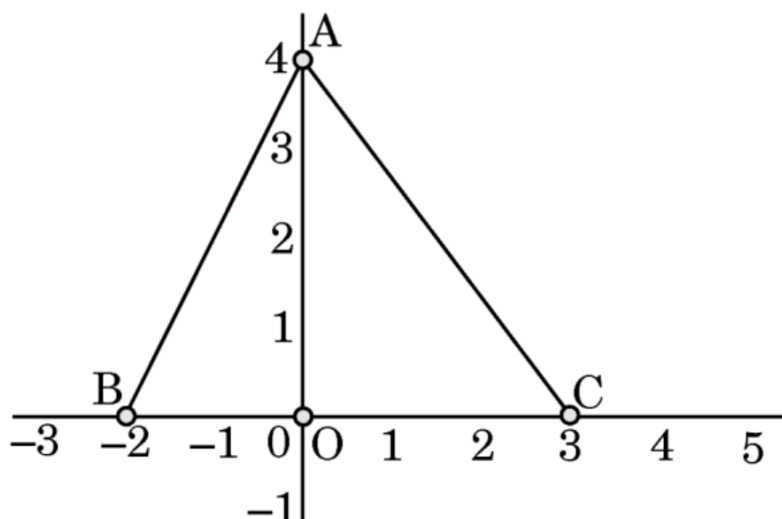
36. There is a triangular park in the society. The park is divided into two sections as shown in the figure.

In the region OAC, children are allowed to play games like cricket, football, while in the region AOB, activities which involve running are not allowed.

The vertices of the triangular park ABC are $A(0, 4)$, $B(-2, 0)$ and $C(3, 0)$.

Based on the above information, answer the following questions :

[4 marks]



36(i). Write the equation of the boundary line AB of the park.

[1 mark]

Solution:

Concept:

- The slope of the line through (x_1, y_1) and (x_2, y_2) is $m = (y_2 - y_1)/(x_2 - x_1)$.
- A line of slope m through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$.

Step 1: Calculate the slope of AB The endpoints are $A(0, 4)$ and $B(-2, 0)$. Hence

$$m_{AB} = \frac{0 - 4}{-2 - 0} = 2.$$

The positive slope agrees with the rise of the boundary from B towards A.

Step 2: Substitute point A in the line equation Using the point-slope form with $A(0, 4)$,

$$y - 4 = 2(x - 0).$$

At $x = -2$, this equation gives $y = 0$, so it also passes through B as required.

Step 3: Conclusion The boundary line AB is $y = 2x + 4$, equivalently $2x - y + 4 = 0$.

Quick Tip to Solve: Since A lies on the y-axis, its ordinate 4 is the intercept; calculate only the slope to write AB immediately.

36(ii). Write the equation of the boundary line AC of the park.

[1 mark]

Solution:

Concept:

- For two distinct points (x_1, y_1) and (x_2, y_2) , the line's slope is $m = (y_2 - y_1)/(x_2 - x_1)$.
- The point-slope form is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the line.

Step 1: Calculate the slope of AC The endpoints are $A(0, 4)$ and $C(3, 0)$. Thus

$$m_{AC} = \frac{0 - 4}{3 - 0} = -\frac{4}{3}.$$

The negative sign agrees with the fall of the boundary from A towards C.

Step 2: Use the point-slope form Taking $A(0, 4)$ as the known point gives

$$y - 4 = -\frac{4}{3}(x - 0).$$

Multiplying by 3 and collecting terms gives $4x + 3y = 12$. The point C satisfies this because $4(3) + 3(0) = 12$.

Step 3: Conclusion The boundary line AC is $\boxed{4x + 3y = 12}$, or $y = 4 - \frac{4x}{3}$.

Quick Tip to Solve: AC has intercepts 3 and 4 on the coordinate axes, so the intercept form $x/3 + y/4 = 1$ gives the equation directly.

36(iii)(a). Using integration, find the area of region OAC, in which children are allowed to play cricket, football.

[2 marks]

Solution:

Concept:

- If $y = f(x) \geq 0$ on $[a, b]$, the area between its graph and the x-axis is $\int_a^b f(x) dx$.
- A vertical strip has area $y dx$. For triangle OAC, the lower boundary is $y = 0$ and the upper boundary is line AC.

Step 1: Express the upper boundary as a function of x Line AC passes through $A(0, 4)$ and $C(3, 0)$. Its slope is $-4/3$, so

$$y = 4 - \frac{4x}{3}.$$

The region OAC extends from $x = 0$ to $x = 3$. The height of each vertical strip above the x-axis is therefore $4 - 4x/3$, which is nonnegative throughout this interval. The region to the left of O belongs to AOB and is excluded.

Quick Tip to Solve: For OAC, integrate only from 0 to 3 under AC; including negative x-values would add the separate region AOB.

Step 2: Integrate the strip heights Adding the strip areas gives

$$\begin{aligned}A_{OAC} &= \int_0^3 \left(4 - \frac{4x}{3}\right) dx \\&= \left[4x - \frac{2x^2}{3}\right]_0^3 \\&= 12 - 6 = 6.\end{aligned}$$

The lower endpoint contributes zero. As a check, the triangle has perpendicular base $OC = 3$ and height $OA = 4$, so its geometric area is $\frac{1}{2}(3)(4)$, consistent with the integral.

Step 3: Conclusion The area in which children can play cricket and football is 6 square units.

Quick Tip to Solve: For OAC, integrate only from 0 to 3 under AC; including negative x-values would add the separate region AOB.

OR

36(iii)(b). Using integration, find the area of region AOB.

[2 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, from $x = a$ to $x = b$, is $\int_a^b [f(x) - g(x)] dx$.
- A straight line through two points has slope $m = (y_2 - y_1)/(x_2 - x_1)$. The region AOB is bounded by AB and the two coordinate axes.

Step 1: Determine the line bounding the region The vertices are $A(0, 4)$, $O(0, 0)$ and $B(-2, 0)$. The slope of AB is

$$m = \frac{4 - 0}{0 - (-2)} = 2.$$

Since AB passes through $A(0, 4)$, its equation is $y = 2x + 4$. Throughout $-2 \leq x \leq 0$, this line lies above the x -axis. Thus a vertical strip in AOB has height $2x + 4$ and width dx .

Step 2: Integrate the vertical strips Use the limits from B to O in increasing order. The lower boundary has ordinate zero, so the required area is calculated by

$$\begin{aligned}\int_{-2}^0 (2x + 4) dx &= [x^2 + 4x]_{-2}^0 \\&= 0 - (4 - 8) = 4.\end{aligned}$$

The integrand is nonnegative over this interval, so this definite integral directly gives the geometric area.

Step 3: Conclusion The area of region AOB is 4 square units.

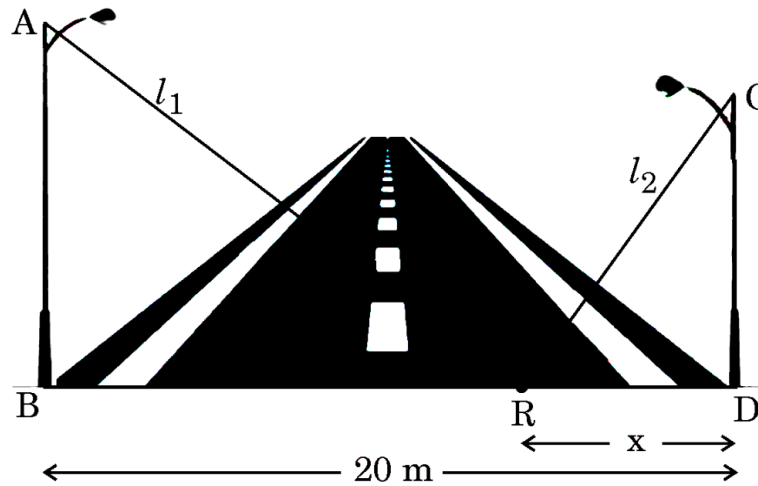
Quick Tip to Solve: For AOB use the interval from -2 to 0 ; the interval from 0 to 3 belongs to OAC instead.

37. Two vertical light poles of height 22 m and 16 m stand on the opposite sides of a 20 m wide road as shown below in the figure.

Two ladders of length l_1 and l_2 are placed from a common point R on the road at a distance of x m from the smaller pole.

Based on the above information, answer the following questions :

[4 marks]



37(i). Express $p(x) = l_1 + l_2$ in terms of x .

[1 mark]

Solution:

Concept:

- For a right triangle, the square of the hypotenuse equals the sum of the squares of its perpendicular sides.
- A ladder length is positive, so take the positive square root when recovering it from its square.

Step 1: Express the horizontal distances The road is 20 m wide and R is x m from the smaller pole. Hence the horizontal distances from R to the 16 m and 22 m poles are respectively x and $20 - x$, where $0 \leq x \leq 20$.

Step 2: Find the two ladder lengths Applying Pythagoras to the two right triangles gives

$$l_1 = \sqrt{(20 - x)^2 + 484}, \quad l_2 = \sqrt{x^2 + 256}.$$

The requested function is the sum of the lengths themselves, so add these positive square roots.

Step 3: Conclusion $p(x) = \sqrt{(20 - x)^2 + 484} + \sqrt{x^2 + 256}$, for $0 \leq x \leq 20$, with lengths measured in metres.

Quick Tip to Solve: Match the distance x to the shorter pole: its ladder contributes $\sqrt{x^2 + 16^2}$.

37(ii). Find $p'(x)$.

[1 mark]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}\sqrt{u(x)} = \frac{u'(x)}{2\sqrt{u(x)}}$ when $u(x) > 0$.
- Differentiate a sum term by term, including the derivative of each inner expression.

Step 1: Differentiate the first ladder length From part (i), $p(x) = \sqrt{(20-x)^2 + 484} + \sqrt{x^2 + 256}$. For the first term the inner derivative is $-2(20-x)$, since $\frac{d}{dx}(20-x) = -1$. Thus

$$\frac{d}{dx}\sqrt{(20-x)^2 + 484} = \frac{x-20}{\sqrt{(20-x)^2 + 484}}.$$

Step 2: Differentiate the second ladder length The inner derivative of $x^2 + 256$ is $2x$. Therefore

$$\frac{d}{dx}\sqrt{x^2 + 256} = \frac{x}{\sqrt{x^2 + 256}}.$$

Both radicands are positive, so these derivatives are defined throughout the physical interval; add the two contributions.

Step 3: Conclusion

$$p'(x) = \frac{x-20}{\sqrt{(20-x)^2 + 484}} + \frac{x}{\sqrt{x^2 + 256}}.$$

Quick Tip to Solve: The 22 m pole contributes a negative derivative because its horizontal distance $20-x$ decreases as x increases.

37(iii)(a). Find the value of x for which $l_1^2 + l_2^2$ is minimum.

[2 marks]

Solution:

Concept:

- Pythagoras' theorem gives the square of each ladder's length as the sum of the squares of its horizontal and vertical distances.
- For a differentiable function $S(x)$, an interior minimum occurs at a stationary point $S'(x) = 0$; $S''(x) > 0$ confirms a minimum.

Step 1: Express the sum of squares The point R is x m from the foot of the 16 m pole, so it is $(20-x)$ m from the foot of the 22 m pole. Thus, for $0 \leq x \leq 20$,

$$\begin{aligned}l_1^2 &= (20-x)^2 + 22^2, \\l_2^2 &= x^2 + 16^2, \\S(x) &= l_1^2 + l_2^2 = 2x^2 - 40x + 1140.\end{aligned}$$

We differentiate the sum of squares requested here, rather than the sum of the ladder lengths.

Quick Tip to Solve: When minimizing squared ladder lengths, the pole heights contribute only constants; differentiate the two squared horizontal distances.

Step 2: Locate and verify the minimum Differentiating with respect to the numerical distance x ,

$$S'(x) = 4x - 40 = 0 \Rightarrow x = 10.$$

Also, $S''(x) = 4 > 0$ throughout the interval. The stationary point lies within the road, and the quadratic is strictly convex, so this is its unique absolute minimum. Equivalently,

$$S(x) = 2(x - 10)^2 + 940,$$

whose squared term cannot be negative.

Step 3: Conclusion The sum $l_1^2 + l_2^2$ is minimum when $x = 10$ m, so the ladders share a foot at the midpoint of the road.

Quick Tip to Solve: When minimizing squared ladder lengths, the pole heights contribute only constants; differentiate the two squared horizontal distances.

OR

37(iii)(b). If the 22 m long pole is also replaced by a 16 m long pole, at what distance from either pole should the ladders be kept so that the sum of squares of lengths of ladders needed to reach the top of the pole is minimum ? [2 marks]

Solution:

Concept:

- For a ladder reaching the top of a vertical pole, Pythagoras' theorem gives $l^2 = d^2 + h^2$, where d is the horizontal distance and h the pole height.
- A stationary point at which the second derivative is positive is a local minimum; a strictly convex quadratic has its unique absolute minimum there.

Step 1: Model the two equal-height poles Both poles are now 16 m high and their feet remain 20 m apart. Let the common foot of the ladders be x m from one pole and $(20 - x)$ m from the other, with $0 \leq x \leq 20$. The sum of the squares of the ladder lengths is

$$\begin{aligned} S(x) &= x^2 + 16^2 + (20 - x)^2 + 16^2 \\ &= 2x^2 - 40x + 912. \end{aligned}$$

The vertical contributions are fixed; moving the common foot changes only the horizontal distances.

Step 2: Differentiate and test the stationary point Set the first derivative equal to zero:

$$S'(x) = 4x - 40 = 0 \Rightarrow x = 10.$$

Since $S''(x) = 4 > 0$ for every x , this is the unique minimum, and it lies inside the allowed interval. The remaining horizontal distance is also $20 - 10 = 10$ m. One can verify the absolute minimum

Quick Tip to Solve: For two equal-height poles, check the midpoint first, then use the positive second derivative to justify that it minimizes the squared lengths.

directly from

$$S(x) = 2(x - 10)^2 + 712.$$

Every displacement from the midpoint adds a positive squared term.

Step 3: Conclusion The ladders should be placed with their common foot 10 m from either pole, at the midpoint of the road.

Quick Tip to Solve: For two equal-height poles, check the midpoint first, then use the positive second derivative to justify that it minimizes the squared lengths.

38. A survey was conducted to find out the success rate of students who qualified the entrance examination by dropping a year after class XII.

As per the data collected, 40% students appearing in the examination were dropouts and the remaining students were regular students of class XII.

Of the dropouts, 5% qualify the examination while 10% of the regular students qualify the examination.

Based on the above information, answer the following questions.

[4 marks]

38(i). Find the probability that a student selected at random is a regular student.

[1 mark]

Solution:

Concept:

- Complementary events exhaust all outcomes and cannot occur together: $P(E^c) = 1 - P(E)$.
- A percentage describes a probability after division by 100; random selection here is from all students appearing in the examination.

Step 1: Identify the complementary categories Let D denote the event that the selected student is a dropout and R the event that the student is regular. The survey divides all candidates into these two categories, so $R = D^c$. The given proportion is

$$P(D) = \frac{40}{100} = 0.40.$$

Step 2: Subtract the dropout proportion Because D and R are mutually exclusive and exhaustive,

$$P(R) = 1 - P(D) = 1 - 0.40 = 0.60.$$

The qualification rates are unnecessary here, since the selection is not restricted to students who qualified.

Step 3: Conclusion The probability that the selected student is regular is $\frac{3}{5} = 0.60$.

Quick Tip to Solve: For a random candidate's category, use the original population proportions; introduce qualification rates only when the event mentions the result.

38(ii). A student is selected at random from a group of dropout students. What is the probability that the student will not qualify the examination ? [1 mark]

Solution:

Concept:

- Let D denote the event that a student is a dropout and Q the event that the student qualifies the examination.
- Within a fixed group, qualification and non-qualification are complementary events: $P(Q^c | D) = 1 - P(Q | D)$.

Step 1: Identify the given conditional probability The student is selected from the dropout group, so the sample space is already restricted to D . The statement that 5% of dropouts qualify therefore gives $P(Q | D) = 0.05$.

Step 2: Apply the complement rule within the dropout group Every dropout either qualifies or does not qualify. Hence,

$$P(Q^c | D) = 1 - 0.05 = 0.95.$$

Multiplying by the overall dropout fraction 0.40 would instead find a joint probability for selection from all students, which is not what is asked here.

Step 3: Conclusion The required probability is $0.95 = \frac{19}{20}$.

Quick Tip to Solve: For selection from a named group, take the complement of the success rate within that same group.

38(iii)(a). A student selected at random qualified the examination. Find the probability that student is not a dropout. [2 marks]

Solution:

Concept:

- Let D , R and Q denote a dropout, a regular student and a student who qualifies, respectively. Here $R = D^c$, since these two student groups exhaust the population.
- The total probability of qualification is $P(Q) = P(D)P(Q | D) + P(R)P(Q | R)$.
- Bayes' theorem reverses the conditioning: $P(R | Q) = \frac{P(R)P(Q | R)}{P(Q)}$.

Step 1: Calculate the qualification probability The data give $P(D) = 0.40$, $P(R) = 0.60$, $P(Q | D) = 0.05$ and $P(Q | R) = 0.10$. Students can qualify through either of the two disjoint groups, so

$$\begin{aligned} P(Q) &= (0.40)(0.05) + (0.60)(0.10) \\ &= 0.02 + 0.06 = 0.08. \end{aligned}$$

Step 2: Restrict selection to qualified students The desired student is not a dropout, which means the student belongs to R . The joint probability of being regular and qualifying is $P(R \cap Q) = 0.60 \times 0.10 =$

Quick Tip to Solve: For a student already known to have qualified, divide the regular-and-qualified fraction by the total qualified fraction.

0.06. Divide this by the total qualified fraction because qualification is already known:

$$P(R | Q) = \frac{0.06}{0.08} = \frac{3}{4}.$$

Step 3: Conclusion The probability that the qualified student is not a dropout is $\boxed{\frac{3}{4}}$.

Quick Tip to Solve: For a student already known to have qualified, divide the regular-and-qualified fraction by the total qualified fraction.

OR

38(iii)(b). A student selected at random did not qualify the examination. Find the probability that the student was a regular student. [2 marks]

Solution:

Concept:

- Let D , R and F denote a dropout, a regular student and a student who does not qualify, respectively. The events D and R partition the student population.
- Failure rates are complements of the corresponding qualification rates: $P(F | D) = 1 - 0.05$ and $P(F | R) = 1 - 0.10$.
- Bayes' theorem gives $P(R | F) = \frac{P(R)P(F | R)}{P(D)P(F | D) + P(R)P(F | R)}$.

Step 1: Find the total probability of non-qualification The group probabilities are $P(D) = 0.40$ and $P(R) = 0.60$. The non-qualification rates are 0.95 for dropouts and 0.90 for regular students. Adding the mutually exclusive routes to non-qualification gives

$$\begin{aligned} P(F) &= (0.40)(0.95) + (0.60)(0.90) \\ &= 0.38 + 0.54 = 0.92. \end{aligned}$$

Step 2: Apply Bayes' theorem for a regular student The probability of being both regular and unsuccessful is $P(R \cap F) = 0.60 \times 0.90 = 0.54$. Since the selected student is known not to have qualified, normalize by the entire unsuccessful group:

$$P(R | F) = \frac{0.54}{0.92} = \frac{54}{92} = \frac{27}{46}.$$

This differs from $P(F | R) = 0.90$, which conditions on being regular rather than on non-qualification.

Step 3: Conclusion The required probability that the unsuccessful student was regular is $\boxed{\frac{27}{46}}$.

Quick Tip to Solve: First complement both qualification rates, then use regular failures divided by failures from both groups.