

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-4-2



Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

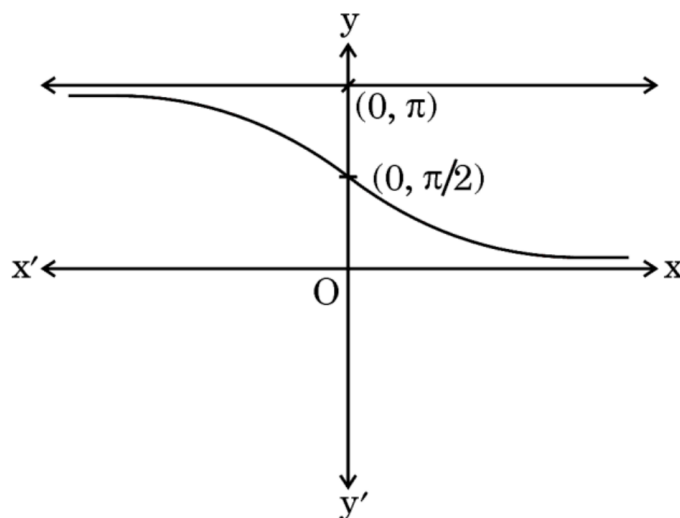
- (i) This question paper contains **38** questions. All questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The following graph represents:

[1 mark]



- (A) $y = \sec^{-1} x$
- (B) $y = \cot^{-1} x$
- (C) $y = \tan^{-1} x$
- (D) $y = \operatorname{cosec}^{-1} x$

Correct Answer: (B) $y = \cot^{-1} x$

Solution:

Concept:

- The principal branch of $y = \cot^{-1} x$ has domain $\mathbb{R}$ and range $(0, \pi)$.
- It is decreasing, passes through $(0, \pi/2)$, and approaches π and 0 at the two ends.

Step 1: Read the intercept and end behaviour The curve is continuous for all real x and decreases from near $y = \pi$ to near $y = 0$. Its intersection with the vertical axis is labelled $(0, \pi/2)$. These are precisely the principal inverse-cotangent features: $\cot(\pi/2) = 0$.

Step 2: Compare all four options Inverse tangent is increasing and passes through the origin, so (C) fails. Inverse secant and inverse cosecant have domains $(-\infty, -1] \cup [1, \infty)$, so neither exists at $x = 0$; this excludes (A) and (D). Only inverse cotangent has both the shown intercept and decreasing shape.

Step 3: Conclusion The graph represents $y = \cot^{-1} x$. Hence, the correct option is (B).

Quick Tip to Solve: Check the vertical-axis intercept first: inverse cotangent gives $\pi/2$, whereas inverse tangent gives zero.

2. Let $A = [a_{ij}]$ be a 2×2 matrix whose elements are given by $a_{ij} = \frac{(2i-j)^2}{3}$. Then A' is: [1 mark]

(A) $\begin{pmatrix} \frac{1}{3} & 3 \\ 0 & \frac{4}{3} \end{pmatrix}$

(B) $\begin{pmatrix} \frac{1}{3} & 0 \\ 3 & \frac{4}{3} \end{pmatrix}$

(C) $\begin{pmatrix} \frac{4}{3} & 3 \\ 0 & \frac{1}{3} \end{pmatrix}$

(D) $\begin{pmatrix} \frac{4}{3} & 0 \\ 3 & \frac{1}{3} \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} \frac{1}{3} & 3 \\ 0 & \frac{4}{3} \end{pmatrix}$

Solution:

Concept:

- The entry a_{ij} uses the row number i and column number j .
- Transposition interchanges rows and columns: $(A')_{ij} = a_{ji}$.

Step 1: Calculate the four entries Use $a_{ij} = (2i-j)^2/3$ with $i, j \in \{1, 2\}$:

$$a_{11} = \frac{1}{3}, \quad a_{12} = 0, \quad a_{21} = 3, \quad a_{22} = \frac{4}{3}.$$

Thus $A = \begin{pmatrix} 1/3 & 0 \\ 3 & 4/3 \end{pmatrix}$.

Step 2: Interchange the rows and columns The diagonal entries remain unchanged, while the off-diagonal entries exchange positions:

$$A' = \begin{pmatrix} 1/3 & 3 \\ 0 & 4/3 \end{pmatrix}.$$

Option (B) is the original matrix rather than its transpose. Options (C) and (D) interchange the diagonal entries, which transposition does not do.

Step 3: Conclusion $A' = \begin{pmatrix} 1/3 & 3 \\ 0 & 4/3 \end{pmatrix}$. Hence, the correct option is (A).

Quick Tip to Solve: Write the entry table before transposing; then keep the diagonal fixed and swap only the two off-diagonal entries.

3. If points $(2, 3)$, $(0, 4)$ and $(p, 2)$ are collinear, then the value of p is:

[1 mark]

- (A) $\frac{4}{7}$
- (B) $-\frac{3}{7}$
- (C) 4
- (D) -4

Correct Answer: (C) 4

Solution:

Concept:

- Three points are collinear when the determinant formed from their coordinates and a column of ones vanishes.
- Equivalently, all three points must satisfy the equation of the same straight line.

Step 1: Apply the collinearity determinant The zero-area condition gives

$$\begin{vmatrix} 2 & 3 & 1 \\ 0 & 4 & 1 \\ p & 2 & 1 \end{vmatrix} = 0.$$

Expanding along the first row yields

$$2(4 - 2) - 3(0 - p) + (0 - 4p) = 0.$$

Step 2: Simplify and check the options The determinant simplifies to $4 + 3p - 4p = 4 - p$, so it vanishes at $p = 4$. The line through $(0, 4)$ and $(2, 3)$ is $y = 4 - x/2$; at $y = 2$, its abscissa is also 4. The other offered values $4/7, -3/7, -4$ leave $4 - p \neq 0$.

Step 3: Conclusion The required value is $p = 4$. Hence, the correct option is (C).

Quick Tip to Solve: Here the line loses one unit in height for every two units in x ; dropping from $y=4$ to $y=2$ therefore needs $x=4$.

4. Differential of e^{e^x} with respect to x is:

[1 mark]

- (A) $\log x$
- (B) e^{e^x}
- (C) $e^x e^{e^x}$
- (D) $(e^x)^2$

Correct Answer: (C) $e^x e^{e^x}$

Solution:

Concept:

- For a differentiable function $u(x)$, the chain rule gives $\frac{d}{dx} e^{u(x)} = e^{u(x)} u'(x)$.

Quick Tip to Solve: For a nested exponential, copy the whole outer exponential and multiply by the derivative of its exponent.

- The inner exponential $u(x) = e^x$ has derivative $u'(x) = e^x$.

Step 1: Identify the inner and outer functions The expression e^{e^x} is an exponential whose exponent is itself the function e^x . Set $u = e^x$. Differentiating the outer exponential with respect to u gives e^u , while differentiating the inner function with respect to x gives e^x .

Step 2: Multiply the two derivatives By the chain rule,

$$\frac{d}{dx} (e^{e^x}) = e^{e^x} e^x.$$

Option (B) omits the inner derivative. Option (D) replaces the nested exponential by another e^x , which is not valid. Option (A), $\log x$, is unrelated to differentiating this composite exponential.

Step 3: Conclusion The derivative is $e^x e^{e^x}$. Hence, the correct option is (C).

Quick Tip to Solve: For a nested exponential, copy the whole outer exponential and multiply by the derivative of its exponent.

5. The principal value of $\sec^{-1}(\sqrt{2}) + 2 \operatorname{cosec}^{-1}(-\sqrt{2})$ is:

[1 mark]

- (A) $-\frac{\pi}{2}$
 (B) $-\frac{\pi}{4}$
 (C) $\frac{\pi}{4}$
 (D) $\frac{\pi}{2}$

Correct Answer: (B) $-\frac{\pi}{4}$

Solution:

Concept:

- The principal range of inverse secant is $[0, \pi] \setminus \{\pi/2\}$.
- The principal range of inverse cosecant is $[-\pi/2, \pi/2] \setminus \{0\}$; negative inputs give negative angles.

Step 1: Find the two principal angles Because $\sec(\pi/4) = \sqrt{2}$, the principal value of $\sec^{-1}(\sqrt{2})$ is $\pi/4$. Also $\operatorname{cosec}(-\pi/4) = -\sqrt{2}$, and $-\pi/4$ belongs to the principal inverse-cosecant range. Therefore $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\pi/4$.

Step 2: Apply the coefficient and add The coefficient 2 multiplies the inverse-cosecant angle, so

$$\frac{\pi}{4} + 2 \left(-\frac{\pi}{4} \right) = -\frac{\pi}{4}.$$

Option (A) keeps only the second term. Options (C) and (D) cannot be obtained when the negative principal inverse-cosecant angle and its factor 2 are both retained.

Step 3: Conclusion The principal value is $-\pi/4$. Hence, the correct option is (B).

Quick Tip to Solve: Find each principal angle before combining them; the negative cosecant input requires the negative principal angle.

6. If the distance travelled by a particle in t seconds is given by $S = 72t + 3t^2 - t^3$, then time taken by the particle to come to rest is: [1 mark]

- (A) 4 seconds
- (B) 6 seconds
- (C) 3 seconds
- (D) 0 seconds

Correct Answer: (B) 6 seconds

Solution:

Concept:

- The instantaneous velocity is $v = dS/dt$, where S is the given displacement function.
- A particle comes to rest when its instantaneous velocity is zero; physical time satisfies $t \geq 0$.

Step 1: Differentiate the position function For $S = 72t + 3t^2 - t^3$, differentiation term by term gives

$$v(t) = 72 + 6t - 3t^2.$$

The condition for rest is imposed on this derivative, rather than on S itself, because zero position does not mean zero velocity.

Step 2: Solve the rest condition Set $72 + 6t - 3t^2 = 0$. Dividing by -3 and factoring gives

$$t^2 - 2t - 24 = (t - 6)(t + 4) = 0.$$

The roots are 6 and -4 , but only the nonnegative time is admissible. At the other listed times 4, 3, 0, the velocities are respectively 48, 63, 72, so none gives rest.

Step 3: Conclusion The particle comes to rest after 6 s. Hence, the correct option is (B).

Quick Tip to Solve: Differentiate first, set the velocity to zero, and discard a negative root when the question asks elapsed time.

7. $\int \frac{dx}{\sqrt{25 - 16x^2}}$ is equal to:

[1 mark]

- (A) $\frac{1}{5} \sin^{-1} 4x + C$
(B) $\frac{1}{25} \sin^{-1} 16x + C$
(C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$
(D) $\frac{1}{16} \sin^{-1} \frac{4x}{5} + C$

Correct Answer: (C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$

Solution:

Concept:

- The standard form is $\int du/\sqrt{1 - u^2} = \sin^{-1} u + C$.
- A substitution must account for the scaling in both the denominator and the differential.

Step 1: Reduce the radical to standard form Factor 25 from the radicand and put $u = 4x/5$, so $dx = (5/4)du$. Then

$$\sqrt{25 - 16x^2} = 5\sqrt{1 - u^2}.$$

This substitution is valid on the real interval $|x| < 5/4$, where the integrand is defined.

Step 2: Integrate and compare the coefficients Therefore

$$\int \frac{dx}{\sqrt{25 - 16x^2}} = \frac{1}{4} \int \frac{du}{\sqrt{1 - u^2}}.$$

The antiderivative has argument $4x/5$ and coefficient $1/4$. Options (A) and (B) use the wrong argument; (D) has the right argument but a coefficient four times too small. Differentiation checks the required scaling.

Step 3: Conclusion The integral is $\frac{1}{4} \sin^{-1}(4x/5) + C$. Hence, the correct option is (C).

Quick Tip to Solve: For a radical of the form $a^2 - b^2 x^2$, use argument bx/a and prefactor $1/b$.

8. If $\int_0^1 \frac{dx}{e^x + e^{-x}} = \tan^{-1} e + k$, then the value of k is:

[1 mark]

- (A) e
- (B) $\frac{\pi}{4}$
- (C) 0
- (D) $-\frac{\pi}{4}$

Correct Answer: (D) $-\frac{\pi}{4}$

Solution:

Concept:

- Multiplying numerator and denominator by e^x converts the integrand to $e^x/(1 + e^{2x})$.
- The substitution $u = e^x$, $du = e^x dx$, leads to $\int du/(1 + u^2) = \tan^{-1} u$.

Step 1: Transform the integral and its limits Write

$$I = \int_0^1 \frac{e^x dx}{1 + e^{2x}}.$$

With $u = e^x$, the lower limit becomes 1 and the upper limit becomes e . Thus $I = \int_1^e du/(1 + u^2)$. Changing the limits avoids substituting back after integration.

Step 2: Evaluate and identify the constant Using the inverse-tangent antiderivative,

$$I = [\tan^{-1} u]_1^e = \tan^{-1} e - \frac{\pi}{4}.$$

Compare this with the given $I = \tan^{-1} e + k$. The lower-limit contribution is subtracted, excluding the positive value in (B). Options (A) and (C) also omit the required value $-\tan^{-1} 1$.

Step 3: Conclusion The constant is $k = -\pi/4$. Hence, the correct option is (D).

Quick Tip to Solve: After substituting $u=e$ to the x , remember the lower limit is 1, so its inverse tangent contributes minus $\pi/4$.

9. If $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is: [1 mark]

- (A) 0
(B) -2
(C) -1
(D) 2

Correct Answer: (D) 2

Solution:

Concept:

- Continuity at a point requires the two-sided limit to equal the value of the function at that point.
- The standard trigonometric limit is $\lim_{x \rightarrow 0} \sin x / x = 1$, with angles in radians.

Step 1: Evaluate the limit away from zero Use the formula defined for nonzero values of x . Since both terms have limits, their sum has the limit

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} + \lim_{x \rightarrow 0} \cos x = 1 + 1 = 2.$$

Direct substitution into the fraction would give an indeterminate expression, so the standard limit must be used.

Step 2: Apply continuity at the specified point The defined value is $f(0) = k$. Continuity therefore requires k to equal the limit calculated above. The choices 0, -2, -1 do not equal the limiting value and would create a discontinuity at zero.

Step 3: Conclusion $k = 2$. Hence, the correct option is (D).

Quick Tip to Solve: For a piecewise definition at zero, compute the limit using the nonzero branch before setting it equal to the assigned value.

10. The area of the region bounded by the curve $y = x$ and x -axis, between $x = 0$ and $x = 2$ is: [1 mark]

- (A) 2 sq. units
(B) $\frac{1}{2}$ sq. unit
(C) 1 sq. unit
(D) 4 sq. units

Correct Answer: (A) 2 sq. units

Solution:

Concept:

- For a nonnegative function on an interval, area between its curve and the x -axis is $\int_a^b f(x) dx$.
- The region under a straight line through the origin can also be checked using the area of a triangle.

Step 1: Identify the area integral On $0 \leq x \leq 2$, the function $y = x$ is nonnegative. Thus the required

Quick Tip to Solve: For the area under $y = x$ from zero to a positive number a , use the triangle shortcut $a^2/2$.

area is obtained directly from

$$A = \int_0^2 x \, dx = \left[\frac{x^2}{2} \right]_0^2.$$

No sign change or splitting of the interval is needed.

Step 2: Evaluate and check geometrically The upper value of the antiderivative is $2^2/2 = 2$ and the lower value is zero. The region has a base of length 2 and height 2, so the triangle formula also gives $\frac{1}{2}(2)(2) = 2$. Option (D) counts the surrounding rectangle; options (B) and (C) do not match the integral.

Step 3: Conclusion $A = 2$ sq. units. Hence, the correct option is (A).

Quick Tip to Solve: For the area under $y = x$ from zero to a positive number a , use the triangle shortcut $a^2/2$.

11. The greatest integer function, $f(x) = [x]$, $0 < x < 3$ is not differentiable at how many points?

[1 mark]

- (A) At only one point
- (B) At only two points
- (C) At no point
- (D) At three points

Correct Answer: (B) At only two points

Solution:

Concept:

- The greatest integer function $[x]$ is constant between successive integers, so its derivative is zero there.
- At an integer it has a jump discontinuity; differentiability implies continuity, so it cannot be differentiable at a jump.

Step 1: Examine the noninteger points Within the given open interval, $f(x) = 0$ for $0 < x < 1$, $f(x) = 1$ for $1 < x < 2$, and $f(x) = 2$ for $2 < x < 3$. Each branch is locally constant at every point inside it, and hence differentiable with derivative zero.

Step 2: Locate the jumps inside the domain At $x = 1$, the left and right limits are 0 and 1; at $x = 2$, they are 1 and 2. Both points are discontinuities. The endpoints 0 and 3 are excluded by $0 < x < 3$, so they are not counted. This rules out one, zero and three points.

Step 3: Conclusion The function is not differentiable at $x = 1, 2$, exactly two points. Hence, the correct option is (B).

Quick Tip to Solve: Count only integers strictly inside the stated interval; excluded endpoints cannot add to the answer.

12. The sum of the order and the degree of the differential equation $4 \left[\frac{d^2y}{dx^2} \right]^2 + 3 \left[1 + \left(\frac{dy}{dx} \right)^2 - y \right] = 0$ is: [1 mark]

- (A) 2
- (B) 3
- (C) not defined
- (D) 4

Correct Answer: (D) 4

Solution:

Concept:

- The order is the order of the highest derivative occurring in the differential equation.
- The degree is the power of that highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Determine the highest derivative The equation contains dy/dx and d^2y/dx^2 . The highest-order derivative is the second derivative, so the order is 2. Squaring this derivative does not change its order.

Step 2: Determine the degree and required sum Writing the equation as

$$4(y'')^2 + 3 + 3(y')^2 - 3y = 0$$

makes its polynomial form explicit. The second derivative occurs to power 2, with nonzero coefficient 4, so the degree is also 2. Degree is defined here because no derivative occurs inside a radical or transcendental function. Options (A) and (B) miss one of the two contributions, while (C) wrongly denies the polynomial form.

Step 3: Conclusion The required sum is $2 + 2 = 4$. Hence, the correct option is (D).

Quick Tip to Solve: First identify the highest-order derivative, then read its exponent; the exponent and derivative order describe different things.

13. If $(3\hat{i} - 2\hat{j} + 5\hat{k}) \times (4\hat{i} + p\hat{j} + q\hat{k}) = \vec{0}$, then the values of p and q are:

[1 mark]

- (A) $p = -\frac{2}{3}, q = \frac{5}{3}$
(B) $p = -\frac{8}{3}, q = \frac{20}{3}$
(C) $p = \frac{20}{3}, q = \frac{8}{3}$
(D) $p = 0, q = 0$

Correct Answer: (B) $p = -\frac{8}{3}, q = \frac{20}{3}$

Solution:

Concept:

- The cross product of two nonzero vectors vanishes precisely when the vectors are parallel.
- Parallel vectors have proportional corresponding components: $\vec{b} = \lambda \vec{a}$ for a scalar λ .

Step 1: Use the known components to find the ratio Let $\vec{a} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{b} = 4\hat{i} + p\hat{j} + q\hat{k}$. Both are nonzero because their first components are nonzero. Since $\vec{a} \times \vec{b} = \vec{0}$, write $\vec{b} = \lambda \vec{a}$. Comparing first components gives $4 = 3\lambda$, so $\lambda = 4/3$.

Step 2: Compare the remaining components and options The same scalar must multiply all components. Therefore

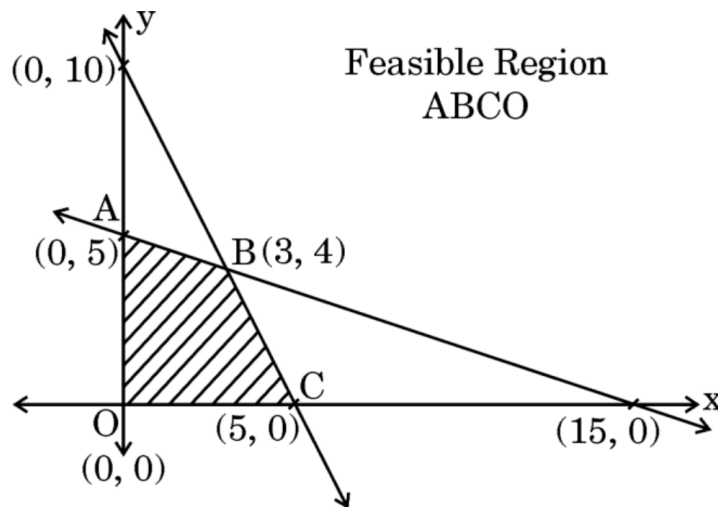
$$p = -2 \left(\frac{4}{3} \right) = -\frac{8}{3}, \quad q = 5 \left(\frac{4}{3} \right) = \frac{20}{3}.$$

Option (A) uses an incorrect scale. Option (C) gives a positive p and an incorrect q , so its components cannot share the multiplier $4/3$. Option (D) would make the second vector parallel to the x -axis alone, which the first vector is not.

Step 3: Conclusion $p = -\frac{8}{3}, q = \frac{20}{3}$. Hence, the correct option is (B).

Quick Tip to Solve: When a cross product is zero, find the common multiplier from a known nonzero component instead of expanding a determinant.

14. In the graph, the feasible region representing the Linear Programming Problem for maximising objective function $Z = px + qy$, $p, q > 0$ is shaded. If all points on segment AB give $\max(Z)$, then which of the following is true? [1 mark]



- (A) $p = 2q$
- (B) $p = 3q$
- (C) $q = 3p$
- (D) $q = 2p$

Correct Answer: (C) $q = 3p$

Solution:

Concept:

- A linear objective function is constant on a line segment when it has equal values at the two endpoints.
- For $Z = px + qy$ with $q > 0$, each objective line has slope $-p/q$.

Step 1: Read the endpoints from the given graph The graph gives $A = (0, 5)$ and $B = (3, 4)$. If all points of AB give the maximum, these endpoints in particular must give the same value. Hence

$$Z(A) = 5q, \quad Z(B) = 3p + 4q, \quad 5q = 3p + 4q.$$

Step 2: Check the relation on the feasible region The equality yields $q = 3p$. The line through A and B has slope $(4 - 5)/(3 - 0) = -1/3$, agreeing with $-p/q$. The other vertices are $O = (0, 0)$ and $C = (5, 0)$; for this relation their objective values are 0 and $5p$, whereas both A and B give $15p$. Since $p > 0$, the edge AB indeed gives the maximum. The other listed relations fail the necessary endpoint equality.

Step 3: Conclusion $q = 3p$. Hence, the correct option is (C).

Quick Tip to Solve: For multiple optimum points on an edge, equate the objective values at its endpoints before checking the other vertices.

15. Three points $A(0, 1, 1)$, $B(2, 0, -1)$ and $C(1, 0, 3)$ form $\triangle ABC$. The $\text{ar}(\triangle ABC)$ is:

[1 mark]

- (A) $\frac{\sqrt{53}}{2}$ sq. units
(B) $\sqrt{53}$ sq. units
(C) $\frac{\sqrt{11}}{2}$ sq. units
(D) $\sqrt{11}$ sq. units

Correct Answer: (A) $\frac{\sqrt{53}}{2}$ sq. units

Solution:

Concept:

- The area of a triangle with adjacent side vectors $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}|\vec{u} \times \vec{v}|$.
- The magnitude of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$.

Step 1: Form two sides with the same starting point Using A as the common vertex gives

$$\overrightarrow{AB} = (2, -1, -2), \quad \overrightarrow{AC} = (1, -1, 2).$$

Subtracting coordinates in this consistent order ensures that the resulting cross product is the vector associated with the parallelogram on these two sides.

Step 2: Compute the cross product and halve its magnitude The cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-4, -6, -1).$$

Thus its magnitude is $\sqrt{16 + 36 + 1} = \sqrt{53}$. The triangle occupies half the parallelogram area. Option (B) omits this factor of one-half, and options (C) and (D) have the wrong squared magnitude.

Step 3: Conclusion $\text{ar}(\triangle ABC) = \frac{\sqrt{53}}{2}$ sq. units. Hence, the correct option is (A).

Quick Tip to Solve: Use two side vectors from the same vertex, take their cross product magnitude, and remember to halve it for a triangle.

16. The region represented by the system of inequations $3x + y \geq 3$, $2x - y \geq -5$, $x, y \geq 0$ is: [1 mark]

- (A) unbounded in 1st quadrant
- (B) bounded in 1st quadrant
- (C) unbounded in 2nd quadrant
- (D) bounded in 2nd quadrant

Correct Answer: (A) unbounded in 1st quadrant

Solution:

Concept:

- Nonnegativity constraints $x, y \geq 0$ restrict a feasible region to the first quadrant, including its boundary axes.
- A feasible region is unbounded if it contains feasible points arbitrarily far from the origin.

Step 1: Rewrite the inequalities to identify allowed points The two line constraints are equivalent to

$$y \geq 3 - 3x, \quad y \leq 2x + 5.$$

Along the nonnegative x -axis, put $y = 0$. The first constraint then requires $x \geq 1$, and the second requires $x \geq -5/2$, which is automatically satisfied for $x \geq 1$.

Step 2: Establish the quadrant and boundedness Every point $(x, 0)$ with $x \geq 1$ is therefore feasible. As x can increase without limit, this ray proves that the region is unbounded. All its points have nonnegative coordinates, so it lies in the first quadrant. The bounded option contradicts this ray, and the second-quadrant options contradict the nonnegativity restrictions.

Step 3: Conclusion The feasible region is unbounded in the first quadrant. Hence, the correct option is (A).

Quick Tip to Solve: To test boundedness quickly, try an axis: if one entire ray satisfies every constraint, the feasible region is unbounded.

17. Which of the following is *not* a Linear Differential Equation?

[1 mark]

- (A) $(1 + x^2) dy + 2xy dx = \cot x dx$
- (B) $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$
- (C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$
- (D) $y dx - (x + 3y^2) dy = 0$

Correct Answer: (C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$

Solution:

Concept:

- A first-order equation is linear in its dependent variable when it has the form $dy/dx + P(x)y = Q(x)$.
- An equation can also be linear in x as a function of y : $dx/dy + P(y)x = Q(y)$. Powers and products of the dependent variable determine linearity, not the appearance of the original equation.

Quick Tip to Solve: Before rejecting an equation because it contains y^2 , check whether it is linear when x is treated as the dependent variable.

Step 1: Check options (A) and (B) For (A), dividing by dx gives $(1 + x^2)y' + 2xy = \cot x$, which is linear in y . In (B), apply the product rule: $d(xy)/dx = y + xy'$. Hence its equation becomes $xy' + 2y = x(\sin x + \log x)$, also linear in y on its domain.

Step 2: Check options (C) and (D) For (C), rearrangement gives $dy/dx = x(1 + y^2)/[y(1 + x^2)]$. The dependent variable occurs as $y + 1/y$; treating x as dependent instead gives $x + 1/x$. Thus neither choice yields a linear equation. For (D), choose x as dependent:

$$\frac{dx}{dy} - \frac{x}{y} = 3y.$$

This is linear in x , so the term $3y^2$ in the original expression does not make (D) the required answer.

Step 3: Conclusion Option (C) is not a linear differential equation. Hence, the correct option is (C).

Quick Tip to Solve: Before rejecting an equation because it contains y^2 , check whether it is linear when x is treated as the dependent variable.

18. The probability that it will rain tomorrow in cities A, B and C is 60%, 70% and 80% respectively. The probability that it will rain tomorrow in at least one of the cities is: [1 mark]

- (A) $\frac{3}{250}$
 (B) $\frac{244}{250}$
 (C) 1
 (D) $\frac{9}{10}$

Correct Answer: (B) $\frac{244}{250}$

Solution:

Concept:

- For an event E , $P(E) = 1 - P(E^c)$; the complement of rain in at least one city is no rain in all three.
- In the independent-city rainfall model, the probability of all three no-rain events is the product of their individual probabilities.

Step 1: Find the no-rain probabilities Let A, B, C denote rainfall in the respective cities. Their complementary probabilities are

$$P(A^c) = 0.40, \quad P(B^c) = 0.30, \quad P(C^c) = 0.20.$$

Multiplication applies to the three independent no-rain events, rather than addition of the rainfall probabilities, which would double-count simultaneous rainfall.

Step 2: Use the complement The probability of no rain anywhere is

$$P(A^c \cap B^c \cap C^c) = 0.4 \times 0.3 \times 0.2 = 0.024 = \frac{6}{250}.$$

Quick Tip to Solve: For at least one among independent events, multiply the failure probabilities first and subtract that product from 1.

Consequently,

$$P(A \cup B \cup C) = 1 - \frac{6}{250} = \frac{244}{250}.$$

Option (A) is not this complement; option (C) incorrectly makes rain certain, and option (D) omits the full three-city calculation.

Step 3: Conclusion The probability is $244/250 = 0.976$. Hence, the correct option is (B).

Quick Tip to Solve: For at least one among independent events, multiply the failure probabilities first and subtract that product from 1.

19. Assertion (A): If A and B are two square matrices such that AB and BA are defined, then it is not necessary that $AB = BA$.

Reason (R): Product of two diagonal matrices of same order is commutative.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- Matrix multiplication is generally not commutative: even for square matrices of the same order, AB need not equal BA .
- For diagonal matrices of the same order, multiplication multiplies corresponding diagonal entries, and real-number multiplication is commutative.

Step 1: Judge the Assertion Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Direct multiplication gives

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Both products exist but differ. Thus the Assertion is true.

Step 2: Judge the Reason If $D = \text{diag}(d_1, \dots, d_n)$ and $E = \text{diag}(e_1, \dots, e_n)$, then

$$DE = \text{diag}(d_1 e_1, \dots, d_n e_n) = ED.$$

Quick Tip to Solve: In an Assertion-Reason matrix question, distinguish a true special case from a reason that proves the general assertion.

The Reason is therefore also true.

Step 3: Test whether the Reason explains the Assertion The Reason describes a special class in which the two products are equal. It does not explain why arbitrary square matrices can have unequal products. Therefore the explanation required in option (A) is absent; (C) and (D) are excluded because neither statement is false.

Step 4: Conclusion Both statements are true, but the Reason is not the correct explanation of the Assertion. Hence, the correct option is (B).

Quick Tip to Solve: In an Assertion-Reason matrix question, distinguish a true special case from a reason that proves the general assertion.

20. Assertion (A): A function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3 + 2, \forall x \in \mathbb{N}$ is one-one but not onto.

Reason (R): Since $\forall y \in \mathbb{N}$ (Codomain), there does not exist $x = (y - 2)^{1/3}$ in $\mathbb{N}$ (Domain) such that $f(x) = x^3 + 2 = y$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ forces $a = b$, and onto if every codomain element has a preimage in the domain.
- To disprove a universal statement, one counterexample is enough. The words $\forall y \in \mathbb{N}$ must be checked exactly as printed.

Step 1: Judge the Assertion For natural numbers a, b , equality $a^3 + 2 = b^3 + 2$ gives $a^3 = b^3$, hence $a = b$. Thus f is one-one. It is not onto because $y = 2$ would require $x = 0$, outside the convention $\mathbb{N} = \{1, 2, \dots\}$. Equivalently, $y = 4$ would require $x = \sqrt[3]{2}$, which is not natural and works regardless of whether zero is included. The Assertion is true.

Step 2: Judge the Reason as printed The Reason says that for every natural y , there does not exist a natural preimage. This is false: choose $y = 3$. Then

$$x = (3 - 2)^{1/3} = 1 \in \mathbb{N}, \quad f(1) = 3.$$

Several codomain elements do have preimages, although some do not.

Step 3: Check the explanatory link The correct test for failure of surjectivity is that at least one codomain element lacks a preimage. The printed Reason instead claims that every element lacks one,

Quick Tip to Solve: For non-surjectivity use one missing output; do not replace "there exists an output with no preimage" by "every output has no preimage".

so it is false and cannot explain the Assertion. Options (A) and (B) require a true Reason, while (D) requires a false Assertion.

Step 4: Conclusion The Assertion is true, but the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: For non-surjectivity use one missing output; do not replace "there exists an output with no preimage" by "every output has no preimage".

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). Let two rods placed on the ground be represented by vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector representing a flag-post of height 5 m that has to be erected perpendicular to both the rods. [2 marks]

Solution:

Concept:

- The cross product $\vec{u} \times \vec{v}$ is perpendicular to both non-parallel vectors $\vec{u}$ and $\vec{v}$.
- To obtain a vector of length h along a nonzero vector $\vec{n}$, use $h\vec{n}/|\vec{n}|$. The opposite sign gives the opposite perpendicular direction.

Step 1: Find a common perpendicular Let $\vec{u} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{v} = -2\hat{i} + \hat{j} - 2\hat{k}$. Their cross product is

$$\begin{aligned}\vec{n} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} \\ &= \hat{i}(2 - 3) - \hat{j}(-8 + 6) + \hat{k}(4 - 2) \\ &= -\hat{i} + 2\hat{j} + 2\hat{k}.\end{aligned}$$

It is nonzero, so it supplies a direction perpendicular to the plane containing the rods.

Step 2: Scale the perpendicular to the required height The magnitude is $|\vec{n}| = \sqrt{1 + 4 + 4} = 3$. Therefore a vector of length 5 m is

$$\vec{p} = \frac{5}{3}\vec{n} = -\frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{10}{3}\hat{k} \quad (\text{m}).$$

Verification gives $\vec{n} \cdot \vec{u} = -4 - 2 + 6 = 0$ and $\vec{n} \cdot \vec{v} = 2 + 2 - 4 = 0$, while $|\vec{p}| = 5$ m. The reversed vector is also perpendicular; the sign is chosen according to the direction in which the post is erected.

Step 3: Conclusion A suitable flag-post vector is $-\frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{10}{3}\hat{k}$ m, with its opposite giving the other normal direction.

Quick Tip to Solve: Compute the cross product for direction, then normalize it before multiplying by the post height; the raw cross product need not have length 5.

21(b). A unit vector $\vec{a}$ is such that it makes an angle $\frac{\pi}{4}$ with x -axis, $\frac{\pi}{3}$ with y -axis and an acute angle θ with z -axis. Find θ and the components of $\vec{a}$. [2 marks]

Solution:

Concept:

- The direction cosines of a vector are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, where the angles are measured with the positive coordinate axes.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$. For a unit vector, they are its components along $\hat{i}$, $\hat{j}$, $\hat{k}$.

Step 1: Use the direction-cosine identity The given angles yield $l = \cos(\pi/4) = 1/\sqrt{2}$, $m = \cos(\pi/3) = 1/2$, and $n = \cos \theta$. Thus

$$\begin{aligned}\cos^2 \theta &= 1 - \cos^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} \\ &= 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.\end{aligned}$$

This identity follows by resolving the unit vector into three perpendicular components and adding the squares of their magnitudes.

Step 2: Choose the sign and write the components Because θ is acute, its cosine is positive, so $\cos \theta = 1/2$ rather than $-1/2$. Hence $\theta = \pi/3$, and

$$\vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}.$$

As a check, the squared magnitude of this vector is $1/2 + 1/4 + 1/4 = 1$. Each coefficient therefore agrees both with the given axis angle and with the requirement that the vector be a unit vector.

Step 3: Conclusion $\theta = \pi/3$ and the components are $(1/\sqrt{2}, 1/2, 1/2)$.

Quick Tip to Solve: After finding $\cos^2 \theta$, use the stated acute angle to select the positive cosine before writing the unit vector.

22. Evaluate: $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) + \tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ [2 marks]

Solution:

Concept:

- The principal-value range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$, and that of $\cot^{-1} x$ is $(0, \pi)$.
- The expression $\tan^{-1}(\tan t)$ equals the representative angle in the principal range, which need not be t . Tangent has period π .

Step 1: Evaluate the first three terms Using the standard trigonometric values and the specified

Quick Tip to Solve: Reduce $2\pi/3$ to $-\pi/3$ before applying inverse tangent; cancel the resulting opposite $\pi/3$ terms to simplify the sum.

principal ranges,

$$\begin{aligned}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) &= -\frac{\pi}{6}, \\ \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) &= \frac{\pi}{3}, \\ \tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) &= \tan^{-1}(-1) = -\frac{\pi}{4}.\end{aligned}$$

The cotangent term is positive because its principal angle lies between zero and π ; it is not the reciprocal of the inverse tangent.

Step 2: Reduce the final angle and add Since $2\pi/3$ lies outside the principal range of inverse tangent, subtract one period:

$$\tan \frac{2\pi}{3} = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

Hence $\tan^{-1}(\tan(2\pi/3)) = -\pi/3$. The sum is therefore

$$\begin{aligned}-\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} - \frac{\pi}{3} &= -\frac{\pi}{6} - \frac{\pi}{4} \\ &= -\frac{2\pi + 3\pi}{12} = -\frac{5\pi}{12}.\end{aligned}$$

The two $\pi/3$ terms cancel after the principal-value reduction.

Step 3: Conclusion The value of the given expression is $\boxed{-5\pi/12}$.

Quick Tip to Solve: Reduce $2\pi/3$ to $-\pi/3$ before applying inverse tangent; cancel the resulting opposite $\pi/3$ terms to simplify the sum.

23. Find the interval(s) in which the function $f(x) = \frac{x}{\log x}$, where $x \in (0, 1) \cup (1, \infty)$, is increasing.

[2 marks]

Solution:

Concept:

- On an interval in its domain, a differentiable function is increasing if its derivative is positive throughout that interval.
- For the natural logarithm, $\frac{d}{dx} \log x = \frac{1}{x}$. The quotient rule gives $(u/v)' = (u'v - uv')/v^2$, where $v \neq 0$.

Step 1: Differentiate on the given domain For $x > 0$, $x \neq 1$, the denominator is nonzero. Applying the quotient rule,

$$f'(x) = \frac{(\log x)(1) - x(1/x)}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}.$$

Thus the derivative exists on both components of the stated domain.

Step 2: Determine the sign of the derivative Since $(\log x)^2 > 0$ everywhere in the domain, the sign of

Quick Tip to Solve: Because the denominator of the derivative is a nonzero square, solve only $\log x - 1 > 0$ and then check the domain.

$f'(x)$ is the sign of $\log x - 1$. Hence

$$f'(x) > 0 \iff \log x > 1 \iff x > e.$$

For $0 < x < 1$ and $1 < x < e$, the derivative is negative, so neither interval contributes an increasing portion. At $x = e$, the derivative is zero; this is the boundary between decrease and increase, not a separate increasing interval. The value $x = 1$ is excluded from the domain.

Step 3: Conclusion The required interval of increase is (e, ∞) .

Quick Tip to Solve: Because the denominator of the derivative is a nonzero square, solve only $\log x - 1 > 0$ and then check the domain.

24. If $(2\hat{i} + 3\hat{j} + \hat{k})$ and $(2\hat{i} + \hat{j} - \hat{k})$ represent the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of $\triangle ABC$, find the vector representing the median through A . [2 marks]

Solution:

Concept:

- A median joins a vertex to the midpoint of the opposite side. Let M be the midpoint of BC ; the required directed vector is $\overrightarrow{AM}$.
- With A as the reference point, the midpoint formula is $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$. Add corresponding components before dividing by two.

Step 1: Express the midpoint vector Write $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{AC}$. Since M bisects BC ,

$$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\vec{c} - \vec{b}).$$

Using the triangle law along A, B, M ,

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \vec{b} + \frac{1}{2}(\vec{c} - \vec{b}) = \frac{1}{2}(\vec{b} + \vec{c}).$$

Step 2: Substitute the two side vectors The given side vectors start at the same vertex A , so their average gives the vector to the midpoint:

$$\begin{aligned}\overrightarrow{AM} &= \frac{1}{2}[(2\hat{i} + 3\hat{j} + \hat{k}) + (2\hat{i} + \hat{j} - \hat{k})] \\ &= \frac{1}{2}(4\hat{i} + 4\hat{j} + 0\hat{k}).\end{aligned}$$

In particular, the opposite $\hat{k}$ components cancel, as expected for the midpoint of endpoints with third components 1 and -1 .

Step 3: Conclusion The vector representing the median from A to BC is $\overrightarrow{AM} = 2\hat{i} + 2\hat{j}$.

Quick Tip to Solve: When both side vectors start at the median's vertex, average them directly; their difference instead gives the opposite side.

25(a). Show that the function $f(x) = \begin{cases} \frac{\cos x}{-x + \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$. [2 marks]

Solution:

Concept:

- A function is continuous at $x = a$ precisely when $\lim_{x \rightarrow a} f(x) = f(a)$; both one-sided limits must agree with the assigned value.
- The standard trigonometric limit is $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$, with angles in radians. Also, $\cos(\pi/2 + h) = -\sin h$.

Step 1: Record the value at the joining point Put $a = \pi/2$. The second branch of the definition applies at $x = a$, and therefore $f(a) = 1$. The first branch is used only for $x \neq a$; its zero denominator at a does not determine the assigned function value.

Step 2: Compute the limit through nearby points Let $h = x - \pi/2$. Then $x \rightarrow \pi/2$ means $h \rightarrow 0$, and for $h \neq 0$,

$$\begin{aligned} f\left(\frac{\pi}{2} + h\right) &= \frac{\cos(\pi/2 + h)}{-(\pi/2 + h) + \pi/2} \\ &= \frac{-\sin h}{-h} = \frac{\sin h}{h}. \end{aligned}$$

Hence $\lim_{x \rightarrow \pi/2} f(x) = 1$. The same standard limit holds whether h approaches zero from the left or the right, so both one-sided limits exist and are equal. They agree with the value found from the branch at the joining point.

Step 3: Conclusion Since $\lim_{x \rightarrow \pi/2} f(x) = 1 = f(\pi/2)$, the function is continuous at $x = \pi/2$.

Quick Tip to Solve: Shift the joining point to zero with $h = x - \pi/2$; the two negative signs cancel and expose the limit $\sin h/h$.

OR

25(b). Find whether the function $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$ at $x = 2$ is differentiable or not. [2 marks]

Solution:

Concept:

- Differentiability at $x = a$ requires the two one-sided difference quotients to have the same finite limit.
- Continuity is necessary for differentiability, but it is not sufficient: two straight-line branches can meet and still have different slopes.

Step 1: Check the joining value and continuity Since 2 belongs to the second branch, $f(2) = 2(2) - 3 =$

Quick Tip to Solve: At a piecewise linear join, first compare the values, then compare the slopes; equal values alone do not prove differentiability.

1. The limits of the two branches are

$$\lim_{x \rightarrow 2^-} (x - 1) = 1, \quad \lim_{x \rightarrow 2^+} (2x - 3) = 1.$$

Thus both limits equal $f(2)$, so the function is continuous at the joining point. We must additionally check its derivatives there.

Step 2: Compare the one-sided derivatives For $h < 0$, $2 + h < 2$, and $f(2 + h) = 1 + h$. For $h > 0$, $f(2 + h) = 1 + 2h$. Therefore

$$f'_-(2) = \lim_{h \rightarrow 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0^-} \frac{h}{h} = 1,$$
$$f'_+(2) = \lim_{h \rightarrow 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0^+} \frac{2h}{h} = 2.$$

The two one-sided slopes differ, so the difference quotient has no common two-sided limit at 2.

Step 3: Conclusion Since $f'_-(2) = 1 \neq 2 = f'_+(2)$, f is not differentiable at $x = 2$, although it is continuous there.

Quick Tip to Solve: At a piecewise linear join, first compare the values, then compare the slopes; equal values alone do not prove differentiability.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. In a school, the probability of holding a debate competition is $\frac{1}{3}$ and that of a quiz competition is $\frac{2}{3}$. In the two participating teams, A has 4 girls and 6 boys and B has 7 girls and 3 boys. If a debate competition is held, the students are selected from team A and for the quiz competition they are selected from team B . If only two students are to be chosen from the teams, then find the probability that one will be a girl and the other a boy. [3 marks]

Solution:

Concept:

- The law of total probability gives $P(E) = P(D)P(E | D) + P(Q)P(E | Q)$, where D and Q are mutually exclusive competition types covering all possibilities.
- Selecting two distinct students without replacement from n students gives $\binom{n}{2}$ equally likely unordered pairs. With g girls and b boys, gb pairs contain one of each.

Step 1: Define the events and choose the correct team Let D denote a debate, Q a quiz, and E the event that the selected pair contains one girl and one boy. The printed probabilities are $P(D) = 1/3$ and $P(Q) = 2/3$. A debate selects both students from team A , whereas a quiz selects both from team B . Each team contains 10 students, so it has $\binom{10}{2} = 45$ possible pairs.

Step 2: Find the conditional probabilities Team A contains 4 girls and 6 boys. A favourable pair is

Quick Tip to Solve: Compute one-girl-one-boy probability within each team first, then weight by the debate and quiz probabilities printed in the question.

formed by choosing one of each, giving $4 \times 6 = 24$ pairs. Thus

$$P(E | D) = \frac{\binom{4}{1} \binom{6}{1}}{\binom{10}{2}} = \frac{24}{45} = \frac{8}{15}.$$

Team B contains 7 girls and 3 boys, so similarly

$$P(E | Q) = \frac{\binom{7}{1} \binom{3}{1}}{\binom{10}{2}} = \frac{21}{45} = \frac{7}{15}.$$

Counting unordered pairs avoids double-counting the girl-boy and boy-girl orders. Equivalently, ordered sampling would use denominator 10×9 and twice the favourable count.

Step 3: Weight by the competition probabilities Since exactly one of the two competitions is held, the two cases are disjoint and exhaustive. Therefore

$$\begin{aligned} P(E) &= \frac{1}{3} \cdot \frac{8}{15} + \frac{2}{3} \cdot \frac{7}{15} \\ &= \frac{8}{45} + \frac{14}{45} = \frac{22}{45}. \end{aligned}$$

This weighted average lies between $7/15$ and $8/15$, which checks the arithmetic and the weighting.

Step 4: Conclusion The probability that the two selected students consist of one girl and one boy is

$$\frac{22}{45}.$$

Quick Tip to Solve: Compute one-girl-one-boy probability within each team first, then weight by the debate and quiz probabilities printed in the question.

27. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, then compute $A^2 - 4A - 5I$.

[3 marks]

Solution:

Concept:

- Matrix multiplication uses row-column products: the entry in position (i, j) of A^2 is $\sum_{k=1}^3 a_{ik} a_{kj}$.
- Matrix addition and subtraction are entrywise. The identity matrix I has 1 on its main diagonal and 0 elsewhere, so subtracting $5I$ changes only diagonal entries.

Step 1: Compute the square of the matrix With $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, multiply each row of the first matrix

Quick Tip to Solve: Exploit the repeated entries: compute one diagonal and one off-diagonal entry of A^2 , then check that $A^2 = 4A + 5I$.

by each column of the second:

$$A^2 = \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}.$$

Thus each diagonal entry is 9, whereas every off-diagonal entry is 8. This also checks the symmetry of the product, since the given matrix is symmetric.

Step 2: Subtract the scalar multiples entrywise First calculate the two matrices to be subtracted:

$$4A = \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix}, \quad 5I = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Hence

$$A^2 - 4A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5I.$$

In the required expression each diagonal entry therefore becomes $9 - 4 - 5 = 0$, and each off-diagonal entry becomes $8 - 8 - 0 = 0$. Notice that $5I$, rather than a matrix with 5 in every position, must be subtracted.

Step 3: Conclusion Therefore, $A^2 - 4A - 5I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the zero matrix of order 3×3 .

Quick Tip to Solve: Exploit the repeated entries: compute one diagonal and one off-diagonal entry of A^2 , then check that $A^2 = 4A + 5I$.

28(a). Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. A function $f : A \rightarrow B$ is defined by $f(x) = \frac{x-2}{x-3}$. Find whether f is one-one and onto. [3 marks]

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for any two elements of its domain.
- A function $f : A \rightarrow B$ is onto if every $y \in B$ has a preimage $x \in A$ satisfying $f(x) = y$. The proposed preimage must belong to the specified domain.

Step 1: Check the domain and codomain For $x \in A = \mathbb{R} \setminus \{3\}$, the denominator $x - 3$ is nonzero. Also,

$$f(x) = \frac{x-2}{x-3} = 1 + \frac{1}{x-3}.$$

Since $1/(x-3) \neq 0$, the value $f(x)$ cannot equal 1. Thus every value belongs to $B = \mathbb{R} \setminus \{1\}$, and the given function is well defined.

Quick Tip to Solve: Rewrite the function as $1 + 1/(x-3)$; then solve for x and check that the inverse never produces the excluded input 3.

Step 2: Prove that the function is one-one Take $x_1, x_2 \in A$ and suppose that $f(x_1) = f(x_2)$. Cross-multiplication is valid because both denominators are nonzero:

$$\begin{aligned}(x_1 - 2)(x_2 - 3) &= (x_2 - 2)(x_1 - 3), \\ x_1x_2 - 3x_1 - 2x_2 + 6 &= x_1x_2 - 3x_2 - 2x_1 + 6, \\ -x_1 + x_2 &= 0.\end{aligned}$$

Therefore $x_1 = x_2$, establishing the one-one property for arbitrary elements of the domain.

Step 3: Prove that every codomain value has a preimage Let $y \in B$, so $y \neq 1$. Solving $y = (x - 2)/(x - 3)$ gives

$$yx - 3y = x - 2, \quad x = \frac{3y - 2}{y - 1}.$$

This is real and $x - 3 = 1/(y - 1) \neq 0$, so $x \in A$. Moreover $x - 2 = y/(y - 1)$, and substituting into the function gives

$$f(x) = \frac{y/(y - 1)}{1/(y - 1)} = y.$$

The construction works for each arbitrary $y \in B$; hence f is onto.

Step 4: Conclusion The function f is both one-one and onto; therefore it is a bijection from A to B .

Quick Tip to Solve: Rewrite the function as $1 + 1/(x - 3)$; then solve for x and check that the inverse never produces the excluded input 3.

OR

28(b). Let n be a fixed positive integer. A relation R is defined in set $\mathbb{Z}$ such that $R = \{(x, y) : (x - y) \text{ is divisible by } n, x, y \in \mathbb{Z}\}$. Determine if R is an equivalence relation. [3 marks]

Solution:

Concept:

- An equivalence relation is a relation that is reflexive, symmetric and transitive on its specified set.
- For integers x, y , the condition $n \mid (x - y)$ means $x - y = nk$ for some integer k , where n is the fixed positive integer in the question.

Step 1: Verify reflexivity on the integers Let x be any integer. Then

$$x - x = 0 = n \cdot 0.$$

Since $0 \in \mathbb{Z}$, the difference is divisible by n . Consequently $(x, x) \in R$ for every $x \in \mathbb{Z}$, so R is reflexive. Divisibility includes a zero difference; the integer multiplier need not be positive.

Step 2: Verify symmetry Take integers x, y such that $(x, y) \in R$. By the definition of R , there is an

Quick Tip to Solve: Replace each divisibility statement by an integer multiple of n ; negate that multiple for symmetry and add two multiples for transitivity.

integer k for which $x - y = nk$. Reversing the order gives

$$y - x = -(x - y) = n(-k).$$

The multiplier $-k$ is also an integer. Therefore $n \mid (y - x)$, which means $(y, x) \in R$. Thus R is symmetric; changing the sign of a multiple of n preserves divisibility.

Step 3: Verify transitivity Let $(x, y) \in R$ and $(y, z) \in R$, with $x, y, z \in \mathbb{Z}$. There exist integers k, ℓ such that

$$x - y = nk, \quad y - z = n\ell.$$

Adding these equalities eliminates the middle element:

$$x - z = (x - y) + (y - z) = n(k + \ell).$$

As $k + \ell \in \mathbb{Z}$, the difference $x - z$ is divisible by n , so $(x, z) \in R$. This proves transitivity for every such pair of related integers.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, R is an equivalence relation on $\mathbb{Z}$, for every fixed positive integer n .

Quick Tip to Solve: Replace each divisibility statement by an integer multiple of n ; negate that multiple for symmetry and add two multiples for transitivity.

29. Solve the following Linear Programming Problem graphically:

Minimize $Z = 20x + 10y$

subject to

$$x + 2y \leq 40$$

$$3x + y \geq 30$$

$$4x + 3y \geq 60$$

$$x, y \geq 0$$

[3 marks]

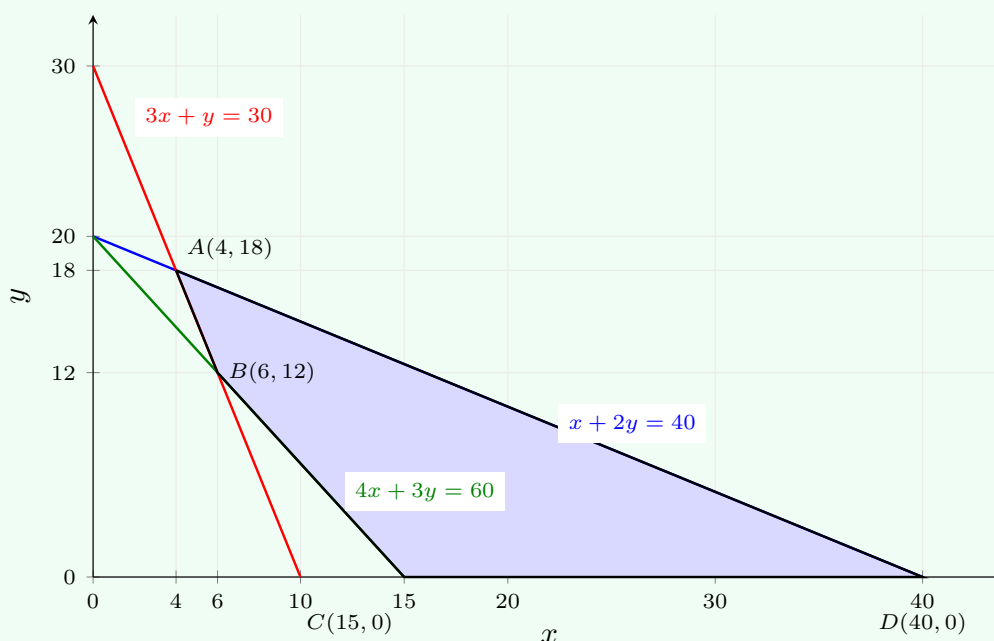
Solution:

Concept:

- Each linear inequality represents a closed half-plane. The feasible region is the intersection of these half-planes with the first quadrant, where $x, y \geq 0$.
- For a linear objective function on a nonempty bounded polygon, a minimum occurs at a corner point. Evaluate $Z = 20x + 10y$ at every feasible vertex.

Step 1: Plot the boundary lines and identify the feasible sides Draw $x + 2y = 40$ through $(40, 0)$ and $(0, 20)$, $3x + y = 30$ through $(10, 0)$ and $(0, 30)$, and $4x + 3y = 60$ through $(15, 0)$ and $(0, 20)$. The origin satisfies the first inequality but fails the other two. Thus take the side below $x + 2y = 40$, and the sides above $3x + y = 30$ and $4x + 3y = 60$, all within the first quadrant. Their common shaded region is the quadrilateral shown below.

Quick Tip to Solve: Check each intersection against all three inequalities before including it in the corner-value table.



The shaded quadrilateral ABCD is the feasible region.

Step 2: Calculate the feasible vertices Solving $x + 2y = 40$ with $3x + y = 30$ gives $A = (4, 18)$. Solving $3x + y = 30$ with $4x + 3y = 60$ gives $B = (6, 12)$. On the x -axis, the inequalities require $x \leq 40$, $x \geq 10$, and $x \geq 15$, so the feasible endpoints are $C = (15, 0)$ and $D = (40, 0)$. The intersection $(0, 20)$ of the first and third lines is infeasible since $3x + y = 20 < 30$; it is not a vertex of the common region.

Step 3: Evaluate the objective function at the vertices The region is nonempty and bounded. Its corner values are

(x, y)	$Z = 20x + 10y$
$(4, 18)$	$80 + 180 = 260$
$(6, 12)$	$120 + 120 = 240$
$(15, 0)$	300
$(40, 0)$	800

Comparing these values locates the minimizing corner. The adjacent edges do not have the same slope as a level line $20x + 10y = k$, so the minimum is attained at a single vertex.

Step 4: Conclusion The minimum value is $Z_{\min} = 240$, attained at $x = 6, y = 12$.

Quick Tip to Solve: Check each intersection against all three inequalities before including it in the corner-value table.

30(a). If $xy = e^{x-y}$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- In an implicit relation, y is treated as a differentiable function of x . The product rule gives $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$.
- By the chain rule, $\frac{d}{dx}e^{x-y} = e^{x-y}\left(1 - \frac{dy}{dx}\right)$. The original equation can then replace the exponential by xy .

Step 1: Differentiate both sides of the printed relation Let $y' = \frac{dy}{dx}$. Starting with $xy = e^{x-y}$, differentiate with respect to x . The left side is a product of x and $y(x)$, while the exponent on the right is also a function of x . Therefore,

$$y + xy' = e^{x-y}(1 - y').$$

The term $-y'$ inside the bracket comes from differentiating $-y$; it must not be replaced by a constant.

Step 2: Collect the derivative terms and simplify Expanding the right side and moving its derivative term to the left gives

$$\begin{aligned} xy' + e^{x-y}y' &= e^{x-y} - y, \\ (x + e^{x-y})y' &= e^{x-y} - y. \end{aligned}$$

Use $e^{x-y} = xy$ from the given relation. Then

$$x(1 + y)y' = y(x - 1).$$

Division is justified on this real curve: $xy = e^{x-y} > 0$, so $x \neq 0$. Also $y = -1$ would require, with $t = -x > 0$, $t = e^{1-t}$. The function $\log t + t$ is strictly increasing and equals 1 only at $t = 1$, yielding the exceptional point $(-1, -1)$. At that point the differentiated equation gives $0 \cdot y' = 2$, so a finite dy/dx does not exist. Away from it, the usual implicit derivative is well defined.

Step 3: Conclusion Wherever the curve admits a finite derivative,

$$\boxed{\frac{dy}{dx} = \frac{e^{x-y} - y}{x + e^{x-y}} = \frac{y(x - 1)}{x(1 + y)}}.$$

Quick Tip to Solve: Differentiate xy as a product first, then replace the exponential by xy only after collecting the derivative terms.

30(b). Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right)$ with respect to $\cos^{-1} x^2$.

[3 marks]

Solution:**Concept:**

- To differentiate u with respect to v , express u directly in terms of v when possible. Here a half-angle substitution simplifies both square roots.
- The identity $\tan(\pi/4 - \alpha) = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha}$ converts the radical ratio into a tangent. Principal inverse-trigonometric ranges must be checked before cancelling $\tan^{-1}$ with $\tan$.

Step 1: Introduce the variable with respect to which differentiation is required Put $v = \cos^{-1}(x^2)$. The radicals require $|x| \leq 1$, and $x^2 \in [0, 1]$, so $v \in [0, \pi/2]$. Since $x^2 = \cos v$, the half-angle identities give

$$\begin{aligned}\sqrt{1+x^2} &= \sqrt{1+\cos v} = \sqrt{2} \cos(v/2), \\ \sqrt{1-x^2} &= \sqrt{1-\cos v} = \sqrt{2} \sin(v/2).\end{aligned}$$

Both half-angle functions are nonnegative on this range, so these square-root simplifications have the correct signs.

Step 2: Simplify the inverse tangent using its principal range Denote the function to be differentiated by u . Substitution into its printed numerator and denominator gives

$$\begin{aligned}u &= \tan^{-1} \left(\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)} \right) \\ &= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{v}{2} \right) \right].\end{aligned}$$

Because $0 \leq v \leq \pi/2$, the angle $\pi/4 - v/2$ lies in $[0, \pi/4]$, inside the principal range of $\tan^{-1}$. Hence $u = \pi/4 - v/2$, with no additional multiple of π .

Step 3: Differentiate the resulting linear relation Differentiating $u = \pi/4 - v/2$ directly with respect to v gives $\frac{du}{dv} = 0 - \frac{1}{2}$. This method avoids introducing derivatives with respect to x and then dividing two expressions containing a common factor of x ; the relation between u and v already gives the requested derivative directly.

Step 4: Conclusion The derivative of the given function with respect to $\cos^{-1}(x^2)$ is $\boxed{-\frac{1}{2}}$.

Quick Tip to Solve: Set the inverse-cosine expression equal to a new angle and use half-angle identities before differentiating the radical ratio.

31(a). Find a point on the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2}$ at a distance of $\sqrt{2}$ units from the point $(1, 2, 3)$. [3 marks]

Solution:

Concept:

- A point on a symmetric-form line is obtained by setting all three ratios equal to a real parameter t .
- The squared distance between (x, y, z) and (a, b, c) is $(x-a)^2 + (y-b)^2 + (z-c)^2$. Squaring the required nonnegative distance removes the radical.

Step 1: Parameterise the given line The printed middle ratio $\frac{1-y}{2}$ equals $\frac{y-1}{-2}$. Set

$$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2} = t.$$

A general point is therefore $P(2+3t, 1-2t, 3+2t)$. Its displacement from $A(1, 2, 3)$ is $(1+3t, -1-2t, 2t)$. The negative sign in the middle coordinate must be retained.

Step 2: Apply the distance condition Using $AP = \sqrt{2}$, we obtain

$$\begin{aligned}(1+3t)^2 + (-1-2t)^2 + (2t)^2 &= 2, \\ 1 + 6t + 9t^2 + 1 + 4t + 4t^2 + 4t^2 &= 2, \\ 17t^2 + 10t &= 0, \\ t(17t + 10) &= 0.\end{aligned}$$

Thus $t = 0$ or $t = -\frac{10}{17}$. Both values are real; the sphere with centre A and radius $\sqrt{2}$ intersects the line in two points.

Step 3: Substitute and check the points At $t = 0$, the point is $(2, 1, 3)$, whose squared distance from A is $1 + 1 = 2$. At $t = -10/17$, substitution gives

$$\left(2 - \frac{30}{17}, 1 + \frac{20}{17}, 3 - \frac{20}{17}\right) = \left(\frac{4}{17}, \frac{37}{17}, \frac{31}{17}\right).$$

The second point has squared distance $\frac{(-13)^2 + 3^2 + (-20)^2}{17^2} = \frac{578}{289} = 2$, so it also satisfies the condition. Either point answers the request for a point on the line.

Step 4: Conclusion One required point is $(2, 1, 3)$. The other possible point is $\left(\frac{4}{17}, \frac{37}{17}, \frac{31}{17}\right)$.

Quick Tip to Solve: First test the base point of the line: here $(2, 1, 3)$ already has the required distance from $(1, 2, 3)$.

31(b). Find the shortest distance between the lines $\vec{r} = (4 + \lambda)\hat{i} + (2\lambda - 1)\hat{j} - 3\lambda\hat{k}$ and $\vec{r} = (1 + 2\mu)\hat{i} + (2 - 5\mu)\hat{k} + (4\mu - 1)\hat{j}$. [3 marks]

Solution:

Concept:

- For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, a common perpendicular has direction $\vec{b}_1 \times \vec{b}_2$.
- For nonparallel directions the shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. This is the magnitude of the projection of the displacement between the lines along their common perpendicular.

Step 1: Identify a point and direction on each line Reading coordinates in the order $\hat{i}, \hat{j}, \hat{k}$, the first line is

$$\vec{r} = (4, -1, 0) + \lambda(1, 2, -3).$$

Thus $\vec{a}_1 = (4, -1, 0)$ and $\vec{b}_1 = (1, 2, -3)$. The coefficient of $\hat{k}$ is -3λ , so its constant part is zero. The second line prints the $\hat{k}$ term before the $\hat{j}$ term; in coordinate order it becomes

$$\vec{r} = (1, -1, 2) + \mu(2, 4, -5).$$

Hence $\vec{a}_2 = (1, -1, 2)$, $\vec{b}_2 = (2, 4, -5)$ and $\vec{a}_2 - \vec{a}_1 = (-3, 0, 2)$.

Step 2: Compute the direction of the common perpendicular The cross product gives

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} \\ &= [2(-5) - (-3)(4)]\hat{i} - [1(-5) - (-3)(2)]\hat{j} + [1(4) - 2(2)]\hat{k} \\ &= 2\hat{i} - \hat{j}.\end{aligned}$$

Its magnitude is $\sqrt{2^2 + (-1)^2} = \sqrt{5}$, which is nonzero. Therefore the lines are not parallel and the scalar triple product formula applies.

Step 3: Project the displacement onto the common perpendicular Evaluate the scalar triple product:

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (-3)(2) + 0(-1) + 2(0) = -6.$$

Its nonzero value also shows that these nonparallel lines are skew. Their shortest distance is therefore

$$d = \frac{|-6|}{\sqrt{5}} = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}.$$

The absolute value removes the sign depending on the chosen order of the vectors, since a distance must be nonnegative.

Step 4: Conclusion The shortest distance between the two lines is $\boxed{\frac{6\sqrt{5}}{5} \text{ units}}.$

Quick Tip to Solve: Collect each coordinate's constant and parameter terms separately, and put both lines into $\hat{i}, \hat{j}, \hat{k}$ order before taking the cross product.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Solve the differential equation $ye^y dx = (y^3 + 2xe^y) dy$, when $y(0) = 1$.

[5 marks]

Solution:

Concept:

- An equation containing dx and dy may become linear by treating x as the dependent variable and y as the independent variable.
- For $\frac{dx}{dy} + P(y)x = Q(y)$, the integrating factor is $I(y) = e^{\int P(y) dy}$. Multiplication gives $\frac{d}{dy}(Ix) = IQ$.
- The condition $y(0) = 1$ means the solution curve passes through $(x, y) = (0, 1)$; use this point to determine the integration constant.

Step 1: Write a linear equation in x Starting from the printed equation,

$$ye^y dx = (y^3 + 2xe^y) dy,$$

treat x as a function of y . Near the prescribed point $y = 1$, division by ye^y is valid, yielding

$$\frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}, \quad \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}.$$

This is linear in x , with $P(y) = -2/y$ and $Q(y) = y^2 e^{-y}$. The relevant interval containing the initial point has $y > 0$.

Step 2: Find the integrating factor For $y > 0$, the integrating factor is

$$I(y) = \exp\left(\int -\frac{2}{y} dy\right) = e^{-2 \ln y} = y^{-2}.$$

Multiplying every term of the linear equation by y^{-2} gives

$$y^{-2} \frac{dx}{dy} - 2y^{-3}x = e^{-y}.$$

By the product rule, the left side is exactly $\frac{d}{dy}(xy^{-2})$. Thus the chosen factor converts the whole left side into a single derivative.

Step 3: Integrate with respect to y The equation now becomes

$$\frac{d}{dy}\left(\frac{x}{y^2}\right) = e^{-y}.$$

Integrating on the interval containing $y = 1$,

$$\frac{x}{y^2} = -e^{-y} + C, \quad x = y^2(C - e^{-y}).$$

Quick Tip to Solve: When x occurs only to the first power, try x as the dependent variable; here the integrating factor y^{-2} cancels the y^2 on the right.

The negative sign arises because the derivative of e^{-y} is $-e^{-y}$. The constant C is determined by the given initial point.

Step 4: Apply the initial condition and verify Substituting $x = 0$, $y = 1$ gives $0 = -e^{-1} + C$, so $C = e^{-1}$. The required implicit relation is therefore

$$x = y^2(e^{-1} - e^{-y}).$$

To check it, differentiate with respect to y :

$$\frac{dx}{dy} = 2y(e^{-1} - e^{-y}) + y^2 e^{-y} = \frac{2x}{y} + y^2 e^{-y}.$$

Multiplication by $ye^y dy$ recovers the original differential equation. The relation also gives $x = 0$ at $y = 1$. At that point $dx/dy = e^{-1} \neq 0$, so it defines a local solution y as a function of x .

Step 5: Conclusion The particular solution satisfying $y(0) = 1$ is $x = y^2 \left(\frac{1}{e} - e^{-y} \right)$, with the branch through $(0, 1)$.

Quick Tip to Solve: When x occurs only to the first power, try x as the dependent variable; here the integrating factor y^{-2} cancels the y^2 on the right.

OR

32(b). Find the general solution of the differential equation $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$. [5 marks]

Solution:

Concept:

- A differential equation is homogeneous when the numerator and denominator in dy/dx are homogeneous polynomials of the same degree. Substituting $y = vx$, with $v = y/x$, gives $dy/dx = v + x dv/dx$.
- After separating the variables, integrate both sides and retain an arbitrary constant. Solutions lost by division must be checked in the original equation.

Step 1: Express the equation in homogeneous form The printed equation is

$$(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy.$$

Where the denominator is nonzero, it gives

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{1 - 3(y/x)^2}{(y/x)^3 - 3(y/x)}.$$

Both polynomials have degree three, so the substitution $y = vx$ reduces this equation to a separable equation on intervals with $x \neq 0$.

Quick Tip to Solve: After setting $y = vx$, use $t = v^2$ in the separated integral: it converts the quartic denominator into simple partial fractions.

Step 2: Substitute and separate the variables Put $y = vx$. Then

$$v + x \frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}.$$

Subtracting v and combining the numerator carefully,

$$x \frac{dv}{dx} = \frac{1 - 3v^2 - v(v^3 - 3v)}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}.$$

Thus, for $v^4 \neq 1$,

$$\frac{v^3 - 3v}{1 - v^4} dv = \frac{dx}{x}.$$

Step 3: Evaluate the integral using partial fractions Set $t = v^2$, so that $dt = 2v dv$. The left-hand integral becomes

$$\int \frac{t - 3}{2(1 - t^2)} dt.$$

The required decomposition is

$$\frac{t - 3}{2(1 - t^2)} = -\frac{1}{2(1 - t)} - \frac{1}{1 + t}.$$

Integrating, with C_1 an arbitrary constant,

$$\frac{1}{2} \ln |1 - v^2| - \ln(1 + v^2) = \ln |x| + C_1.$$

Since $1 + v^2 > 0$ for real v , no absolute value is needed in its logarithm.

Step 4: Return to the original variables and check excluded solutions Multiply the logarithmic equation by two, combine logarithms and exponentiate. Absorbing the sign and the integration constant into a real constant C gives

$$\frac{1 - v^2}{(1 + v^2)^2} = Cx^2.$$

Substituting $v = y/x$ and cancelling x^2 yields

$$x^2 - y^2 = C(x^2 + y^2)^2.$$

The separation excluded the real constant ratios $v = 1$ and $v = -1$. Direct substitution shows that both $y = x$ and $y = -x$ satisfy the original differential equation; they are already included by setting $C = 0$. As a verification, differentiating $(x^2 - y^2)/(x^2 + y^2)^2$ away from the origin gives

$$d\left(\frac{x^2 - y^2}{(x^2 + y^2)^2}\right) = \frac{-2[(x^3 - 3xy^2)dx - (y^3 - 3x^2y)dy]}{(x^2 + y^2)^3}.$$

Hence the implicit family satisfies the printed equation.

Step 5: Conclusion The general solution is $x^2 - y^2 = C(x^2 + y^2)^2$, where C is an arbitrary real constant. The straight-line solutions $y = \pm x$ occur for $C = 0$.

Quick Tip to Solve: After setting $y = vx$, use $t = v^2$ in the separated integral: it converts the quartic denominator into simple partial fractions.

33. Using integration, find the area of the region bounded by the curve $y = x|x|$, x -axis, $x = -2$ and $x = 2$. [5 marks]

Solution:

Concept:

- The geometric area between $y = f(x)$ and the x -axis on $[a, b]$ is $A = \int_a^b |f(x)| dx$. A definite integral of f itself gives signed area, so portions below the axis require a change of sign.
- Absolute value is evaluated piecewise: $|x| = -x$ for $x < 0$, and $|x| = x$ for $x \geq 0$. The power rule is $\int x^2 dx = x^3/3 + C$.

Step 1: Write the curve separately on the two sides of the origin For the given curve $y = x|x|$, the sign of x determines its expression:

$$y = \begin{cases} -x^2, & -2 \leq x < 0, \\ x^2, & 0 \leq x \leq 2. \end{cases}$$

The curve meets the x -axis at $x = 0$. It lies below the axis on $[-2, 0)$ and above it on $(0, 2]$. The vertical lines $x = -2$ and $x = 2$ close the two portions of the region. We therefore split the area integral at the origin, where the sign changes.

Step 2: Find the area on the negative side On $[-2, 0]$, the upper boundary is $y = 0$ and the lower boundary is $y = -x^2$. The vertical distance between them is $0 - (-x^2) = x^2$. Hence

$$\begin{aligned} A_1 &= \int_{-2}^0 [0 - (-x^2)] dx \\ &= \int_{-2}^0 x^2 dx \\ &= \left[\frac{x^3}{3} \right]_{-2}^0 \\ &= 0 - \frac{(-2)^3}{3} = \frac{8}{3}. \end{aligned}$$

This is a positive area even though the curve has negative ordinates on this interval.

Step 3: Find the area on the positive side On $[0, 2]$, the upper boundary is $y = x^2$, and the lower boundary is $y = 0$. Thus

$$\begin{aligned} A_2 &= \int_0^2 (x^2 - 0) dx \\ &= \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{2^3}{3} - 0 = \frac{8}{3}. \end{aligned}$$

The two portions have equal areas because their heights at opposite values of x have the same magnitude.

Step 4: Add the geometric areas and check the sign convention The bounded region consists of both

Quick Tip to Solve: For $y = x|x|$, the height above or below the axis is x^2 ; use $2 \int_0^2 x^2 dx$ after identifying the two equal portions.

portions, so their positive areas must be added:

$$A = A_1 + A_2 = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.$$

Equivalently, $|y| = |x| |x| = x^2$, so $A = \int_{-2}^2 x^2 dx$. As a separate check, integrating $x|x|$ without taking its absolute value would give $-8/3 + 8/3 = 0$. That cancellation describes the signed integral, not the area of the bounded region, and explains why splitting at $x = 0$ is necessary.

Step 5: Conclusion The required area is $\boxed{\frac{16}{3} \text{ square units}}$.

Quick Tip to Solve: For $y = x|x|$, the height above or below the axis is x^2 ; use $2 \int_0^2 x^2 dx$ after identifying the two equal portions.

34(a). Find: $\int \frac{x dx}{(x-1)(x^2+4)}$

[5 marks]

Solution:

Concept:

- A rational function whose denominator has a linear factor and an irreducible quadratic factor is decomposed into partial fractions of the form $A/(x-1) + (Bx+C)/(x^2+4)$, where A, B, C are constants.
- The standard integrals are $\int dx/(x-a) = \ln|x-a| + C_0$ and $\int dx/(x^2+a^2) = a^{-1} \tan^{-1}(x/a) + C_0$, where $a > 0$ in the second formula.
- A numerator proportional to the derivative of a denominator gives a logarithm: $\int f'(x)/f(x) dx = \ln|f(x)| + C_0$. Here the derivative of x^2+4 is $2x$.

Step 1: Set up the partial fractions Let the required integral be I . Since the numerator has smaller degree than the denominator, no polynomial division is necessary. The numerator over the quadratic factor must be linear, so write

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Multiplying by the common denominator gives the polynomial identity

$$x = A(x^2+4) + (Bx+C)(x-1).$$

This identity determines the constants independently of the value of x .

Step 2: Determine the three constants Expanding the right side and collecting equal powers of x , we obtain

$$x = (A+B)x^2 + (C-B)x + (4A-C).$$

Comparing coefficients yields $A+B=0$, $C-B=1$ and $4A-C=0$. Thus $B=-A$ and $C=4A$;

Quick Tip to Solve: After partial fractions, split the quadratic numerator into an x -term and a constant: the former produces a logarithm and the latter an inverse tangent.

substitution in $C - B = 1$ gives $5A = 1$. Therefore

$$A = \frac{1}{5}, \quad B = -\frac{1}{5}, \quad C = \frac{4}{5}.$$

Consequently the integrand equals

$$\frac{1}{5(x-1)} - \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)}.$$

Step 3: Integrate each term By linearity of integration,

$$I = \frac{1}{5} \int \frac{dx}{x-1} - \frac{1}{5} \int \frac{x dx}{x^2+4} + \frac{4}{5} \int \frac{dx}{x^2+4}.$$

For the middle term, put $t = x^2 + 4$, so $dt = 2x dx$; its integral is $\frac{1}{2} \ln(x^2 + 4)$. For the last term use the standard inverse-tangent formula with $a = 2$; its integral is $\frac{1}{2} \tan^{-1}(x/2)$. The first term gives $\ln|x-1|$. The absolute value is needed only for the linear factor, because $x^2 + 4$ is positive for every real x .

Step 4: Check the antiderivative Differentiating the three obtained terms gives respectively $1/[5(x-1)]$, $-x/[5(x^2+4)]$ and $4/[5(x^2+4)]$. Their sum, on taking a common denominator, has numerator

$$\frac{1}{5} [(x^2+4) + (-x+4)(x-1)] = x.$$

Thus the derivative equals the original integrand. The antiderivative is valid on each interval of its domain, which excludes $x = 1$.

Step 5: Conclusion

$$I = \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + C_0.$$

Here C_0 is an arbitrary constant of integration.

Quick Tip to Solve: After partial fractions, split the quadratic numerator into an x -term and a constant: the former produces a logarithm and the latter an inverse tangent.

OR

34(b). Evaluate: $\int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$

[5 marks]

Solution:

Concept:

- Integration by parts for definite integrals is $\int_a^b u dv = [uv]_a^b - \int_a^b v du$. Choose the inverse-tangent factor as u , because its derivative is the simpler rational function $1/(1+x^2)$.
- The chain rule gives $d(1+x^2)^{-1/2}/dx = -x(1+x^2)^{-3/2}$, so the remaining factor has an immediate

Quick Tip to Solve: Differentiate the inverse tangent and integrate the other factor in integration by parts; the resulting integral simplifies immediately with $x = \tan \theta$.

antiderivative.

- To integrate $(1+x^2)^{-3/2}$, one can set $x = \tan \theta$. Then $1+x^2 = \sec^2 \theta$ and $dx = \sec^2 \theta d\theta$, with the limits transformed along with the variable.

Step 1: Choose the factors for integration by parts Denote the required integral by I . Put

$$u = \tan^{-1} x, \quad dv = \frac{x}{(1+x^2)^{3/2}} dx.$$

Then $du = dx/(1+x^2)$. To obtain v , substitute $t = 1+x^2$, which gives $dt = 2x dx$. Hence

$$v = \frac{1}{2} \int t^{-3/2} dt = -t^{-1/2} = -\frac{1}{\sqrt{1+x^2}}.$$

We may take this antiderivative without adding a constant for the integration-by-parts calculation.

Step 2: Apply integration by parts with the given limits Substitution in the definite integration-by-parts formula gives

$$\begin{aligned} I &= \left[-\frac{\tan^{-1} x}{\sqrt{1+x^2}} \right]_0^1 + \int_0^1 \frac{dx}{(1+x^2)^{3/2}} \\ &= -\frac{\pi}{4\sqrt{2}} + J, \end{aligned}$$

where $J = \int_0^1 (1+x^2)^{-3/2} dx$. The lower boundary term vanishes because $\tan^{-1} 0 = 0$. At the upper limit, $\tan^{-1} 1 = \pi/4$ and $\sqrt{1+1^2} = \sqrt{2}$. The sign of J is positive because the minus sign in the formula cancels the negative sign of v .

Step 3: Evaluate the remaining definite integral In J , let $x = \tan \theta$. Since x increases from 0 to 1, the new limits are $\theta = 0$ and $\theta = \pi/4$. On this interval $\sec \theta > 0$, so $(\sec^2 \theta)^{3/2} = \sec^3 \theta$. Therefore

$$\begin{aligned} J &= \int_0^{\pi/4} \frac{\sec^2 \theta}{\sec^3 \theta} d\theta \\ &= \int_0^{\pi/4} \cos \theta d\theta = [\sin \theta]_0^{\pi/4} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Equivalently, the antiderivative $x/\sqrt{1+x^2}$ gives the same endpoint values.

Step 4: Combine and check the sign Inserting the value of J into the integration-by-parts expression yields

$$I = -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{1 - \pi/4}{\sqrt{2}}.$$

The original integrand is non-negative throughout $[0, 1]$ and positive inside the interval. Our expression has positive numerator because $\pi < 4$, so its sign is consistent with the integral. No arbitrary constant is added to a definite integral.

Step 5: Conclusion

$$\boxed{\int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx = \frac{4 - \pi}{4\sqrt{2}}}.$$

Quick Tip to Solve: Differentiate the inverse tangent and integrate the other factor in integration by parts; the resulting integral simplifies immediately with $x = \tan \theta$.

35. Find the vector and cartesian equations of the line passing through the point of intersection of the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$, and parallel to the line $\frac{x-1}{-2} = \frac{7-y}{-3} = z$. [5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.
- At an intersection, all three corresponding coordinates of the two parametric lines must agree; the parameters on the two lines need not be equal.
- Direction ratios are the coefficients of the common parameter. A numerator such as $7 - y$ must first be rewritten carefully to identify the sign of the direction ratio of y .

Step 1: Express the intersecting lines in coordinates For the first line, expand the given vector equation to obtain

$$x = 1 + 3\lambda, \quad y = 1 - \lambda, \quad z = -1.$$

For the second line, the corresponding coordinates are

$$x = 4 + 2\mu, \quad y = 0, \quad z = -1 + 3\mu.$$

These equations show that the first line lies in the plane $z = -1$, while every point of the second line has $y = 0$. Their intersection must satisfy both of these coordinate conditions, as well as the equality of their x -coordinates.

Step 2: Find and verify the point of intersection Equating the y -coordinates gives $1 - \lambda = 0$, so $\lambda = 1$. Equating the z -coordinates gives $-1 = -1 + 3\mu$, so $\mu = 0$. With these values, the two expressions for x also agree:

$$1 + 3(1) = 4, \quad 4 + 2(0) = 4.$$

Thus the intersection is $P = (4, 0, -1)$, with position vector $\vec{a} = 4\hat{i} - \hat{k}$. Checking all three coordinates is essential: the values of two parameters obtained from two coordinates would not establish an intersection if the remaining coordinates disagreed.

Step 3: Obtain the direction of the required parallel line Put the common value in the given symmetric equation equal to s :

$$\frac{x-1}{-2} = \frac{7-y}{-3} = z = s.$$

Therefore $x = 1 - 2s$, $7 - y = -3s$, and $z = s$. Solving the middle equation gives $y = 7 + 3s$. Hence the direction vector is

$$\vec{b} = -2\hat{i} + 3\hat{j} + \hat{k}.$$

In particular, the second direction ratio is $+3$, because the printed numerator is $7 - y$. The direction ratios $(-2, -3, 1)$ would represent a different direction.

Step 4: Construct both equations and check the requirements Use $P = (4, 0, -1)$ as the point on the required line and $\vec{b}$ as its direction vector. Its parametric coordinates are

$$x = 4 - 2t, \quad y = 3t, \quad z = -1 + t.$$

Quick Tip to Solve: Rewrite a reversed numerator such as $7 - y$ in parametric form before reading direction ratios; its sign changes the y -direction ratio.

At $t = 0$, these give exactly the intersection P . Increasing t changes the coordinates in the ratio $-2 : 3 : 1$, the same direction ratios as the given parallel line. Eliminating t yields the symmetric cartesian equation, while collecting the point and direction vectors yields the vector equation.

Step 5: Conclusion The required vector equation is

$$\vec{r} = (4\hat{i} - \hat{k}) + t(-2\hat{i} + 3\hat{j} + \hat{k}), \quad t \in \mathbb{R}.$$

The required cartesian equation is

$$\frac{x-4}{-2} = \frac{y}{3} = \frac{z+1}{1}.$$

Quick Tip to Solve: Rewrite a reversed numerator such as $7 - y$ in parametric form before reading direction ratios; its sign changes the y -direction ratio.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

An NGO organises a charity event in which they decide to distribute woollen caps to protect children from winter. The caps to be distributed are in three separate boxes, Box I has 30 red caps, Box II has 20 red and 10 green caps, and Box III has 30 green caps. The probability that a Box i is selected and a cap picked out is $\frac{i}{6}$, where $i = 1, 2, 3$.

Based on the above information, answer the following questions:

A person selects a cap.

[4 marks]

36(i). What is the probability that he selects a red cap?

[2 marks]

Solution:

Concept:

- The law of total probability gives $P(R) = \sum_{i=1}^3 P(B_i)P(R | B_i)$, where B_i means selection of Box i and R means selection of a red cap.
- Each box contains 30 caps. The given selection probabilities are $P(B_1) = 1/6$, $P(B_2) = 2/6$ and $P(B_3) = 3/6$; these add to one.

Step 1: Find the probability of red within each box Box I contains only red caps, Box II contains 20 red caps among its 30 caps, and Box III contains only green caps. Therefore,

$$P(R | B_1) = 1, \quad P(R | B_2) = \frac{20}{30} = \frac{2}{3}, \quad P(R | B_3) = 0.$$

These are conditional probabilities after a particular box has been selected.

Step 2: Weight the three possibilities by box selection A red cap can come from Box I or Box II, and the two possibilities are disjoint. Box III makes no contribution. Thus the total-probability calculation

Quick Tip to Solve: Write the box weights as $1/6, 2/6, 3/6$ first, then multiply each by that box's red fraction; the all-green box contributes zero.

is

$$\begin{aligned}P(R) &= \frac{1}{6}(1) + \frac{2}{6}\left(\frac{2}{3}\right) + \frac{3}{6}(0) \\&= \frac{1}{6} + \frac{2}{9} = \frac{3+4}{18}.\end{aligned}$$

Counting red caps across all boxes without these weights would incorrectly treat the boxes as equally likely.

Step 3: Conclusion The probability that he selects a red cap is $\boxed{\frac{7}{18}}$.

Quick Tip to Solve: Write the box weights as $1/6, 2/6, 3/6$ first, then multiply each by that box's red fraction; the all-green box contributes zero.

36(ii). If he selects a green cap, what is the probability that the cap has come from Box II? [2 marks]

Solution:

Concept:

- Bayes' theorem gives $P(B_2 | G) = P(B_2)P(G | B_2)/P(G)$, where G denotes a green cap and B_2 denotes Box II.
- The total probability $P(G)$ is found by weighting each box's green fraction by its given box-selection probability, $P(B_i) = i/6$.

Step 1: Calculate the total probability of a green cap Box I has no green caps, Box II has 10 green caps out of 30, and Box III has 30 green caps out of 30. Hence

$$P(G | B_1) = 0, \quad P(G | B_2) = \frac{1}{3}, \quad P(G | B_3) = 1.$$

Applying total probability to these mutually exclusive box choices,

$$P(G) = \frac{1}{6}(0) + \frac{2}{6}\left(\frac{1}{3}\right) + \frac{3}{6}(1) = \frac{1}{9} + \frac{1}{2} = \frac{11}{18}.$$

Step 2: Apply Bayes' theorem to Box II The event that the cap is green and came from Box II has probability

$$P(B_2 \cap G) = P(B_2)P(G | B_2) = \frac{2}{6} \times \frac{1}{3} = \frac{1}{9}.$$

We now restrict the sample space to green-cap outcomes, so divide this joint probability by the total green-cap probability:

$$P(B_2 | G) = \frac{1/9}{11/18} = \frac{1}{9} \times \frac{18}{11}.$$

The requested reverse conditional probability is different from the green fraction within Box II.

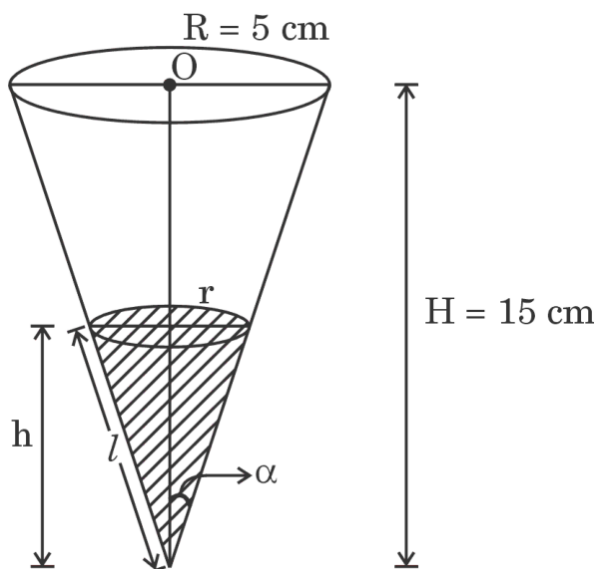
Step 3: Conclusion Given that the selected cap is green, the probability that it came from Box II is $\boxed{\frac{2}{11}}$.

Quick Tip to Solve: For a reverse conditional question, divide the target box's weighted green contribution by the sum of all weighted green contributions.

37. At a birthday party, children are being served orange juice in conical cups, as shown in the figure. Each cup is 15 cm deep and has a radius 5 cm. The juice is being poured into this cup at a rate of $0.1 \text{ cm}^3/\text{s}$.

On the basis of the above information, answer the following questions:

[4 marks]



37(i). Establish a relation between the height h of the juice in the cup and radius r of the surface of the juice in the cup, if the semi-vertical angle of the cone is α .

[1 mark]

Solution:

Concept:

- The semi-vertical angle α is the angle between the cone's axis and its sloping side. In an axial right triangle, $\tan \alpha$ equals radius divided by height.
- The cone occupied by juice is similar to the complete cup, so its radius-to-height ratio equals $R/H = 5/15$.

Step 1: Express the ratio using the semi-vertical angle Consider an axial section of the juice cone. Relative to angle α , the opposite side is the surface radius r and the adjacent side is the juice height h . Therefore,

$$\tan \alpha = \frac{r}{h}, \quad r = h \tan \alpha.$$

Step 2: Use the dimensions of the cup The whole cup has the same semi-vertical angle as the juice cone. Consequently,

$$\tan \alpha = \frac{5}{15} = \frac{1}{3}, \quad \frac{r}{h} = \frac{1}{3}.$$

This ratio remains constant as the juice level changes because the cup's shape is fixed.

Step 3: Conclusion The relation is $r = h \tan \alpha$. For this cup it becomes $h = 3r$, or equivalently $r = h/3$.

Quick Tip to Solve: Measure the semi-vertical angle from the axis: its tangent is radius over height, so the cup dimensions immediately give $r = h/3$.

37(ii). At what rate is the juice level in the cup rising when the juice is 6 cm deep?

[1 mark]

Solution:

Concept:

- For similar cones, the radius r and depth h satisfy $r/h = R/H$, where the cup has $R = 5$ cm and $H = 15$ cm.
- The juice volume is $V = \frac{1}{3}\pi r^2 h$. Differentiate with respect to time t after expressing V in terms of h .

Step 1: Express volume in terms of depth Similarity gives $r = h/3$. Thus

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}.$$

Both radius and depth change as juice is poured, so the radius cannot be held fixed.

Step 2: Use the given inflow rate Differentiating with respect to time gives

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

At $h = 6$ cm, use $dV/dt = 0.1$ cm³/s:

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s}.$$

The positive derivative agrees with a rising juice level.

Step 3: Conclusion The juice level rises at $\boxed{\frac{1}{40\pi} \text{ cm/s}}$ when the depth is 6 cm.

Quick Tip to Solve: First replace r by $h/3$; differentiating a one-variable volume avoids treating the changing radius as constant.

37(iii)(a). When the juice is 6 cm deep, then find at what rate is the upper surface area of juice increasing?

[2 marks]

Solution:

Concept:

- Similar cones give $r/h = 5/15 = 1/3$, where r is the radius of the circular upper juice surface and h is its depth.
- The upper surface is a circle of area $A = \pi r^2$. Its time derivative is $dA/dt = 2\pi r dr/dt$.

Step 1: Obtain the rates of depth and radius The juice volume is

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{27}, \quad \frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

Quick Tip to Solve: For the upper surface, use πr^2 . The slant height belongs to the wetted conical surface, not to this circular area.

With $h = 6$ cm and $dV/dt = 0.1 \text{ cm}^3/\text{s}$,

$$\frac{dh}{dt} = \frac{0.1}{4\pi} = \frac{1}{40\pi} \text{ cm/s}.$$

Since $r = h/3$, at this instant $r = 2$ cm and $dr/dt = 1/(120\pi) \text{ cm/s}$.

Step 2: Differentiate the circular surface area The upper surface area of the juice is $A = \pi r^2$; it is the exposed circular area, not the conical side in contact with the cup. Differentiation therefore gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(2) \left(\frac{1}{120\pi} \right) = \frac{1}{30} \text{ cm}^2/\text{s}.$$

Equivalently, $A = \pi h^2/9$ gives the same derivative directly. The positive sign confirms that the upper surface expands as the depth increases.

Step 3: Conclusion The upper surface area increases at $\boxed{\frac{1}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: For the upper surface, use πr^2 . The slant height belongs to the wetted conical surface, not to this circular area.

OR

37(iii)(b). When the juice is 6 cm deep, then find the rate at which the wetted surface area of the cup is increasing. [2 marks]

Solution:

Concept:

- The wetted area is the curved area $S = \pi r\ell$ of the smaller cone occupied by juice, where ℓ is its slant height.
- Similarity gives $r = h/3$; Pythagoras gives $\ell = \sqrt{r^2 + h^2}$. Express S using depth h before differentiating.

Step 1: Express the wetted area using depth For the cup, $R/H = 5/15 = 1/3$, so $r = h/3$. The slant height of the juice cone is

$$\ell = \sqrt{h^2 + \left(\frac{h}{3}\right)^2} = \frac{\sqrt{10}}{3}h.$$

Hence the area of the cup wall in contact with juice is

$$S = \pi r\ell = \frac{\pi\sqrt{10}}{9}h^2.$$

The circular upper surface is exposed to air and is therefore excluded from the wetted area.

Quick Tip to Solve: Use the curved area $\pi r\ell$ for wetted surface; substitute both $r = h/3$ and $\ell = \sqrt{10}h/3$ before differentiation.

Step 2: Find the depth rate and differentiate the wetted area From the volume $V = \pi h^3/27$,

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

At $h = 6$ cm, the inflow $0.1 \text{ cm}^3/\text{s}$ gives $dh/dt = 1/(40\pi) \text{ cm/s}$. Thus

$$\begin{aligned}\frac{dS}{dt} &= \frac{2\pi\sqrt{10}}{9} h \frac{dh}{dt} \\ &= \frac{2\pi\sqrt{10}}{9} (6) \left(\frac{1}{40\pi} \right) \\ &= \frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}.\end{aligned}$$

The result is positive because progressively more of the inner conical wall is covered.

Step 3: Conclusion The wetted surface area increases at $\boxed{\frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Use the curved area $\pi r\ell$ for wetted surface; substitute both $r = h/3$ and $\ell = \sqrt{10}h/3$ before differentiation.

38. Case Study - 3

A carpenter needs to design a wooden box in the shape of a cuboid such that the sum of its length and breadth is 3 cm more than its height. Twice of its length, thrice of its breadth and its height add up to 10 cm. Its breadth added to 7 times its height is 1 cm less than 3 times its length.

On the basis of the above information, answer the following questions:

[4 marks]

38(i). Write the equations representing the various dimensions and express them as the matrix equation $AX = B$.

[1 mark]

Solution:

Concept:

- Let x, y, z be the length, breadth and height, respectively, measured in centimetres.
- In $AX = B$, each row of A contains the coefficients of one equation in the fixed order x, y, z .

Step 1: Translate the three conditions The sum of length and breadth is 3 cm more than the height, so $x + y = z + 3$. The second condition gives $2x + 3y + z = 10$. The last condition says $y + 7z = 3x - 1$, since the sum is 1 cm less than three times the length. Rearranging gives

$$x + y - z = 3, \quad 2x + 3y + z = 10, \quad 3x - y - 7z = 1.$$

Step 2: Arrange the coefficients and constants Keep the same variable order in every row and place

Quick Tip to Solve: Rewrite the last condition as $3x - y - 7z = 1$ before extracting its coefficient row; this keeps the right-hand constant positive.

the corresponding constants in a column:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}.$$

Multiplying each row of A by X recovers exactly one of the three equations above.

Step 3: Conclusion The required matrix equation is

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}.$$

Quick Tip to Solve: Rewrite the last condition as $3x - y - 7z = 1$ before extracting its coefficient row; this keeps the right-hand constant positive.

38(ii). Find if A^{-1} exists. Justify your answer.

[1 mark]

Solution:

Concept:

- A square matrix is invertible exactly when its determinant is nonzero.
- Expand a determinant along the first row with alternating signs $+, -, +$.

Step 1: Use the coefficient matrix from part (i) With length, breadth and height ordered as x, y, z , the equations give

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}.$$

It is square of order three, so the determinant criterion applies.

Step 2: Evaluate the determinant Expanding along the first row,

$$\begin{aligned} |A| &= 1\{3(-7) - 1(-1)\} - 1\{2(-7) - 1(3)\} \\ &\quad + (-1)\{2(-1) - 3(3)\} \\ &= -20 + 17 + 11 = 8. \end{aligned}$$

In particular, the negative third entry contributes positively because its minor is also negative.

Step 3: Conclusion Since $|A| = 8 \neq 0$, A^{-1} exists.

Quick Tip to Solve: Compute only the determinant to decide existence; finding all cofactors or the full inverse is unnecessary here.

Solution:**Concept:**

- For an invertible square matrix, $A^{-1} = \frac{\text{adj } A}{|A|}$.
- A cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j ; the adjugate is the transpose of the cofactor matrix.

Step 1: Recall the matrix and determinant Using the coefficient matrix obtained from the equations in part (i),

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}, \quad |A| = -20 + 17 + 11 = 8.$$

The determinant is nonzero, so the adjugate formula can be used.

Step 2: Calculate the cofactors Evaluating the two-by-two determinants with their cofactor signs gives

$$\begin{aligned} C_{11} &= -21 + 1 = -20, & C_{12} &= -(-14 - 3) = 17, \\ C_{13} &= -2 - 9 = -11, & C_{21} &= -(-7 - 1) = 8, \\ C_{22} &= -7 + 3 = -4, & C_{23} &= -(-1 - 3) = 4, \\ C_{31} &= 1 + 3 = 4, & C_{32} &= -(1 + 2) = -3, \\ C_{33} &= 3 - 2 = 1. \end{aligned}$$

Hence the cofactor matrix and its transpose are

$$C = \begin{pmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

As a check, the first row of A multiplied by the columns of the adjugate gives $(8, 0, 0)$, as required by $A \text{adj } A = 8I$.

Step 3: Conclusion Thus the required inverse is

$$A^{-1} = \frac{1}{8} \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

Quick Tip to Solve: After finding cofactors, transpose them before dividing by 8; using the cofactor matrix directly gives the wrong inverse.

38(iii)(b). Find $A^2 + 7I$.

[2 marks]

Solution:**Concept:**

- Matrix multiplication uses the row-column rule: $(A^2)_{ij} = \sum_{k=1}^3 a_{ik}a_{kj}$.
- The identity matrix I has ones on the main diagonal and zeros elsewhere, so adding $7I$ changes only the diagonal entries.

Step 1: Multiply the coefficient matrix by itself From part (i), $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}$. Use each row of the first factor with each column of the second factor:

$$A^2 = \begin{pmatrix} 1+2-3 & 1+3+1 & -1+1+7 \\ 2+6+3 & 2+9-1 & -2+3-7 \\ 3-2-21 & 3-3+7 & -3-1+49 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 7 \\ 11 & 10 & -6 \\ -20 & 7 & 45 \end{pmatrix}.$$

For instance, the bottom-left entry is $3(1) + (-1)(2) + (-7)(3) = -20$; squaring individual entries would not produce this matrix.

Step 2: Add seven times the identity Since $7I = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, matrix addition adds corresponding entries:

$$A^2 + 7I = \begin{pmatrix} 0+7 & 5 & 7 \\ 11 & 10+7 & -6 \\ -20 & 7 & 45+7 \end{pmatrix}.$$

All six off-diagonal entries remain as calculated in A^2 .

Step 3: Conclusion The required matrix is

$$A^2 + 7I = \begin{pmatrix} 7 & 5 & 7 \\ 11 & 17 & -6 \\ -20 & 7 & 52 \end{pmatrix}.$$

Quick Tip to Solve: First compute A^2 by row-column products, then add 7 only to its three diagonal entries; do not add 7 to every entry.