

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-4-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

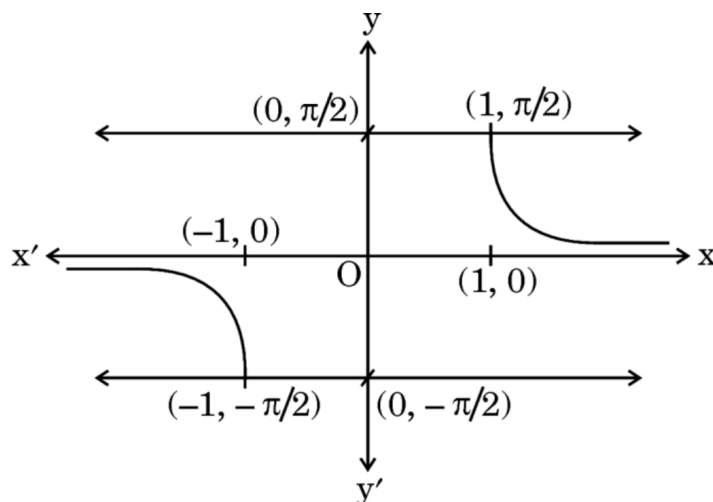
- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The following graph represents:

[1 mark]



- (A) $y = \cos^{-1} x$
- (B) $y = \sec^{-1} x$
- (C) $y = \tan^{-1} x$
- (D) $y = \operatorname{cosec}^{-1} x$

Correct Answer: (D) $y = \operatorname{cosec}^{-1} x$

Solution:

Concept:

- The principal range of $\operatorname{cosec}^{-1} x$ is $[-\pi/2, 0) \cup (0, \pi/2]$.
- Its domain is $(-\infty, -1] \cup [1, \infty)$, since $|\sin y| \leq 1$.

Step 1: Read the branch endpoints The graph has separate branches outside $(-1, 1)$, passing through $(1, \pi/2)$ and $(-1, -\pi/2)$. On the positive branch the value decreases towards zero as x increases; the negative branch approaches zero from below.

Step 2: Compare all four functions For $y = \operatorname{cosec}^{-1} x$, $\sin y = 1/x$, which gives precisely these endpoints and branches. The inverse cosine is defined only on $[-1, 1]$. The inverse tangent is continuous for every real x and passes through the origin. The inverse secant has range $[0, \pi] \setminus \{\pi/2\}$, so it cannot have the negative branch shown.

Step 3: Conclusion The graph represents $y = \operatorname{cosec}^{-1} x$. Hence, the correct option is (D).

Quick Tip to Solve: Check the points at $x=1$ and $x=-1$ before comparing the shapes of inverse-function graphs.

2. If $A = [a_{ij}]$ is a 2×2 matrix whose elements are given by $a_{ij} = \frac{|i - 3j|}{2}$, then A' is:

[1 mark]

(A) $\begin{pmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{pmatrix}$

(C) $\begin{pmatrix} 2 & \frac{5}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{5}{2} & 1 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{pmatrix}$

Solution:

Concept:

- The entry a_{ij} is in row i , column j ; substitute both indices.
- Transposition interchanges rows and columns: $(A')_{ij} = a_{ji}$.

Step 1: Compute the four entries Using the printed formula,

$$a_{11} = \frac{|1 - 3|}{2} = 1, \quad a_{12} = \frac{|1 - 6|}{2} = \frac{5}{2}, \quad a_{21} = \frac{|2 - 3|}{2} = \frac{1}{2}, \quad a_{22} = \frac{|2 - 6|}{2} = 2.$$

Thus $A = \begin{pmatrix} 1 & 5/2 \\ 1/2 & 2 \end{pmatrix}$.

Step 2: Transpose and check the options The first column of A becomes the first row of A' , and its second column becomes the second row. Therefore

$$A' = \begin{pmatrix} 1 & 1/2 \\ 5/2 & 2 \end{pmatrix}.$$

Option (A) leaves the off-diagonal entries unswapped. Options (C) and (D) also change the diagonal entries, which a transpose does not do.

Step 3: Conclusion $A' = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Calculate a_{12} and a_{21} separately: the row index i is not a constant in this formula.

3. The principal value of $\sec^{-1}(\sqrt{2}) + 2 \operatorname{cosec}^{-1}(-\sqrt{2})$ is:

[1 mark]

- (A) $-\frac{\pi}{2}$
- (B) $-\frac{\pi}{4}$
- (C) $\frac{\pi}{4}$
- (D) $\frac{\pi}{2}$

Correct Answer: (B) $-\frac{\pi}{4}$

Solution:

Concept:

- Use principal ranges when evaluating inverse trigonometric functions.
- For inverse cosecant, the principal range excludes zero and lies between $-\pi/2$ and $\pi/2$.

Step 1: Evaluate the principal angles Since $\sec(\pi/4) = \sqrt{2}$, the principal value of $\sec^{-1}(\sqrt{2})$ is $\pi/4$. Also $\sin(-\pi/4) = -1/\sqrt{2}$, so $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\pi/4$. The negative sign must be retained; a positive angle would have positive cosecant.

Step 2: Apply the coefficient and compare The coefficient 2 multiplies the entire second principal angle. Hence

$$\frac{\pi}{4} + 2\left(-\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}.$$

Option (A) is the second term alone, while options (C) and (D) have positive signs and do not equal this sum.

Step 3: Conclusion The principal value is $-\pi/4$. Hence, the correct option is (B).

Quick Tip to Solve: Evaluate the two principal angles first, then apply the coefficient 2 and add.

4. If points $(2, 3)$, $(0, 4)$ and $(p, 2)$ are collinear, then the value of p is:

[1 mark]

- (A) $\frac{4}{7}$
- (B) $-\frac{3}{7}$
- (C) 4
- (D) -4

Correct Answer: (C) 4

Solution:

Concept:

- Collinear points have equal slopes whenever the denominators are nonzero.
- A determinant with rows $(x, y, 1)$ is zero for three collinear points.

Step 1: Find the common line The slope through $(2, 3)$ and $(0, 4)$ is $(3 - 4)/(2 - 0) = -1/2$. Since its vertical intercept is 4, the line is $y = 4 - x/2$. Any third point collinear with these two must satisfy this same equation.

Quick Tip to Solve: Use the point with $x=0$ to read the intercept, then substitute the unknown point into the line.

Step 2: Substitute the third point At $(p, 2)$, $2 = 4 - p/2$, so $p/2 = 2$ and $p = 4$. As a check, substituting 4 gives $y = 2$, as required. The other listed values $4/7, -3/7, -4$ do not yield $y = 2$ on this line, so they fail the collinearity condition.

Step 3: Conclusion $p = 4$. Hence, the correct option is (C).

Quick Tip to Solve: Use the point with $x=0$ to read the intercept, then substitute the unknown point into the line.

5. Differential of e^{e^x} with respect to x is:

[1 mark]

(A) $\log x$

(B) e^{e^x}

(C) $e^x e^{e^x}$

(D) $(e^x)^2$

Correct Answer: (C) $e^x e^{e^x}$

Solution:

Concept:

- The chain rule is $\frac{d}{dx} e^{u(x)} = e^{u(x)} u'(x)$.
- For the inner function $u(x) = e^x$, its derivative is also e^x .

Step 1: Differentiate the outer exponential Set $y = e^{e^x}$ and $u = e^x$. The derivative of e^u with respect to u is e^u . This produces the outer factor e^{e^x} , but the chain rule still requires multiplying by the derivative of the exponent.

Step 2: Include the inner derivative As $du/dx = e^x$,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^{e^x} e^x.$$

Option (B) omits the inner derivative. Option (D) replaces the outer exponential by e^x , and option (A), $\log x$, is unrelated to differentiating this nested exponential. Only option (C) contains both required factors.

Step 3: Conclusion $\frac{d}{dx} e^{e^x} = e^x e^{e^x}$. Hence, the correct option is (C).

Quick Tip to Solve: For a nested exponential, keep the original outer exponential and multiply by the derivative of its exponent.

6. The surface area of a sphere when its volume changes at the same rate as its radius is: [1 mark]

- (A) 4π sq. units
- (B) 1 sq. unit
- (C) 4 sq. units
- (D) π sq. units

Correct Answer: (B) 1 sq. unit

Solution:

Concept:

- A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$.
- The chain rule gives $dV/dt = (dV/dr)(dr/dt)$.

Step 1: Relate the two rates Differentiating the volume with respect to time gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

The question states that the volume changes at the same rate as the radius, so $dV/dt = dr/dt$. For the changing sphere, the common rate is nonzero, allowing cancellation.

Step 2: Identify the surface area Therefore $4\pi r^2 = 1$. The left side is exactly the surface area of the sphere. Thus the required area is one square unit, rather than 4π , 4, or π . Those alternatives would require different ratios of volume rate to radius rate; the stated equal-rate ratio is exactly 1.

Step 3: Conclusion The surface area is 1 square unit. Hence, the correct option is (B).

Quick Tip to Solve: Remember that dV/dr equals the surface area of a sphere; equal nonzero rates make this derivative 1.

7. If $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is: [1 mark]

- (A) 0
- (B) -2
- (C) -1
- (D) 2

Correct Answer: (D) 2

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0)$.
- The standard radian limit is $\lim_{x \rightarrow 0} \sin x/x = 1$, and cosine is continuous.

Step 1: Evaluate the punctured limit For $x \neq 0$, use the given expression rather than the separate

Quick Tip to Solve: For continuity of a piecewise function, evaluate the limit from the formula valid away from the joining point.

value k . The two-sided limits of its terms both exist:

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} + \cos x \right) = 1 + \cos 0 = 1 + 1 = 2.$$

Directly putting $x = 0$ into $\sin x/x$ would give an undefined quotient and is not a valid evaluation.

Step 2: Match the assigned value The piecewise definition assigns $f(0) = k$. For continuity this value must equal the limit 2. The values 0, -2 and -1 in the other options all differ from the limiting value and would leave a discontinuity at the origin.

Step 3: Conclusion $k = 2$. Hence, the correct option is (D).

Quick Tip to Solve: For continuity of a piecewise function, evaluate the limit from the formula valid away from the joining point.

8. The greatest integer function, $f(x) = [x]$, $0 < x < 3$ is not differentiable at how many points?

[1 mark]

- (A) At only one point
- (B) At only two points
- (C) At no point
- (D) At three points

Correct Answer: (B) At only two points

Solution:

Concept:

- The greatest integer function $[x]$ gives the largest integer not exceeding x .
- Differentiability implies continuity, while a constant function is differentiable with derivative zero.

Step 1: Check the noninteger intervals On $(0, 1)$, $(1, 2)$ and $(2, 3)$, the function has constant values 0, 1 and 2, respectively. Thus at every noninteger point in the given domain it is differentiable, with derivative zero.

Step 2: Count the jumps within the domain At $x = 1$, the left-hand limit is 0 but the function value and right-hand limit are 1. At $x = 2$, the left-hand limit is 1 but the function value and right-hand limit are 2. Both are jump discontinuities and hence points of nondifferentiability. The endpoints 0 and 3 are excluded by $0 < x < 3$, so neither is counted. This gives two points, ruling out the counts one, zero and three.

Step 3: Conclusion The function is not differentiable at exactly two points, $x = 1$ and $x = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Count only integers strictly inside the stated open interval; exclude its endpoints.

9. $\int \frac{dx}{\sqrt{25 - 16x^2}}$ is equal to:

[1 mark]

- (A) $\frac{1}{5} \sin^{-1} 4x + C$
(B) $\frac{1}{25} \sin^{-1} 16x + C$
(C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$
(D) $\frac{1}{16} \sin^{-1} \frac{4x}{5} + C$

Correct Answer: (C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$

Solution:

Concept:

- The inverse-sine standard form is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, for $a > 0$.
- A substitution must also transform the differential: if $u = 4x$, then $dx = du/4$.

Step 1: Make the substitution Put $u = 4x$. The denominator becomes $\sqrt{25 - u^2}$, so

$$I = \frac{1}{4} \int \frac{du}{\sqrt{5^2 - u^2}}.$$

The outside factor is essential; it comes from the differential, not from the square root.

Step 2: Apply and check the standard form Thus $I = \frac{1}{4} \sin^{-1}(4x/5) + C$. Differentiating gives $\frac{1}{4} \frac{4/5}{\sqrt{1 - 16x^2/25}} = 1/\sqrt{25 - 16x^2}$. Options (A) and (B) have incorrect arguments; option (D) has one quarter of the required coefficient.

Step 3: Conclusion $\boxed{\frac{1}{4} \sin^{-1}(4x/5) + C}$. Hence, the correct option is (C).

Quick Tip to Solve: Normalize the argument to $4x/5$, then retain $dx = du/4$ to avoid the factor-of-four trap.

10. If $\int_0^1 \frac{dx}{e^x + e^{-x}} = \tan^{-1} e + k$, then the value of k is:

[1 mark]

- (A) e
(B) $\frac{\pi}{4}$
(C) 0
(D) $-\frac{\pi}{4}$

Correct Answer: (D) $-\frac{\pi}{4}$

Solution:

Concept:

- For a definite integral, change both limits when substituting.
- The identity $\int dt/(1 + t^2) = \tan^{-1} t + C$ applies after setting $t = e^x$.

Quick Tip to Solve: After substituting $t = e^x$, use lower limit 1; subtract $\tan^{-1} 1 = \pi/4$.

Step 1: Transform the integrand and limits Multiply numerator and denominator by e^x :

$$I = \int_0^1 \frac{e^x dx}{1 + e^{2x}}.$$

Set $t = e^x$, so $dt = e^x dx$. The lower and upper limits become 1 and e , respectively.

Step 2: Evaluate and compare Therefore

$$I = \int_1^e \frac{dt}{1 + t^2} = \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}.$$

Comparing with the given expression identifies the additive constant. Option (B) incorrectly adds the lower-limit value; (C) omits it; (A) confuses the upper limit with the constant.

Step 3: Conclusion $k = -\pi/4$. Hence, the correct option is (D).

Quick Tip to Solve: After substituting $t = e^x$, use lower limit 1; subtract $\tan^{-1} 1 = \pi/4$.

11. The area of the region bounded by the curve $y = x$ and x -axis, between $x = 0$ and $x = 2$ is: [1 mark]

- (A) 2 sq. units
- (B) $\frac{1}{2}$ sq. unit
- (C) 1 sq. unit
- (D) 4 sq. units

Correct Answer: (A) 2 sq. units

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$ and the x -axis on $[a, b]$ is $\int_a^b f(x) dx$.
- The line $y = x$ is above the axis throughout $0 \leq x \leq 2$, so no sign change is needed.

Step 1: Set up the area The lower boundary is $y = 0$, and the upper boundary is $y = x$. A vertical strip therefore has height x , giving

$$A = \int_0^2 (x - 0) dx.$$

Both end points must be used, since the region is restricted to this interval.

Step 2: Evaluate and check geometrically

$$A = \left[\frac{x^2}{2} \right]_0^2 = \frac{4}{2} - 0 = 2.$$

The same region is a right triangle with base 2 and height 2, so $\frac{1}{2}(2)(2) = 2$. Option (D) is the enclosing rectangle area, while (B) and (C) do not use both dimensions of the triangle.

Step 3: Conclusion $A = 2$ square units. Hence, the correct option is (A).

Quick Tip to Solve: Use half the product of base 2 and height 2 as a quick check of the integral.

12. Product of the order and degree of differential equation $1 + \left(\frac{dy}{dx}\right)^3 = \lambda \left(\frac{d^3y}{dx^3}\right)^2$ is: [1 mark]

- (A) 5
- (B) 6
- (C) 2
- (D) 3

Correct Answer: (B) 6

Solution:

Concept:

- The order is the order of the highest derivative present in the differential equation.
- The degree is the power of the highest-order derivative after the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The printed equation is

$$1 + \left(\frac{dy}{dx}\right)^3 = \lambda \left(\frac{d^3y}{dx^3}\right)^2.$$

Its highest derivative is d^3y/dx^3 . Therefore its order is 3; the exponent of the first derivative does not determine the order.

Step 2: Identify the degree and multiply The equation already contains polynomial powers of derivatives, with no derivative inside a radical or transcendental function. The highest-order derivative is squared, so the degree is 2. Their product is $3 \times 2 = 6$. Options (C) and (D) give the degree and order separately; option (A) gives their sum instead of their product.

Step 3: Conclusion Order $\times$ degree = 6. Hence, the correct option is (B).

Quick Tip to Solve: Read the derivative order first, then its exponent: third derivative squared means 3 times 2.

13. Which of the following is not a Linear Differential Equation? [1 mark]

- (A) $(1 + x^2) dy + 2xy dx = \cot x dx$
- (B) $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$
- (C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$
- (D) $y dx - (x + 3y^2) dy = 0$

Correct Answer: (C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$

Solution:

Concept:

- A first-order linear equation in $y(x)$ has form $dy/dx + P(x)y = Q(x)$.
- An equation may instead be linear in $x(y)$, with form $dx/dy + P(y)x = Q(y)$.

Step 1: Check options (A) and (B) Option (A) becomes $dy/dx + [2x/(1+x^2)]y = \cot x/(1+x^2)$, which is

Quick Tip to Solve: Before rejecting an equation containing y squared, try treating x as the dependent variable.

linear in y . In (B), the product rule gives $d(xy)/dx = y + x dy/dx$. Thus $x dy/dx + 2y = x(\sin x + \log x)$, also linear in y on its domain.

Step 2: Check options (C) and (D) Option (C) gives $dy/dx = x(1 + y^2)/[y(1 + x^2)]$, which involves both y and $1/y$. Reversing the variables gives $dx/dy = y(1 + x^2)/[x(1 + y^2)]$, involving x and $1/x$; neither is linear. In (D), using x as dependent variable gives $dx/dy - x/y = 3y$, which is linear in x . The appearance of y^2 in (D) therefore does not make it the required choice.

Step 3: Conclusion Option (C) is not a linear differential equation.

Quick Tip to Solve: Before rejecting an equation containing y squared, try treating x as the dependent variable.

14. The region represented by the system of inequations $3x + y \geq 3$, $2x - y \geq -5$, $x, y \geq 0$ is: [1 mark]

- (A) unbounded in 1st quadrant
- (B) bounded in 1st quadrant
- (C) unbounded in 2nd quadrant
- (D) bounded in 2nd quadrant

Correct Answer: (A) unbounded in 1st quadrant

Solution:

Concept:

- The restrictions $x \geq 0$, $y \geq 0$ confine the feasible points to the first quadrant, including its axes.
- A feasible region is unbounded if it contains feasible points arbitrarily far from the origin.

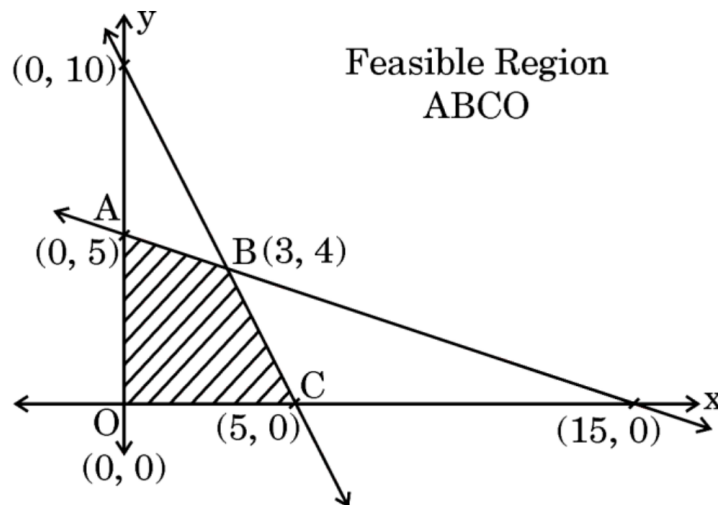
Step 1: Rewrite the half-plane conditions The inequalities become $y \geq 3 - 3x$ and $y \leq 2x + 5$, together with $x, y \geq 0$. These describe the common part of the corresponding half-planes in the first quadrant.

Step 2: Exhibit an unlimited feasible ray On the positive x -axis, take $y = 0$. For every $x \geq 1$, $3x + 0 \geq 3$ and $2x - 0 \geq -5$, and both coordinates are non-negative. Thus the entire ray $(x, 0)$, $x \geq 1$, is feasible and extends indefinitely. This rules out boundedness, and the coordinate restrictions rule out the second-quadrant options (C) and (D).

Step 3: Conclusion The region is unbounded in the first quadrant. Hence, the correct option is (A).

Quick Tip to Solve: Test $y=0$: if all sufficiently large positive x satisfy every inequality, the region is unbounded.

15. In the graph, the feasible region representing the Linear Programming Problem for maximising objective function $Z = px + qy$, $p, q > 0$ is shaded. If all points on segment AB give $\max(Z)$, then which of the following is true? [1 mark]



- (A) $p = 2q$
- (B) $p = 3q$
- (C) $q = 3p$
- (D) $q = 2p$

Correct Answer: (C) $q = 3p$

Solution:

Concept:

- If a linear objective is constant on a segment, its values at both endpoints are equal.
- The official graph gives $A = (0, 5)$ and $B = (3, 4)$; use these labelled coordinates.

Step 1: Evaluate at the two endpoints For $Z = px + qy$, the endpoint values are

$$Z(A) = 5q, \quad Z(B) = 3p + 4q.$$

Since all points of AB give the same maximum, both endpoints must give equal objective values.

Step 2: Find the coefficient relation and check Equating gives $5q = 3p + 4q$, hence $q = 3p$. Equivalently the slope of AB is $(4 - 5)/(3 - 0) = -1/3$; the objective lines have slope $-p/q$, so parallelism requires $p/q = 1/3$. Options (A), (B) and (D) give different coefficient ratios and therefore unequal endpoint values.

Step 3: Conclusion $q = 3p$. Hence, the correct option is (C).

Quick Tip to Solve: Equate Z at $A(0,5)$ and $B(3,4)$; endpoint equality avoids guessing the line slope.

16. If $(3\hat{i} - 2\hat{j} + 5\hat{k}) \times (4\hat{i} + p\hat{j} + q\hat{k}) = \vec{0}$, then the values of p and q are:

[1 mark]

- (A) $p = -\frac{2}{3}, q = \frac{5}{3}$
(B) $p = -\frac{8}{3}, q = \frac{20}{3}$
(C) $p = \frac{20}{3}, q = -\frac{8}{3}$
(D) $p = 0, q = 0$

Correct Answer: (B) $p = -\frac{8}{3}, q = \frac{20}{3}$

Solution:

Concept:

- Two nonzero vectors have zero cross product exactly when they are parallel.
- Parallel vectors are scalar multiples, so corresponding components have the same ratio.

Step 1: Determine the proportionality factor Let $(4, p, q) = t(3, -2, 5)$. Comparing the first components gives $4 = 3t$, so $t = 4/3$. Both vectors are nonzero because their first components are nonzero.

Step 2: Calculate and verify the remaining components Thus $p = -2t = -8/3$ and $q = 5t = 20/3$. A cross-product check gives the components $-2q - 5p$, $20 - 3q$, and $3p + 8$, all zero for these values. Option (A) incorrectly uses factor $1/3$; (C) interchanges the components and their signs; (D) leaves $3p + 8 = 8 \neq 0$.

Step 3: Conclusion $p = -8/3, q = 20/3$. Hence, the correct option is (B).

Quick Tip to Solve: Use the known i-components to get the factor $4/3$, then multiply both other components by it.

17. Three points $A(0, 1, 1)$, $B(2, 0, -1)$ and $C(1, 0, 3)$ form $\triangle ABC$. The $\text{ar}(\triangle ABC)$ is:

[1 mark]

- (A) $\frac{\sqrt{53}}{2}$ sq. units
(B) $\sqrt{53}$ sq. units
(C) $\frac{\sqrt{11}}{2}$ sq. units
(D) $\sqrt{11}$ sq. units

Correct Answer: (A) $\frac{\sqrt{53}}{2}$ sq. units

Solution:

Concept:

- The area of a triangle with adjacent vectors $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}|\vec{u} \times \vec{v}|$.
- The cross-product magnitude gives the parallelogram area; the triangle occupies half of it.

Step 1: Form adjacent side vectors Using the same initial point A , obtain $\vec{AB} = (2, -1, -2)$ and $\vec{AC} = (1, -1, 2)$. Subtract coordinates component by component; using position vectors directly

Quick Tip to Solve: First subtract the coordinates of the same vertex from both other vertices, then halve the cross-product magnitude.

would give the wrong area.

Step 2: Calculate the cross product and compare options

$$\vec{AB} \times \vec{AC} = (-4, -6, -1), \quad |\vec{AB} \times \vec{AC}| = \sqrt{16 + 36 + 1} = \sqrt{53}.$$

Thus the triangle area is half this magnitude. Option (B) omits this factor, while (C) and (D) use an incorrect squared magnitude 11.

Step 3: Conclusion $\text{ar}(\triangle ABC) = \frac{\sqrt{53}}{2}$ sq. units. Hence, the correct option is (A).

Quick Tip to Solve: First subtract the coordinates of the same vertex from both other vertices, then halve the cross-product magnitude.

18. A box contains 4 red, 5 blue and 1 green marble. A child randomly takes out a marble from the box, notes down the colour and puts it back in the box. If the activity is repeated 3 times, what is the probability that at least one marble is red? [1 mark]

- (A) $\frac{27}{125}$
(B) $\frac{8}{125}$
(C) $\frac{2}{125}$
(D) $\frac{98}{125}$

Correct Answer: (D) $\frac{98}{125}$

Solution:

Concept:

- Replacement makes the three draws independent with unchanged colour probabilities.
- For an event E , $P(E) = 1 - P(E^c)$; the complement of at least one red is no red.

Step 1: Find the probability of a non-red draw There are $4 + 5 + 1 = 10$ marbles, of which $5 + 1 = 6$ are non-red. Hence $P(\text{non-red}) = 6/10 = 3/5$, and $P(\text{red}) = 2/5$. Returning each marble restores the original composition.

Step 2: Use the complement for three independent draws

$$P(\text{no red in three draws}) = \left(\frac{3}{5}\right)^3 = \frac{27}{125}.$$

Therefore $P(\text{at least one red}) = 1 - 27/125 = 98/125$. Option (A) is the complement and (B) is the probability that all three are red; (C) is neither the event nor its complement.

Step 3: Conclusion The required probability is $\frac{98}{125}$. Hence, the correct option is (D).

Quick Tip to Solve: For at least one success, calculate the probability of zero successes and subtract it from one.

19. Assertion (A): If A and B are two square matrices such that AB and BA are defined, then it is not necessary that $AB = BA$.

Reason (R): Product of two diagonal matrices of same order is commutative.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- Matrix multiplication is generally noncommutative: the order of the factors matters.
- Diagonal matrices of the same order multiply entrywise along their diagonals.

Step 1: Judge the assertion Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $AB = B$ but $BA = 0$, so $AB \neq BA$ even though both products exist. Thus the assertion is true.

Step 2: Judge the reason For diagonal matrices $D = \text{diag}(d_1, \dots, d_n)$ and $E = \text{diag}(e_1, \dots, e_n)$, $DE = \text{diag}(d_1e_1, \dots, d_ne_n) = ED$, since scalar multiplication commutes. Thus the reason is true.

Step 3: Check the explanatory connection The reason identifies a special class in which multiplication does commute. It does not explain why arbitrary square matrices need not commute, which the counterexample establishes. Thus (A) wrongly asserts an explanation, while (C) and (D) wrongly deny one of the true statements.

Step 4: Conclusion Both statements are true, but the reason does not explain the assertion. Hence, the correct option is (B).

Quick Tip to Solve: Use a small counterexample for the general assertion, then test whether the special case actually explains it.

20. Assertion (A): A function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3 + 2, \forall x \in \mathbb{N}$ is one-one but not onto.

Reason (R): Since $\forall y \in \mathbb{N}$ (Codomain), there does not exist $x = (y - 2)^{1/3}$ in $\mathbb{N}$ (Domain) such that $f(x) = x^3 + 2 = y$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A function is one-one when equal outputs force equal inputs.
- Onto means every codomain element has a preimage; one missing preimage disproves it.

Step 1: Judge the assertion If $f(a) = f(b)$, then $a^3 + 2 = b^3 + 2$, so $a^3 = b^3$ and $a = b$. Thus f is one-one. The natural number 1 is not an output, since $x^3 + 2 \geq 3$ for positive natural x . Therefore f is not onto, and the assertion is true.

Step 2: Judge the reason as printed The printed reason claims that for every natural y there is no natural preimage. This is false: choose $y = 3$, for which $x = (3 - 2)^{1/3} = 1$ is natural and $f(1) = 3$. Failure of surjectivity requires some codomain elements to lack preimages, not all of them.

Step 3: Select the assertion-reason relation Since the assertion is true and the reason is false, the reason cannot explain the assertion. Options (A) and (B) incorrectly call the reason true, and option (D) incorrectly calls the assertion false.

Step 4: Conclusion The assertion is true, but the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Check universal wording with a single actual output: $y=3$ immediately disproves the printed reason.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Evaluate: $\tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) + \cot^{-1} \left(\frac{1}{\sqrt{3}} \right) + \tan^{-1} \left(\sin \left(-\frac{\pi}{2} \right) \right) + \tan^{-1} \left(\tan \frac{2\pi}{3} \right)$ [2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$, while that of $\cot^{-1}$ is $(0, \pi)$.
- Inverse trigonometric values must lie in their principal ranges, even when the original angle does not.

Step 1: Evaluate the first two terms Because $\tan(-\pi/6) = -1/\sqrt{3}$ and $-\pi/6$ belongs to the prin-

Quick Tip to Solve: Reduce each tangent angle into $(-\pi/2, \pi/2)$ before applying the inverse tangent.

principal tangent-inverse range, $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Also $\cot(\pi/3) = 1/\sqrt{3}$ and $\pi/3 \in (0, \pi)$, so $\cot^{-1}(1/\sqrt{3}) = \pi/3$.

Step 2: Evaluate the remaining terms in the principal range We have $\sin(-\pi/2) = -1$, giving $\tan^{-1}(-1) = -\pi/4$. Further, $\tan(2\pi/3) = -\sqrt{3} = \tan(-\pi/3)$. Since $2\pi/3$ lies outside $(-\pi/2, \pi/2)$, the fourth term is $-\pi/3$, rather than $2\pi/3$.

Step 3: Combine the four values

$$\begin{aligned} S &= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} - \frac{\pi}{3} \\ &= -\frac{\pi}{6} - \frac{\pi}{4} = -\frac{2\pi + 3\pi}{12} = -\frac{5\pi}{12}. \end{aligned}$$

The equal positive and negative $\pi/3$ terms cancel before taking a common denominator.

Step 4: Conclusion The given expression evaluates to $\boxed{-\frac{5\pi}{12}}$.

Quick Tip to Solve: Reduce each tangent angle into $(-\pi/2, \pi/2)$ before applying the inverse tangent.

22(a). Show that the function $f(x) = \begin{cases} \frac{\cos x}{-x + \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$ **is continuous at** $x = \frac{\pi}{2}$. [2 marks]

Solution:

Concept:

- Continuity at a means $\lim_{x \rightarrow a} f(x) = f(a)$.
- The standard radian limit is $\lim_{h \rightarrow 0} \sin h/h = 1$, valid from both sides.

Step 1: Read the value at the point The special branch gives $f(\pi/2) = 1$. To test continuity, use the other branch only for points approaching $\pi/2$. Direct substitution into that quotient gives $0/0$, an indeterminate form, rather than a value of the limit.

Step 2: Transform the nearby quotient Put $h = x - \pi/2$, so $h \rightarrow 0$ when $x \rightarrow \pi/2$. Using $\cos(\pi/2 + h) = -\sin h$ and $-x + \pi/2 = -h$, obtain

$$f(\pi/2 + h) = \frac{-\sin h}{-h} = \frac{\sin h}{h}, \quad h \neq 0.$$

Both minus signs cancel; retaining only one would incorrectly produce a limit of -1 .

Step 3: Compare the two-sided limit with the function value The standard limit applies for both $h \rightarrow 0^-$ and $h \rightarrow 0^+$. Consequently

$$\lim_{x \rightarrow \pi/2} f(x) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 = f(\pi/2).$$

Thus both one-sided limits exist, agree, and equal the defined value at the point.

Quick Tip to Solve: Shift the variable by $\pi/2$; the quotient becomes the standard $\sin(h)/h$ limit.

Step 4: Conclusion Hence f is continuous at $x = \frac{\pi}{2}$.

Quick Tip to Solve: Shift the variable by $\pi/2$; the quotient becomes the standard $\sin(h)/h$ limit.

OR

22(b). Find whether the function $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$ at $x = 2$ is differentiable or not. [2 marks]

Solution:

Concept:

- Differentiability at a joining point requires equal finite left and right derivatives.
- The derivatives at the point are obtained from $[f(2+h) - f(2)]/h$ as h approaches zero from each side.

Step 1: Find the value and check continuity The branch for $x \geq 2$ gives $f(2) = 2(2) - 3 = 1$. From the left, $\lim_{x \rightarrow 2^-} (x - 1) = 1$; from the right, $\lim_{x \rightarrow 2^+} (2x - 3) = 1$. Thus the function is continuous at the join. Continuity alone, however, does not guarantee differentiability.

Step 2: Compute the left derivative For $h < 0$, the point $2+h$ lies in the first branch, so $f(2+h) = 1+h$. Hence

$$f'_-(2) = \lim_{h \rightarrow 0^-} \frac{(1+h) - 1}{h} = 1.$$

This uses the function value at the join, not a separately chosen branch value.

Step 3: Compute the right derivative and compare For $h > 0$, $f(2+h) = 2(2+h) - 3 = 1 + 2h$. Hence

$$f'_+(2) = \lim_{h \rightarrow 0^+} \frac{(1+2h) - 1}{h} = 2.$$

The two slopes are finite but unequal, so a unique tangent slope does not exist at the joining point.

Step 4: Conclusion Since $f'_-(2) = 1 \neq 2 = f'_+(2)$, the function is not differentiable at $x = 2$.

Quick Tip to Solve: At a continuous piecewise linear join, compare the two branch slopes to test differentiability.

23. Find the values of x for which $f(x) = x^x$, $x > 0$ is increasing. [2 marks]

Solution:

Concept:

- For a differentiable function, a positive derivative on an interval implies that the function is strictly increasing there.
- For a variable base and exponent, logarithmic differentiation is useful: if $y = x^x$, then $\log y = x \log x$, for $x > 0$.

Quick Tip to Solve: When differentiating x^x , take logarithms first; its derivative is not the fixed-power formula $x x^{x-1}$.

Step 1: Differentiate logarithmically Put $y = x^x$. Since $x > 0$, both $\log x$ and $\log y$ are defined. Taking natural logarithms and differentiating with respect to x , we get

$$\log y = x \log x, \quad \frac{1}{y} \frac{dy}{dx} = \log x + 1.$$

Thus

$$f'(x) = x^x (\log x + 1).$$

The factor x^x is positive throughout the given domain, so it does not affect the sign of the derivative.

Step 2: Determine the sign of the derivative The required condition is

$$f'(x) > 0 \iff \log x + 1 > 0 \iff \log x > -1 \iff x > e^{-1}.$$

For $0 < x < e^{-1}$, the derivative is negative, while at $x = e^{-1}$ it is zero. Thus the function decreases before this stationary point and increases after it. The zero derivative at the single endpoint does not prevent strict increase on an interval that includes that endpoint.

Step 3: Conclusion $f(x) = x^x$ is increasing for $x > \frac{1}{e}$. Its maximal interval of strict increase can also be written as $[1/e, \infty)$.

Quick Tip to Solve: When differentiating x^x , take logarithms first; its derivative is not the fixed-power formula $x x^{x-1}$.

24. If the position vectors of three points A , B and C are $3\hat{i} + \hat{j}$, $5\hat{i} + 6\hat{j} - 3\hat{k}$ and $4\hat{j}$ respectively, then show that they form an isosceles triangle. [2 marks]

Solution:

Concept:

- The vector from a point with position vector $\vec{a}$ to a point with position vector $\vec{b}$ is $\vec{b} - \vec{a}$.
- The length of $u\hat{i} + v\hat{j} + w\hat{k}$ is $\sqrt{u^2 + v^2 + w^2}$. A triangle is isosceles if two of its side lengths are equal.

Step 1: Find the side vectors The given points have coordinates $A = (3, 1, 0)$, $B = (5, 6, -3)$ and $C = (0, 4, 0)$. Subtracting position vectors in a consistent order gives

$$\begin{aligned}\overrightarrow{AB} &= 2\hat{i} + 5\hat{j} - 3\hat{k}, \\ \overrightarrow{AC} &= -3\hat{i} + 3\hat{j}, \\ \overrightarrow{BC} &= -5\hat{i} - 2\hat{j} + 3\hat{k}.\end{aligned}$$

These are displacement vectors, so their magnitudes give the side lengths.

Quick Tip to Solve: Compare squared side lengths first; equal squared lengths prove equality without simplifying square roots.

Step 2: Compare the side lengths Using the sum of the squares of the components,

$$|AB|^2 = 2^2 + 5^2 + (-3)^2 = 38,$$

$$|AC|^2 = (-3)^2 + 3^2 = 18,$$

$$|BC|^2 = (-5)^2 + (-2)^2 + 3^2 = 38.$$

Hence $|AB| = |BC| = \sqrt{38}$, whereas $|AC| = 3\sqrt{2}$. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel: the former has a nonzero $\hat{k}$ -component and the latter has zero $\hat{k}$ -component. Therefore the three points do form a nondegenerate triangle.

Step 3: Conclusion $AB = BC = \sqrt{38}$; therefore $\triangle ABC$ is isosceles, with equal sides meeting at B and base AC .

Quick Tip to Solve: Compare squared side lengths first; equal squared lengths prove equality without simplifying square roots.

25(a). Let two rods placed on the ground be represented by vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector representing a flag-post of height 5 m that has to be erected perpendicular to both the rods. [2 marks]

Solution:

Concept:

- The cross product of two nonparallel vectors is perpendicular to both. If $\vec{n} \neq \vec{0}$, then $\vec{n}/|\vec{n}|$ is a unit vector in its direction.
- A vector of magnitude h perpendicular to both $\vec{u}$ and $\vec{v}$ is $\pm h(\vec{u} \times \vec{v})/|\vec{u} \times \vec{v}|$.

Step 1: Find a common perpendicular Let $\vec{u} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{v} = -2\hat{i} + \hat{j} - 2\hat{k}$. Their cross product is

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} \\ &= -\hat{i} + 2\hat{j} + 2\hat{k}.\end{aligned}$$

Its magnitude is $\sqrt{1 + 4 + 4} = 3$, so it is nonzero and supplies a direction normal to the plane of the rods.

Step 2: Scale to the required height and verify A vector in this direction with length 5 m is

$$\vec{p} = \frac{5}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) \text{ m}.$$

Indeed, $|\vec{p}| = (5/3) \times 3 = 5$ m. Perpendicularity can be checked directly using the unscaled normal:

$$(-1, 2, 2) \cdot (4, -1, 3) = -4 - 2 + 6 = 0,$$

$$(-1, 2, 2) \cdot (-2, 1, -2) = 2 + 2 - 4 = 0.$$

Quick Tip to Solve: Normalize the cross product before multiplying by the height; the cross product alone usually has the wrong magnitude.

Multiplying by $5/3$ preserves both zero dot products. The negative of this vector represents the opposite normal direction; the upward choice depends on the coordinate orientation.

Step 3: Conclusion One required vector is $\vec{p} = \frac{5}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) \text{ m}$. Its negative also has the required magnitude and perpendicularity.

Quick Tip to Solve: Normalize the cross product before multiplying by the height; the cross product alone usually has the wrong magnitude.

OR

25(b). A unit vector $\vec{a}$ is such that it makes an angle $\frac{\pi}{4}$ with x -axis, $\frac{\pi}{3}$ with y -axis and an acute angle θ with z -axis. Find θ and the components of $\vec{a}$. [2 marks]

Solution:

Concept:

- The direction cosines l, m, n of a vector satisfy $l^2 + m^2 + n^2 = 1$.
- For a unit vector, its Cartesian components equal its direction cosines: $\vec{a} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k}$.

Step 1: Use the direction-cosine identity The given angles with the positive coordinate axes are $\alpha = \pi/4$, $\beta = \pi/3$ and $\gamma = \theta$. Therefore

$$\cos^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{3} + \cos^2 \theta = 1.$$

Substituting the known values,

$$\frac{1}{2} + \frac{1}{4} + \cos^2 \theta = 1 \implies \cos^2 \theta = \frac{1}{4}.$$

This gives two algebraic possibilities, $\cos \theta = \pm 1/2$, before the angle restriction is used.

Step 2: Select the acute angle and write the vector Since θ is acute, $0 < \theta < \pi/2$ and consequently $\cos \theta > 0$. We must take $\cos \theta = 1/2$, which gives $\theta = \pi/3$. As $\vec{a}$ is a unit vector, its components are

$$\left(\cos \frac{\pi}{4}, \cos \frac{\pi}{3}, \cos \frac{\pi}{3} \right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right).$$

Their squared sum is $1/2 + 1/4 + 1/4 = 1$, confirming the unit magnitude and consistency with all three given angles.

Step 3: Conclusion $\theta = \frac{\pi}{3}$, and $\vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$.

Quick Tip to Solve: The acute-angle condition fixes the positive square root for the missing direction cosine; do not keep both signs.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26(a). Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. A function $f : A \rightarrow B$ is defined by $f(x) = \frac{x-2}{x-3}$. Find whether f is one-one and onto. [3 marks]

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all domain elements.
- A function $f : A \rightarrow B$ is onto if every $y \in B$ has a preimage $x \in A$. Solving $y = f(x)$ constructs that preimage.

Step 1: Check that the function takes values in the stated codomain For every $x \in A = \mathbb{R} \setminus \{3\}$, the denominator $x - 3$ is nonzero, so $f(x)$ is defined. Moreover,

$$f(x) = \frac{x-2}{x-3} = 1 + \frac{1}{x-3}.$$

Since $1/(x-3) \neq 0$, the value $f(x)$ cannot be 1. Thus each image belongs to $B = \mathbb{R} \setminus \{1\}$, as required.

Step 2: Prove that the function is one-one Take $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Both denominators are nonzero, so cross multiplication is legitimate:

$$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \implies (x_1-2)(x_2-3) = (x_2-2)(x_1-3).$$

Expanding and cancelling the common terms gives

$$-3x_1 - 2x_2 = -3x_2 - 2x_1 \implies x_1 = x_2.$$

Thus different domain elements cannot have the same image.

Step 3: Prove that every codomain element has a preimage Let $y \in B$ be arbitrary, so $y \neq 1$. Solving $y = (x-2)/(x-3)$, we obtain

$$yx - 3y = x - 2 \implies x = \frac{3y-2}{y-1}.$$

This real number belongs to A , because

$$x-3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Also $x-2 = y/(y-1)$, so substituting this preimage gives

$$f(x) = \frac{y/(y-1)}{1/(y-1)} = y.$$

Therefore every element of the entire stated codomain is attained, establishing surjectivity.

Quick Tip to Solve: For a rational-function onto proof, solve for the preimage and explicitly check that it avoids the excluded domain value.

Step 4: Conclusion f is both one-one and onto; hence it is a bijection from $\mathbb{R} \setminus \{3\}$ to $\mathbb{R} \setminus \{1\}$.

Quick Tip to Solve: For a rational-function onto proof, solve for the preimage and explicitly check that it avoids the excluded domain value.

OR

26(b). Let n be a fixed positive integer. A relation R is defined in set $\mathbb{Z}$ such that $R = \{(x, y) : (x - y) \text{ is divisible by } n, x, y \in \mathbb{Z}\}$. Determine if R is an equivalence relation. [3 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive. Each property must hold for all elements concerned, not merely for selected examples.
- For the fixed positive integer n , divisibility means $n \mid (x - y)$ exactly when $x - y = kn$ for some integer k . Integer multiples of n are closed under negation and addition.

Step 1: Check reflexivity Let $x \in \mathbb{Z}$ be arbitrary. Then $x - x = 0 = 0 \cdot n$. Since zero is divisible by every positive integer, $n \mid (x - x)$, so $(x, x) \in R$. This holds for every integer x ; hence R is reflexive.

Step 2: Check symmetry Suppose $(x, y) \in R$. By the definition of R , there is an integer k such that $x - y = kn$. Reversing the order gives

$$y - x = -(x - y) = (-k)n.$$

As $-k$ is also an integer, $n \mid (y - x)$, and therefore $(y, x) \in R$. Thus R is symmetric.

Step 3: Check transitivity Suppose $(x, y) \in R$ and $(y, z) \in R$, with $x, y, z \in \mathbb{Z}$. There exist integers k, l satisfying $x - y = kn$ and $y - z = ln$. Adding these equations eliminates y :

$$x - z = (x - y) + (y - z) = (k + l)n.$$

Since $k + l \in \mathbb{Z}$, $n \mid (x - z)$. Therefore $(x, z) \in R$, establishing transitivity for arbitrary related pairs.

Step 4: Conclusion R is reflexive, symmetric and transitive on $\mathbb{Z}$. Hence R is an equivalence relation.

Quick Tip to Solve: Express each difference as an integer multiple of n ; negation proves symmetry and addition proves transitivity.

27. If $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, then compute $A^2 - 7A + 10I$.

[3 marks]

Solution:

Concept:

- The square $A^2 = A \cdot A$ is obtained by multiplying each row of the first matrix by each column of the second; it is not the matrix of squared entries.
- In $A^2 - 7A + 10I$, I denotes the identity matrix of order three. Thus $10I$ contributes 10 to each diagonal entry and zero to every off-diagonal entry.

Step 1: Multiply the matrix by itself Using the given matrix, row-column multiplication gives

$$A^2 = \begin{pmatrix} 3 \cdot 3 + 2 \cdot 1 & 3 \cdot 2 + 2 \cdot 4 & 0 \\ 1 \cdot 3 + 4 \cdot 1 & 1 \cdot 2 + 4 \cdot 4 & 0 \\ 0 & 0 & 5 \cdot 5 \end{pmatrix} = \begin{pmatrix} 11 & 14 & 0 \\ 7 & 18 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

The zeros in the third row and third column separate the upper two-by-two block from the final diagonal entry. In particular, the off-diagonal entries 14 and 7 arise from two products each.

Step 2: Subtract and add entry by entry The remaining terms are

$$7A = \begin{pmatrix} 21 & 14 & 0 \\ 7 & 28 & 0 \\ 0 & 0 & 35 \end{pmatrix}, \quad 10I = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

Hence

$$A^2 - 7A + 10I = \begin{pmatrix} 11 - 21 + 10 & 14 - 14 & 0 \\ 7 - 7 & 18 - 28 + 10 & 0 \\ 0 & 0 & 25 - 35 + 10 \end{pmatrix}.$$

Every off-diagonal entry cancels. On the diagonal the three arithmetic checks are $11 - 21 + 10 = 0$, $18 - 28 + 10 = 0$, and $25 - 35 + 10 = 0$. This verifies the expression directly without assuming a matrix polynomial identity.

Step 3: Conclusion

$$A^2 - 7A + 10I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Quick Tip to Solve: Compute the upper two-by-two block first; the isolated entry 5 gives $25 - 35 + 10$ immediately.

28(a). If $xy = e^{x-y}$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- For an implicit equation in x and y , regard y as a differentiable function of x . The product rule gives $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$.
- The chain rule gives $\frac{d}{dx}e^{x-y} = e^{x-y}\left(1 - \frac{dy}{dx}\right)$. The derivative of the exponent includes the derivative of y .

Step 1: Differentiate both sides The printed equation is $xy = e^{x-y}$. Write $y' = dy/dx$. Differentiate the product on the left and the exponential on the right:

$$y + xy' = e^{x-y}(1 - y').$$

Expanding the right-hand side gives

$$y + xy' = e^{x-y} - e^{x-y}y'.$$

Both occurrences of y' must be collected before dividing, because y appears on both sides of the original equation.

Step 2: Collect the derivative and simplify Move the term containing y' on the right to the left and move y to the right:

$$(x + e^{x-y})y' = e^{x-y} - y.$$

Therefore, at a point where the coefficient is nonzero,

$$y' = \frac{e^{x-y} - y}{x + e^{x-y}}.$$

Substitute the original relation $e^{x-y} = xy$ to obtain

$$y' = \frac{xy - y}{x + xy} = \frac{y(x - 1)}{x(1 + y)}.$$

Since $e^{x-y} > 0$, the relation forces $xy > 0$, so $x \neq 0$. The formula describes a differentiable branch wherever $x(1 + y) \neq 0$; a zero coefficient cannot be divided out.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{e^{x-y} - y}{x + e^{x-y}} = \frac{y(x - 1)}{x(1 + y)}}$$

wherever the denominator is nonzero.

Quick Tip to Solve: Differentiate xy as a product, then replace the exponential by xy only after collecting all derivative terms.

28(b). Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}x^2$.

[3 marks]

Solution:

Concept:

- To differentiate a function u with respect to another function v , first express u directly in terms of v , if possible. A trigonometric substitution can avoid lengthy quotient differentiation.
- The principal value $v = \cos^{-1}(x^2)$ lies in $[0, \pi/2]$ for real $|x| \leq 1$. On this interval the half-angle square roots have nonnegative signs.

Step 1: Use the requested variable as an angle Let $v = \cos^{-1}(x^2)$, so that $x^2 = \cos v$ and $0 \leq v \leq \pi/2$. The half-angle identities give

$$\sqrt{1+x^2} = \sqrt{1+\cos v} = \sqrt{2} \cos \frac{v}{2}, \quad \sqrt{1-x^2} = \sqrt{1-\cos v} = \sqrt{2} \sin \frac{v}{2}.$$

Both expressions are nonnegative on this angle range. Their sum is positive, so the fraction appearing in the question is well-defined throughout the real domain.

Step 2: Simplify the inverse tangent If u denotes the given inverse-tangent expression, its argument becomes

$$\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)} = \tan\left(\frac{\pi}{4} - \frac{v}{2}\right).$$

This follows from the tangent subtraction formula after dividing numerator and denominator by $\cos(v/2)$, which is positive here. Moreover, $0 \leq \pi/4 - v/2 \leq \pi/4$, entirely within the principal range of $\tan^{-1}$. Consequently

$$u = \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{v}{2}\right)\right] = \frac{\pi}{4} - \frac{v}{2}.$$

Thus the outer inverse tangent introduces no branch adjustment.

Step 3: Differentiate in the new variable The expression is now a linear function of the variable with respect to which differentiation is requested. Its constant term $\pi/4$ differentiates to zero, and the coefficient of v is $-1/2$. This direct method also avoids a ratio of two derivatives that would have removable difficulties at particular values of x .

Step 4: Conclusion

$$\frac{d}{d(\cos^{-1}x^2)} \tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right) = -\frac{1}{2}.$$

Quick Tip to Solve: Set the inverse-cosine variable equal to an angle and use half-angle identities; check the inverse-tangent range before cancelling.

29. Solve the following Linear Programming Problem graphically:

Maximise $Z = 200x + 120y$

subject to the constraints

$$x + y \leq 300$$

$$3x + y \leq 600$$

$$x - y \geq -100$$

$$x, y \geq 0$$

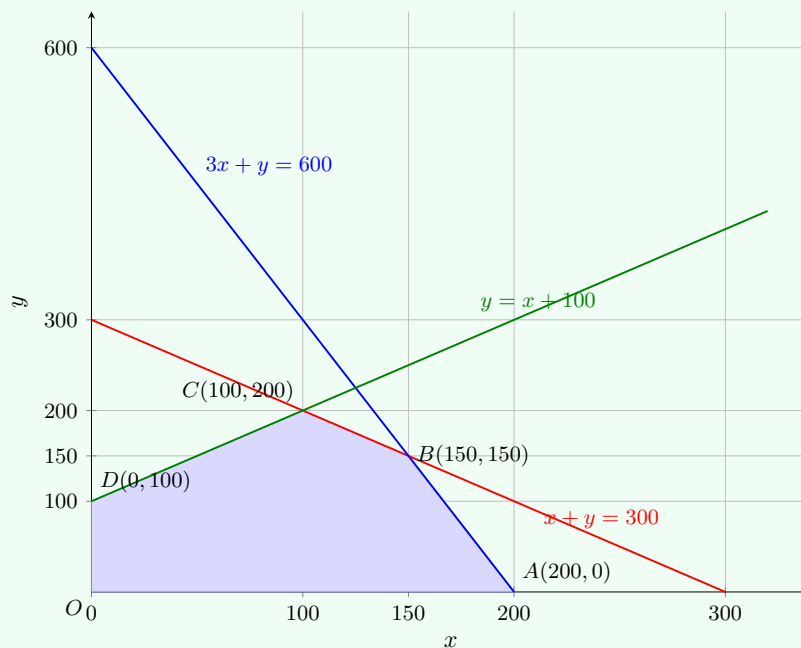
[3 marks]

Solution:

Concept:

- Each inequality defines a half-plane. The feasible region is their intersection with the first quadrant $x, y \geq 0$.
- A linear objective $Z = 200x + 120y$, where x, y are decision variables, attains its maximum over a bounded feasible polygon at a vertex.

Step 1: Plot the boundaries and shade the common region The line $x + y = 300$ has intercepts $(300, 0), (0, 300)$; retain the side containing the origin. The line $3x + y = 600$ has intercepts $(200, 0), (0, 600)$; retain the side containing the origin. Rewrite the third constraint as $y \leq x + 100$, and shade below the line through $(0, 100), (100, 200)$. The intersection of all three regions in the first quadrant is the shaded polygon $OABCD$.



Boundary lines and shaded feasible polygon $OABCD$

Step 2: Find feasible vertices The axis vertices are $O = (0, 0)$, $A = (200, 0)$, $D = (0, 100)$. Subtracting $x + y = 300$ from $3x + y = 600$ gives $2x = 300$, so their intersection is $B = (150, 150)$. Combining $x + y = 300$ with $y = x + 100$ gives $2x = 200$, hence $C = (100, 200)$. The remaining pair intersects at $(125, 225)$, which violates $x + y \leq 300$; therefore it is not feasible.

Step 3: Compare the objective values At every feasible vertex, substitute the coordinates into the objective. At $O = (0, 0)$, $Z = 0$. At $A = (200, 0)$, $Z = 200(200) + 120(0) = 40000$. At $B = (150, 150)$,

Quick Tip to Solve: Check every intersection against all constraints before putting it in the objective table.

$Z = 200(150) + 120(150) = 48000$. At $C = (100, 200)$, $Z = 200(100) + 120(200) = 44000$. At $D = (0, 100)$, $Z = 12000$. The common region is closed and bounded, and these are all its vertices. Thus comparison of these five values determines the maximum over the entire shaded polygon.

Step 4: Conclusion The maximum is $Z = 48000$ at $x = 150, y = 150$.

Quick Tip to Solve: Check every intersection against all constraints before putting it in the objective table.

30(a). Find a point on the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2}$ at a distance of $\sqrt{2}$ units from the point $(1, 2, 3)$. [3 marks]

Solution:

Concept:

- A point on a line in symmetric form is obtained by setting each ratio equal to a common real parameter t .
- The distance between $P(x, y, z)$ and $Q(a, b, c)$ obeys $PQ^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$. Squaring a specified positive distance produces a quadratic in the parameter.

Step 1: Write a general point on the printed line The printed middle ratio $(1-y)/2$ is equivalent to $(y-1)/(-2)$. Set

$$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2} = t.$$

Then $x = 2 + 3t, y = 1 - 2t, z = 3 + 2t$. Thus every point on the line is $P = (2 + 3t, 1 - 2t, 3 + 2t)$; the three coordinates must use the same parameter.

Step 2: Impose the required distance Let $Q = (1, 2, 3)$. Since $PQ = \sqrt{2}$,

$$\begin{aligned} 2 &= (1 + 3t)^2 + (-1 - 2t)^2 + (2t)^2 \\ &= (1 + 6t + 9t^2) + (1 + 4t + 4t^2) + 4t^2 \\ &= 2 + 10t + 17t^2. \end{aligned}$$

Hence $17t^2 + 10t = t(17t + 10) = 0$, giving $t = 0$ or $t = -10/17$. Both roots are real and give the stated positive distance.

Step 3: Substitute and verify the points For $t = 0$, the point is $(2, 1, 3)$, whose squared distance from Q is $1^2 + (-1)^2 = 2$. For $t = -10/17$, the point is $(4/17, 37/17, 31/17)$. Its displacement from Q is $(-13/17, 3/17, -20/17)$, whose squared length is $(169 + 9 + 400)/289 = 2$. Each point also satisfies the line because it comes from its parameterisation.

Step 4: Conclusion One required point is $(2, 1, 3)$. The other possible point is $(4/17, 37/17, 31/17)$.

Quick Tip to Solve: For a point at a fixed distance, solve the parameter quadratic and check both roots.

30(b). Find the shortest distance between the lines $\vec{r} = (4 + \lambda)\hat{i} + (2\lambda - 1)\hat{j} - 3\hat{k}$ and $\vec{r} = (1 + 2\mu)\hat{i} + (2 - 5\mu)\hat{k} + (4\mu - 1)\hat{j}$. [3 marks]

Solution:

Concept:

- For nonparallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the shortest distance is $d = |(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)| / |\vec{b}_1 \times \vec{b}_2|$.
- The cross product is perpendicular to both directions, so projecting the displacement between points on the two lines onto this common normal gives the shortest separation. The absolute value makes the distance nonnegative.

Step 1: Read coordinates in the standard order Write all vector coefficients in the order $\hat{i}, \hat{j}, \hat{k}$. The first line has $\vec{a}_1 = (4, -1, -3)$, $\vec{b}_1 = (1, 2, 0)$. The second printed line lists its $\hat{k}$ term before its $\hat{j}$ term; therefore $\vec{a}_2 = (1, -1, 2)$, $\vec{b}_2 = (2, 4, -5)$. The displacement between these chosen points is $\vec{a}_2 - \vec{a}_1 = (-3, 0, 5)$.

Step 2: Calculate a common normal

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 4 & -5 \end{vmatrix} = -10\hat{i} + 5\hat{j} + 0\hat{k}.$$

Its magnitude is $\sqrt{100 + 25} = 5\sqrt{5}$. Since this vector is nonzero, the lines are not parallel and the scalar-triple-product distance formula applies.

Step 3: Project the displacement onto the unit normal The scalar triple product is

$$(-3, 0, 5) \cdot (-10, 5, 0) = 30.$$

Consequently

$$d = \frac{|30|}{5\sqrt{5}} = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}.$$

The nonzero numerator also shows that the two lines are skew rather than intersecting. Dividing by the magnitude of the normal is necessary: the dot product alone depends on the chosen length of that normal.

Step 4: Conclusion The shortest distance is $\boxed{6/\sqrt{5}}$ units, equivalently $\boxed{6\sqrt{5}/5}$ units.

Quick Tip to Solve: Reorder the second line into i, j, k coefficients before taking its cross product.

31. In a school, the probability of holding a debate competition is $\frac{1}{3}$ and that of a quiz competition is $\frac{2}{3}$. In the two participating teams, A has 4 girls and 6 boys and B has 7 girls and 3 boys. If a debate competition is held, the students are selected from team A and for the quiz competition they are selected from team B . If only two students are to be chosen from the teams, then find the probability that one will be a girl and the other a boy. [3 marks]

Solution:

Concept:

- Total probability: for mutually exclusive competitions D, Q , the probability of a mixed pair E is $P(E) = P(D)P(E | D) + P(Q)P(E | Q)$.
- Two distinct students are selected without replacement. There are $\binom{10}{2} = 45$ unordered pairs in either team. The favourable pairs are counted by choosing one girl and one boy.

Step 1: Identify the competition cases Let D denote a debate and Q a quiz. Let E mean that the selected pair has one girl and one boy. From the printed question, $P(D) = 1/3$ and $P(Q) = 2/3$. These probabilities add to one. If D occurs, both students are selected from team A ; if Q occurs, both are selected from team B . The team probabilities must therefore be weighted by the corresponding competition probabilities.

Step 2: Find the conditional probabilities Team A has four girls and six boys. Its favourable pairs number $4 \times 6 = 24$, since each girl can be paired with any boy. Team B has seven girls and three boys, giving $7 \times 3 = 21$ favourable pairs. Hence

$$P(E | D) = \frac{\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} = \frac{24}{45} = \frac{8}{15}, \quad P(E | Q) = \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{21}{45} = \frac{7}{15}.$$

Both denominators are the same because each team contains ten students. Unordered-pair counting includes either possible selection order, so no extra factor of two is inserted.

Step 3: Apply the law of total probability The debate and quiz cases are disjoint, so their contributions can be added:

$$\begin{aligned} P(E) &= \frac{1}{3} \cdot \frac{8}{15} + \frac{2}{3} \cdot \frac{7}{15} \\ &= \frac{8}{45} + \frac{14}{45} = \frac{22}{45}. \end{aligned}$$

As a check, this weighted average lies between the two conditional probabilities $7/15$ and $8/15$, as required.

Step 4: Conclusion The required probability is $\boxed{\frac{22}{45}}$.

Quick Tip to Solve: Count mixed pairs as girls times boys, divide by all unordered pairs, and then weight by the competition probabilities.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Find: $\int \frac{x \, dx}{(x-1)(x^2+4)}$

[5 marks]

Solution:

Concept:

- A rational function whose denominator is a linear factor times an irreducible quadratic can be decomposed as $\frac{A}{x-1} + \frac{Bx+C}{x^2+4}$. The quadratic numerator must allow a term in x and a constant.
- Standard antiderivatives are $\int dx/(x-a) = \ln|x-a|$, $\int f'(x)/f(x) \, dx = \ln|f(x)|$, and $\int dx/(x^2+a^2) = a^{-1} \tan^{-1}(x/a)$, up to additive constants.

Step 1: Set up the partial fractions The numerator has degree one and the denominator degree three, so polynomial division is unnecessary. Since x^2+4 has no real root, use

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Multiplying by the common denominator gives a polynomial identity:

$$\begin{aligned} x &= A(x^2+4) + (Bx+C)(x-1) \\ &= (A+B)x^2 + (C-B)x + (4A-C). \end{aligned}$$

This identity determines constants independent of x , so it can be used throughout each interval where the original integrand is defined.

Step 2: Determine the constants Equating coefficients of x^2 , x , and the constant term yields

$$A+B=0, \quad C-B=1, \quad 4A-C=0.$$

The first and third equations give $B=-A$ and $C=4A$. Substitution into the middle equation gives $4A+A=1$. Thus $A=1/5$, $B=-1/5$, and $C=4/5$. Consequently,

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)}.$$

Step 3: Integrate the three terms Write the requested integral as

$$I = \frac{1}{5} \int \frac{dx}{x-1} - \frac{1}{5} \int \frac{x \, dx}{x^2+4} + \frac{4}{5} \int \frac{dx}{x^2+4}.$$

For the second integral put $u = x^2+4$, so $du = 2x \, dx$, obtaining $\frac{1}{2} \ln(x^2+4)$. The expression inside this logarithm is strictly positive. For the third use the standard inverse-tangent formula with $a=2$, obtaining $\frac{1}{2} \tan^{-1}(x/2)$. The first integral is $\ln|x-1|$; its absolute value is required on both sides of $x=1$. Combining these primitives leaves one arbitrary constant K .

Quick Tip to Solve: After partial fractions, split the quadratic numerator into its derivative term and constant term to obtain a logarithm and an inverse tangent.

Step 4: Verify by differentiation Differentiating the combined primitive gives

$$\frac{1}{5(x-1)} - \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)}.$$

Its common-denominator numerator is

$$\frac{1}{5}\{x^2 + 4 + (-x + 4)(x - 1)\} = \frac{1}{5}(5x) = x,$$

confirming the original integrand. The primitive applies separately on intervals excluding $x = 1$.

Step 5: Conclusion

$$\int \frac{x dx}{(x-1)(x^2+4)} = \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + K.$$

Quick Tip to Solve: After partial fractions, split the quadratic numerator into its derivative term and constant term to obtain a logarithm and an inverse tangent.

OR

32(b). Evaluate: $\int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$

[5 marks]

Solution:

Concept:

- Integration by parts gives $\int u dv = uv - \int v du$. Choose the inverse tangent as u , because its derivative is $1/(1+x^2)$.
- Since $\frac{d}{dx}(1+x^2)^{-1/2} = -x(1+x^2)^{-3/2}$, the remaining factor has an immediate antiderivative.
- A definite integration by parts includes the boundary term $[uv]_0^1$. The auxiliary antiderivative is $x/\sqrt{1+x^2}$.

Step 1: Choose the factors Denote the required integral by I , and take

$$u = \tan^{-1} x, \quad dv = \frac{x}{(1+x^2)^{3/2}} dx, \quad du = \frac{dx}{1+x^2}.$$

To integrate dv , put $t = 1+x^2$, with $dt = 2x dx$. This gives

$$v = \frac{1}{2} \int t^{-3/2} dt = -t^{-1/2} = -\frac{1}{\sqrt{1+x^2}}.$$

The minus sign is crucial: differentiation of this expression recovers the positive factor in the original integrand. Differentiating the inverse tangent will remove the inverse-trigonometric function from the remaining integral.

Step 2: Apply integration by parts and evaluate the boundary term Applying the definite formula

Quick Tip to Solve: Differentiate the inverse tangent in integration by parts and recognize the other factor as minus the derivative of $1/\sqrt{1+x^2}$.

gives

$$I = \left[-\frac{\tan^{-1} x}{\sqrt{1+x^2}} \right]_0^1 + \int_0^1 \frac{dx}{(1+x^2)^{3/2}}.$$

At $x = 1$, the numerator is $\pi/4$ and the denominator is $\sqrt{2}$. At $x = 0$, the inverse tangent vanishes. Thus the boundary contribution equals $-\pi/(4\sqrt{2})$. The remaining integral has a positive sign, because the subtraction of $v du$ cancels the negative sign of v .

Step 3: Evaluate the auxiliary integral Set $x = \tan \theta$, so $dx = \sec^2 \theta d\theta$. The limits become 0 and $\pi/4$. Throughout this interval $\sec \theta$ is positive, so $(1 + \tan^2 \theta)^{3/2} = \sec^3 \theta$. Therefore

$$\begin{aligned} J &= \int_0^1 \frac{dx}{(1+x^2)^{3/2}} \\ &= \int_0^{\pi/4} \cos \theta d\theta = [\sin \theta]_0^{\pi/4} = \frac{1}{\sqrt{2}}. \end{aligned}$$

As an independent check, the product rule gives

$$\frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) = \frac{1+x^2-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}.$$

Its values at the two endpoints are $1/\sqrt{2}$ and zero, confirming J . Insert this value into the integration-by-parts expression.

Step 4: Conclusion Hence

$$I = \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}.$$

Quick Tip to Solve: Differentiate the inverse tangent in integration by parts and recognize the other factor as minus the derivative of $1/\sqrt{1+x^2}$.

33. Using integration, find the area of the region enclosed by the curve $y = |x - 6|$, the x -axis, and between $x = 4$ and $x = 8$.

[5 marks]

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$ and the x -axis over $[a, b]$ is $\int_a^b f(x) dx$.
- The modulus satisfies $|x - 6| = 6 - x$ for $x \leq 6$ and $|x - 6| = x - 6$ for $x \geq 6$. Split the integral where its argument changes sign.
- The two straight-line pieces meet the x -axis at $x = 6$. The vertical boundaries $x = 4$ and $x = 8$ specify the outer limits of the enclosed region.

Step 1: Identify the curve on each part of the interval The argument of the modulus vanishes when $x - 6 = 0$, giving $x = 6$, which lies between the prescribed boundaries. On $[4, 6]$, we have $x - 6 \leq 0$, so the curve is $y = 6 - x$. On $[6, 8]$, we have $x - 6 \geq 0$, so the curve is $y = x - 6$. Both expressions are non-negative on their respective intervals, and meet at $(6, 0)$. The endpoint heights are $y(4) = 2$ and

Quick Tip to Solve: Split the integral at $x = 6$, the zero of the modulus argument, and add the two positive areas; symmetry confirms they are equal.

$y(8) = 2$. Thus the area is the sum of the areas under these two pieces; no signed contribution should be subtracted from the other.

Step 2: Set up the area integral with the correct split Let A denote the required area. Taking a vertical strip of width dx , its height is the curve's ordinate minus the ordinate of the x -axis, namely $|x - 6| - 0$. Therefore

$$A = \int_4^8 |x - 6| dx = \int_4^6 (6 - x) dx + \int_6^8 (x - 6) dx.$$

Splitting at 6 allows the modulus to be replaced by its correct linear expression on each side. Integrating $x - 6$ over the whole interval would instead calculate a signed integral, allowing cancellation and failing to give the enclosed area.

Step 3: Calculate the left-hand area Integrate the first expression and substitute its limits:

$$\begin{aligned} A_1 &= \int_4^6 (6 - x) dx = \left[6x - \frac{x^2}{2} \right]_4^6 \\ &= (36 - 18) - (24 - 8) = 18 - 16 = 2. \end{aligned}$$

This is positive, as expected because the curve lies above the axis on the entire left interval.

Step 4: Calculate the right-hand area and combine The second portion similarly gives

$$\begin{aligned} A_2 &= \int_6^8 (x - 6) dx = \left[\frac{x^2}{2} - 6x \right]_6^8 \\ &= (32 - 48) - (18 - 36) = -16 + 18 = 2. \end{aligned}$$

Both values agree because the outer boundaries are equally distant from $x = 6$ and the heights on the two sides are symmetric. Each portion also has base 2 and height 2, providing the check $\frac{1}{2}(2)(2) = 2$. Adding the integrals, rather than their signed linear extensions, gives $A = A_1 + A_2$.

Step 5: Conclusion The required enclosed area is 4 square units.

Quick Tip to Solve: Split the integral at $x = 6$, the zero of the modulus argument, and add the two positive areas; symmetry confirms they are equal.

34(a). Solve the differential equation $ye^y dx = (y^3 + 2xe^y) dy$, when $y(0) = 1$.

[5 marks]

Solution:

Concept:

- Treat x as a function of y . A linear equation has the form $dx/dy + P(y)x = Q(y)$, where P and Q depend only on y .
- Multiplication by the integrating factor $e^{\int P(y) dy}$ makes the left side a product derivative, allowing direct integration.
- The initial condition $y(0) = 1$ means that the required solution curve passes through $(x, y) = (0, 1)$.

Step 1: Identify the linear equation Starting from $ye^y dx = (y^3 + 2xe^y) dy$, regard y as the independent

Quick Tip to Solve: If the coefficient of dy is linear in x , try taking x as the dependent variable; here this immediately gives the integrating factor y^{-2} .

variable. Dividing by ye^y ,

$$\frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}, \quad \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}.$$

The division is valid near the initial value $y = 1$, since $y \neq 0$ there. The coefficients are $P(y) = -2/y$ and $Q(y) = y^2 e^{-y}$. This is linear in x , although the original equation need not appear linear when written for dy/dx .

Step 2: Calculate the integrating factor We obtain

$$\text{IF} = e^{\int (-2/y) dy} = e^{-2 \log |y|} = \frac{1}{y^2}.$$

Multiplying the differential equation by this factor gives

$$\frac{1}{y^2} \frac{dx}{dy} - \frac{2x}{y^3} = e^{-y} \implies \frac{d}{dy} \left(\frac{x}{y^2} \right) = e^{-y}.$$

The equality on the right follows from the product rule applied to xy^{-2} . The factor cancels the y^2 on the right side, leaving an elementary exponential integral.

Step 3: Integrate and determine the constant Integration with respect to y gives

$$\frac{x}{y^2} = -e^{-y} + C.$$

At $(x, y) = (0, 1)$, we have $0 = -e^{-1} + C$, hence $C = e^{-1}$. Substituting this constant determines the particular curve through the initial point. An implicit relation in x and y is sufficient; there is no need to invert it to an explicit elementary expression for y .

Step 4: Check the obtained relation Differentiate $x = y^2(e^{-1} - e^{-y})$ to obtain

$$\frac{dx}{dy} = 2y(e^{-1} - e^{-y}) + y^2 e^{-y} = \frac{2x}{y} + y^2 e^{-y}.$$

Multiplication by ye^y recovers the given differential equation. At $y = 1$, the relation gives $x = 0$. Moreover, $dx/dy = e^{-1} \neq 0$ at this point, so the curve gives a local function $y(x)$ satisfying the prescribed condition.

Step 5: Conclusion The required particular solution is

$$x = y^2 (e^{-1} - e^{-y}).$$

Quick Tip to Solve: If the coefficient of dy is linear in x , try taking x as the dependent variable; here this immediately gives the integrating factor y^{-2} .

34(b). Find the general solution of the differential equation $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$. [5 marks]

Solution:

Concept:

- For a homogeneous differential equation, the ratio dy/dx is a function of y/x . Substitution $y = vx$ gives $dy/dx = v + x dv/dx$.
- Separating variables reduces the equation to an integral in v and an integral in x . Factorisation and partial fractions simplify the rational integral.
- A substitution or division can exclude straight-line solutions. Check constant values of v separately and include them when the final family permits.

Step 1: Recognise the homogeneous equation From the given equation,

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}.$$

Both the numerator and denominator are homogeneous polynomials of degree three. Dividing them by x^3 expresses the quotient entirely in terms of y/x , wherever this representation is defined. We therefore put $y = vx$ and differentiate using the product rule; v depends on x , so its derivative must be retained. The resulting implicit relation will also describe portions of the solution curves where this coordinate substitution is unavailable.

Step 2: Substitute $y = vx$ and separate variables The original equation gives

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{1 - 3v^2}{v^3 - 3v}.$$

Since $dy/dx = v + x dv/dx$, simplification yields

$$x \frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v}, \quad \frac{v^3 - 3v}{1 - v^4} dv = \frac{dx}{x}.$$

Factor $1 - v^4 = (1 - v^2)(1 + v^2)$ and decompose the separated coefficient:

$$\frac{v^3 - 3v}{1 - v^4} = -\frac{v}{1 - v^2} - \frac{2v}{1 + v^2}.$$

Step 3: Integrate and restore the original variables Hence

$$\frac{1}{2} \log |1 - v^2| - \log(1 + v^2) = \log |x| + C_1.$$

Multiplying by two and rearranging logarithms gives

$$\log \left| \frac{1 - v^2}{x^2(1 + v^2)^2} \right| = C_2.$$

Restore $v = y/x$, absorb the constant and the possible sign, and obtain the relation $x^2 - y^2 =$

Quick Tip to Solve: After substituting $y=vx$, factor $1 - v^4$ into $(1 - v^2)(1 + v^2)$; the separated fraction then integrates using two logarithms.

$C(x^2 + y^2)^2$. The separation excluded $v = \pm 1$; the straight-line solutions $y = \pm x$ satisfy the original equation and are recovered by $C = 0$.

Step 4: Conclusion The general solution is

$$x^2 - y^2 = C(x^2 + y^2)^2,$$

where C is an arbitrary real constant.

Quick Tip to Solve: After substituting $y=vx$, factor $1 - v^4$ into $(1 - v^2)(1 + v^2)$; the separated fraction then integrates using two logarithms.

35. Find the equation of a line (in vector and cartesian form) that passes through the point of intersection of lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$, and is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$. [5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.
- In the symmetric equation $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$, the common parameter gives coordinates $x = x_0 + lt$, $y = y_0 + mt$, $z = z_0 + nt$. An intersection must satisfy all three coordinates of both lines.

Step 1: Write the given lines using independent parameters Let the common ratio for the first line be t . Its coordinates are

$$x = 1 + 2t, \quad y = 2 + 3t, \quad z = 3 + 4t.$$

Let the common ratio for the second line be s . Since the last expression in that line is z , it means $z = s$, with zero initial coordinate and direction coefficient 1. Thus its coordinates are

$$x = 4 + 5s, \quad y = 1 + 2s, \quad z = s.$$

The two parameters are independent: their values need not be equal at a point common to the lines.

Step 2: Determine and check the intersection point Equating the first two coordinates at the intersection gives

$$\begin{aligned} 2t - 5s &= 3, \\ 3t - 2s &= -1. \end{aligned}$$

Multiply the first equation by 3 and the second by 2. Subtracting gives $-11s = 11$, hence $s = -1$. Substitution into $3t - 2s = -1$ yields $3t + 2 = -1$, so $t = -1$. The first line then gives

$$P = (1 - 2, 2 - 3, 3 - 4) = (-1, -1, -1).$$

This also satisfies the third-coordinate condition: $3 + 4t = -1 = s$. For a direct check in the second line, $(x - 4)/5 = -1$, $(y - 1)/2 = -1$, and $z = -1$. Thus all three coordinates agree; solving just the

Quick Tip to Solve: Solve for the two line parameters using two coordinates, then always check the third coordinate before using the resulting point.

first two equations has indeed produced an intersection.

Step 3: Use the specified direction vector The position vector of P is $-\hat{i} - \hat{j} - \hat{k}$. The required parallel direction is $3\hat{i} + 2\hat{j} - 8\hat{k}$, so the parametric coordinates of the required line are

$$x = -1 + 3\lambda, \quad y = -1 + 2\lambda, \quad z = -1 - 8\lambda.$$

At $\lambda = 0$ the line passes through P ; increasing λ changes its coordinates in the ratio $3 : 2 : -8$, as required. Eliminating λ gives the symmetric cartesian form, keeping the negative sign in the third denominator.

Step 4: Conclusion The vector equation is $\vec{r} = -\hat{i} - \hat{j} - \hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k}), \lambda \in \mathbb{R}$. The cartesian equation is $\frac{x+1}{3} = \frac{y+1}{2} = \frac{z+1}{-8}$.

Quick Tip to Solve: Solve for the two line parameters using two coordinates, then always check the third coordinate before using the resulting point.

Section E (12 marks)

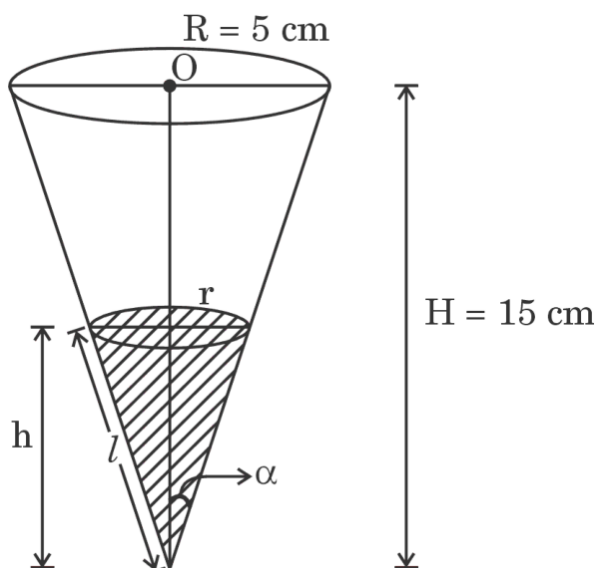
This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

At a birthday party, children are being served orange juice in conical cups, as shown in the figure. Each cup is 15 cm deep and has a radius 5 cm. The juice is being poured into this cup at a rate of $0.1 \text{ cm}^3/\text{s}$.

On the basis of the above information, answer the following questions:

[4 marks]



36(i). Establish a relation between the height h of the juice in the cup and radius r of the surface of the juice in the cup, if the semi-vertical angle of the cone is α .

[1 mark]

Solution:

Concept:

- The semi-vertical angle α is the angle between the cone's axis and its slant side. In an axial section, the radius is opposite this angle and the height is adjacent, so $\tan \alpha = r/h$.
- The juice surface is parallel to the cup's rim. Therefore the cone occupied by the juice is similar to the whole cup and has the same radius-to-height ratio.

Step 1: Apply the tangent ratio to the juice cone Consider the right triangle formed by the vertex, the centre of the juice surface and a point on its circular boundary. Its perpendicular sides have lengths h and r , and the angle at the vertex between the axis and slant side is α . Hence

$$\tan \alpha = \frac{r}{h}, \quad r = h \tan \alpha.$$

Step 2: Use the fixed dimensions of the cup For the whole cup, the corresponding radius is 5 cm and height is 15 cm. Similarity gives $\tan \alpha = 5/15 = 1/3$, which is constant as the juice level changes. Thus the relation can also be written $r = h/3$. Using the slant height in place of h would give a sine ratio and would not establish the requested radius-height relation.

Quick Tip to Solve: Measure the semi-vertical angle from the axis: radius is opposite and vertical height is adjacent, so use tangent.

Step 3: Conclusion $r = h \tan \alpha$; for this cup, $r = h/3$, or equivalently $h = 3r$.

Quick Tip to Solve: Measure the semi-vertical angle from the axis: radius is opposite and vertical height is adjacent, so use tangent.

36(ii). At what rate is the juice level in the cup rising when the juice is 6 cm deep?

[1 mark]

Solution:

Concept:

- The juice forms a cone similar to the whole cup. If its depth is h and surface radius is r , then $r/h = 5/15 = 1/3$.
- The conical volume is $V = \frac{1}{3}\pi r^2 h$. Related rates follow by differentiating with respect to time t .

Step 1: Express volume using the depth Substituting $r = h/3$ eliminates the changing radius:

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}.$$

Both the radius and depth increase; treating the radius as constant would give an incorrect rate.

Step 2: Differentiate and substitute the given depth The inflow gives $dV/dt = 0.1 \text{ cm}^3/\text{s}$. Thus

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}, \quad 0.1 = \frac{\pi(6)^2}{9} \frac{dh}{dt} = 4\pi \frac{dh}{dt}.$$

The positive inflow gives a positive height rate.

Step 3: Conclusion The juice level rises at $\frac{1}{40\pi} \text{ cm/s}$.

Quick Tip to Solve: First replace the radius by $h/3$; then differentiate the resulting single-variable volume.

36(iii)(a). When the juice is 6 cm deep, then find at what rate is the upper surface area of juice increasing?

[2 marks]

Solution:

Concept:

- Similar cones give $r = h/3$, where r is the surface radius and h is the juice depth.
- The upper surface is a horizontal circle with area $A = \pi r^2$; its rate is distinct from the wetted conical wall area.
- The inflow rate is $dV/dt = 0.1 \text{ cm}^3/\text{s}$, and $V = \pi h^3/27$.

Step 1: Find the depth rate at the specified instant Differentiate the volume before putting $h = 6 \text{ cm}$:

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

Quick Tip to Solve: For the upper surface use πr^2 , and replace r by $h/3$ before differentiating.

Therefore, at the required instant,

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s.}$$

This step accounts for the simultaneous increase of radius and height.

Step 2: Differentiate the circular upper surface area Substitute the similar-cone relation into the area formula:

$$A = \pi \left(\frac{h}{3}\right)^2 = \frac{\pi h^2}{9}, \quad \frac{dA}{dt} = \frac{2\pi h}{9} \frac{dh}{dt}.$$

At $h = 6$ cm, the radius is 2 cm, so

$$\frac{dA}{dt} = \frac{12\pi}{9} \cdot \frac{1}{40\pi} = \frac{1}{30} \text{ cm}^2/\text{s}.$$

The units are area per unit time, as required.

Step 3: Conclusion The upper surface area increases at $\boxed{\frac{1}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: For the upper surface use πr^2 , and replace r by $h/3$ before differentiating.

OR

36(iii)(b). When the juice is 6 cm deep, then find the rate at which the wetted surface area of the cup is increasing. [2 marks]

Solution:

Concept:

- The wetted part is the curved conical wall below the juice surface. Its area is $S = \pi r l$, with l the slant height.
- Similarity gives $r = h/3$, and Pythagoras gives $l = \sqrt{h^2 + r^2} = h\sqrt{10}/3$.
- The inflow rate determines dh/dt through $V = \pi h^3/27$.

Step 1: Find the instantaneous depth rate The cup has radius 5 cm and depth 15 cm, so the ratio $r/h = 1/3$ stays constant as it fills. Differentiating the volume gives

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

At depth 6 cm,

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s.}$$

Step 2: Differentiate the wetted curved surface area Using both the changing radius and slant height,

$$S = \pi \left(\frac{h}{3}\right) \left(\frac{h\sqrt{10}}{3}\right) = \frac{\pi\sqrt{10}}{9} h^2.$$

Quick Tip to Solve: Use curved area $\pi r l$; express both r and l in terms of depth before differentiating.

Thus

$$\frac{dS}{dt} = \frac{2\pi\sqrt{10}}{9}h \frac{dh}{dt} = \frac{12\pi\sqrt{10}}{9} \cdot \frac{1}{40\pi} = \frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}.$$

The circular top is the liquid surface, not part of the wetted cup wall, so it is not added to S .

Step 3: Conclusion The wetted surface area increases at $\boxed{\frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Use curved area πrl ; express both r and l in terms of depth before differentiating.

37. Case Study - 2

A carpenter needs to design a wooden box in the shape of a cuboid such that the sum of its length and breadth is 3 cm more than its height. Twice of its length, thrice of its breadth and its height add up to 10 cm. Its breadth added to 7 times its height is 1 cm less than 3 times its length.

On the basis of the above information, answer the following questions:

[4 marks]

37(i). Write the equations representing the various dimensions and express them as the matrix equation $AX = B$.

[1 mark]

Solution:

Concept:

- Write each verbal relation as a linear equation, keeping the unknowns in the same order.
- In $AX = B$, rows of A are the coefficients of the unknowns and B contains the constants.

Step 1: Define the dimensions and translate the statements Let length, breadth and height be l, b, h , respectively, in centimetres. The sum statement gives $l + b = h + 3$. The second statement gives $2l + 3b + h = 10$. Finally, $b + 7h = 3l - 1$, so moving the terms gives $3l - b - 7h = 1$.

Step 2: Collect the coefficients Thus the equations are

$$l + b - h = 3, \quad 2l + 3b + h = 10, \quad 3l - b - 7h = 1.$$

Using the column order l, b, h , write

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}, \quad X = \begin{pmatrix} l \\ b \\ h \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}.$$

Each row of the product reproduces its corresponding equation.

Step 3: Conclusion The required matrix equation is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix} \begin{pmatrix} l \\ b \\ h \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}.$$

Quick Tip to Solve: Keep the column order length, breadth, height fixed; the last relation becomes $3l - b - 7h = 1$.

37(ii). Find if A^{-1} exists. Justify your answer.

[1 mark]

Solution:

Concept:

- A square matrix has an inverse if and only if its determinant is nonzero.
- Expand a third-order determinant along a row, using the alternating signs $+, -, +$.

Step 1: Use the coefficient matrix With dimensions ordered as length, breadth and height, the coefficient matrix from the three relations is

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}.$$

In particular the final relation is $3l - b - 7h = 1$, which fixes the signs in the last row.

Step 2: Evaluate the determinant Expanding along the first row gives

$$\begin{aligned} |A| &= 1(-21 + 1) - 1(-14 - 3) + (-1)(-2 - 9) \\ &= -20 + 17 + 11 = 8. \end{aligned}$$

The determinant is nonzero, so the matrix is nonsingular and the inverse criterion applies.

Step 3: Conclusion Since $|A| = 8 \neq 0$, A^{-1} exists.

Quick Tip to Solve: Use the first row to expand: its three entries are only 1, 1, -1, reducing arithmetic.

37(iii)(a). Find A^{-1} .

[2 marks]

Solution:

Concept:

- For an invertible square matrix, $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.

Step 1: Determine the determinant The coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}.$$

Expanding along its first row,

$$|A| = 1(-21 + 1) - 1(-14 - 3) - 1(-2 - 9) = 8.$$

Therefore division by the determinant is allowed.

Step 2: Compute the cofactors and transpose The first row of cofactors is $(-20, 17, -11)$. For example,

Quick Tip to Solve: Transpose the cofactor matrix before dividing by 8; cofactors alone do not give the inverse.

the minor used for C_{21} is $1(-7) - (-1)(-1) = -8$, so $C_{21} = 8$. The remaining cofactors give

$$C = \begin{pmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{pmatrix}.$$

Hence

$$\text{adj } A = C^T = \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

Step 3: Apply the inverse formula Divide every entry of the adjugate by the determinant 8. As a check, the first row of A dotted with the three columns of the adjugate gives 8, 0, 0; the corresponding products for the other rows give 0, 8, 0 and 0, 0, 8. Thus $A \text{adj } A = 8I$.

Step 4: Conclusion

$$A^{-1} = \frac{1}{8} \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by 8; cofactors alone do not give the inverse.

OR

37(iii)(b). Find $A^2 + 7I$.

[2 marks]

Solution:

Concept:

- Matrix squaring means $A^2 = A \cdot A$; each entry is a row-column dot product.
- For a third-order matrix, I has diagonal entries 1 and other entries 0, so adding $7I$ changes only diagonal entries.

Step 1: Multiply the first row Use

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}.$$

The first-row products are

$$(A^2)_{11} = 1 + 2 - 3 = 0,$$

$$(A^2)_{12} = 1 + 3 + 1 = 5,$$

$$(A^2)_{13} = -1 + 1 + 7 = 7.$$

These calculations use columns of A , rather than multiplying entries individually.

Step 2: Complete the remaining rows The second-row dot products give $2 + 6 + 3 = 11$, $2 + 9 - 1 = 10$, and $-2 + 3 - 7 = -6$. The third-row products give $3 - 2 - 21 = -20$, $3 - 3 + 7 = 7$, and $-3 - 1 + 49 = 45$.

Quick Tip to Solve: Compute A^2 by row-column products, then add 7 only to its three diagonal entries.

Therefore

$$A^2 = \begin{pmatrix} 0 & 5 & 7 \\ 11 & 10 & -6 \\ -20 & 7 & 45 \end{pmatrix}.$$

Step 3: Add the scalar multiple of the identity Since $7I = \text{diag}(7, 7, 7)$, the diagonal entries become $0 + 7$, $10 + 7$, and $45 + 7$. All six off-diagonal entries retain their values. This is matrix addition, so the scalar is not added to every entry.

Step 4: Conclusion

$$A^2 + 7I = \begin{pmatrix} 7 & 5 & 7 \\ 11 & 17 & -6 \\ -20 & 7 & 52 \end{pmatrix}.$$

Quick Tip to Solve: Compute A^2 by row-column products, then add 7 only to its three diagonal entries.

38. Case Study - 3

An NGO organises a charity event in which they decide to distribute woollen caps to protect children from winter. The caps to be distributed are in three separate boxes, Box I has 30 red caps, Box II has 20 red and 10 green caps, and Box III has 30 green caps. The probability that a Box i is selected and a cap picked out is $\frac{i}{6}$, where $i = 1, 2, 3$.

Based on the above information, answer the following questions:

A person selects a cap.

[4 marks]

38(i). What is the probability that he selects a red cap?

[2 marks]

Solution:

Concept:

- Let B_i denote selection of Box i , and let R denote selection of a red cap. The three box events are mutually exclusive and exhaustive.
- The law of total probability gives $P(R) = \sum_{i=1}^3 P(B_i)P(R | B_i)$; each term is the probability of selecting that box and then a red cap from it.

Step 1: Find the box and colour probabilities From the given rule, the probabilities of choosing the three boxes are

$$P(B_1) = \frac{1}{6}, \quad P(B_2) = \frac{2}{6} = \frac{1}{3}, \quad P(B_3) = \frac{3}{6} = \frac{1}{2}.$$

Box I contains only red caps, Box II contains 20 red caps out of 30, and Box III contains no red caps. Thus

$$P(R | B_1) = 1, \quad P(R | B_2) = \frac{20}{30} = \frac{2}{3}, \quad P(R | B_3) = 0.$$

Step 2: Add the contributions from all boxes A red cap can come from Box I or Box II. These possibilities cannot occur together, so their probabilities add. Including the zero contribution from

Quick Tip to Solve: Multiply each box's red-cap fraction by its selection probability; the all-green Box III contributes zero.

Box III gives

$$\begin{aligned}P(R) &= \frac{1}{6}(1) + \frac{1}{3}\left(\frac{2}{3}\right) + \frac{1}{2}(0) \\&= \frac{1}{6} + \frac{2}{9} = \frac{3}{18} + \frac{4}{18}.\end{aligned}$$

The boxes are not equally likely, so the colour fractions must be weighted by the stated box-selection probabilities.

Step 3: Conclusion The probability that the person selects a red cap is $\boxed{\frac{7}{18}}$.

Quick Tip to Solve: Multiply each box's red-cap fraction by its selection probability; the all-green Box III contributes zero.

38(ii). If he selects a green cap, what is the probability that the cap has come from Box II? [2 marks]

Solution:

Concept:

- Let B_i denote selection of Box i , and G the event of picking a green cap. The box-selection probabilities are the prior probabilities $P(B_i) = i/6$.
- Bayes' theorem gives $P(B_2 | G) = \frac{P(B_2)P(G | B_2)}{P(G)}$, where the denominator includes green caps from every possible box.

Step 1: Calculate the total probability of a green cap The probabilities of choosing Boxes I, II and III are respectively $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$. Box I has no green caps, Box II has 10 green caps out of 30, and all caps in Box III are green. Therefore

$$P(G | B_1) = 0, \quad P(G | B_2) = \frac{1}{3}, \quad P(G | B_3) = 1.$$

Using the law of total probability,

$$\begin{aligned}P(G) &= \frac{1}{6}(0) + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{2}(1) \\&= \frac{1}{9} + \frac{1}{2} = \frac{11}{18}.\end{aligned}$$

Step 2: Apply Bayes' theorem to Box II Selecting Box II and then a green cap has joint probability

$$P(B_2 \cap G) = P(B_2)P(G | B_2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

Since it is known that the selected cap is green, divide this joint probability by the probability of the observed green-cap event:

$$P(B_2 | G) = \frac{1/9}{11/18} = \frac{1}{9} \cdot \frac{18}{11}.$$

The denominator must also count green caps selected from Box III, because that is another way the

Quick Tip to Solve: For Bayes' theorem, divide the weighted green contribution of Box II by the total weighted green contributions of all boxes.

observed event can occur.

Step 3: Conclusion Given that the cap is green, the probability that it came from Box II is $\frac{2}{11}$.

Quick Tip to Solve: For Bayes' theorem, divide the weighted green contribution of Box II by the total weighted green contributions of all boxes.