

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

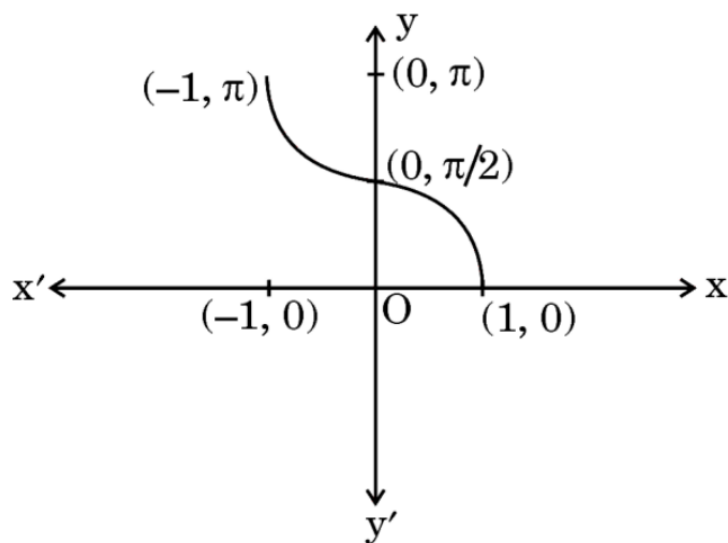
- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The following graph represents:

[1 mark]



- (A) $y = \cos^{-1} x$
- (B) $y = \sec^{-1} x$
- (C) $y = \sin^{-1} x$
- (D) $y = \tan^{-1} x$

Correct Answer: (A) $y = \cos^{-1} x$

Solution:

Concept:

- The principal function $y = \cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$.
- It decreases as x increases, with endpoint values $\cos^{-1}(-1) = \pi$ and $\cos^{-1}(1) = 0$.

Step 1: Match the labelled points The graph passes through $(-1, \pi)$, $(0, \pi/2)$ and $(1, 0)$. These agree with $\cos^{-1}(-1) = \pi$, $\cos^{-1}(0) = \pi/2$ and $\cos^{-1}(1) = 0$, respectively. Its decreasing shape also agrees with the principal inverse cosine function.

Step 2: Check the other choices Inverse sine and inverse tangent are increasing and both have value zero at $x = 0$; hence neither matches the graph. Inverse secant is defined for $|x| \geq 1$, so it cannot provide the continuous curve shown on $[-1, 1]$.

Step 3: Conclusion The graph represents $y = \cos^{-1} x$. Hence, the correct option is (A).

Quick Tip to Solve: Check the intercept at $x = 0$: a value of $\pi/2$, together with a decreasing curve on $[-1, 1]$, identifies inverse cosine.

2. Let $A = [a_{ij}]$ be a 2×2 matrix whose elements are given by $a_{ij} = \frac{(i+2j)^2}{3}$. Then A' is given by: [1 mark]

(A) $\begin{pmatrix} 12 & \frac{25}{3} \\ \frac{16}{3} & 3 \end{pmatrix}$

(B) $\begin{pmatrix} 12 & \frac{16}{3} \\ \frac{25}{3} & 3 \end{pmatrix}$

(C) $\begin{pmatrix} 3 & \frac{25}{3} \\ \frac{16}{3} & 12 \end{pmatrix}$

(D) $\begin{pmatrix} 3 & \frac{16}{3} \\ \frac{25}{3} & 12 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} 3 & \frac{16}{3} \\ \frac{25}{3} & 12 \end{pmatrix}$

Solution:

Concept:

- The indices i and j specify the row and column of a_{ij} , respectively.
- Transposition interchanges rows and columns: $(A')_{ij} = a_{ji}$.

Step 1: Calculate the four entries Substitute $i, j \in \{1, 2\}$ into the given formula:

$$a_{11} = \frac{(1+2)^2}{3} = 3, \quad a_{12} = \frac{(1+4)^2}{3} = \frac{25}{3},$$

$$a_{21} = \frac{(2+2)^2}{3} = \frac{16}{3}, \quad a_{22} = \frac{(2+4)^2}{3} = 12.$$

Thus $A = \begin{pmatrix} 3 & 25/3 \\ 16/3 & 12 \end{pmatrix}$.

Step 2: Transpose and compare Keep the diagonal entries in place and exchange the two off-diagonal entries:

$$A' = \begin{pmatrix} 3 & 16/3 \\ 25/3 & 12 \end{pmatrix}.$$

Options (A) and (B) reverse the diagonal entries. Option (C) gives A itself, without exchanging its unequal off-diagonal entries.

Step 3: Conclusion $A' = \begin{pmatrix} 3 & 16/3 \\ 25/3 & 12 \end{pmatrix}$. Hence, the correct option is (D).

Quick Tip to Solve: Write the four entries in row order first; transposition swaps only the off-diagonal entries of this square matrix.

3. The greatest integer function, $f(x) = [x]$, $0 < x < 3$ is not differentiable at how many points?

[1 mark]

- (A) At only one point
- (B) At only two points
- (C) At no point
- (D) At three points

Correct Answer: (B) At only two points

Solution:

Concept:

- The greatest integer function $[x]$ is constant between consecutive integers.
- Differentiability implies continuity, so a jump discontinuity prevents differentiability.

Step 1: Examine non-integer points For any non-integer x in $(0, 3)$, choose a sufficiently small neighbourhood containing no integer. The function is constant throughout this neighbourhood, so its derivative exists and equals zero. Thus the non-integer points are all differentiable points.

Step 2: Count the jumps inside the domain The only integers strictly between 0 and 3 are 1 and 2. At an integer n , the left-hand limit is $n - 1$, while $[n] = n$, so the function jumps and is not differentiable. The endpoints 0 and 3 are excluded. Therefore the choices of one, zero or three such points are incorrect.

Step 3: Conclusion The function is not differentiable at exactly two points, $x = 1$ and $x = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Count integers strictly inside the given interval; do not include either endpoint when the interval is open.

4. The area of the region bounded by the curve $y = x$ and x -axis, between $x = 0$ and $x = 2$ is: [1 mark]

- (A) 2 sq. units
- (B) $\frac{1}{2}$ sq. unit
- (C) 1 sq. unit
- (D) 4 sq. units

Correct Answer: (A) 2 sq. units

Solution:

Concept:

- When $f(x) \geq 0$ on $[a, b]$, the area between its curve and the x -axis is $\int_a^b f(x) dx$.
- Here $y = x$ is non-negative throughout $[0, 2]$.

Step 1: Set up the area integral The lower boundary is $y = 0$ and the upper boundary is $y = x$. A

Quick Tip to Solve: For a straight line through the origin, check the integral quickly using one half of the base times the endpoint height.

vertical strip has height x and width dx . Summing these strips from $x = 0$ to $x = 2$ gives

$$A = \int_0^2 (x - 0) dx.$$

Step 2: Evaluate and check the choices

$$A = \left[\frac{x^2}{2} \right]_0^2 = \frac{4}{2} - 0 = 2.$$

This also agrees with a triangle of base 2 and height 2, whose area is $\frac{1}{2}(2)(2) = 2$. The values 1/2, 1 and 4 do not equal this definite integral; 4 is the enclosing rectangle area.

Step 3: Conclusion The required area is 2 square units. Hence, the correct option is (A).

Quick Tip to Solve: For a straight line through the origin, check the integral quickly using one half of the base times the endpoint height.

5. If $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is: [1 mark]

- (A) 0
- (B) -2
- (C) -1
- (D) 2

Correct Answer: (D) 2

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0)$.
- The standard limits are $\lim_{x \rightarrow 0} (\sin x)/x = 1$ and $\lim_{x \rightarrow 0} \cos x = 1$, with angles measured in radians.

Step 1: Find the limiting value Use the branch specified for $x \neq 0$. Both of its terms have finite two-sided limits, so their limits can be added:

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} + \cos x \right) = 1 + 1 = 2.$$

The fact that the first expression is undefined at zero does not prevent its limit from existing.

Step 2: Apply continuity at zero The separately defined value is $f(0) = k$. It must equal the limiting value 2 for the graph to have no break at zero. The alternatives 0, -2 and -1 each differ from the two-sided limit and hence cannot make the function continuous.

Step 3: Conclusion $k = 2$. Hence, the correct option is (D).

Quick Tip to Solve: Evaluate the limit from the nonzero branch first, then equate it to the separately assigned value $f(0) = k$.

6. A cylindrical tank is being filled with sand at a rate of $314 \text{ m}^3/\text{h}$. If the radius of the tank is 10 m , then the height of sand in the tank increases at the rate of: [1 mark]

- (A) 1.1 m/h
- (B) 1 m/h
- (C) $\pi \text{ m/h}$
- (D) $\frac{\pi}{2} \text{ m/h}$

Correct Answer: (B) 1 m/h

Solution:

Concept:

- The volume of a cylinder is $V = \pi r^2 h$, where r is its radius and h is the sand height.
- For a fixed radius, $dV/dt = \pi r^2 dh/dt$.

Step 1: Relate the rates The tank radius stays fixed at $r = 10 \text{ m}$; only the sand height changes with time. Differentiating $V = 100\pi h$ with respect to t gives

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}.$$

The given filling rate is $314 \text{ m}^3/\text{h}$.

Step 2: Substitute and compare

$$\frac{dh}{dt} = \frac{314}{100\pi} \text{ m/h} \approx 1 \text{ m/h}.$$

Using the standard numerical approximation $\pi = 3.14$, this becomes $314/314 = 1 \text{ m/h}$. With π retained exactly the rate is approximately 0.9995 m/h , selecting the same option. The other printed choices 1.1 , π and $\pi/2 \text{ m/h}$ do not match the given volume rate.

Step 3: Conclusion The height increases at 1 m/h to the precision used in the paper. Hence, the correct option is (B).

Quick Tip to Solve: Divide the volume filling rate by the fixed base area; keep the radius constant when differentiating the cylinder volume.

7. The principal value of $\sec^{-1}(\sqrt{2}) + 2 \operatorname{cosec}^{-1}(-\sqrt{2})$ is:

[1 mark]

- (A) $-\frac{\pi}{2}$
- (B) $-\frac{\pi}{4}$
- (C) $\frac{\pi}{4}$
- (D) $\frac{\pi}{2}$

Correct Answer: (B) $-\frac{\pi}{4}$

Solution:

Concept:

- The principal range of $\sec^{-1} x$ is $[0, \pi] \setminus \{\pi/2\}$.
- The principal range of $\operatorname{cosec}^{-1} x$ is $[-\pi/2, \pi/2] \setminus \{0\}$.

Step 1: Find each principal value Since $\sec(\pi/4) = \sqrt{2}$ and $\pi/4$ belongs to the principal range, $\sec^{-1}(\sqrt{2}) = \pi/4$. Similarly $\operatorname{cosec}(-\pi/4) = -\sqrt{2}$, and $-\pi/4$ lies in the principal range, so $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\pi/4$.

Step 2: Combine with the given coefficient

$$\sec^{-1}(\sqrt{2}) + 2 \operatorname{cosec}^{-1}(-\sqrt{2}) = \frac{\pi}{4} + 2 \left(-\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}.$$

The value $-\pi/2$ omits the positive inverse-secant term. The positive choices $\pi/4$ and $\pi/2$ cannot equal this signed sum.

Step 3: Conclusion The principal value is $-\pi/4$. Hence, the correct option is (B).

Quick Tip to Solve: Fix the principal angle of each inverse function before multiplying by the coefficient; the negative cosecant argument gives a negative principal angle.

8. If points $(2, 3)$, $(0, 4)$ and $(p, 2)$ are collinear, then the value of p is:

[1 mark]

- (A) $\frac{4}{7}$
- (B) $-\frac{3}{7}$
- (C) 4
- (D) -4

Correct Answer: (C) 4

Solution:

Concept:

- Three points are collinear exactly when their triangle has zero area.
- For coordinates (x_i, y_i) , this requires $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

Quick Tip to Solve: After the determinant gives a candidate value, substitute it into the line through the first two points to check the sign.

Step 1: Form the determinant Use the three printed points in the collinearity condition:

$$\begin{vmatrix} 2 & 3 & 1 \\ 0 & 4 & 1 \\ p & 2 & 1 \end{vmatrix} = 0.$$

Expanding along the first row gives

$$2(4 - 2) - 3(0 - p) + (0 - 4p) = 0.$$

Step 2: Solve and verify The equation simplifies to $4 + 3p - 4p = 0$, or $4 - p = 0$. As a check, the first two points define a line with slope $(4 - 3)/(0 - 2) = -1/2$, whose equation is $y = 4 - x/2$. Setting $y = 2$ gives $x = 4$. The other values $4/7$, $-3/7$ and -4 do not satisfy $4 - p = 0$.

Step 3: Conclusion $p = 4$. Hence, the correct option is (C).

Quick Tip to Solve: After the determinant gives a candidate value, substitute it into the line through the first two points to check the sign.

9. Differential of e^{e^x} with respect to x is:

[1 mark]

- (A) $\log x$
- (B) e^{e^x}
- (C) $e^x e^{e^x}$
- (D) $(e^x)^2$

Correct Answer: (C) $e^x e^{e^x}$

Solution:

Concept:

- The chain rule gives $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.
- The derivative of the exponential is $\frac{d}{du} e^u = e^u$, and $\frac{d}{dx} e^x = e^x$.

Step 1: Identify the inner and outer functions Let $u = e^x$, so that the given function is $y = e^u$. Differentiation must account for both layers: the outer exponential contributes e^u , while its input contributes $du/dx = e^x$.

Step 2: Apply the chain rule and compare

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^u e^x = e^{e^x} e^x.$$

Option (B) retains the outer exponential but omits the inner derivative e^x . Option (A), $\log x$, is unrelated to either derivative. Option (D) incorrectly replaces the outer factor e^{e^x} by another e^x .

Step 3: Conclusion The derivative is $e^x e^{e^x}$. Hence, the correct option is (C).

Quick Tip to Solve: For nested exponentials, retain the entire outer exponential and multiply by the derivative of its exponent.

10. $\int \frac{dx}{\sqrt{25 - 16x^2}}$ is equal to:

[1 mark]

- (A) $\frac{1}{5} \sin^{-1} 4x + C$
(B) $\frac{1}{25} \sin^{-1} 16x + C$
(C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$
(D) $\frac{1}{16} \sin^{-1} \frac{4x}{5} + C$

Correct Answer: (C) $\frac{1}{4} \sin^{-1} \frac{4x}{5} + C$

Solution:

Concept:

- The standard integral is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, for $a > 0$.
- A linear substitution must transform both the expression under the square root and the differential.

Step 1: Make a linear substitution Set $u = 4x$, so $du = 4 dx$ and $dx = du/4$. Then

$$I = \int \frac{dx}{\sqrt{25 - 16x^2}} = \frac{1}{4} \int \frac{du}{\sqrt{5^2 - u^2}}.$$

Step 2: Apply the inverse sine formula The standard formula gives

$$I = \frac{1}{4} \sin^{-1}\left(\frac{u}{5}\right) + C = \frac{1}{4} \sin^{-1}\left(\frac{4x}{5}\right) + C.$$

Options (A) and (B) use the wrong argument as well as the wrong prefactor. Option (D) has the right argument but a prefactor four times too small; differentiating it gives only one quarter of the integrand.

Step 3: Conclusion $I = \frac{1}{4} \sin^{-1}\left(\frac{4x}{5}\right) + C$. Hence, the correct option is (C).

Quick Tip to Solve: In $\sqrt{a^2 - b^2x^2}$, use argument bx/a and prefactor $1/b$; here $a = 5$ and $b = 4$.

11. If $\int_0^1 \frac{dx}{e^x + e^{-x}} = \tan^{-1} e + k$, then the value of k is:

[1 mark]

- (A) e
- (B) $\frac{\pi}{4}$
- (C) 0
- (D) $-\frac{\pi}{4}$

Correct Answer: (D) $-\frac{\pi}{4}$

Solution:

Concept:

- Multiplying numerator and denominator by e^x exposes the substitution $t = e^x$.
- The inverse tangent formula is $\int \frac{dt}{1+t^2} = \tan^{-1} t + C$.

Step 1: Transform the definite integral Write the integral as

$$I = \int_0^1 \frac{e^x dx}{1 + e^{2x}}.$$

With $t = e^x$, $dt = e^x dx$. The lower limit $x = 0$ becomes $t = 1$, and the upper limit $x = 1$ becomes $t = e$.

Step 2: Evaluate and isolate the constant

$$I = \int_1^e \frac{dt}{1+t^2} = [\tan^{-1} t]_1^e = \tan^{-1} e - \frac{\pi}{4}.$$

Comparing this with $\tan^{-1} e + k$ shows that the lower-limit contribution is subtracted. Option (B) uses the wrong sign, option (C) omits the lower limit, and option (A) incorrectly treats the upper substitution limit as the constant.

Step 3: Conclusion $k = -\frac{\pi}{4}$. Hence, the correct option is (D).

Quick Tip to Solve: After substituting e^x , change the limits immediately; $x = 0$ gives $t = 1$, so subtract $\tan^{-1} 1$.

12. The sum of the order and the degree of the differential equation $y = x \frac{dy}{dx} + 3 \left[2 - \left(\frac{dy}{dx} \right)^2 \right]$ is: [1 mark]

- (A) not defined
- (B) 3
- (C) 1
- (D) 2

Correct Answer: (B) 3

Solution:

Concept:

- The order is the order of the highest derivative appearing in the differential equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in derivatives.

Step 1: Write the equation as a polynomial in the derivative Expanding and rearranging the given equation yields

$$y = x \frac{dy}{dx} + 6 - 3 \left(\frac{dy}{dx} \right)^2, \quad 3 \left(\frac{dy}{dx} \right)^2 - x \frac{dy}{dx} + y - 6 = 0.$$

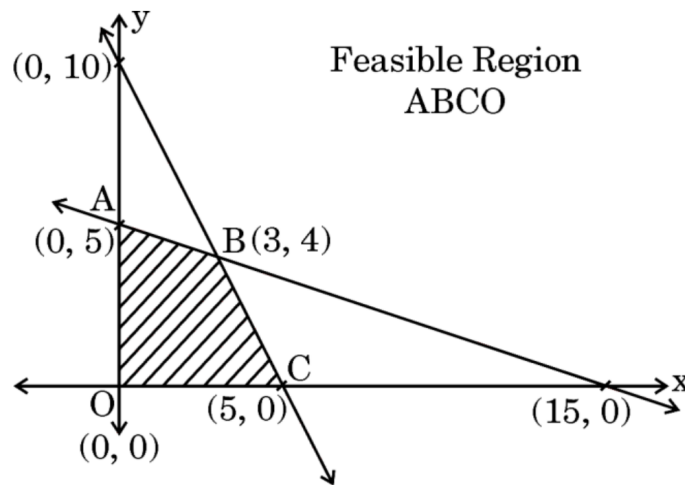
The only derivative present is dy/dx ; its square does not introduce a second derivative.

Step 2: Identify order and degree separately The highest derivative is the first derivative, so the order is 1. The equation is already polynomial in that derivative, with highest power 2, so its degree is 2. Thus the required sum is $1 + 2 = 3$. Option (A) is inapplicable because the degree exists. Options (C) and (D) give an individual value instead of their sum.

Step 3: Conclusion The sum of order and degree is 3. Hence, the correct option is (B).

Quick Tip to Solve: Distinguish $(dy/dx)^2$ from d^2y/dx^2 : the former raises the degree, while the latter raises the order.

13. In the graph, the feasible region representing the Linear Programming Problem for maximising objective function $Z = px + qy$, $p, q > 0$ is shaded. If all points on segment AB give $\max(Z)$, then which of the following is true? [1 mark]



- (A) $p = 2q$
- (B) $p = 3q$
- (C) $q = 3p$
- (D) $q = 2p$

Correct Answer: (C) $q = 3p$

Solution:

Concept:

- A linear objective function has the same value throughout a line segment when it has the same value at both endpoints.
- For $Z = px + qy$, evaluate Z at each vertex of the feasible region to identify the maximum.

Step 1: Compare the values at the endpoints The printed graph gives $A = (0, 5)$ and $B = (3, 4)$. Hence

$$Z(A) = 5q, \quad Z(B) = 3p + 4q.$$

Since every point of AB attains the same maximum, its endpoints must agree:

$$5q = 3p + 4q \implies q = 3p.$$

Step 2: Check that the common value is maximal With $q = 3p$, the common endpoint value is $15p$. At the other vertices $C = (5, 0)$ and $O = (0, 0)$, the values are $5p$ and 0 , respectively. Since $p > 0$, both are smaller. Linearity then gives $Z = 15p$ everywhere on AB . None of the other three positive-coefficient relations makes $5q = 3p + 4q$, so they cannot give a constant maximum on this segment.

Step 3: Conclusion $q = 3p$. Hence, the correct option is (C).

Quick Tip to Solve: For an entire optimal edge, equate the objective values at its two endpoints before comparing the remaining vertices.

14. Three points $A(0, 1, 1)$, $B(2, 0, -1)$ and $C(1, 0, 3)$ form $\triangle ABC$. The $\text{ar}(\triangle ABC)$ is:

[1 mark]

- (A) $\frac{\sqrt{53}}{2}$ sq. units
- (B) $\sqrt{53}$ sq. units
- (C) $\frac{\sqrt{11}}{2}$ sq. units
- (D) $\sqrt{11}$ sq. units

Correct Answer: (A) $\frac{\sqrt{53}}{2}$ sq. units

Solution:

Concept:

- The area of triangle ABC is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.
- For a vector $a\hat{i} + b\hat{j} + c\hat{k}$, its magnitude is $\sqrt{a^2 + b^2 + c^2}$.

Step 1: **Form two sides from the same vertex** Subtract the coordinates of A from those of B and C :

$$\vec{AB} = (2, -1, -2), \quad \vec{AC} = (1, -1, 2).$$

Using a common starting vertex ensures that their cross product represents the associated parallelogram.

Step 2: **Calculate the cross product and triangle area**

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} \\ &= -4\hat{i} - 6\hat{j} - \hat{k}, \\ |\vec{AB} \times \vec{AC}| &= \sqrt{16 + 36 + 1} = \sqrt{53}.\end{aligned}$$

The triangle has half this area. Option (B) gives the parallelogram area, while options (C) and (D) use an incorrect cross-product magnitude $\sqrt{11}$.

Step 3: **Conclusion** $\text{ar}(\triangle ABC) = \frac{\sqrt{53}}{2}$ square units. Hence, the correct option is (A).

Quick Tip to Solve: Compute the cross product first, square its three components to find its magnitude, and halve only at the final area step.

15. If $(3\hat{i} - 2\hat{j} + 5\hat{k}) \times (4\hat{i} + p\hat{j} + q\hat{k}) = \vec{0}$, then the values of p and q are:

[1 mark]

- (A) $p = -\frac{2}{3}, q = \frac{5}{3}$
(B) $p = -\frac{8}{3}, q = \frac{20}{3}$
(C) $p = \frac{20}{3}, q = \frac{8}{3}$
(D) $p = 0, q = 0$

Correct Answer: (B) $p = -\frac{8}{3}, q = \frac{20}{3}$

Solution:

Concept:

- Two nonzero vectors have zero cross product precisely when they are parallel.
- Parallel vectors are scalar multiples; their corresponding components have the same proportionality factor.

Step 1: Find the proportionality factor Both vectors are nonzero because their $\hat{i}$ -components are 3 and 4. Hence write

$$4\hat{i} + p\hat{j} + q\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 5\hat{k}).$$

Equating the $\hat{i}$ -components gives $4 = 3\lambda$, so $\lambda = 4/3$.

Step 2: Compare the remaining components

$$p = -2\lambda = -\frac{8}{3}, \quad q = 5\lambda = \frac{20}{3}.$$

Option (A) uses the scale factor $1/3$, inconsistent with the first component. Option (C) makes p positive and assigns the wrong value to q , so its component ratios are unequal. In option (D), a vector along the x -axis cannot be parallel to a vector having nonzero y - and z -components.

Step 3: Conclusion $p = -\frac{8}{3}$ and $q = \frac{20}{3}$. Hence, the correct option is (B).

Quick Tip to Solve: When a cross product vanishes, use one known component ratio to fix the common scalar multiplier before finding the unknown components.

16. Which of the following is *not* a Linear Differential Equation?

[1 mark]

(A) $(1 + x^2) dy + 2xy dx = \cot x dx$

(B) $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

(C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$

(D) $y dx - (x + 3y^2) dy = 0$

Correct Answer: (C) $x(1 + y^2) dx - y(1 + x^2) dy = 0$

Solution:

Concept:

- A first-order equation is linear in y when it has the form $dy/dx + P(x)y = Q(x)$.
- An equation may instead be linear in x as a function of y : $dx/dy + P(y)x = Q(y)$.

Step 1: Check options A and B Option (A), after division by $(1 + x^2) dx$, becomes

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{\cot x}{1 + x^2},$$

which is linear in y . For option (B), use the product rule $d(xy)/dx = y + x dy/dx$. On intervals with $x \neq 0$, this gives

$$\frac{dy}{dx} + \frac{2}{x}y = \sin x + \log x,$$

also linear in y .

Step 2: Check options C and D in both orientations Option (C) gives

$$\frac{dy}{dx} = \frac{x(1 + y^2)}{y(1 + x^2)}.$$

Its right-hand side contains both y and $1/y$, so it is nonlinear in y . Reversing the roles gives $dx/dy = y(1 + x^2)/[x(1 + y^2)]$, containing both x and $1/x$; it is nonlinear in x too. Option (D), however, is linear in x as a function of y :

$$\frac{dx}{dy} - \frac{x}{y} = 3y \quad (y \neq 0).$$

The presence of y^2 in the original option (D) does not prevent this linear form.

Step 3: Conclusion Option (C) is not a linear differential equation. Hence, the correct option is (C).

Quick Tip to Solve: Before classifying an equation containing dx and dy , check whether taking x as the dependent variable makes it linear.

17. The region represented by the system of inequations $3x + y \geq 3$, $2x - y \geq -5$, $x, y \geq 0$ is: [1 mark]

- (A) unbounded in 1st quadrant
- (B) bounded in 1st quadrant
- (C) unbounded in 2nd quadrant
- (D) bounded in 2nd quadrant

Correct Answer: (A) unbounded in 1st quadrant

Solution:

Concept:

- The feasible region is the intersection of all the given half-planes. The conditions $x, y \geq 0$ restrict it to the first quadrant, including its axes.
- A region is unbounded if feasible points can lie arbitrarily far from the origin; it need not extend in every direction.

Step 1: Rewrite the inequalities The two oblique boundary conditions are $y \geq 3 - 3x$ and $y \leq 2x + 5$. They must hold together with $x \geq 0$ and $y \geq 0$. Thus no feasible point has a negative coordinate; the second-quadrant options are excluded.

Step 2: Test a ray of feasible points On the positive x -axis set $y = 0$. Then $3x \geq 3$ and $2x \geq -5$ both hold for every $x \geq 1$. Hence the entire ray $\{(x, 0) : x \geq 1\}$ is feasible. Its distance from the origin can grow without limit, so the bounded-region option is excluded.

Step 3: Conclusion The feasible region is unbounded in the first quadrant. Hence, the correct option is (A).

Quick Tip to Solve: To prove unboundedness quickly, set one coordinate to zero and check whether the other can grow without limit.

18. The probability that a particular item is available in three shops A , B and C is $\frac{4}{5}$, $\frac{3}{4}$ and $\frac{2}{3}$ respectively. If a person visits all the three shops to buy the item, then what is the probability that it will be available in at least one of the shops? [1 mark]

- (A) $\frac{59}{60}$
- (B) 1
- (C) $\frac{1}{60}$
- (D) $\frac{9}{60}$

Correct Answer: (A) $\frac{59}{60}$

Solution:

Concept:

- Use the complement: at least one shop has the item exactly when it is not unavailable in all three shops.

Quick Tip to Solve: For independent shops, multiply the three failure probabilities first, then subtract that product from one.

- For independent shop availability, multiply the probabilities of unavailability to find the probability that none has the item.

Step 1: Find the unavailability probabilities If A, B, C denote availability at the respective shops, their complements have probabilities

$$P(A^c) = 1 - \frac{4}{5} = \frac{1}{5}, \quad P(B^c) = 1 - \frac{3}{4} = \frac{1}{4}, \quad P(C^c) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Step 2: Use the complement of no availability Using the independent-shop model implicit in the listed options,

$$P(A^c \cap B^c \cap C^c) = \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{60}.$$

Therefore $P(A \cup B \cup C) = 1 - \frac{1}{60} = \frac{59}{60}$. Option (C) is the probability of none, not at least one; option (B) ignores the nonzero chance of failure. Option (D) also does not equal the complement calculation.

Step 3: Conclusion The probability that the item is available in at least one shop is $\frac{59}{60}$. Hence, the correct option is (A).

Quick Tip to Solve: For independent shops, multiply the three failure probabilities first, then subtract that product from one.

19. Assertion (A): If A and B are two square matrices such that AB and BA are defined, then it is not necessary that $AB = BA$.

Reason (R): Product of two diagonal matrices of same order is commutative.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

Solution:

Concept:

- Matrix multiplication is generally not commutative: equal orders ensure the products are defined, but do not ensure $AB = BA$.
- Diagonal matrices of the same order multiply coordinatewise on their diagonals, so their products do commute.

Quick Tip to Solve: A true statement about diagonal matrices is a special case; use a non-diagonal example to test general commutativity.

Step 1: Evaluate the assertion Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here $AB = B$, whereas BA is the zero matrix. Thus $AB \neq BA$, even though both products exist. The assertion is true.

Step 2: Evaluate the reason For $D = \text{diag}(d_1, \dots, d_n)$ and $E = \text{diag}(e_1, \dots, e_n)$, both DE and ED have diagonal entries $d_i e_i = e_i d_i$, with all other entries zero. The reason is therefore true.

Step 3: Check the explanatory link The reason describes the special case in which commutativity holds. It does not explain why general square matrices need not commute. Thus option (A) is excluded; options (C) and (D) wrongly declare one of the true statements false.

Step 4: Conclusion Both the assertion and the reason are true, but R is not the correct explanation of A. Hence, the correct option is (B).

Quick Tip to Solve: A true statement about diagonal matrices is a special case; use a non-diagonal example to test general commutativity.

20. Assertion (A): A function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3 + 2$, $\forall x \in \mathbb{N}$ is one-one but not onto.

Reason (R): Since $\forall y \in \mathbb{N}$ (Codomain), there does not exist $x = (y - 2)^{1/3}$ in $\mathbb{N}$ (Domain) such that $f(x) = x^3 + 2 = y$. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ forces $a = b$. It is onto if every element of the codomain has a preimage.
- To refute a universal statement, one counterexample suffices. Distinguish "some values have no preimage" from "every value has no preimage".

Step 1: Evaluate the assertion If $f(a) = f(b)$, then $a^3 + 2 = b^3 + 2$, so $a^3 = b^3$ and hence $a = b$. Thus f is one-one. The natural number 1 has no natural-number preimage because $x^3 + 2 = 1$ would require $x^3 = -1$. Therefore f is not onto. The assertion is true.

Step 2: Read and test the printed reason The reason says that for every natural number y , no natural number $x = (y - 2)^{1/3}$ exists with $f(x) = y$. This universal claim is false: take $y = 3$. Then $x = 1 \in \mathbb{N}$

Quick Tip to Solve: Check the quantifier in R: substituting $y = f(1) = 3$ immediately disproves its claim about every natural number.

and $f(1) = 1^3 + 2 = 3$. Some codomain values do have preimages.

Step 3: Check whether the reason explains the assertion Failure of surjectivity requires at least one missing codomain value, not absence of preimages for every value. Since the printed reason is false, it cannot explain the true assertion. Options (A) and (B) incorrectly call R true; option (D) incorrectly calls A false.

Step 4: Conclusion The assertion is true, but the reason, with its printed universal quantifier, is false. Hence, the correct option is (C).

Quick Tip to Solve: Check the quantifier in R: substituting $y = f(1) = 3$ immediately disproves its claim about every natural number.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21(a). Show that the function $f(x) = \begin{cases} \frac{\cos x}{-x + \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$. [2 marks]

Solution:

Concept:

- A function is continuous at a when $\lim_{x \rightarrow a} f(x) = f(a)$; a finite common two-sided limit is required.
- The standard trigonometric limit is $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$, with angles measured in radians.

Step 1: Evaluate the value at the junction The second branch defines the value at the point directly:

$$f\left(\frac{\pi}{2}\right) = 1.$$

The expression in the first branch has numerator and denominator both tending to zero as $x \rightarrow \pi/2$, so direct substitution in that fraction does not determine the limit.

Step 2: Transform the limit to the standard form Put $h = x - \pi/2$. Then $h \rightarrow 0$, $\cos x = \cos(\pi/2 + h) = -\sin h$, and $-x + \pi/2 = -h$. For $h \neq 0$,

$$\frac{\cos x}{-x + \pi/2} = \frac{-\sin h}{-h} = \frac{\sin h}{h}.$$

Hence

$$\lim_{x \rightarrow \pi/2} f(x) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

This limit is the same from either side, since the standard sine limit is two-sided.

Step 3: Conclusion The limit equals the assigned function value: $\lim_{x \rightarrow \pi/2} f(x) = 1 = f(\pi/2)$. Therefore f is continuous at $x = \pi/2$.

Quick Tip to Solve: Near $\pi/2$, shift the variable by $h = x - \pi/2$; both minus signs cancel and the fraction becomes $\sin h/h$.

21(b). Find whether the function $f(x) = \begin{cases} x - 1, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$ at $x = 2$ is differentiable or not. [2 marks]

Solution:

Concept:

- Differentiability at a junction requires the left and right difference-quotient limits to exist and be equal.
- Continuity is necessary for differentiability but is not sufficient; joining two line segments can give a corner with different slopes.

Step 1: Check continuity and the junction value Since $x = 2$ belongs to the second branch, $f(2) = 2(2) - 3 = 1$. Also,

$$\lim_{x \rightarrow 2^-} (x - 1) = 1, \quad \lim_{x \rightarrow 2^+} (2x - 3) = 1.$$

Both one-sided function limits agree with $f(2)$, so the function is continuous at the junction. We must still compare its one-sided derivatives.

Step 2: Calculate the one-sided derivatives Using the definition at $x = 2$, for $h < 0$ we have $f(2 + h) = 1 + h$, whereas for $h > 0$ we have $f(2 + h) = 1 + 2h$. Consequently,

$$f'_-(2) = \lim_{h \rightarrow 0^-} \frac{(1 + h) - 1}{h} = 1,$$

$$f'_+(2) = \lim_{h \rightarrow 0^+} \frac{(1 + 2h) - 1}{h} = 2.$$

These finite limits differ. Therefore the two-sided difference quotient has no limit, even though the function values join continuously.

Step 3: Conclusion Since $f'_-(2) = 1 \neq 2 = f'_+(2)$, the function is not differentiable at $x = 2$.

Quick Tip to Solve: For a piecewise linear function, first check the values join, then compare the slopes of the adjoining branches.

22(a). Let two rods placed on the ground be represented by vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$. Find a vector representing a flag-post of height 5 m that has to be erected perpendicular to both the rods. [2 marks]

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is perpendicular to both nonparallel vectors $\vec{a}$ and $\vec{b}$.
- To obtain a vector of length 5, first divide a perpendicular vector by its magnitude, then multiply the resulting unit vector by 5.

Step 1: Find a common perpendicular direction Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$. Their cross

Quick Tip to Solve: Do not use the cross product directly as the flag-post vector: normalise it before multiplying by the requested height.

product is

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} \\ &= \hat{i}(2 - 3) - \hat{j}(-8 + 6) + \hat{k}(4 - 2) \\ &= -\hat{i} + 2\hat{j} + 2\hat{k}.\end{aligned}$$

Its magnitude is $\sqrt{(-1)^2 + 2^2 + 2^2} = 3$, so it is a nonzero common perpendicular.

Step 2: Scale the direction to the required height The corresponding unit vector is $\frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}$. Multiplying by the required height gives

$$\vec{p} = \frac{5}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) \text{ m.}$$

Indeed $|\vec{p}| = 5 \text{ m}$. The unscaled direction has dot products $-4 - 2 + 6 = 0$ with $\vec{a}$, and $2 + 2 - 4 = 0$ with $\vec{b}$, confirming both perpendicularities. Reversing the sign gives the other normal direction; either sign represents a vector of the prescribed length.

Step 3: Conclusion One required flag-post vector is $\vec{p} = -\frac{5}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{10}{3}\hat{k} \text{ m}$. Its negative is also perpendicular to both rods and has the same length.

Quick Tip to Solve: Do not use the cross product directly as the flag-post vector: normalise it before multiplying by the requested height.

OR

22(b). A unit vector $\vec{a}$ is such that it makes an angle $\frac{\pi}{4}$ with x -axis, $\frac{\pi}{3}$ with y -axis and an acute angle θ with z -axis. Find θ and the components of $\vec{a}$. [2 marks]

Solution:

Concept:

- The direction cosines of a vector are the cosines of its angles with the positive coordinate axes. They satisfy $l^2 + m^2 + n^2 = 1$.
- For a unit vector, the components equal the direction cosines: $\vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$. An acute angle with the positive z -axis requires $n > 0$.

Step 1: Apply the direction-cosine identity Here $l = \cos(\pi/4) = 1/\sqrt{2}$ and $m = \cos(\pi/3) = 1/2$. Therefore

$$\cos^2 \theta = 1 - l^2 - m^2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

This identity follows because the squared components of a unit vector sum to one; there is no requirement that the three angles themselves add to a fixed angle.

Step 2: Select the positive root The squared equation permits $\cos \theta = \pm 1/2$. However, θ is acute, so $\cos \theta = 1/2$ and $\theta = \pi/3$. The components are consequently $(1/\sqrt{2}, 1/2, 1/2)$. As a check, their

Quick Tip to Solve: Use the sum of squares of direction cosines, then use the acute-angle condition to select the positive root.

squared magnitudes sum to $1/2 + 1/4 + 1/4 = 1$, confirming that the resulting vector is a unit vector.

Step 3: Conclusion $\theta = \pi/3, \quad \vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}.$

Quick Tip to Solve: Use the sum of squares of direction cosines, then use the acute-angle condition to select the positive root.

23. Find the interval(s) for which the function $f(x) = \frac{x}{3} + \frac{3}{x}$, $x \neq 0$ is increasing. [2 marks]

Solution:

Concept:

- A differentiable function is increasing on each interval on which $f'(x) > 0$. The domain must be considered before solving the derivative inequality.
- Here $x = 0$ is excluded and $\frac{d}{dx}(x^{-1}) = -x^{-2}$; multiplication of an inequality by $x^2 > 0$ does not reverse its sign.

Step 1: Differentiate the given function For $f(x) = x/3 + 3/x$, the domain is $\mathbb{R} \setminus \{0\}$. Differentiating each term gives

$$f'(x) = \frac{1}{3} - \frac{3}{x^2} = \frac{x^2 - 9}{3x^2}.$$

The denominator is strictly positive wherever the function is defined, so the sign of the derivative is exactly the sign of $x^2 - 9$.

Step 2: Solve the positivity condition The condition for an increasing interval is

$$x^2 - 9 > 0 \iff |x| > 3.$$

Hence $x < -3$ or $x > 3$. Between -3 and 3 , excluding zero, the derivative is negative and those intervals are decreasing. At $x = \pm 3$ the derivative vanishes; these are the boundary points of the intervals obtained from strict positivity.

Step 3: Conclusion The increasing intervals are $(-\infty, -3)$ and $(3, \infty)$.

Quick Tip to Solve: Since the derivative denominator is a positive square, solve the inequality using only its numerator.

24. If in a parallelogram $PQRS$, $\vec{PQ} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{QR} = \hat{i} - 2\hat{j} + \hat{k}$, then find the unit vectors parallel to the diagonals $\vec{PR}$ and $\vec{SQ}$. [2 marks]

Solution:

Concept:

- For successive sides of a parallelogram, the diagonal from the starting vertex is the sum of the two side vectors.

Quick Tip to Solve: Use the sum for PR and the first side minus the second for SQ; normalize only after finding both vectors.

- A unit vector in the direction of a nonzero vector $\vec{v}$ is $\vec{v}/|\vec{v}|$. Reversing direction changes the sign, while preserving parallelism.

Step 1: Find the diagonal vectors Write $\vec{a} = \overrightarrow{PQ} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \overrightarrow{QR} = \hat{i} - 2\hat{j} + \hat{k}$. Traversing from P through Q to R gives

$$\overrightarrow{PR} = \vec{a} + \vec{b} = 3\hat{i} + \hat{j} + 2\hat{k}.$$

Since $\overrightarrow{PS} = \vec{b}$, traversing from S through P to Q gives

$$\overrightarrow{SQ} = -\vec{b} + \vec{a} = \hat{i} + 5\hat{j}.$$

This order matters: $\overrightarrow{QS}$ would have the opposite sign.

Step 2: Normalize each diagonal The magnitudes are

$$|\overrightarrow{PR}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}, \quad |\overrightarrow{SQ}| = \sqrt{1^2 + 5^2} = \sqrt{26}.$$

Dividing by the respective magnitudes gives vectors of length one directed along the indicated arrows. Their negatives are also unit vectors parallel to the same diagonal lines, but point in the reverse directions.

Step 3: Conclusion Unit vectors along the indicated directions are $\frac{3\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{14}}$, $\frac{\hat{i} + 5\hat{j}}{\sqrt{26}}$, respectively. If either orientation is allowed, include the negative of each vector as well.

Quick Tip to Solve: Use the sum for PR and the first side minus the second for SQ; normalize only after finding both vectors.

25. Evaluate: $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) + \tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ [2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$, whereas the principal range of $\cot^{-1}$ is $(0, \pi)$.
- An inverse trigonometric function selects its principal value; in general $\tan^{-1}(\tan x)$ does not equal x outside the principal range.

Step 1: Evaluate the first three terms The principal angle whose tangent is $-1/\sqrt{3}$ is $-\pi/6$. The angle in $(0, \pi)$ whose cotangent is $1/\sqrt{3}$ is $\pi/3$. Also, $\sin(-\pi/2) = -1$, so $\tan^{-1}(\sin(-\pi/2)) = \tan^{-1}(-1) = -\pi/4$. Each chosen value belongs to the appropriate principal range, so no supplementary angle is needed for these terms.

Step 2: Reduce the last angle and add Although $2\pi/3$ is outside the principal range of inverse tangent, subtracting π preserves its tangent. Thus

$$\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = -\frac{\pi}{3}.$$

Quick Tip to Solve: Reduce $2\pi/3$ by π before applying inverse tangent, and use the principal cotangent range $(0, \pi)$.

Adding all four values gives

$$-\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{6} - \frac{\pi}{4} = -\frac{5\pi}{12}.$$

The two terms of magnitude $\pi/3$ cancel exactly.

Step 3: Conclusion The required value is $\boxed{-5\pi/12}$.

Quick Tip to Solve: Reduce $2\pi/3$ by π before applying inverse tangent, and use the principal cotangent range $(0, \pi)$.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26(a). Find a point on the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2}$ at a distance of $\sqrt{2}$ units from the point $(1, 2, 3)$. [3 marks]

Solution:

Concept:

- Every point on a line in symmetric form is obtained by setting its common ratio equal to a real parameter t .
- The distance between (x, y, z) and (a, b, c) is $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. Squaring a prescribed positive distance produces an algebraic equation for the parameter.

Step 1: Parametrize the line Using the printed form

$$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z-3}{2} = t,$$

a general point is $P(t) = (2+3t, 1-2t, 3+2t)$. In particular, the minus sign in the middle coordinate follows from $1-y=2t$. The reference point for the distance is $A = (1, 2, 3)$, which need not lie on the line.

Step 2: Impose the distance condition The displacement from A to $P(t)$ has components $(1+3t, -1-2t, 2t)$. Since the required distance is $\sqrt{2}$,

$$\begin{aligned}(1+3t)^2 + (-1-2t)^2 + (2t)^2 &= 2, \\(1+6t+9t^2) + (1+4t+4t^2) + 4t^2 &= 2, \\17t^2 + 10t &= 0.\end{aligned}$$

Factoring gives $t(17t+10)=0$, so $t=0$ or $t=-10/17$. Both roots are real and can be tested in the original distance condition.

Step 3: Find and verify the possible points For $t=0$, the point is $(2, 1, 3)$, whose squared distance from A is $1^2 + (-1)^2 + 0^2 = 2$. For $t=-10/17$, the point is $(4/17, 37/17, 31/17)$. Its displacement from A is $(-13/17, 3/17, -20/17)$, whose squared length is $(169+9+400)/289 = 2$. Each point also

Quick Tip to Solve: Set all ratios equal to t , retain the minus sign in y , and solve the squared distance equation for every root.

satisfies the symmetric line equation by its construction, so both are valid.

Step 4: Conclusion One required point is $(2, 1, 3)$. The other possible point is $(4/17, 37/17, 31/17)$.

Quick Tip to Solve: Set all ratios equal to t , retain the minus sign in y , and solve the squared distance equation for every root.

OR

26(b). Find the shortest distance between the lines $\vec{r} = (4 + \lambda)\hat{i} + (2\lambda - 1)\hat{j} - 3\hat{k}$ and $\vec{r} = (1 + 2\mu)\hat{i} + (2 - 5\mu)\hat{k} + (4\mu - 1)\hat{j}$. [3 marks]

Solution:

Concept:

- A line in vector form is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{a}$ is the position vector of a point on the line and $\vec{b}$ is its direction vector.
- For two non-parallel lines, the shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. The cross product gives a direction perpendicular to both lines.

Step 1: Identify the points and direction vectors Collect the constant terms and the coefficients of the parameters in the usual $\hat{i}, \hat{j}, \hat{k}$ order. The second line prints its $\hat{k}$ term before its $\hat{j}$ term, so their coordinates must be placed carefully:

$$\begin{aligned}\vec{a}_1 &= (4, -1, -3), & \vec{b}_1 &= (1, 2, 0), \\ \vec{a}_2 &= (1, -1, 2), & \vec{b}_2 &= (2, 4, -5).\end{aligned}$$

Thus the vector joining the selected points is $\vec{a}_2 - \vec{a}_1 = (-3, 0, 5)$.

Step 2: Find a common perpendicular direction The cross product is

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 4 & -5 \end{vmatrix} = -10\hat{i} + 5\hat{j}.$$

Its magnitude is $\sqrt{100 + 25} = 5\sqrt{5}$. Since this is nonzero, the two lines are not parallel and the scalar-triple-product formula applies.

Step 3: Project the joining vector onto the perpendicular The component of the joining vector along the common perpendicular gives the separation of the lines:

$$d = \frac{|(-3, 0, 5) \cdot (-10, 5, 0)|}{5\sqrt{5}} = \frac{|30|}{5\sqrt{5}} = \frac{6}{\sqrt{5}}.$$

The nonzero scalar triple product also confirms that the two non-parallel lines do not intersect.

Quick Tip to Solve: Reorder the second line into i, j, k coordinates before taking the cross product; its printed k term appears before its j term.

Step 4: Conclusion The shortest distance between the lines is $\frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$ units.

Quick Tip to Solve: Reorder the second line into i, j, k coordinates before taking the cross product; its printed k term appears before its j term.

27. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, then compute $A^2 - 5A + 4I$.

[3 marks]

Solution:

Concept:

- Matrix multiplication uses rows of the first matrix and columns of the second: $(A^2)_{ij} = \sum_{k=1}^3 a_{ik}a_{kj}$. Squaring a matrix does not mean squaring its individual entries.
- Addition and subtraction of matrices of the same order are performed entry by entry. The identity matrix I has diagonal entries equal to one and all off-diagonal entries equal to zero.

Step 1: Compute the square by row-column products Multiply the given matrix by itself:

$$A^2 = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}.$$

For example, the first row gives $2(2)+0(2)+1(1) = 5$, $2(0)+0(1)+1(-1) = -1$, and $2(1)+0(3)+1(0) = 2$. Computing all nine entries similarly,

$$A^2 = \begin{pmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{pmatrix}.$$

Step 2: Substitute the scalar multiples Multiplication by five scales every entry of A , whereas $4I$ contributes four only on the main diagonal:

$$5A = \begin{pmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{pmatrix}, \quad 4I = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Therefore entrywise subtraction and addition yield

$$A^2 - 5A + 4I = \begin{pmatrix} 5-10+4 & -1-0 & 2-5 \\ 9-10 & -2-5+4 & 5-15 \\ 0-5 & -1-(-5) & -2-0+4 \end{pmatrix}.$$

In particular, the middle entry in the last row becomes $-1 + 5 = 4$; subtracting a negative entry

Quick Tip to Solve: Compute A squared by row-column multiplication, then add four only to the diagonal when including $4I$.

changes its sign.

Step 3: Conclusion The required matrix is $\begin{pmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{pmatrix}$.

Quick Tip to Solve: Compute A squared by row-column multiplication, then add four only to the diagonal when including $4I$.

28. In a school, the probability of holding a debate competition is $\frac{1}{3}$ and that of a quiz competition is $\frac{2}{3}$. In the two participating teams, A has 4 girls and 6 boys and B has 7 girls and 3 boys. If a debate competition is held, the students are selected from team A and for the quiz competition they are selected from team B . If only two students are to be chosen from the teams, then find the probability that one will be a girl and the other a boy. [3 marks]

Solution:

Concept:

- If D and Q are mutually exclusive and exhaustive competition events, then $P(E) = P(D)P(E | D) + P(Q)P(E | Q)$, where E is the event of selecting one girl and one boy.
- Selecting two distinct students without replacement from a team of ten gives $\binom{10}{2}$ equally likely unordered pairs. From a team with g girls and b boys, gb pairs contain one student of each gender.

Step 1: Calculate the probability for a debate Let D denote a debate and Q a quiz. The competition probabilities are $P(D) = \frac{1}{3}$ and $P(Q) = \frac{2}{3}$. If a debate is held, both selected students come from team A , which contains four girls and six boys, making ten students altogether. There are $\binom{10}{2} = 45$ possible pairs and $\binom{4}{1}\binom{6}{1} = 24$ favourable pairs. Hence

$$P(E | D) = \frac{24}{45} = \frac{8}{15}.$$

Using unordered pairs automatically includes both possible orders in which a girl and a boy might be selected.

Step 2: Calculate the probability for a quiz If a quiz is held, selection is from team B , whose seven girls and three boys again give ten students. The total number of pairs remains 45, while the number with one girl and one boy is $\binom{7}{1}\binom{3}{1} = 21$. Therefore

$$P(E | Q) = \frac{21}{45} = \frac{7}{15}.$$

The conditional probabilities differ because the two teams have different gender compositions, even though their total sizes are equal.

Step 3: Weight the two cases by competition probabilities Debate and quiz are the two alternative

Quick Tip to Solve: Count girl-boy pairs as girls times boys in each team, divide by 45, and then weight by the debate and quiz probabilities.

cases and their probabilities add to one. Applying the law of total probability,

$$\begin{aligned}P(E) &= P(D)P(E | D) + P(Q)P(E | Q) \\&= \frac{1}{3} \cdot \frac{8}{15} + \frac{2}{3} \cdot \frac{7}{15} \\&= \frac{8}{45} + \frac{14}{45} = \frac{22}{45}.\end{aligned}$$

Thus each team's selection probability is weighted by how likely its corresponding competition is to occur.

Step 4: Conclusion The required probability of selecting one girl and one boy is $\boxed{\frac{22}{45}}$.

Quick Tip to Solve: Count girl-boy pairs as girls times boys in each team, divide by 45, and then weight by the debate and quiz probabilities.

29. Solve the following Linear Programming Problem graphically:

Maximize $Z = 20x + 10y$

subject to constraints

$$x + 2y \leq 28$$

$$3x + y \leq 24$$

$$x \geq 2$$

$$x, y \geq 0$$

[3 marks]

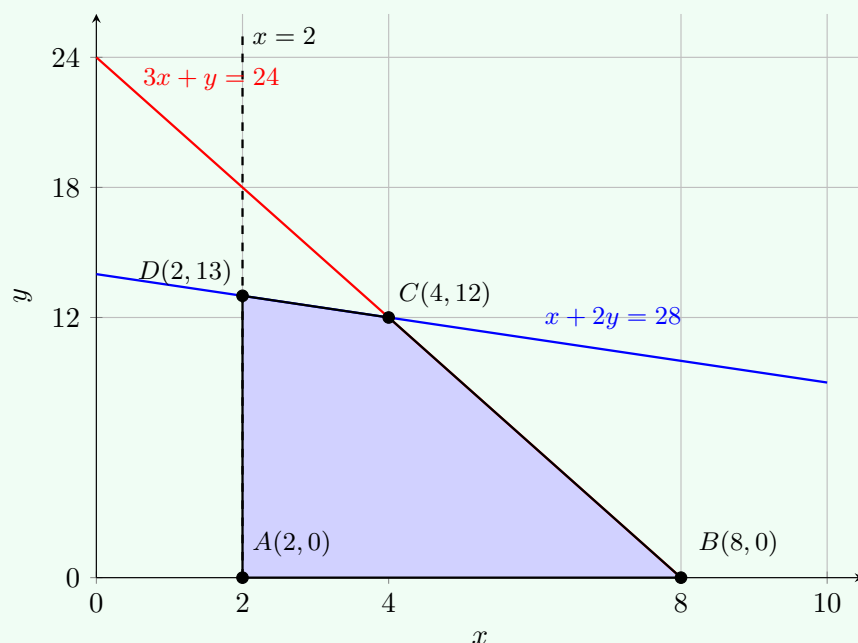
Solution:

Concept:

- A feasible region consists of all points satisfying every constraint simultaneously. Here it lies in the first quadrant, to the right of $x = 2$, and below both constraint lines.
- The corner-point theorem states that a linear objective $Z = 20x + 10y$ attains its maximum over a nonempty bounded polygon at a vertex. Compare Z at every vertex.

Step 1: Plot the constraint lines and identify the common region For $x + 2y = 28$, the intercepts are (28, 0) and (0, 14). For $3x + y = 24$, they are (8, 0) and (0, 24). Each inequality contains the origin, so take the side below its line. The condition $x \geq 2$ restricts the region to the right of the vertical line $x = 2$, while $y \geq 0$ restricts it to the upper half-plane. Their common portion is the shaded quadrilateral $ABCD$ shown in the graph.

Quick Tip to Solve: Keep the smaller upper bound on y when fixing $x = 2$; this gives (2, 13), not the infeasible intersection (2, 18).



The shaded feasible region $ABCD$ and its four corner points.

Step 2: Calculate the vertices of the feasible region On the horizontal axis, the feasible endpoints are $A = (2, 0)$ and $B = (8, 0)$. The intersection of the two sloping boundaries is obtained from

$$\begin{aligned} x + 2y &= 28, & 3x + y &= 24, \\ y &= 24 - 3x, & x + 2(24 - 3x) &= 28, \\ -5x &= -20, & x &= 4, \quad y = 12. \end{aligned}$$

Thus $C = (4, 12)$. At $x = 2$, the first line gives $y = 13$, whereas the second gives $y = 18$. The stricter upper bound is 13, giving $D = (2, 13)$. Each listed vertex satisfies all constraints, so these are precisely the four feasible corners.

Step 3: Evaluate the objective at every corner Since the feasible region is bounded, its maximum is obtained by comparing the following values:

Vertex (x, y)	$Z = 20x + 10y$
$A(2, 0)$	40
$B(8, 0)$	160
$C(4, 12)$	200
$D(2, 13)$	170

The value at C exceeds the values at the other three vertices.

Step 4: Conclusion The maximum value is $Z_{\max} = 200$, attained at $x = 4, y = 12$.

Quick Tip to Solve: Keep the smaller upper bound on y when fixing $x = 2$; this gives $(2, 13)$, not the infeasible intersection $(2, 18)$.

30(a). Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. A function $f : A \rightarrow B$ is defined by $f(x) = \frac{x-2}{x-3}$. Find whether f is one-one and onto. [3 marks]

Solution:

Concept:

- A function $f : A \rightarrow B$ is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for every $x_1, x_2 \in A$.
- It is onto if every $y \in B$ equals $f(x)$ for some $x \in A$. Both the expression for this preimage and its membership in A must be checked.

Step 1: Verify that the mapping takes values in the stated codomain For $x \in A = \mathbb{R} \setminus \{3\}$, the denominator is nonzero, so $f(x)$ exists. Also,

$$f(x) = \frac{x-2}{x-3} = 1 + \frac{1}{x-3}.$$

Since $1/(x-3) \neq 0$, the value $f(x)$ cannot equal 1. Thus it belongs to $B = \mathbb{R} \setminus \{1\}$, as required.

Step 2: Prove that equal images have equal preimages Take $x_1, x_2 \in A$ and assume $f(x_1) = f(x_2)$. The nonzero denominators permit cross-multiplication:

$$\begin{aligned}\frac{x_1-2}{x_1-3} &= \frac{x_2-2}{x_2-3}, \\ (x_1-2)(x_2-3) &= (x_2-2)(x_1-3), \\ x_1x_2 - 3x_1 - 2x_2 + 6 &= x_1x_2 - 3x_2 - 2x_1 + 6, \\ -x_1 + x_2 &= 0.\end{aligned}$$

Therefore $x_1 = x_2$, which proves that f is one-one.

Step 3: Construct a preimage for each element of the codomain Let $y \in B$ be arbitrary. Then $y \neq 1$. Solve the equation $y = f(x)$:

$$\begin{aligned}y(x-3) &= x-2, \\ x(y-1) &= 3y-2, \\ x &= \frac{3y-2}{y-1}.\end{aligned}$$

This is a real number because $y-1 \neq 0$, and

$$x-3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Hence $x \in A$. For this choice, $x-2 = y/(y-1)$ and $x-3 = 1/(y-1)$, so

$$f(x) = \frac{y/(y-1)}{1/(y-1)} = y.$$

Every element of B therefore has a preimage in A , proving that f is onto.

Step 4: Conclusion The function f is both one-one and onto; therefore $f : A \rightarrow B$ is a bijection.

Quick Tip to Solve: Write $f(x) = 1 + 1/(x-3)$ to see why 1 is excluded; when proving onto, always check that the resulting preimage is not 3.

30(b). Let n be a fixed positive integer. A relation R is defined in set $\mathbb{Z}$ such that $R = \{(x, y) : (x - y) \text{ is divisible by } n, x, y \in \mathbb{Z}\}$. Determine if R is an equivalence relation. [3 marks]

Solution:

Concept:

- An equivalence relation must be reflexive, symmetric and transitive on its underlying set. Here that set is $\mathbb{Z}$, the integers.
- For the fixed positive integer n , divisibility means $n \mid (x - y)$ if and only if $x - y = kn$ for an integer k . The negative and the sum of integer multiples of n are also multiples of n .

Step 1: Check reflexivity for every integer Let $x \in \mathbb{Z}$ be arbitrary. The difference between x and itself is

$$x - x = 0 = 0 \cdot n.$$

Thus $n \mid (x - x)$, since 0 is an integer multiplier. This gives $(x, x) \in R$ for every $x \in \mathbb{Z}$, establishing reflexivity. The positivity of n does not exclude divisibility of zero.

Step 2: Check symmetry by reversing the difference Suppose $(x, y) \in R$, where $x, y \in \mathbb{Z}$. By the definition, there is an integer k such that

$$x - y = kn.$$

Reversing the order gives

$$y - x = -(x - y) = (-k)n.$$

Since $-k \in \mathbb{Z}$, this difference is divisible by n . Therefore $(y, x) \in R$ whenever $(x, y) \in R$, establishing symmetry.

Step 3: Check transitivity by adding the differences Suppose $(x, y) \in R$ and $(y, z) \in R$, with $x, y, z \in \mathbb{Z}$. There are integers k and l such that

$$x - y = kn, \quad y - z = ln.$$

Add the two equations to eliminate the intermediate integer y :

$$x - z = (x - y) + (y - z) = kn + ln = (k + l)n.$$

Because $k + l$ is an integer, $n \mid (x - z)$, and so $(x, z) \in R$. This establishes transitivity for every pair of related differences.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, R is an equivalence relation on $\mathbb{Z}$ for every fixed positive integer n .

Quick Tip to Solve: Express each divisible difference as an integer multiple of n : zero proves reflexivity, negation proves symmetry, and addition proves transitivity.

31(a). If $xy = e^{x-y}$, then find $\frac{dy}{dx}$.

[3 marks]

Solution:

Concept:

- For an implicit relation between x and y , regard y as a function of x and differentiate both sides with respect to x .
- Use the product rule $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$, and the chain rule $\frac{d}{dx}e^{x-y} = e^{x-y} \left(1 - \frac{dy}{dx}\right)$.

Step 1: Differentiate the given implicit equation The given relation is

$$xy = e^{x-y}.$$

Write $y' = \frac{dy}{dx}$. Since both x and y occur in the exponent, its derivative is $1 - y'$, rather than simply 1. Differentiating the left side by the product rule and the right side by the chain rule gives

$$y + xy' = e^{x-y}(1 - y').$$

Step 2: Collect the derivative terms and simplify Expand the right side, then bring all terms containing y' to the left:

$$\begin{aligned}y + xy' &= e^{x-y} - e^{x-y}y', \\(x + e^{x-y})y' &= e^{x-y} - y, \\y' &= \frac{e^{x-y} - y}{x + e^{x-y}}.\end{aligned}$$

Now use the original equation $e^{x-y} = xy$ to eliminate the exponential:

$$y' = \frac{xy - y}{x + xy} = \frac{y(x - 1)}{x(1 + y)}.$$

The division applies wherever the denominator is nonzero. Also, $x \neq 0$ on the given relation because the exponential is strictly positive.

Step 3: Conclusion Thus, wherever the implicit derivative is given by this quotient,

$$\boxed{\frac{dy}{dx} = \frac{y(x - 1)}{x(1 + y)}}.$$

Quick Tip to Solve: Differentiate the exponent as $1 - y'$, collect the y' terms first, and only then replace e^{x-y} by xy .

31(b). Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right)$ with respect to $\cos^{-1} x^2$.

[3 marks]

Solution:**Concept:**

- To differentiate one function u with respect to another v , express u in terms of v , or use $\frac{du}{dv} = \frac{du/dx}{dv/dx}$ where the ratio is defined.
- The half-angle identities are $1 + \cos v = 2 \cos^2(v/2)$ and $1 - \cos v = 2 \sin^2(v/2)$. Also, $\tan(\pi/4 - a) = \frac{\cos a - \sin a}{\cos a + \sin a}$.

Step 1: Substitute the function with respect to which differentiation is required Let

$$v = \cos^{-1}(x^2), \quad u = \tan^{-1} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right).$$

For the real-valued expression, $-1 \leq x \leq 1$, so $0 \leq x^2 \leq 1$ and $0 \leq v \leq \pi/2$. Since $x^2 = \cos v$, the half-angle identities give

$$\sqrt{1+x^2} = \sqrt{2} \cos \frac{v}{2}, \quad \sqrt{1-x^2} = \sqrt{2} \sin \frac{v}{2}.$$

Both half-angle functions are nonnegative on this interval, so no sign ambiguity is introduced when taking square roots.

Step 2: Simplify the inverse tangent and differentiate Cancel $\sqrt{2}$ in the numerator and denominator. The subtraction formula for tangent then gives

$$\begin{aligned} u &= \tan^{-1} \left(\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)} \right) \\ &= \tan^{-1} \left(\tan \left(\frac{\pi}{4} - \frac{v}{2} \right) \right). \end{aligned}$$

Here $0 \leq \pi/4 - v/2 \leq \pi/4$, which lies within the principal range of $\tan^{-1}$. Therefore the inverse tangent returns that angle directly:

$$u = \frac{\pi}{4} - \frac{v}{2}.$$

Differentiate this linear relation with respect to v . This also avoids the indeterminate ratio of the two x -derivatives at $x = 0$.

Step 3: Conclusion The required derivative is

$$\boxed{\frac{d}{d(\cos^{-1} x^2)} \left[\tan^{-1} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right) \right] = -\frac{1}{2}}.$$

Quick Tip to Solve: Put $v = \cos^{-1} x^2$ so that the two square roots become half-angle sine and cosine functions; check the principal range before cancelling inverse functions.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32(a). Find: $\int \frac{x \, dx}{(x-1)(x^2+4)}$

[5 marks]

Solution:

Concept:

- A rational function whose denominator contains a linear factor and an irreducible quadratic factor is decomposed as $\frac{A}{x-1} + \frac{Bx+C}{x^2+4}$, with constant coefficients A, B, C .
- Use $\int \frac{dx}{x-a} = \ln|x-a| + C_0$, $\int \frac{x \, dx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2) + C_0$, and $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}(x/a) + C_0$.

Step 1: Set up the partial fraction decomposition Let the required integral be I . The numerator has smaller degree than the denominator, so no polynomial division is needed. Since x^2+4 has no real linear factors, its partial fraction numerator must be linear:

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Multiplying through by the common denominator gives a polynomial identity,

$$x = A(x^2+4) + (Bx+C)(x-1).$$

Step 2: Determine the coefficients Expand the identity and collect equal powers of x :

$$x = (A+B)x^2 + (C-B)x + (4A-C).$$

Comparing coefficients with $0x^2 + 1x + 0$, we obtain

$$A+B=0, \quad C-B=1, \quad 4A-C=0.$$

Thus $B=-A$ and $C=4A$. Substituting into $C-B=1$ gives $5A=1$, and hence

$$A = \frac{1}{5}, \quad B = -\frac{1}{5}, \quad C = \frac{4}{5}.$$

Step 3: Integrate the three standard terms Substitute the coefficients and separate the integrals:

$$I = \frac{1}{5} \int \frac{dx}{x-1} - \frac{1}{5} \int \frac{x \, dx}{x^2+4} + \frac{4}{5} \int \frac{dx}{x^2+4}.$$

For the middle integral, put $t = x^2 + 4$, so $dt = 2x \, dx$; it becomes $\frac{1}{2} \ln(x^2+4)$. For the last integral use $a = 2$, giving $\frac{1}{2} \tan^{-1}(x/2)$. Therefore

$$I = \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1} \frac{x}{2} + C_0.$$

Quick Tip to Solve: Use a linear numerator above x^2+4 ; after finding the coefficients, split it into an x term for the logarithm and a constant term for the inverse tangent.

Since $x^2 + 4 > 0$ for every real x , no absolute-value sign is needed around that logarithm's argument. The antiderivative applies on intervals excluding $x = 1$.

Step 4: Check the antiderivative Differentiating the expression obtained above gives

$$\frac{1}{5(x-1)} - \frac{x}{5(x^2+4)} + \frac{4}{5(x^2+4)} = \frac{(x^2+4) + (-x+4)(x-1)}{5(x-1)(x^2+4)}.$$

The numerator simplifies to $5x$, so the derivative is precisely $\frac{x}{(x-1)(x^2+4)}$, confirming the coefficients and integration factors.

Step 5: Conclusion Hence,

$$\int \frac{x dx}{(x-1)(x^2+4)} = \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1} \frac{x}{2} + C_0.$$

Quick Tip to Solve: Use a linear numerator above $x^2 + 4$; after finding the coefficients, split it into an x term for the logarithm and a constant term for the inverse tangent.

OR

32(b). Evaluate: $\int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$

[5 marks]

Solution:

Concept:

- Integration by parts gives $\int u dv = uv - \int v du$. Choose the inverse tangent as u , since its derivative is the simple rational function $1/(1+x^2)$.
- The chain rule gives $\frac{d}{dx}(1+x^2)^{-1/2} = -x(1+x^2)^{-3/2}$, which supplies an antiderivative of the remaining factor.
- For a definite integral, evaluate the boundary term at both limits. The standard derivative $\frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) = (1+x^2)^{-3/2}$ evaluates the integral left after integration by parts.

Step 1: Choose the factors for integration by parts Denote the required integral by

$$I = \int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx.$$

Set $u = \tan^{-1} x$ and $dv = x(1+x^2)^{-3/2} dx$. Then

$$du = \frac{dx}{1+x^2}, \quad v = -\frac{1}{\sqrt{1+x^2}}.$$

To check the sign of v , differentiate it: the two negative signs from the outer power and the leading minus sign cancel, giving exactly $x(1+x^2)^{-3/2}$. All these functions are continuous on $[0, 1]$, so

Quick Tip to Solve: Differentiate the inverse tangent and integrate the factor containing x ; this turns the remaining integral into the derivative of x divided by the square root of 1 plus x squared.

integration by parts applies on the full interval.

Step 2: Separate the boundary term and the remaining integral Applying the definite form of integration by parts,

$$\begin{aligned} I &= \left[-\frac{\tan^{-1} x}{\sqrt{1+x^2}} \right]_0^1 + \int_0^1 \frac{1}{(1+x^2)^{3/2}} dx \\ &= -\frac{\pi}{4\sqrt{2}} + J, \end{aligned}$$

where J denotes the last integral. Here $\tan^{-1} 1 = \pi/4$ and $\tan^{-1} 0 = 0$, with angles expressed in radians. The remaining integral has a positive sign because v is negative in the term $-\int v du$.

Step 3: Evaluate the remaining integral Differentiating $x/\sqrt{1+x^2}$ explicitly,

$$\begin{aligned} \frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) &= \frac{1}{\sqrt{1+x^2}} - \frac{x^2}{(1+x^2)^{3/2}} \\ &= \frac{1}{(1+x^2)^{3/2}}. \end{aligned}$$

Hence

$$J = \left[\frac{x}{\sqrt{1+x^2}} \right]_0^1 = \frac{1}{\sqrt{2}}.$$

Substitute this value into the expression for I , then combine the terms using the common denominator $4\sqrt{2}$. The integrand is nonnegative on the interval, which agrees with the positive value obtained since $4 > \pi$.

Step 4: Conclusion Therefore, $\int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx = \boxed{\frac{4-\pi}{4\sqrt{2}}}.$

Quick Tip to Solve: Differentiate the inverse tangent and integrate the factor containing x ; this turns the remaining integral into the derivative of x divided by the square root of 1 plus x squared.

33. Using integration, find the area of the region bounded by $y = 5x + 4$, $y = 0$, $x = -1$ and $x = 1$.

[5 marks]

Solution:

Concept:

- The geometric area between $y = f(x)$ and the x -axis over $[a, b]$ is $\int_a^b |f(x)| dx$. Area is nonnegative even when the function is below the axis.
- If a curve crosses the x -axis inside the interval, split the integral at each zero. On each subinterval, integrate the upper boundary minus the lower boundary.
- For the straight line $y = 5x + 4$, an antiderivative is $F(x) = \frac{5x^2}{2} + 4x$, and $\int_a^b (5x + 4) dx = F(b) - F(a)$.

Step 1: Locate the intercept and identify the two regions The given vertical boundaries are $x = -1$ and $x = 1$, and the horizontal boundary is $y = 0$. To find where the line meets this boundary, solve

$$5x + 4 = 0 \implies x = -\frac{4}{5}.$$

Quick Tip to Solve: Find the x -intercept before integrating for area; because it lies inside the given interval, reverse the sign on the portion below the x -axis and add the two positive integrals.

This value lies strictly between -1 and 1 . At the left boundary the line has ordinate $5(-1) + 4 = -1$, while at the right boundary its ordinate is $5(1) + 4 = 9$. Thus the line lies below the axis on $[-1, -4/5]$ and above it on $[-4/5, 1]$. The bounded region consists of these two portions, and their positive areas must be added.

Step 2: Integrate the portion below the axis On the first interval, the upper boundary is $y = 0$ and the lower boundary is $y = 5x + 4$. Therefore,

$$\begin{aligned} A_1 &= \int_{-1}^{-4/5} [0 - (5x + 4)] dx \\ &= - \left[\frac{5x^2}{2} + 4x \right]_{-1}^{-4/5}. \end{aligned}$$

The required endpoint values are

$$F\left(-\frac{4}{5}\right) = \frac{8}{5} - \frac{16}{5} = -\frac{8}{5}, \quad F(-1) = \frac{5}{2} - 4 = -\frac{3}{2}.$$

Hence $A_1 = -(-8/5 + 3/2) = 1/10$. The minus sign in the area integral converts the negative ordinate of the line into a positive vertical distance from the axis.

Step 3: Integrate the portion above the axis On the second interval, the line is the upper boundary and the axis is the lower boundary. Consequently,

$$\begin{aligned} A_2 &= \int_{-4/5}^1 [(5x + 4) - 0] dx \\ &= \left[\frac{5x^2}{2} + 4x \right]_{-4/5}^1 \\ &= \frac{13}{2} - \left(-\frac{8}{5}\right) = \frac{81}{10}. \end{aligned}$$

Add A_1 and A_2 to obtain the entire geometric area. Integrating $5x + 4$ directly from -1 to 1 would instead give the signed area 8 , which subtracts the small portion below the axis. The split integration prevents this cancellation and includes both bounded portions.

Step 4: Conclusion The required area is $A = A_1 + A_2 = \frac{1}{10} + \frac{81}{10} = \boxed{\frac{41}{5} \text{ square units}}$.

Quick Tip to Solve: Find the x-intercept before integrating for area; because it lies inside the given interval, reverse the sign on the portion below the x-axis and add the two positive integrals.

34(a). Solve the differential equation $ye^y dx = (y^3 + 2xe^y) dy$, when $y(0) = 1$.

[5 marks]

Solution:

Concept:

- A differential equation can be linear in x as a function of y , even if it is not linear in y as a function of x . Its form is $\frac{dx}{dy} + P(y)x = Q(y)$.
- The integrating factor for this form is $IF = e^{\int P(y) dy}$. Multiplication by it makes the left side the derivative of $x IF$.
- An initial condition $y(0) = 1$ specifies the point $(x, y) = (0, 1)$. Substitute this point in the integrated relation to determine the constant.

Step 1: Treat x as the dependent variable The given equation is

$$ye^y dx = (y^3 + 2xe^y) dy.$$

Since the initial point has $y = 1$, we work on a neighbourhood in which $y \neq 0$. Dividing by $ye^y dy$ and collecting the term containing x , we obtain

$$\frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}.$$

Thus this is a first-order linear equation for $x(y)$, with $P(y) = -2/y$ and $Q(y) = y^2 e^{-y}$. Choosing the dependent variable this way is essential: the factor e^y makes direct linear treatment of $y(x)$ unsuitable.

Step 2: Calculate the integrating factor and integrate The integrating factor is

$$IF = e^{\int (-2/y) dy} = e^{-2 \ln |y|} = \frac{1}{y^2}.$$

Multiplying the linear equation by $1/y^2$ gives

$$\frac{1}{y^2} \frac{dx}{dy} - \frac{2x}{y^3} = e^{-y}.$$

By the product rule, the left side is $\frac{d}{dy}(xy^{-2})$. Therefore

$$\frac{d}{dy} \left(\frac{x}{y^2} \right) = e^{-y}, \quad \frac{x}{y^2} = -e^{-y} + C.$$

In this integration the negative sign is necessary, because $\frac{d}{dy}(-e^{-y}) = e^{-y}$.

Step 3: Apply and check the initial condition At $x = 0$, the specified value of y is 1. Hence

$$0 = -e^{-1} + C, \quad C = e^{-1}.$$

The relation is consequently $x = y^2(e^{-1} - e^{-y})$. At $y = 1$, its right side is zero, as required.

Quick Tip to Solve: When the equation contains dx multiplied by a function of y , first test whether $x(y)$ makes it linear; here the integrating factor cancels the entire y -squared factor on the right.

Differentiating this relation gives

$$\frac{dx}{dy} = 2y(e^{-1} - e^{-y}) + y^2 e^{-y} = \frac{2x}{y} + y^2 e^{-y}.$$

Multiplication by ye^y recovers $ye^y dx = (2xe^y + y^3) dy$, confirming that the relation satisfies the original equation.

Step 4: Conclusion The particular solution satisfying $y(0) = 1$ is

$$x = y^2 \left(\frac{1}{e} - e^{-y} \right).$$

Quick Tip to Solve: When the equation contains dx multiplied by a function of y , first test whether $x(y)$ makes it linear; here the integrating factor cancels the entire y -squared factor on the right.

OR

34(b). Find the general solution of the differential equation $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$. [5 marks]

Solution:

Concept:

- An equation $dy/dx = f(y/x)$ is homogeneous. The substitution $y = vx$, with $v = y/x$, gives $dy/dx = v + x dv/dx$ and reduces it to a separable equation.
- In separating variables, any factor divided out must be checked for constant-ratio solutions. Logarithms use absolute values on intervals where their arguments do not vanish.
- Partial fractions reduce a rational integral to standard logarithmic integrals. An implicit algebraic relation can express the resulting family without logarithms.

Step 1: Reduce the homogeneous equation From the printed equation,

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{1 - 3(y/x)^2}{(y/x)^3 - 3(y/x)}.$$

On a working interval with $x \neq 0$, put $y = vx$. Then

$$v + x \frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}.$$

Subtracting v and simplifying the numerator carefully gives

$$x \frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v}.$$

The terms in v^2 cancel: $1 - 3v^2 - v(v^3 - 3v) = 1 - v^4$.

Quick Tip to Solve: After $y=vx$, the numerator collapses to $1-v$ to the fourth power; use $u=v$ squared for the integral, and restore the $v=$ plus or minus 1 solutions after separation.

Step 2: Separate variables and evaluate the integral For $v^4 \neq 1$, separation gives

$$\frac{v^3 - 3v}{1 - v^4} dv = \frac{dx}{x}.$$

Put $u = v^2$, so that $du = 2v dv$. The left integral becomes

$$\begin{aligned}\int \frac{v(v^2 - 3)}{1 - v^4} dv &= \frac{1}{2} \int \frac{u - 3}{(1 - u)(1 + u)} du \\ &= \frac{1}{2} \int \left(-\frac{1}{1 - u} - \frac{2}{1 + u} \right) du \\ &= \frac{1}{2} \ln |1 - u| - \ln |1 + u|.\end{aligned}$$

Hence

$$\frac{1}{2} \ln |1 - v^2| - \ln(1 + v^2) = \ln |x| + C_1.$$

The absolute value on $1 + v^2$ may be omitted since it is positive for real v .

Step 3: Express the result in x and y Multiply the logarithmic equation by two and combine logarithms:

$$\ln \left| \frac{1 - v^2}{(1 + v^2)^2} \right| = 2 \ln |x| + C_2.$$

Exponentiation and absorption of the fixed sign into an arbitrary real constant give

$$\frac{1 - v^2}{(1 + v^2)^2} = Cx^2.$$

Substituting $v = y/x$ simplifies this to

$$\frac{x^2 - y^2}{(x^2 + y^2)^2} = C.$$

The substitution and separation temporarily excluded $v = 1$ and $v = -1$. Direct substitution into the original differential equation verifies both $y = x$ and $y = -x$; they are included by taking $C = 0$ in the final algebraic relation.

Step 4: Verify the implicit family Differentiating the invariant, away from $(0, 0)$, gives

$$d \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right) = \frac{-2[(x^3 - 3xy^2) dx + (3x^2y - y^3) dy]}{(x^2 + y^2)^3}.$$

Setting this derivative to zero reproduces the given differential equation, including the sign on its right-hand side. Thus the algebraic relation describes the solution curves on regular intervals.

Step 5: Conclusion The general solution is

$$\boxed{x^2 - y^2 = C(x^2 + y^2)^2},$$

where C is an arbitrary real constant. The straight-line solutions $y = \pm x$ correspond to $C = 0$.

Quick Tip to Solve: After $y=vx$, the numerator collapses to $1-v$ to the fourth power; use $u=v$ squared for the integral, and restore the $v=\pm 1$ solutions after separation.

35. Find the length of the perpendicular drawn from the point $P(1, 2, 3)$ to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$. Also, find the equation of the perpendicular line joining P and the foot of the perpendicular.

[5 marks]

Solution:

Concept:

- A line through A with direction vector $\mathbf{d}$ has equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$.
- If H is the foot of the perpendicular from P , then H lies on the given line and $\overrightarrow{PH} \cdot \mathbf{d} = 0$. This condition determines the parameter of the foot.
- The perpendicular distance is $|\overrightarrow{PH}|$. The line joining P to H passes through P and has direction vector $\overrightarrow{PH}$.

Step 1: Parametrise the given line Set the three ratios equal to t :

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = t.$$

Thus a general point of the line is $H = (6+3t, 7+2t, 7-2t)$. The given line passes through $A = (6, 7, 7)$ and has direction vector $\mathbf{d} = (3, 2, -2)$.

Step 2: Locate the foot using perpendicularity Since $P = (1, 2, 3)$, the vector from P to the general point H is

$$\overrightarrow{PH} = (5+3t, 5+2t, 4-2t).$$

For H to be the foot, this vector must be perpendicular to $\mathbf{d}$. Hence

$$\begin{aligned} 3(5+3t) + 2(5+2t) - 2(4-2t) &= 0, \\ 15 + 9t + 10 + 4t - 8 + 4t &= 0, \\ 17 + 17t &= 0. \end{aligned}$$

This gives $t = -1$, so the foot is $H = (3, 5, 9)$.

Step 3: Calculate the perpendicular distance The displacement from P to the foot is $\overrightarrow{PH} = (2, 3, 6)$. Its scalar product with the line direction is $2(3) + 3(2) + 6(-2) = 0$, confirming perpendicularity. Its magnitude is

$$PH = \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2} = \sqrt{4+9+36} = \sqrt{49} = 7.$$

Step 4: Write the equation of the perpendicular line Use the point $P = (1, 2, 3)$ and direction ratios $(2, 3, 6)$ of PH . Its parametric equations are $x = 1 + 2s$, $y = 2 + 3s$, $z = 3 + 6s$, with $s \in \mathbb{R}$. At $s = 0$ it passes through P , and at $s = 1$ it reaches H . Eliminating s gives the required symmetric form.

Step 5: Conclusion The foot of the perpendicular is $H = (3, 5, 9)$, its length is $\boxed{7}$ units, and the required perpendicular line is

$$\boxed{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}}.$$

Quick Tip to Solve: Parametrise the given line first, then impose one dot-product equation to locate the foot; the foot-to-point vector supplies both the distance and the perpendicular's direction ratios.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

A carpenter needs to design a wooden box in the shape of a cuboid such that the sum of its length and breadth is 3 cm more than its height. Twice of its length, thrice of its breadth and its height add up to 10 cm. Its breadth added to 7 times its height is 1 cm less than 3 times its length.

On the basis of the above information, answer the following questions:

[4 marks]

36(i). Write the equations representing the various dimensions and express them as the matrix equation $AX = B$.

[1 mark]

Solution:

Concept:

- Translate each condition into a linear equation after assigning variables to the dimensions. Keep one fixed variable order throughout the equations and the matrix.
- In $AX = B$, each row of A contains the coefficients of one equation, X is the column of unknowns and B is the column of constants.

Step 1: Form the equations from the three conditions Let the length, breadth and height, in centimetres, be x, y, z , respectively. The sum of length and breadth is 3 more than height, giving $x + y = z + 3$. The second condition gives $2x + 3y + z = 10$. The breadth plus seven times height is one less than three times length, so $y + 7z = 3x - 1$. With all variable terms on the left, these become

$$\begin{aligned}x + y - z &= 3, \\2x + 3y + z &= 10, \\3x - y - 7z &= 1.\end{aligned}$$

Step 2: Arrange the coefficient and constant matrices Use the variable order x, y, z in every row. The matrix equation is

$$\underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}}_B.$$

Multiplying the rows of A by X reproduces the three equations, including the negative coefficients of breadth and height in the last row.

Step 3: Conclusion The equations are $x + y - z = 3$, $2x + 3y + z = 10$, $3x - y - 7z = 1$; therefore $AX = B$ with

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}.$$

Quick Tip to Solve: Translate 'one less than three times the length' as $3x - 1$ before rearranging; then read coefficients in the same order x, y, z for every row.

36(ii). Find if A^{-1} exists. Justify your answer.

[1 mark]

Solution:

Concept:

- A square matrix is invertible if and only if its determinant is nonzero. The coefficient matrix is formed with a fixed order of the unknown dimensions.
- For dimensions l, b, h , use the order $X = (l, b, h)^T$ in $AX = B$.

Step 1: Identify the coefficient matrix The printed relations are $l + b - h = 3$, $2l + 3b + h = 10$ and $3l - b - 7h = 1$. Thus $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}$. The signs in the third row follow by rearranging $b + 7h = 3l - 1$.

Step 2: Evaluate the determinant Expanding along the first row,

$$|A| = 1(-21 + 1) - 1(-14 - 3) + (-1)(-2 - 9) = -20 + 17 + 11 = 8.$$

This value is nonzero, so the criterion for invertibility is satisfied.

Step 3: Conclusion A^{-1} exists because $|A| = 8 \neq 0$.

Quick Tip to Solve: Keep the dimensions in the order length, breadth, height; rearrange the third equation before taking its coefficients.

36(iii)(a). Find A^{-1} .

[2 marks]

Solution:

Concept:

- For a nonsingular matrix, $A^{-1} = \frac{\text{adj } A}{|A|}$. The adjugate is the transpose of the cofactor matrix.
- Each cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j .

Step 1: Write the matrix and determinant The equations $l + b - h = 3$, $2l + 3b + h = 10$, $3l - b - 7h = 1$ give $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}$. Expanding along its first row gives

$$|A| = (-21 + 1) - (-14 - 3) - (-2 - 9) = 8.$$

Since this is nonzero, the inverse formula applies.

Step 2: Calculate the cofactors and transpose The first row of cofactors is $(-20, 17, -11)$; the second row is $(8, -4, 4)$, and the third row is $(4, -3, 1)$. For example, $C_{12} = -[2(-7) - 3] = 17$ and

Quick Tip to Solve: Apply the alternating cofactor signs, then transpose the complete cofactor matrix before dividing by 8.

$C_{23} = -[1(-1) - 3] = 4$. Hence

$$C = \begin{pmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{pmatrix}, \quad \text{adj } A = C^T = \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

Dividing this adjugate by the determinant gives the required inverse. The transpose is essential: the cofactor matrix itself is not the adjugate.

Step 3: Conclusion

$$A^{-1} = \frac{1}{8} \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}.$$

Quick Tip to Solve: Apply the alternating cofactor signs, then transpose the complete cofactor matrix before dividing by 8.

OR

36(iii)(b). Find $A^2 + 7I$.

[2 marks]

Solution:

Concept:

- Matrix squaring means matrix multiplication: $(A^2)_{ij} = \sum_{k=1}^3 a_{ik}a_{kj}$. It does not mean squaring each entry.
- The identity matrix I has ones on its main diagonal and zeros elsewhere. Adding $7I$ increases only the diagonal entries by seven.

Step 1: Recover the coefficient matrix Using the variable order (l, b, h) , the relations $l + b - h = 3$, $2l + 3b + h = 10$ and $3l - b - 7h = 1$ give $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{pmatrix}$. Use this same matrix in the required expression.

Step 2: Multiply rows by columns The first row of A^2 is $(1 + 2 - 3, 1 + 3 + 1, -1 + 1 + 7) = (0, 5, 7)$. The second row is $(2 + 6 + 3, 2 + 9 - 1, -2 + 3 - 7) = (11, 10, -6)$. The third row is $(3 - 2 - 21, 3 - 3 + 7, -3 - 1 + 49) = (-20, 7, 45)$. Therefore

$$A^2 = \begin{pmatrix} 0 & 5 & 7 \\ 11 & 10 & -6 \\ -20 & 7 & 45 \end{pmatrix}.$$

Step 3: Add the scalar multiple of the identity Now $7I = \text{diag}(7, 7, 7)$. Add these three entries to the diagonal of A^2 , retaining every off-diagonal entry. The new diagonal entries are $0 + 7 = 7$, $10 + 7 = 17$ and $45 + 7 = 52$.

Quick Tip to Solve: Compute all nine row-column products first; when adding $7I$, change only the three diagonal entries.

Step 4: Conclusion

$$A^2 + 7I = \begin{pmatrix} 7 & 5 & 7 \\ 11 & 17 & -6 \\ -20 & 7 & 52 \end{pmatrix}.$$

Quick Tip to Solve: Compute all nine row-column products first; when adding $7I$, change only the three diagonal entries.

37. Case Study - 2

An NGO organises a charity event in which they decide to distribute woollen caps to protect children from winter. The caps to be distributed are in three separate boxes, Box I has 30 red caps, Box II has 20 red and 10 green caps, and Box III has 30 green caps. The probability that a Box i is selected and a cap picked out is $\frac{i}{6}$, where $i = 1, 2, 3$.

Based on the above information, answer the following questions:

A person selects a cap.

[4 marks]

37(i). What is the probability that he selects a red cap?

[2 marks]

Solution:**Concept:**

- Let B_i denote selection of Box i , and let R denote selection of a red cap. The given probabilities are $P(B_1) = 1/6$, $P(B_2) = 2/6$ and $P(B_3) = 3/6$.
- Since the three boxes form mutually exclusive and exhaustive alternatives, the law of total probability gives $P(R) = \sum_{i=1}^3 P(B_i)P(R | B_i)$.

Step 1: Find the conditional probabilities within each box Box I contains only 30 red caps, so $P(R | B_1) = 30/30 = 1$. Box II contains 20 red and 10 green caps, hence $P(R | B_2) = 20/(20 + 10) = 2/3$. Box III contains only green caps, so $P(R | B_3) = 0$. The boxes are not equally likely to be chosen, even though each contains the same total number of caps.

Step 2: Weight the red-cap probabilities by box selection The given box probabilities sum to $(1 + 2 + 3)/6 = 1$. Therefore,

$$\begin{aligned} P(R) &= \frac{1}{6}(1) + \frac{2}{6}\left(\frac{2}{3}\right) + \frac{3}{6}(0) \\ &= \frac{1}{6} + \frac{2}{9} = \frac{3}{18} + \frac{4}{18} = \frac{7}{18}. \end{aligned}$$

Thus pooling the caps and using 50/90 would be inappropriate: the experiment first selects a box with the specified unequal probabilities.

Step 3: Conclusion The probability of selecting a red cap is $\boxed{\frac{7}{18}}$.

Quick Tip to Solve: Multiply each box's red-cap fraction by its given selection probability before adding; equal box sizes do not mean equal box probabilities.

37(ii). If he selects a green cap, what is the probability that the cap has come from Box II? [2 marks]

Solution:

Concept:

- Let G denote selection of a green cap and B_i denote selection of Box i . The given prior probabilities are $P(B_1) = 1/6$, $P(B_2) = 2/6$ and $P(B_3) = 3/6$.
- Bayes' theorem gives $P(B_2 | G) = P(B_2)P(G | B_2)/P(G)$, where the denominator includes all ways of obtaining a green cap.

Step 1: Calculate the probability of selecting a green cap Box I has no green caps, so $P(G | B_1) = 0$. Box II has 10 green caps among 30, giving $P(G | B_2) = 1/3$. All 30 caps in Box III are green, giving $P(G | B_3) = 1$. By total probability,

$$P(G) = \frac{1}{6}(0) + \frac{2}{6}\left(\frac{1}{3}\right) + \frac{3}{6}(1) = \frac{1}{9} + \frac{1}{2} = \frac{11}{18}.$$

Step 2: Condition on the observed green cap The probability of selecting Box II and then obtaining a green cap is

$$P(B_2 \cap G) = P(B_2)P(G | B_2) = \frac{2}{6} \cdot \frac{1}{3} = \frac{1}{9}.$$

Restricting the sample space to outcomes with a green cap, Bayes' theorem gives

$$P(B_2 | G) = \frac{1/9}{11/18} = \frac{1}{9} \cdot \frac{18}{11} = \frac{2}{11}.$$

This conditional probability differs from $P(G | B_2) = 1/3$: knowing the colour is a different condition from knowing the box.

Step 3: Conclusion Given that the selected cap is green, the probability that it came from Box II is

$$\frac{2}{11}.$$

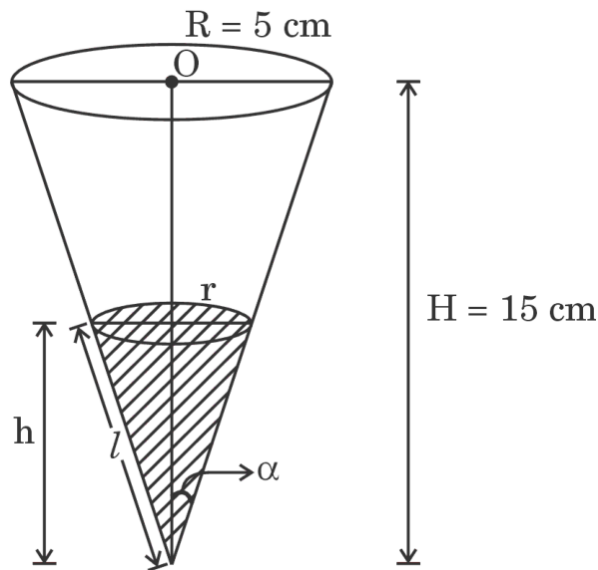
Quick Tip to Solve: In Bayes' theorem, divide the Box II-and-green probability by the total green probability, including the contribution from Box III.

38. Case Study - 3

At a birthday party, children are being served orange juice in conical cups, as shown in the figure. Each cup is 15 cm deep and has a radius 5 cm. The juice is being poured into this cup at a rate of $0.1 \text{ cm}^3/\text{s}$.

On the basis of the above information, answer the following questions:

[4 marks]



38(i). Establish a relation between the height h of the juice in the cup and radius r of the surface of the juice in the cup, if the semi-vertical angle of the cone is α .

[1 mark]

Solution:

Concept:

- The semi-vertical angle α is the angle between the cone's axis and its slant side. In an axial cross-section, height h and radius r form the adjacent and opposite sides of a right triangle.
- For a fixed conical cup, the juice cone is similar to the full cup, so $r/h = R/H$, where $R = 5 \text{ cm}$ and $H = 15 \text{ cm}$.

Step 1: Apply the tangent ratio The right triangle has vertical side h and horizontal side r . Consequently,

$$\tan \alpha = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{r}{h}.$$

The angle is measured from the vertical axis, so reversing this ratio would instead give $\cot \alpha$.

Step 2: Use the dimensions of the fixed cup Similarity with the full cone gives

$$\tan \alpha = \frac{R}{H} = \frac{5}{15} = \frac{1}{3}.$$

Thus the ratio of juice radius to juice depth remains constant as the cup fills; both lengths are measured in the same units.

Step 3: Conclusion The required relation is $r = h \tan \alpha$, equivalently $h = r \cot \alpha$. For this cup, $r = h/3$, or $h = 3r$.

Quick Tip to Solve: Measure the semi-vertical angle from the axis: radius is opposite and height is adjacent, giving tangent as radius divided by height.

38(ii). At what rate is the juice level in the cup rising when the juice is 6 cm deep?

[1 mark]

Solution:

Concept:

- Similar cones have proportional radii and heights. Here $r/h = 5/15 = 1/3$, where r and h are the radius and depth of juice.
- The juice volume is $V = \frac{1}{3}\pi r^2 h$. Its inflow rate is $dV/dt = 0.1 \text{ cm}^3/\text{s}$.

Step 1: Express volume using the juice depth The juice and cup share the same vertex and semi-vertical angle, so $r = h/3$. Eliminate the changing radius before differentiating:

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}.$$

Step 2: Differentiate with respect to time Both the depth and volume vary with time. The chain rule gives

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

At $h = 6 \text{ cm}$, the coefficient is $4\pi \text{ cm}^2$. Thus

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s}.$$

Step 3: Conclusion The juice level is rising at $\boxed{\frac{1}{40\pi} \text{ cm/s}}$.

Quick Tip to Solve: First replace r by $h/3$; treating the radius as constant would give the wrong related-rate equation.

38(iii)(a). When the juice is 6 cm deep, then find at what rate is the upper surface area of juice increasing?

[2 marks]

Solution:

Concept:

- By similarity of the juice cone and the cup, $r/h = 5/15 = 1/3$, where h is the juice depth and r its surface radius.
- The upper surface of juice is a horizontal circle of area $A = \pi r^2$. The changing volume is $V = \frac{1}{3}\pi r^2 h$, with $dV/dt = 0.1 \text{ cm}^3/\text{s}$.

Step 1: Find the rate of increase of depth Using $r = h/3$, express the volume in terms of the depth alone:

$$V = \frac{\pi h^3}{27}, \quad \frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

Quick Tip to Solve: Use $A = \pi r^2$ for the upper surface; combine $r = h/3$ with the volume rate before differentiating the area.

Substitute the given inflow rate and the instant $h = 6$ cm:

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s}.$$

Step 2: Differentiate the circular upper area The upper surface is the exposed circular face of the juice, so its area is

$$A = \pi \left(\frac{h}{3}\right)^2 = \frac{\pi h^2}{9}.$$

Differentiate with respect to time and then substitute the instantaneous depth and its rate:

$$\begin{aligned} \frac{dA}{dt} &= \frac{2\pi h}{9} \frac{dh}{dt} \\ &= \frac{2\pi \cdot 6}{9} \cdot \frac{1}{40\pi} = \frac{1}{30} \text{ cm}^2/\text{s}. \end{aligned}$$

The rate is positive because the surface radius increases as the level rises.

Step 3: Conclusion The upper surface area of juice is increasing at $\boxed{\frac{1}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Use $A = \pi r^2$ for the upper surface; combine $r = h/3$ with the volume rate before differentiating the area.

OR

38(iii)(b). When the juice is 6 cm deep, then find the rate at which the wetted surface area of the cup is increasing. [2 marks]

Solution:

Concept:

- Similar cones give $r/h = 5/15 = 1/3$, where h and r are the depth and radius of the juice cone.
- The wetted cup surface is the curved surface in contact with the juice: $S = \pi r l$, with slant height $l = \sqrt{h^2 + r^2}$.
- The volume $V = \frac{1}{3}\pi r^2 h$ increases at $dV/dt = 0.1 \text{ cm}^3/\text{s}$; the chain rule connects this to the depth and area rates.

Step 1: Express the wetted area in terms of depth Similarity gives $r = h/3$. By Pythagoras, the slant height from the vertex to the juice level is

$$l = \sqrt{h^2 + \frac{h^2}{9}} = \frac{\sqrt{10}}{3}h.$$

The wetted region is the conical wall below the juice level. The horizontal upper circle is not part of the cup wall. Hence

$$S = \pi \left(\frac{h}{3}\right) \left(\frac{\sqrt{10}h}{3}\right) = \frac{\pi\sqrt{10}}{9}h^2.$$

Quick Tip to Solve: For the wetted cone wall use $\pi r l$, not πr^2 ; substitute both $r = h/3$ and $l = \sqrt{10}h/3$ first.

Step 2: Obtain the depth rate from the inflow Eliminate the radius in the volume formula before differentiating:

$$V = \frac{\pi h^3}{27}, \quad \frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}.$$

At $h = 6$ cm, this gives

$$0.1 = 4\pi \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{40\pi} \text{ cm/s}.$$

Step 3: Calculate the wetted-area rate Differentiate S with respect to time, allowing its slant height as well as radius to increase:

$$\begin{aligned} \frac{dS}{dt} &= \frac{2\pi\sqrt{10}h}{9} \frac{dh}{dt} \\ &= \frac{2\pi\sqrt{10} \cdot 6}{9} \cdot \frac{1}{40\pi} = \frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}. \end{aligned}$$

Step 4: Conclusion The wetted surface area of the cup increases at $\boxed{\frac{\sqrt{10}}{30} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: For the wetted cone wall use $\pi r l$, not πr^2 ; substitute both $r = h/3$ and $l = \sqrt{10}h/3$ first.