

CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-5-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1** to **18** are Multiple Choice Questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21** to **25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32** to **35** are Long Answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

Section A (20 marks)

This section comprises Multiple Choice Questions (MCQs) of 1 mark each.

1. If matrix $A = \begin{pmatrix} -p & q \\ r & p \end{pmatrix}$ is such that $A^2 = I$, then:

[1 mark]

(A) $1 + p^2 + qr = 0$

(B) $1 - p^2 - qr = 0$

(C) $1 - p^2 + qr = 0$

(D) $1 + p^2 - qr = 0$

Correct Answer: (B) $1 - p^2 - qr = 0$

Solution:

Concept:

- Matrix multiplication uses the dot product of a row of the first matrix and a column of the second matrix.
- The identity matrix I has diagonal entries one and off-diagonal entries zero; corresponding entries of equal matrices must agree.

Step 1: Multiply the matrix by itself With the printed diagonal entries $-p$ and p , the off-diagonal products cancel:

$$A^2 = \begin{pmatrix} (-p)^2 + qr & -pq + qp \\ -rp + pr & rq + p^2 \end{pmatrix} = \begin{pmatrix} p^2 + qr & 0 \\ 0 & p^2 + qr \end{pmatrix}.$$

Step 2: Compare with the identity Since $A^2 = I$, either diagonal entry gives $p^2 + qr = 1$. Moving both terms to the other side gives $1 - p^2 - qr = 0$. Option (A) would instead require their sum to be -1 ; (C) and (D) use a difference of these terms, which matrix multiplication does not produce.

Step 3: Conclusion The required relation is $1 - p^2 - qr = 0$. Hence, the correct option is (B).

Quick Tip to Solve: For a two-by-two matrix with opposite diagonal entries, calculate one diagonal entry of its square and check that the off-diagonal entries cancel.

2. If A is a square matrix such that $A^2 = A$, then $(A - I)^3 - A$ is equal to:

[1 mark]

(A) I

(B) $-I$

(C) A

(D) A^2

Correct Answer: (B) $-I$

Solution:

Concept:

- A matrix satisfying $A^2 = A$ is idempotent. Multiplying this relation by A gives $A^3 = A$.

Quick Tip to Solve: Reduce every positive power of an idempotent matrix to the matrix itself before collecting coefficients.

- The identity matrix commutes with A , so the binomial expansion can be used for powers of $A - I$.

Step 1: Expand the expression Use $AI = IA = A$ and $I^2 = I$ to obtain

$$(A - I)^3 - A = A^3 - 3A^2 + 3A - I - A.$$

The final subtraction of A must be retained separately from the cubic expansion.

Step 2: Apply idempotency and compare the options Substitute $A^3 = A$ and $A^2 = A$:

$$(A - I)^3 - A = A - 3A + 3A - I - A = -I.$$

Thus the identity term has a negative sign, ruling out (A). Options (C) and (D) both represent A under the given condition, so neither is the required constant matrix.

Step 3: Conclusion The expression equals $-I$. Hence, the correct option is (B).

Quick Tip to Solve: Reduce every positive power of an idempotent matrix to the matrix itself before collecting coefficients.

3. For the inverse trigonometric functions, which of the following Principal Value Branch is *not* correctly defined? [1 mark]

(A) $\tan^{-1} : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(B) $\sec^{-1} : \mathbb{R} - (-1, 1) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(C) $\cot^{-1} : \mathbb{R} \rightarrow (0, \pi)$

(D) $\operatorname{cosec}^{-1} : \mathbb{R} - (-1, 1) \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Correct Answer: (D) $\operatorname{cosec}^{-1} : \mathbb{R} - (-1, 1) \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution:

Concept:

- The principal value branch restricts the range of an inverse trigonometric function so that the corresponding trigonometric function is one-one.
- Because $\operatorname{cosec} \theta = 1/\sin \theta$ is undefined at $\theta = 0$, the principal range of $\operatorname{cosec}^{-1}$ excludes zero.

Step 1: Check the first three principal branches The stated ranges $(-\pi/2, \pi/2)$ for $\tan^{-1}$ and $(0, \pi)$ for $\cot^{-1}$ are correct. For $\sec^{-1}$, the domain is $\mathbb{R} - (-1, 1)$, and the printed range $[0, \pi] - \{\pi/2\}$ is correct because secant is undefined at $\pi/2$. Thus (A), (B) and (C) are correctly defined.

Step 2: Examine the cosecant inverse range The correct principal range is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Option (D) includes zero and therefore fails to state this principal branch correctly. No real input can have principal inverse cosecant equal to zero, since cosecant itself is undefined there.

Step 3: Conclusion The incorrectly defined principal value branch is that in (D). Hence, the correct

Quick Tip to Solve: In reciprocal inverse-trigonometric ranges, exclude the angle where the original reciprocal function is undefined: zero for cosecant and $\pi/2$ for secant.

option is (D).

Quick Tip to Solve: In reciprocal inverse-trigonometric ranges, exclude the angle where the original reciprocal function is undefined: zero for cosecant and pi over two for secant.

4. Let $A = \begin{pmatrix} 0 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$. If $A + B + C = O$, then matrix C is: [1 mark]

(A) $\begin{pmatrix} -3 & -3 & 5 \\ 3 & 4 & 2 \end{pmatrix}$

(B) $\begin{pmatrix} 3 & 3 & 5 \\ -3 & -4 & -2 \end{pmatrix}$

(C) $\begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$

(D) $\begin{pmatrix} -3 & -3 & -5 \\ 3 & 4 & 2 \end{pmatrix}$

Correct Answer: (C) $\begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$

Solution:

Concept:

- Matrices of the same order are added entry by entry, keeping the row and column of each entry fixed.
- From $A + B + C = O$, the unknown matrix is the additive inverse $C = -(A + B)$; every entry changes sign.

Step 1: Add the two given matrices Use the negative entry -3 in the first row of A :

$$A + B = \begin{pmatrix} 0 - 3 & -3 + 0 & 4 + 1 \\ 1 + 2 & 0 + 4 & 2 + 0 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 5 \\ 3 & 4 & 2 \end{pmatrix}.$$

Step 2: Take the additive inverse Negating all six entries gives

$$C = \begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}.$$

Adding this matrix to $A + B$ makes every entry zero. Option (A) is the sum itself, (B) fails to negate the entry 5, and (D) leaves the other five entries with incorrect signs.

Step 3: Conclusion Thus $C = \begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$. Hence, the correct option is (C).

Quick Tip to Solve: First calculate the sum matrix, then change every sign; verify any doubtful entry by adding the three corresponding entries to zero.

5. If A is a non-singular matrix, then which of the following is *not* true?

[1 mark]

- (A) $\text{adj } A$ is singular
- (B) $(\text{adj } A)^{-1} = \text{adj}(A^{-1})$
- (C) $|A| \neq 0$
- (D) A^{-1} exists

Correct Answer: (A) $\text{adj } A$ is singular

Solution:

Concept:

- A square matrix A is non-singular exactly when $|A| \neq 0$, and then its inverse A^{-1} exists.
- The adjoint identity gives $\text{adj } A = |A|A^{-1}$. A nonzero scalar multiple of an invertible matrix is invertible.

Step 1: Determine whether the adjoint is singular Since $|A| \neq 0$, the matrix $\text{adj } A = |A|A^{-1}$ has inverse

$$(\text{adj } A)^{-1} = \frac{1}{|A|}A.$$

Hence the adjoint is non-singular, so the statement in (A) is false.

Step 2: Verify the remaining statements Apply the same adjoint identity to A^{-1} :

$$\text{adj}(A^{-1}) = |A^{-1}|(A^{-1})^{-1} = \frac{1}{|A|}A.$$

This is exactly $(\text{adj } A)^{-1}$, proving (B). Statements (C) and (D) are both consequences of non-singularity and are therefore true.

Step 3: Conclusion The false statement is that $\text{adj } A$ is singular. Hence, the correct option is (A).

Quick Tip to Solve: Replace the adjoint by determinant times inverse; non-singularity then settles both the adjoint and its inverse immediately.

6. If $f(x) = \begin{cases} \frac{x^2 - 4x - 5}{x + 1}, & x \neq -1 \\ k, & x = -1 \end{cases}$ is continuous at $x = -1$, then the value of k is:

[1 mark]

- (A) Any real value
- (B) 6
- (C) -1
- (D) -6

Correct Answer: (D) -6

Solution:

Concept:

- Continuity at $x = a$ requires $\lim_{x \rightarrow a} f(x) = f(a)$, with both one-sided limits equal.

Quick Tip to Solve: For a removable discontinuity, cancel the vanishing factor first and assign the resulting limiting value at the missing point.

- A common factor can be cancelled for $x \neq a$ when evaluating a limit, even if the original quotient is undefined at a .

Step 1: Factor the numerator The quadratic factorises as $x^2 - 4x - 5 = (x - 5)(x + 1)$. Therefore, for $x \neq -1$,

$$f(x) = \frac{(x - 5)(x + 1)}{x + 1} = x - 5.$$

Direct substitution into the original quotient gives $0/0$, so cancellation is needed before taking the limit.

Step 2: Match the limit to the assigned value Both one-sided limits are obtained from the polynomial $x - 5$:

$$\lim_{x \rightarrow -1} f(x) = -1 - 5 = -6.$$

As $f(-1) = k$, continuity fixes k uniquely. It cannot be arbitrary, 6 reverses the sign, and -1 is the input rather than the limiting output.

Step 3: Conclusion Continuity requires $k = -6$. Hence, the correct option is (D).

Quick Tip to Solve: For a removable discontinuity, cancel the vanishing factor first and assign the resulting limiting value at the missing point.

7. If the area of $\triangle ABC$ with vertices $A(3, 1)$, $B(-2, 1)$ and $C(0, k)$ is 5 sq. units, then values of k are: [1 mark]

- (A) 3, 1
- (B) $-1, 3$
- (C) $-1, 2$
- (D) 0, 2

Correct Answer: (B) $-1, 3$

Solution:

Concept:

- The area of a triangle is half the product of a base length and its perpendicular height; the height is a nonnegative distance.
- Points with the same ordinate lie on a horizontal line. The distance of (x, k) from the line $y = 1$ is $|k - 1|$.

Step 1: Identify a convenient base and height Both $A(3, 1)$ and $B(-2, 1)$ lie on $y = 1$. Thus the horizontal base length is $AB = |3 - (-2)| = 5$, and the perpendicular height from $C(0, k)$ to that base is $|k - 1|$.

Step 2: Use the given area and retain both signs Substitute in the area formula:

$$\frac{1}{2} \times 5 \times |k - 1| = 5 \implies |k - 1| = 2.$$

Therefore $k - 1 = 2$ or $k - 1 = -2$, giving $k = 3$ or $k = -1$. Values 0 and 2 give height 1, while 1

Quick Tip to Solve: When two vertices have the same ordinate, use their horizontal separation as the base and an absolute difference of ordinates as the height.

gives zero height; hence the pairs in (A), (C) and (D) fail the area condition.

Step 3: Conclusion The two values are $k = -1, 3$. Hence, the correct option is (B).

Quick Tip to Solve: When two vertices have the same ordinate, use their horizontal separation as the base and an absolute difference of ordinates as the height.

8. Derivative of $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$, with respect to x is:

[1 mark]

(A) -1

(B) 1

(C) $\frac{\pi}{4}$

(D) $-\frac{\pi}{4}$

Correct Answer: (A) -1

Solution:

Concept:

- The addition formula gives $(\sin x + \cos x)/\sqrt{2} = \cos(\pi/4 - x)$.
- The principal range of $\cos^{-1}$ is $[0, \pi]$, so $\cos^{-1}(\cos \theta) = \theta$ whenever $0 \leq \theta \leq \pi$.

Step 1: Check the angle against the principal range From $-\pi/4 < x < \pi/4$, subtract x from $\pi/4$ to obtain

$$0 < \frac{\pi}{4} - x < \frac{\pi}{2}.$$

This entire interval lies in the principal range of inverse cosine, so the inverse and cosine cancel without changing sign or adding a multiple of π .

Step 2: Simplify and differentiate Hence the function is

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{4} - x\right)\right) = \frac{\pi}{4} - x, \quad \frac{dy}{dx} = -1.$$

Option (B) has the wrong slope sign. Options (C) and (D) incorrectly retain an angle constant after differentiation; the derivative of $\pi/4$ is zero.

Step 3: Conclusion The derivative is -1 throughout the given interval. Hence, the correct option is (A).

Quick Tip to Solve: Rewrite the sine-cosine sum as a single cosine, then check its angle lies in the inverse cosine principal range before cancelling.

9. Absolute minimum value of $f(x) = (x - 2)^2 + 5$ in the interval $[-3, 2]$ is:

[1 mark]

- (A) -3
- (B) 2
- (C) 5
- (D) 30

Correct Answer: (C) 5

Solution:

Concept:

- An absolute minimum is the smallest function value attained anywhere in the specified interval, including its endpoints.
- For every real number x , the square $(x - 2)^2$ is non-negative, and it is zero only at $x = 2$.

Step 1: Find the lower bound Since the squared term cannot be negative,

$$f(x) = (x - 2)^2 + 5 \geq 5.$$

Thus neither -3 nor 2 can be a value of this function, much less its minimum.

Step 2: Check whether the bound is attained The point $x = 2$ belongs to the closed interval $[-3, 2]$, so the lower bound is attained. The endpoint values are

$$f(-3) = 25 + 5 = 30, \quad f(2) = 0 + 5 = 5.$$

The value 30 is the other endpoint value, not the minimum.

Step 3: Conclusion The absolute minimum value is 5 , attained at $x = 2$. Hence, the correct option is (C).

Quick Tip to Solve: For a squared expression plus a constant, first check whether the point making the square zero lies in the given interval.

10. $\int \frac{1}{\sqrt{1 + \cos 2x}} dx$ is equal to:

[1 mark]

- (A) $\log \cos x + C$
(B) $\frac{1}{\sqrt{2}} \log |\sec x + \tan x| + C$
(C) $\frac{1}{\sqrt{2}} \log |\sec x - \tan x| + C$
(D) $\log \sin 2x + C$

Correct Answer: No unique option without a domain restriction: (B) applies where $\cos x > 0$, while (C) applies where $\cos x < 0$.

Solution:

Concept:

- The double-angle identity is $1 + \cos 2x = 2 \cos^2 x$. Hence the square-root denominator is $\sqrt{2} |\cos x|$.
- The standard primitive is $\int \sec x dx = \log |\sec x + \tan x| + C$, on intervals where it is defined.

Step 1: Simplify the radical On an interval where $\cos x > 0$, such as $(-\pi/2, \pi/2)$, the integral reduces to

$$I = \frac{1}{\sqrt{2}} \int \sec x dx.$$

The absolute value in the square root must be respected when considering other intervals.

Step 2: Integrate and compare the choices The standard secant formula gives

$$I = \frac{1}{\sqrt{2}} \log |\sec x + \tan x| + C.$$

Differentiation gives $\sec x / \sqrt{2}$, as required on this interval. Option (C) differentiates to $-\sec x / \sqrt{2}$; options (A) and (D) give $-\tan x$ and $2 \cot 2x$, respectively. On an interval where $\cos x < 0$, the primitive instead has the opposite sign, equivalently the expression in (C).

Step 3: Conclusion The printed question does not specify an interval. Option (B) is valid where $\cos x > 0$, and option (C) is valid where $\cos x < 0$. Therefore there is no uniquely correct unrestricted option; (B) is the intended answer if a positive-cosine interval is assumed.

Quick Tip to Solve: Use the double-angle identity under the radical and remember that $\sqrt{\cos^2 x} = |\cos x|$; check the sign on the interval in use.

11. The value of $\int_{-5}^{-1} \frac{1}{x} dx$ is equal to:

[1 mark]

(A) $-\log 5$

(B) x^6

(C) $\log(-5)$

(D) x^{-6}

Correct Answer: (A) $-\log 5$

Solution:

Concept:

- For $x \neq 0$, an antiderivative of $1/x$ is $\log |x|$. The absolute value makes the formula valid for negative inputs.
- The fundamental theorem evaluates a definite integral as the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Check the interval and primitive The interval $[-5, -1]$ contains no zero, so $1/x$ is continuous throughout it and the ordinary definite integral exists. We use

$$F(x) = \log |x|.$$

Since the integrand is negative on this interval, the result must be negative.

Step 2: Substitute the limits Applying the upper limit first gives

$$\int_{-5}^{-1} \frac{1}{x} dx = \log |-1| - \log |-5| = 0 - \log 5.$$

The expression $\log(-5)$ is not real. The printed expressions x^6 and x^{-6} are variable-dependent and are not the value of this definite integral.

Step 3: Conclusion The integral equals $-\log 5$. Hence, the correct option is (A).

Quick Tip to Solve: For an integral of $1/x$ over negative limits, use $\log |x|$ and check that the result has the integrand's negative sign.

12. An ant is observed crawling on a sheet of paper along a straight line given by equation $y = 2x - 4$. Area of the surface covered by the ant bounded by y -axis, x -axis and $x = 1$ is: [1 mark]

- (A) 1 sq. unit
- (B) 3 sq. units
- (C) 2 sq. units
- (D) 4 sq. units

Correct Answer: (B) 3 sq. units

Solution:

Concept:

- Geometrical area between two curves is the integral of upper ordinate minus lower ordinate over the given horizontal interval.
- The y -axis is $x = 0$, the x -axis is $y = 0$, and the remaining vertical boundary is $x = 1$.

Step 1: Identify the upper and lower boundaries The ant's line is $y = 2x - 4$. For $0 \leq x \leq 1$, its ordinates lie between -4 and -2 , so the line is below the x -axis. The height of a vertical strip is therefore

$$0 - (2x - 4) = 4 - 2x.$$

Step 2: Integrate the positive strip height The bounded area is

$$A = \int_0^1 (4 - 2x) dx = [4x - x^2]_0^1 = 4 - 1 = 3.$$

Equivalently, the parallel vertical sides have lengths 4 and 2 and separation 1, giving the same trapezium area. Using either endpoint height alone gives 4 or 2, neither of which is the area.

Step 3: Conclusion The required area is 3 square units. Hence, the correct option is (B).

Quick Tip to Solve: When the line lies below the x -axis, integrate zero minus the line; a negative signed integral is not a geometrical area.

13. The order and degree of the differential equation $1 + \left(\frac{d^3y}{dx^3}\right)^3 = \lambda \frac{d^2y}{dx^2}$ is: [1 mark]

- (A) Order = 3, Degree = 3
- (B) Order = 2, Degree = 2
- (C) Order = 3, Degree = 1
- (D) Order = 2, Degree = 1

Correct Answer: (A) Order = 3, Degree = 3

Solution:

Concept:

- The order of a differential equation is the order of the highest derivative present in it.

Quick Tip to Solve: First circle the highest-order derivative, then read only its exponent to find the degree after expressing the equation polynomially.

- When the equation is polynomial in derivatives, its degree is the power of the highest-order derivative, not the power of any lower-order derivative.

Step 1: Identify the highest derivative The equation is

$$1 + \left(\frac{d^3 y}{dx^3} \right)^3 = \lambda \frac{d^2 y}{dx^2}.$$

It contains second and third derivatives. The third derivative is the highest, so the order is 3, regardless of the value of the parameter λ .

Step 2: Read its polynomial power Bringing all terms to the left gives

$$\left(\frac{d^3 y}{dx^3} \right)^3 - \lambda \frac{d^2 y}{dx^2} + 1 = 0.$$

This is already polynomial in the derivatives. The third derivative is cubed, so the degree is 3. Option (C) incorrectly uses the first power of the second derivative; options (B) and (D) overlook the third derivative.

Step 3: Conclusion The order is 3 and the degree is 3. Hence, the correct option is (A).

Quick Tip to Solve: First circle the highest-order derivative, then read only its exponent to find the degree after expressing the equation polynomially.

14. The general solution for the differential equation $\frac{dy}{dx} = e^{3x-y}$ is:

[1 mark]

- (A) $3e^y = e^{3x} + C$
- (B) $\log(3x - y) = C$
- (C) $e^{3x-y} = C$
- (D) $-e^y + 3e^{3x} = C$

Correct Answer: (A) $3e^y = e^{3x} + C$

Solution:

Concept:

- For a separable differential equation, collect all factors involving y with dy and all factors involving x with dx .
- The exponential identities $e^{3x-y} = e^{3x}e^{-y}$ and $\int e^{3x} dx = \frac{1}{3}e^{3x}$ allow direct integration.

Step 1: Separate the variables Multiply the given equation by $e^y dx$. As e^y is nonzero, this does not lose solutions:

$$e^y dy = e^{3x} dx.$$

Each side now contains only its own variable.

Quick Tip to Solve: After separating, retain the factor $1/3$ from integrating e^{3x} ; multiply through and rename the arbitrary constant only at the end.

Step 2: Integrate and absorb the constant Integrating both sides gives

$$e^y = \frac{1}{3}e^{3x} + C_1, \quad 3e^y = e^{3x} + C,$$

where $C = 3C_1$ is arbitrary. Differentiating this relation recovers the original equation. Options (B) and (C) would force y to have constant slope 3, while (D) differentiates to $\frac{dy}{dx} = 9e^{3x-y}$, with the wrong factor.

Step 3: Conclusion The general solution is $3e^y = e^{3x} + C$. Hence, the correct option is (A).

Quick Tip to Solve: After separating, retain the factor $1/3$ from integrating e^{3x} ; multiply through and rename the arbitrary constant only at the end.

15. The corner points of the feasible region determined by the system of linear constraints are $(0, 0)$, $(0, 40)$, $(20, 40)$, $(60, 20)$ and $(60, 0)$. If the objective function of an LPP is $Z = 4x + 3y$, then the maximum value is: [1 mark]

- (A) 200
- (B) 300
- (C) 240
- (D) 120

Correct Answer: (B) 300

Solution:

Concept:

- For a linear objective function on a bounded polygonal feasible region, an optimum is attained at a corner point.
- Here the objective is $Z = 4x + 3y$, so every listed vertex must be tested using the same coefficients.

Step 1: Evaluate the objective at all vertices Substitute each ordered pair carefully:

$$\begin{aligned} Z(0, 0) &= 0, & Z(0, 40) &= 120, \\ Z(20, 40) &= 80 + 120 = 200, & Z(60, 20) &= 240 + 60 = 300, \\ Z(60, 0) &= 240. \end{aligned}$$

Checking only the greatest x -coordinate would not distinguish the last two vertices.

Step 2: Compare the candidate values The five values are 0, 120, 200, 300, 240. The largest is attained at $(60, 20)$. The other printed choices 200, 240, and 120 are genuine objective values at other vertices, but all are smaller than 300.

Step 3: Conclusion The maximum value is $Z_{\max} = 300$, attained at $(60, 20)$. Hence, the correct option is (B).

Quick Tip to Solve: Make a short list of objective values for every supplied corner; a vertex with the largest coordinate need not give the largest weighted sum.

16. If position vector $\vec{p}$ of a point $(24, n)$ is such that $|\vec{p}| = 25$, then the value of n is:

[1 mark]

- (A) ± 49
- (B) ± 5
- (C) ± 1
- (D) ± 7

Correct Answer: (D) ± 7

Solution:

Concept:

- The position vector of (x, y) is $x\hat{i} + y\hat{j}$, measured from the origin.
- Its magnitude is $\sqrt{x^2 + y^2}$; hence the square of its length equals the sum of the squares of its components.

Step 1: Use the magnitude equation For the point $(24, n)$, the position vector is $\vec{p} = 24\hat{i} + n\hat{j}$. The prescribed magnitude gives

$$\sqrt{24^2 + n^2} = 25, \quad 576 + n^2 = 625.$$

Squaring is valid because both sides of the magnitude equation are non-negative.

Step 2: Solve for both possible signs Subtracting 576 gives

$$n^2 = 49, \quad n = \pm 7.$$

Both signs give length 25; the points lie on opposite sides of the x -axis. Option (A) mistakes n^2 for n , while ± 5 and ± 1 do not satisfy the magnitude equation.

Step 3: Conclusion The possible values of n are ± 7 . Hence, the correct option is (D).

Quick Tip to Solve: Recognise the 7, 24, 25 right-triangle triple, but keep both signs of the unknown coordinate because a magnitude does not fix its direction.

17. If vectors $\vec{a} = 3\hat{i} + 2\hat{j} + \lambda\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, represent the two strips of the Red Cross sign placed outside a doctor's clinic, then the value of λ is:

[1 mark]

- (A) 1
- (B) $\frac{5}{2}$
- (C) $\frac{2}{5}$
- (D) 0

Correct Answer: (C) $\frac{2}{5}$

Solution:

Concept:

- The two strips of a cross meet at right angles, so their direction vectors are perpendicular.

Quick Tip to Solve: For perpendicular vector directions, multiply matching components and set their sum equal to zero.

- For nonzero vectors, perpendicularity is equivalent to $\vec{a} \cdot \vec{b} = 0$; the dot product is the sum of products of corresponding components.

Step 1: Translate the geometry into a dot product The vectors representing the two strips must satisfy

$$(3)(2) + (2)(-4) + (\lambda)(5) = 0.$$

The negative sign of the second component of $\vec{b}$ is essential here.

Step 2: Solve and check the options Thus $6 - 8 + 5\lambda = 0$, giving $5\lambda = 2$. The dot product is $5\lambda - 2$. Substituting the four listed values gives, respectively, 3, $21/2$, 0 and -2 . Only the third value makes the strips perpendicular; taking the reciprocal $5/2$ would not satisfy the equation.

Step 3: Conclusion $\lambda = \frac{2}{5}$. Hence, the correct option is (C).

Quick Tip to Solve: For perpendicular vector directions, multiply matching components and set their sum equal to zero.

18. If $3P(A) = P(B) = \frac{3}{5}$ and $P(A | B) = \frac{2}{3}$, then $P(A \cup B)$ is:

[1 mark]

- (A) $\frac{3}{5}$
- (B) $\frac{1}{5}$
- (C) $\frac{2}{15}$
- (D) $\frac{2}{5}$

Correct Answer: No valid probability solution: the printed data are inconsistent. Formal substitution gives option (D).

Solution:

Concept:

- Conditional probability gives $P(A \cap B) = P(A | B)P(B)$, provided $P(B) > 0$.
- The addition law is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Step 1: Read the given probabilities separately From $3P(A) = P(B) = 3/5$, the individual probabilities are $P(A) = 1/5$ and $P(B) = 3/5$. Using the supplied conditional probability gives

$$P(A \cap B) = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}.$$

Step 2: Apply the union identity and check consistency Formal substitution into the addition law gives

$$P(A \cup B) = \frac{1}{5} + \frac{3}{5} - \frac{2}{5} = \frac{2}{5}.$$

Thus options (A), (B) and (C), namely $3/5$, $1/5$ and $2/15$, do not match the calculation. There is an

Quick Tip to Solve: Divide $3P(A)$ by three before applying the union rule, and check that the intersection never exceeds either event.

inconsistency in the printed data: $P(A \cap B) = 2/5 > P(A) = 1/5$, which is impossible for actual events. Accordingly, the numerical choice is the intended algebraic result, although no actual pair of events can satisfy all the stated probabilities.

Step 3: Conclusion No option is a valid probability answer because the printed data are inconsistent. Formal substitution yields $\frac{2}{5}$, corresponding to the intended option (D).

Quick Tip to Solve: Divide $3P(A)$ by three before applying the union rule, and check that the intersection never exceeds either event.

19. Assertion (A): A relation R on the set $\{1, 2, 3\}$ defined as $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ is an equivalence relation.

Reason (R): A relation that is reflexive, symmetric and transitive is an equivalence relation. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- An equivalence relation must be reflexive, symmetric and transitive.
- A relation consisting of all ordered pairs within each of a collection of disjoint blocks satisfies these three properties.

Step 1: Test the assertion The pairs $(1, 1), (2, 2), (3, 3)$ give reflexivity. The only unequal entries are $(1, 2)$ and $(2, 1)$, which occur together, giving symmetry. Also,

$$R = (\{1, 2\} \times \{1, 2\}) \cup \{(3, 3)\}.$$

If aRb and bRc , all three elements lie in the same block, either $\{1, 2\}$ or $\{3\}$; hence aRc . This proves transitivity, so the assertion is true.

Step 2: Test the reason The reason states the defining criterion for an equivalence relation and is true: reflexivity, symmetry and transitivity together are sufficient and necessary.

Step 3: Check the explanatory connection The three verified properties of this particular relation imply the assertion by exactly the criterion in the reason. Therefore the reason explains the assertion. Option (B) incorrectly denies this link, while (C) and (D) incorrectly declare one statement false.

Step 4: Conclusion Both statements are true, and the reason correctly explains the assertion. Hence, the correct option is (A).

Quick Tip to Solve: Recognise the complete blocks 1,2 and 3; any two related elements stay inside one block, proving transitivity.

20. Assertion (A): Consider a Linear Programming Problem with minimise $Z = x + 2y$ subject to constraints $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$ which gives minimum Z at infinitely many points. The corner points of feasible region are $(0, 3)$ and $(6, 0)$.

Reason (R): If two corner points produce the same minimum value of the objective function, then every point on the line segment joining the points will give the same minimum value. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true, but Reason (R) is false.
(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- A feasible point satisfies every constraint. A constraint identical to the objective can provide a direct bound on its minimum.
- A feasible region is convex, and a linear objective is constant on the segment between two optimal points having the same value.

Step 1: Test the assertion using the constraints Every feasible point has $Z = x + 2y \geq 6$. On the segment $x + 2y = 6$ in the first quadrant, write $y = 3 - x/2$ with $0 \leq x \leq 6$. Then

$$2x + y = 3 + \frac{3}{2}x \geq 3.$$

Thus the entire segment from $(0, 3)$ to $(6, 0)$ is feasible and attains the lower bound $Z = 6$. These are the corner points: the other boundary line $2x + y = 3$ touches the feasible region only at $(0, 3)$. Hence the assertion is true.

Step 2: Test the reason If feasible points P, Q both give the minimum m , then for $0 \leq t \leq 1$, their convex combination is feasible and linearity gives

$$Z((1-t)P + tQ) = (1-t)m + tm = m.$$

Therefore every point of the connecting segment has that same minimum, so the reason is true.

Step 3: Check whether the reason explains the assertion Here both distinct corner points give $Z = 6$. The reason therefore explains why all the infinitely many points joining them minimise the objective. Option (B) wrongly denies this explanation; options (C) and (D) wrongly reject a true statement.

Step 4: Conclusion The minimum is $Z = 6$ at every point of the segment joining $(0, 3)$ and $(6, 0)$. Both statements are true and the reason explains the assertion. The correct option is (A).

Quick Tip to Solve: When a constraint reads $x+2y$ at least 6 and the objective is $x+2y$, test the equality boundary for feasible minimisers.

Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Evaluate $\sin \left[\tan^{-1} \left(\tan \left(\frac{3\pi}{4} \right) \right) \right]$. [2 marks]

Solution:

Concept:

- The principal-value range of $\tan^{-1} t$ is $(-\pi/2, \pi/2)$. Inverse tangent must return an angle in this interval.
- Tangent has period π , so $\tan(\theta - \pi) = \tan \theta$; also $\sin(-\theta) = -\sin \theta$.

Step 1: Evaluate the inner tangent The angle $3\pi/4$ lies in the second quadrant, where tangent is negative. Its reference angle is $\pi/4$, so

$$\tan \left(\frac{3\pi}{4} \right) = \tan \left(\pi - \frac{\pi}{4} \right) = -1.$$

This angle itself is outside the principal-value range of inverse tangent, so directly cancelling $\tan^{-1}$ and $\tan$ would be invalid.

Step 2: Choose the principal angle and take its sine The angle in $(-\pi/2, \pi/2)$ whose tangent is -1 is $-\pi/4$. Equivalently, subtracting one period from $3\pi/4$ gives $3\pi/4 - \pi = -\pi/4$. Hence

$$\begin{aligned} \sin \left[\tan^{-1} \left(\tan \frac{3\pi}{4} \right) \right] &= \sin \left(-\frac{\pi}{4} \right) \\ &= -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}. \end{aligned}$$

The negative sign follows from the principal angle lying in the fourth quadrant.

Step 3: Conclusion The required value is $-\frac{1}{\sqrt{2}}$, or equivalently $-\frac{\sqrt{2}}{2}$.

Quick Tip to Solve: Before cancelling inverse tangent with tangent, shift the angle by a multiple of π into the principal interval.

22(a). Differentiate x^x with respect to $x \log x$. [2 marks]

Solution:

Concept:

- For $x > 0$, logarithmic differentiation of $y = x^x$ starts from $\log y = x \log x$, where $\log$ denotes the natural logarithm.
- If $t = t(x)$, the chain rule gives $dy/dt = (dy/dx)/(dt/dx)$ whenever $dt/dx \neq 0$; an explicit relation between y and t can also be differentiated directly.

Quick Tip to Solve: Recognise x raised to x as the exponential of $x \log x$; taking the given denominator expression as t makes $y = e$ to the t .

Step 1: Differentiate both expressions with respect to x Put $y = x^x$ and $t = x \log x$. Taking logarithms, $\log y = x \log x$. Differentiating by the product rule,

$$\frac{1}{y} \frac{dy}{dx} = \log x + 1, \quad \frac{dy}{dx} = x^x (\log x + 1).$$

Similarly, $dt/dx = \log x + 1$. The positivity condition $x > 0$ permits both logarithms in this real-variable calculation.

Step 2: Differentiate with respect to the requested expression For $\log x + 1 \neq 0$, the required derivative is

$$\frac{dy}{dt} = \frac{x^x (\log x + 1)}{\log x + 1} = x^x.$$

There is also a direct verification: since $t = x \log x$, we have $y = e^{x \log x} = e^t$, so $dy/dt = e^t$. This relation avoids dividing by zero at $x = e^{-1}$, where both derivatives with respect to x vanish, and gives the same derivative of the exponential relation there.

Step 3: Conclusion

$$\frac{d(x^x)}{d(x \log x)} = x^x.$$

Quick Tip to Solve: Recognise x raised to x as the exponential of $x \log x$; taking the given denominator expression as t makes $y = e^t$.

OR

22(b). If $y = P \cos(ux) + Q \sin(ux)$, then show that $\frac{d^2 y}{dx^2} + u^2 y = 0$.

[2 marks]

Solution:

Concept:

- Treat P , Q and u as constants. The chain rule gives $d(\cos ux)/dx = -u \sin ux$ and $d(\sin ux)/dx = u \cos ux$.
- A second differentiation contributes another factor of u and changes the sign of each original sine or cosine term.

Step 1: Compute the first derivative Starting from $y = P \cos(ux) + Q \sin(ux)$, differentiate each term separately. Since the derivative of the inner argument ux is u , the chain rule yields

$$\frac{dy}{dx} = -Pu \sin(ux) + Qu \cos(ux).$$

The minus sign belongs to the derivative of cosine; the derivative of sine has a positive cosine term. The constants P and Q remain unchanged.

Quick Tip to Solve: Each differentiation of $\sin(ux)$ or $\cos(ux)$ supplies a factor u ; after two derivatives the original function returns with a minus sign.

Step 2: Compute the second derivative and substitute y Differentiating once more gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= -Pu^2 \cos(ux) - Qu^2 \sin(ux) \\ &= -u^2 [P \cos(ux) + Q \sin(ux)].\end{aligned}$$

The bracket is precisely the original expression for y . Therefore $d^2y/dx^2 = -u^2y$. Moving this term to the left proves the requested identity for every x . It also covers $u = 0$, when y is constant and both terms in the identity vanish.

Step 3: Conclusion Hence $\boxed{\frac{d^2y}{dx^2} + u^2y = 0}$, as required.

Quick Tip to Solve: Each differentiation of $\sin(ux)$ or $\cos(ux)$ supplies a factor u ; after two derivatives the original function returns with a minus sign.

23. Determine the values of x for which $f(x) = \frac{x-3}{x+1}$, $x \neq -1$ is an increasing function. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- The quotient rule is $(u/v)' = (vu' - uv')/v^2$, provided $v \neq 0$. Domain breaks must be excluded before stating intervals of increase.

Step 1: Differentiate the given quotient Here $u = x - 3$, $v = x + 1$, and $u' = v' = 1$. Thus, at every point in the domain,

$$f'(x) = \frac{(x+1)(1) - (x-3)(1)}{(x+1)^2} = \frac{4}{(x+1)^2}.$$

Step 2: Check the sign on each domain interval The numerator is positive, while the squared denominator is positive for every $x \neq -1$. Hence $f'(x) > 0$ both when $x < -1$ and when $x > -1$. The excluded point divides the domain into two intervals; the derivative test is applied separately to each of them. We cannot claim increase across the discontinuity at $x = -1$.

Step 3: Conclusion The function is strictly increasing on each of $\boxed{(-\infty, -1)}$ and $\boxed{(-1, \infty)}$.

Quick Tip to Solve: Rewrite the function as $1 - 4/(x+1)$ to differentiate quickly, but keep the domain break at $x = -1$ in the answer.

24(a). Three honey bees were found flying along the vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = 4\hat{j} - 2\hat{k}$ and $\vec{c} = 3\hat{i} + 2\hat{k}$ respectively. Find the value of λ such that the path for $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$. [2 marks]

Solution:

Concept:

- Two nonzero vectors are perpendicular if and only if their scalar product is zero.
- For Cartesian vectors, the scalar product is the sum of the products of corresponding components:
 $(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3$.

Step 1: Form the resultant direction vector Multiply each component of $\vec{b}$ by λ and add to $\vec{a}$. Since $\vec{b} = 4\hat{j} - 2\hat{k}$, we obtain

$$\vec{a} + \lambda\vec{b} = 2\hat{i} + (-3 + 4\lambda)\hat{j} + (1 - 2\lambda)\hat{k}.$$

This vector is nonzero for every λ , because its $\hat{i}$ component is always 2.

Step 2: Apply the perpendicularity condition Take the scalar product with $\vec{c} = 3\hat{i} + 2\hat{k}$, whose $\hat{j}$ component is zero:

$$\begin{aligned}(\vec{a} + \lambda\vec{b}) \cdot \vec{c} &= 0, \\ 3(2) + 0(-3 + 4\lambda) + 2(1 - 2\lambda) &= 0, \\ 8 - 4\lambda &= 0.\end{aligned}$$

At the resulting value, the direction vector is $2\hat{i} + 5\hat{j} - 3\hat{k}$, and its product with $\vec{c}$ is $6 - 6 = 0$, as required.

Step 3: Conclusion The required value is $\lambda = 2$.

Quick Tip to Solve: Use $\lambda = -(\vec{a} \cdot \vec{c})/(\vec{b} \cdot \vec{c})$ here: the scalar products are 8 and -4 , respectively.

OR

24(b). If A , B and C be three non-collinear points such that $\vec{AB} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{AC} = 2\hat{i} - 3\hat{j}$, then find the area of $\triangle ABC$. [2 marks]

Solution:

Concept:

- For two sides $\vec{AB}$ and $\vec{AC}$ issuing from the same vertex, the triangle's area is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.
- The magnitude of a vector $p\hat{i} + q\hat{j} + r\hat{k}$ is $\sqrt{p^2 + q^2 + r^2}$; the cross product itself represents an oriented area vector.

Step 1: Calculate the cross product Insert the zero $\hat{k}$ component of $\vec{AC}$ and expand the determinant,

Quick Tip to Solve: Use the two sides starting at A , take the cross-product magnitude, and halve it; the vector's sign does not affect the area.

remembering the minus sign on the middle cofactor:

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -3 & 0 \end{vmatrix} \\ &= \hat{i}(0 - 3) - \hat{j}(0 + 2) + \hat{k}(-3 - 4) \\ &= -3\hat{i} - 2\hat{j} - 7\hat{k}.\end{aligned}$$

Step 2: Convert the parallelogram area to triangle area The magnitude of this cross product gives the area of the parallelogram spanned by the two sides:

$$|\vec{AB} \times \vec{AC}| = \sqrt{(-3)^2 + (-2)^2 + (-7)^2} = \sqrt{62}.$$

A diagonal divides that parallelogram into two equal triangles, so the required area is one half of this magnitude. The nonzero magnitude also confirms that the given points are non-collinear.

Step 3: Conclusion The area of $\triangle ABC$ is $\boxed{\frac{\sqrt{62}}{2}}$ square units.

Quick Tip to Solve: Use the two sides starting at A , take the cross-product magnitude, and halve it; the vector's sign does not affect the area.

25. Find the angle between the following pair of lines:

$$\frac{x-2}{3} = \frac{y+5}{2} = \frac{1-z}{-6} \text{ and } \frac{x-7}{1} = \frac{y}{2} = \frac{6-z}{-2}.$$

[2 marks]

Solution:

Concept:

- In symmetric form $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the denominators (a, b, c) are direction ratios of the line.
- The acute angle θ between lines with direction vectors $\vec{u}, \vec{v}$ satisfies $\cos \theta = |\vec{u} \cdot \vec{v}| / (|\vec{u}||\vec{v}|)$.

Step 1: Put the third coordinate fractions in standard form Reverse both signs in each third fraction:

$$\frac{1-z}{-6} = \frac{z-1}{6}, \quad \frac{6-z}{-2} = \frac{z-6}{2}.$$

Thus the direction vectors may be chosen as $\vec{u} = (3, 2, 6)$ and $\vec{v} = (1, 2, 2)$. The constants in the numerators locate points on the lines and do not determine their angle.

Step 2: Evaluate the scalar-product ratio Compute the scalar product and the two lengths:

$$\begin{aligned}\vec{u} \cdot \vec{v} &= 3(1) + 2(2) + 6(2) = 19, \\ |\vec{u}| &= \sqrt{9 + 4 + 36} = 7, \\ |\vec{v}| &= \sqrt{1 + 4 + 4} = 3.\end{aligned}$$

Quick Tip to Solve: Before reading direction ratios, rewrite every numerator as coordinate minus constant; reversing its sign also reverses the denominator's sign.

Therefore the cosine of the required angle is $19/(7 \times 3)$. The positive scalar product already gives the acute angle, so the absolute value makes no further change.

Step 3: Conclusion The angle between the two lines is $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

Quick Tip to Solve: Before reading direction ratios, rewrite every numerator as coordinate minus constant; reversing its sign also reverses the denominator's sign.

Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

26. A spherical balloon loses its volume due to escape of air from it in such a way that decrease of volume at any instant is proportional to its surface area. Show that the radius is decreasing at a constant rate. [3 marks]

Solution:

Concept:

- For a sphere of radius r , the volume is $V = \frac{4}{3}\pi r^3$ and the surface area is $S = 4\pi r^2$.
- A decreasing volume has $dV/dt < 0$. If its rate of decrease is proportional to surface area, write $-dV/dt = kS$, where $k > 0$ is the constant of proportionality.
- Since the radius depends on time, differentiation uses the chain rule: $dV/dt = (dV/dr)(dr/dt)$.

Step 1: Translate the escape condition into an equation Let $r(t) > 0$ be the balloon's radius at time t . The phrase 'decrease of volume at any instant' describes a rate, so the given proportionality becomes

$$-\frac{dV}{dt} = k(4\pi r^2), \quad k > 0.$$

Equivalently, $dV/dt = -4\pi k r^2$. The minus sign records that air escapes and the volume is becoming smaller, while k is independent of time and radius.

Step 2: Express volume change in terms of radius change Differentiate $V = \frac{4}{3}\pi r^3$ with respect to time, treating r as a function of t :

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Equating the two expressions for dV/dt gives

$$4\pi r^2\frac{dr}{dt} = -4\pi k r^2.$$

While the balloon has positive radius, $4\pi r^2 \neq 0$, so division by that common factor is valid. The resulting radius derivative has neither a time factor nor a radius factor.

Step 3: Conclusion Thus $\frac{dr}{dt} = -k$, a negative constant. Hence the radius is decreasing at the

Quick Tip to Solve: For a sphere, dV/dr equals its surface area; substituting the chain rule therefore cancels the entire radius-dependent factor immediately.

constant rate k while the balloon has positive radius.

Quick Tip to Solve: For a sphere, dV/dr equals its surface area; substituting the chain rule therefore cancels the entire radius-dependent factor immediately.

27(a). Find: $\int \frac{x - \sin x}{1 - \cos x} dx$.

[3 marks]

Solution:

Concept:

- Half-angle identities give $1 - \cos x = 2 \sin^2(x/2)$ and $\sin x = 2 \sin(x/2) \cos(x/2)$.
- Integration by parts is $\int u dv = uv - \int v du$. Also, $\frac{d}{dx} \cot(x/2) = -\frac{1}{2} \csc^2(x/2)$.

Step 1: Rewrite using half-angle identities Let the required integral be I . Split its numerator and simplify each quotient with the same half-angle identities:

$$I = \int \frac{x}{2} \csc^2(x/2) dx - \int \cot(x/2) dx.$$

This rewriting is valid on any interval where the original denominator is nonzero, so $x \neq 2k\pi$ for integers k .

Step 2: Integrate the first term by parts Choose $u = x$ and $dv = \frac{1}{2} \csc^2(x/2) dx$. Then $du = dx$ and $v = -\cot(x/2)$; the factor one-half is absorbed by the derivative of the half-angle. Therefore,

$$\int \frac{x}{2} \csc^2(x/2) dx = -x \cot(x/2) + \int \cot(x/2) dx.$$

The remaining integral is exactly the one subtracted in the expression for I , and the two terms cancel.

Step 3: Check the antiderivative Differentiating the surviving product verifies both its sign and its half-angle factor:

$$\begin{aligned} \frac{d}{dx} \{-x \cot(x/2)\} &= -\cot(x/2) + \frac{x}{2} \csc^2(x/2) \\ &= \frac{x - \sin x}{1 - \cos x}. \end{aligned}$$

Step 4: Conclusion The required integral is $\boxed{-x \cot(x/2) + C}$, where C is an arbitrary constant on each interval of definition.

Quick Tip to Solve: With $1 - \cos x$ in the denominator, try half-angle identities first; here integration by parts makes the two cotangent integrals cancel.

27(b). Evaluate: $\int_0^2 \frac{1}{\sqrt{x^2 + 2x + 3}} dx$.

[3 marks]

Solution:

Concept:

- Completing the square rewrites $x^2 + 2x + 3$ as $(x + 1)^2 + 2$, which is a sum of squares.
- For a positive constant a , $\int \frac{dt}{\sqrt{t^2 + a^2}} = \ln |t + \sqrt{t^2 + a^2}| + C$. A substitution in a definite integral also changes its limits.

Step 1: Complete the square and substitute Denote the required integral by I . Since $x^2 + 2x + 3 = (x + 1)^2 + 2$, put $t = x + 1$, so $dt = dx$. The lower limit $x = 0$ becomes $t = 1$, and the upper limit $x = 2$ becomes $t = 3$. Thus,

$$I = \int_1^3 \frac{dt}{\sqrt{t^2 + 2}}.$$

The quantity inside the square root is strictly positive throughout the interval, so the integrand is continuous there.

Step 2: Apply the standard logarithmic integral Use the standard formula with $a^2 = 2$. On $[1, 3]$, the argument of the logarithm is positive, so absolute-value signs may be omitted:

$$\begin{aligned} I &= \left[\ln \left(t + \sqrt{t^2 + 2} \right) \right]_1^3 \\ &= \ln(3 + \sqrt{11}) - \ln(1 + \sqrt{3}). \end{aligned}$$

Here the upper endpoint contributes $\sqrt{3^2 + 2} = \sqrt{11}$, while the lower endpoint contributes $\sqrt{1^2 + 2} = \sqrt{3}$. Subtract the entire lower-endpoint logarithm.

Step 3: Combine the logarithms Both logarithm arguments are positive, so $\ln A - \ln B = \ln(A/B)$ applies. Since $3 + \sqrt{11} > 1 + \sqrt{3}$, the resulting value is positive, as expected for the integral of a positive function over an increasing interval.

Step 4: Conclusion $\int_0^2 \frac{dx}{\sqrt{x^2 + 2x + 3}} = \ln \left(\frac{3 + \sqrt{11}}{1 + \sqrt{3}} \right).$

Quick Tip to Solve: Complete the square before choosing a formula, and change both limits immediately after $t = x + 1$ to avoid mixing variables at the endpoints.

28. Solve the differential equation $(x + 2y^3) dy = y dx$.

[3 marks]

Solution:

Concept:

- A first-order linear equation for x as a function of y has the form $\frac{dx}{dy} + P(y)x = Q(y)$, with integrating factor $e^{\int P(y) dy}$.

Quick Tip to Solve: When the coefficient of dy is linear in x , try dx/dy ; here it immediately gives a linear equation with integrating factor $1/y$.

- Multiplication by the integrating factor converts the left side into a product derivative. Any value excluded by division must be checked separately in the original equation.

Step 1: Treat x as the dependent variable The equation is $(x + 2y^3) dy = y dx$. On an interval with $y \neq 0$, divide by $y dy$ and rearrange:

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2.$$

This is linear in x , with $P(y) = -1/y$ and $Q(y) = 2y^2$. Treating y as dependent at this stage would hide the simpler linear form.

Step 2: Find and apply the integrating factor An integrating factor on either interval $y > 0$ or $y < 0$ is $1/y$. Indeed, $e^{-\ln|y|} = 1/|y|$, and a nonzero constant multiple of an integrating factor is also valid. Multiplication by $1/y$ gives

$$\begin{aligned}\frac{1}{y} \frac{dx}{dy} - \frac{x}{y^2} &= 2y, \\ \frac{d}{dy} \left(\frac{x}{y} \right) &= 2y.\end{aligned}$$

The product derivative includes the negative sign from differentiating y^{-1} .

Step 3: Integrate and check the excluded solution Integrate with respect to y :

$$\frac{x}{y} = y^2 + C.$$

For verification, this relation gives $dx/dy = 3y^2 + C$, which equals $(x + 2y^3)/y$ after substituting $x/y = y^2 + C$. Division excluded $y = 0$; the constant function $y(x) = 0$ also satisfies the original equation because $dy = 0$ and $y dx = 0$.

Step 4: Conclusion The nonzero solution family is $x = y^3 + Cy$, where C is an arbitrary constant. In addition, $y = 0$ is a constant solution.

Quick Tip to Solve: When the coefficient of dy is linear in x , try dx/dy ; here it immediately gives a linear equation with integrating factor $1/y$.

29. Solve the following Linear Programming Problem graphically:

Maximize $Z = \frac{2x}{5} + \frac{3y}{10}$ **subject to constraints**

$$2x + y \leq 1000$$

$$x + y \leq 800$$

$$x, y \geq 0.$$

[3 marks]

Solution:

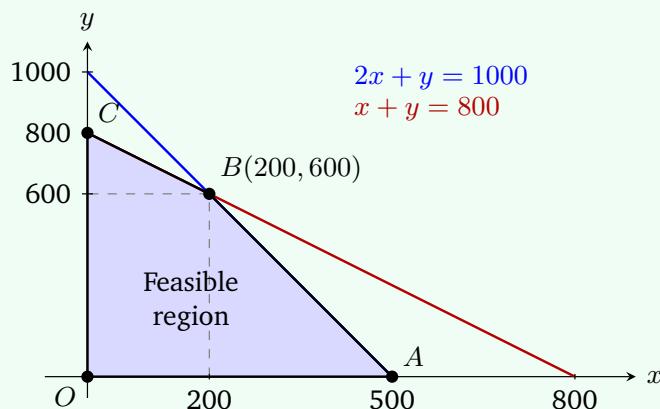
Concept:

- Each linear inequality represents a half-plane. Their common region in the first quadrant is the feasible region, since $x \geq 0$ and $y \geq 0$.
- For a bounded nonempty feasible polygon, a linear objective function attains its maximum at a

Quick Tip to Solve: Check both axis bounds before listing vertices: $(800, 0)$ and $(0, 1000)$ are line intercepts but violate one constraint and are not feasible vertices.

corner point. Evaluate $Z = \frac{2}{5}x + \frac{3}{10}y$ at every vertex.

Step 1: Plot the boundary lines and shade the common region For $2x + y = 1000$, the intercepts are $(500, 0)$ and $(0, 1000)$. For $x + y = 800$, the intercepts are $(800, 0)$ and $(0, 800)$. Draw the solid boundary lines through these intercepts. The origin satisfies both inequalities: shade their common region in the first quadrant, including the boundary. This gives the bounded polygon $OABC$.



The shaded polygon $OABC$ is the feasible region; both boundary lines are included.

Step 2: Find all the corner points On the x -axis the stricter bound is $x \leq 500$, and on the y -axis it is $y \leq 800$. Thus three vertices are $O(0, 0)$, $A(500, 0)$ and $C(0, 800)$. To obtain the fourth vertex, subtract the boundary equations:

$$\begin{aligned} 2x + y &= 1000, & x + y &= 800, \\ x &= 200, & y &= 600. \end{aligned}$$

Their intersection is the feasible vertex $B(200, 600)$.

Step 3: Compare the objective values at the vertices Substitute each vertex into the objective function:

(x, y)	$(0, 0)$	$(500, 0)$	$(200, 600)$	$(0, 800)$
Z	0	200	$80 + 180 = 260$	240

The corner-point theorem applies because the feasible polygon is bounded.

Step 4: Conclusion The maximum value is $Z_{\max} = 260$, attained at $x = 200, y = 600$.

Quick Tip to Solve: Check both axis bounds before listing vertices: $(800, 0)$ and $(0, 1000)$ are line intercepts but violate one constraint and are not feasible vertices.

30(a). Let three toys A , B and C be placed in the same straight line. If the position vectors of A , B and C are $55\hat{i} - 2\hat{j}$, $5\hat{i} + 8\hat{j}$ and $a\hat{i} - 52\hat{j}$ respectively, find the value of ' a '. [3 marks]

Solution:

Concept:

- The displacement from a point P to a point Q is $\overrightarrow{PQ} = \vec{r}_Q - \vec{r}_P$, where $\vec{r}_P$ and $\vec{r}_Q$ are their position vectors.

Quick Tip to Solve: First use the known $\hat{j}$ components to find the scalar multiplier, then apply that multiplier to the $\hat{i}$ components. Do not compare the position vectors directly.

- Three distinct points are collinear if two displacement vectors from the same point are scalar multiples. Their position vectors need not themselves be parallel, since the common line need not pass through the origin.

Step 1: Form two displacement vectors from A The given position vectors correspond to $A(55, -2)$, $B(5, 8)$ and $C(a, -52)$. Subtract the position vector of A from each of the other two:

$$\begin{aligned}\overrightarrow{AB} &= (5 - 55)\hat{i} + [8 - (-2)]\hat{j} \\ &= -50\hat{i} + 10\hat{j}, \\ \overrightarrow{AC} &= (a - 55)\hat{i} + [-52 - (-2)]\hat{j} \\ &= (a - 55)\hat{i} - 50\hat{j}.\end{aligned}$$

Using a common initial point ensures that the two vectors test the direction of the same straight line.

Step 2: Apply the condition for collinearity Since the toys lie on one straight line, write $\overrightarrow{AC} = \lambda \overrightarrow{AB}$. Comparing the $\hat{j}$ components determines the multiplier without involving the unknown:

$$-50 = 10\lambda \quad \Rightarrow \quad \lambda = -5.$$

Now compare the $\hat{i}$ components using this same multiplier:

$$a - 55 = (-5)(-50) = 250.$$

The negative multiplier means that $\overrightarrow{AC}$ points in the opposite direction to $\overrightarrow{AB}$; it does not contradict collinearity. As a check, the resulting displacement $250\hat{i} - 50\hat{j}$ is exactly -5 times $-50\hat{i} + 10\hat{j}$.

Step 3: Conclusion Therefore, $a = 305$.

Quick Tip to Solve: First use the known $\hat{j}$ components to find the scalar multiplier, then apply that multiplier to the $\hat{i}$ components. Do not compare the position vectors directly.

OR

30(b). If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors, then prove that $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \leq 9$.

[3 marks]

Solution:

Concept:

- For vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}$. A unit vector has squared magnitude 1.
- The squared magnitude of every real vector is nonnegative. Applying this to $\vec{a} + \vec{b} + \vec{c}$ bounds the sum of the three pairwise dot products.

Step 1: Expand the sum of squared differences Let S denote the expression to be bounded and put

$$T = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}.$$

Quick Tip to Solve: For the cyclic sum of three squared vector differences, expand the squared magnitude of their sum: it combines all three dot products and gives the sharp bound immediately.

Expand each squared difference using the dot product. Every squared magnitude occurs twice, so

$$\begin{aligned} S &= |\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \\ &= 2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2) - 2T \\ &= 6 - 2T. \end{aligned}$$

The last equality uses the fact that all three given vectors are unit vectors.

Step 2: Bound the dot products using a squared magnitude The three dot products are linked because they come from the same three vectors. Expand the squared magnitude of their sum:

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2T \\ &= 3 + 2T \geq 0. \end{aligned}$$

Thus $T \geq -\frac{3}{2}$. Equivalently, eliminating T between the two expansions gives the useful identity

$$S = 9 - |\vec{a} + \vec{b} + \vec{c}|^2.$$

Since the subtracted term cannot be negative, this identity supplies the required upper bound. Equality is possible precisely when the sum of the three unit vectors is the zero vector, for example when they lie in a plane with pairwise angles of 120° .

Step 3: Conclusion Therefore,

$$|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \leq 9.$$

Quick Tip to Solve: For the cyclic sum of three squared vector differences, expand the squared magnitude of their sum: it combines all three dot products and gives the sharp bound immediately.

31(a). A die is rolled. Consider events: $A = \{1, 2, 5\}$, $B = \{3, 5\}$, $C = \{2, 3, 4, 5\}$ **and hence find:**

(i) $P(A|C)$ and $P(C|A)$

(ii) $P(A \cap B|C)$ and $P(A \cup B|C)$

[3 marks]

Solution:

Concept:

- For events E, F with $P(F) > 0$, conditional probability is $P(E | F) = P(E \cap F)/P(F)$. The conditioning event becomes the restricted sample space.
- For a fair die, all six outcomes are equally likely. Consequently, $P(E | F) = n(E \cap F)/n(F)$, where n denotes the number of outcomes in a set.

Step 1: List the relevant sets The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The given sets have sizes

Quick Tip to Solve: For conditioning on C , cross out outcomes outside $\{2, 3, 4, 5\}$ first; count the favourable outcomes out of these four only.

$n(A) = 3$, $n(B) = 2$ and $n(C) = 4$. Their intersections and union needed below are

$$\begin{aligned}A \cap C &= \{2, 5\}, & A \cap B &= \{5\}, \\A \cup B &= \{1, 2, 3, 5\}.\end{aligned}$$

Conditioning changes which outcomes form the denominator, so reversing the order of the events need not give the same probability.

Step 2: Evaluate the two probabilities in part (i) Given C , the possible outcomes are 2, 3, 4, 5; two belong to A . Given A , the possible outcomes are 1, 2, 5; two belong to C . Therefore,

$$\begin{aligned}P(A | C) &= \frac{P(A \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}, \\P(C | A) &= \frac{P(A \cap C)}{P(A)} = \frac{2/6}{3/6} = \frac{2}{3}.\end{aligned}$$

Step 3: Evaluate the intersection and union in part (ii) Intersect each target event with the conditioning event C :

$$(A \cap B) \cap C = \{5\}, \quad (A \cup B) \cap C = \{2, 3, 5\}.$$

Both conditional probabilities use $P(C) = 4/6$ in the denominator. Hence,

$$\begin{aligned}P(A \cap B | C) &= \frac{1/6}{4/6} = \frac{1}{4}, \\P(A \cup B | C) &= \frac{3/6}{4/6} = \frac{3}{4}.\end{aligned}$$

Step 4: Conclusion (i) $P(A | C) = \frac{1}{2}$ and $P(C | A) = \frac{2}{3}$. (ii) $P(A \cap B | C) = \frac{1}{4}$ and $P(A \cup B | C) = \frac{3}{4}$.

Quick Tip to Solve: For conditioning on C , cross out outcomes outside $\{2, 3, 4, 5\}$ first; count the favourable outcomes out of these four only.

OR

31(b). A box contains 6 cards numbered 1 to 6. A student is asked to pick up two cards, one by one after replacement and note down the numbers on the cards. Let A be the event of getting sum of the numbers on two cards as 10, and B , the event of a number other than 4 on the first card selected. Find $P(A \text{ and } B)$ and find whether the events A and B are independent events or not. [3 marks]

Solution:

Concept:

- With replacement, each of the six numbers can occur on each draw. The ordered pairs (i, j) , where i is the first number and j the second, form $6 \times 6 = 36$ equally likely outcomes.
- Events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$. Independence of the two

Quick Tip to Solve: List the three ordered pairs summing to 10, then discard only the pair starting with 4; replacement permits the repeated pair $(5, 5)$.

draws does not automatically imply independence of events defined using those draws.

Step 1: Identify outcomes with sum ten The two numbers must both lie between 1 and 6. The ordered pairs whose sum is 10 are

$$A = \{(4, 6), (5, 5), (6, 4)\}.$$

Thus $n(A) = 3$ out of the 36 equally likely outcomes, giving

$$P(A) = \frac{3}{36} = \frac{1}{12}.$$

The order must be retained because B specifically refers to the first selected card. Also, $(5, 5)$ is possible because the first card is replaced.

Step 2: Count the outcomes for the other event and its intersection For B , the first number is any of 1, 2, 3, 5, 6, while the second number has six choices. Therefore,

$$n(B) = 5 \times 6 = 30, \quad P(B) = \frac{30}{36} = \frac{5}{6}.$$

Of the three outcomes in A , only $(4, 6)$ has 4 on the first card. Removing that outcome gives

$$A \cap B = \{(5, 5), (6, 4)\}, \quad P(A \text{ and } B) = \frac{2}{36} = \frac{1}{18}.$$

Step 3: Apply the independence criterion Multiply the individual probabilities and compare with the intersection probability:

$$P(A)P(B) = \frac{1}{12} \times \frac{5}{6} = \frac{5}{72}, \quad P(A \cap B) = \frac{1}{18} = \frac{4}{72}.$$

These values differ. Equivalently, knowing that the sum is 10 leaves three possible pairs, of which two satisfy B , so $P(B | A) = 2/3$, differing from $P(B) = 5/6$.

Step 4: Conclusion $P(A \text{ and } B) = \boxed{\frac{1}{18}}$. The events A and B are not independent, since $P(A \cap B) \neq P(A)P(B)$.

Quick Tip to Solve: List the three ordered pairs summing to 10, then discard only the pair starting with 4; replacement permits the repeated pair $(5, 5)$.

Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

32. A man goes to buy fruits from the market. The shopkeeper informs him that 4 apples, 3 oranges and 2 bananas cost Rs. 60; 2 apples, 4 oranges and 6 bananas cost Rs. 90; whereas 6 apples, 2 oranges and 3 bananas cost Rs. 70. Using matrix method, find the cost of one fruit of each kind. [5 marks]

Solution:

Concept:

- A system of three linear equations can be written as $AX = B$, where A contains the coefficients, X the unknowns and B the given totals.
- If $|A| \neq 0$, the unique solution is $X = A^{-1}B$, with $A^{-1} = \text{adj } A / |A|$. The adjugate is the transpose of the cofactor matrix.

Step 1: Translate the fruit costs into equations Let the costs of one apple, one orange and one banana, in rupees, be x, y, z , respectively. Each purchase gives an equation by multiplying the number of each fruit by its unit cost:

$$4x + 3y + 2z = 60,$$

$$2x + 4y + 6z = 90,$$

$$6x + 2y + 3z = 70.$$

In matrix form, $AX = B$, where

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}.$$

Step 2: Check that the coefficient matrix is invertible Expand the determinant along its first row, retaining the alternating signs:

$$\begin{aligned} |A| &= 4(4 \cdot 3 - 6 \cdot 2) - 3(2 \cdot 3 - 6 \cdot 6) + 2(2 \cdot 2 - 4 \cdot 6) \\ &= 4(0) - 3(-30) + 2(-20) \\ &= 50 \neq 0. \end{aligned}$$

The inverse therefore exists, and the given purchases determine a unique set of fruit costs.

Step 3: Calculate the adjugate and inverse For each entry, delete its row and column and apply the cofactor sign $(-1)^{i+j}$. The cofactor matrix is

$$C = \begin{pmatrix} 12 - 12 & -(6 - 36) & 4 - 24 \\ -(9 - 4) & 12 - 12 & -(8 - 18) \\ 18 - 8 & -(24 - 4) & 16 - 6 \end{pmatrix} = \begin{pmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{pmatrix}.$$

Quick Tip to Solve: Keep the column order apple, orange, banana fixed throughout the matrix calculation, and transpose the cofactor matrix before multiplying by the cost totals.

Transposing this matrix gives the adjugate. Thus,

$$\text{adj } A = C^T = \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix}, \quad A^{-1} = \frac{1}{50} \text{adj } A.$$

Step 4: Multiply by the totals and verify Premultiplying $AX = B$ by A^{-1} , we obtain

$$\begin{aligned} X &= \frac{1}{50} \begin{pmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{pmatrix} \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix} \\ &= \frac{1}{50} \begin{pmatrix} -450 + 700 \\ 1800 - 1400 \\ -1200 + 900 + 700 \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 250 \\ 400 \\ 400 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}. \end{aligned}$$

Substitution checks all three purchases:

$$4(5) + 3(8) + 2(8) = 60,$$

$$2(5) + 4(8) + 6(8) = 90,$$

$$6(5) + 2(8) + 3(8) = 70.$$

Step 5: Conclusion One apple costs Rs. 5, one orange costs Rs. 8, and one banana costs Rs. 8.

Quick Tip to Solve: Keep the column order apple, orange, banana fixed throughout the matrix calculation, and transpose the cofactor matrix before multiplying by the cost totals.

33(a). If $y\sqrt{x^2 + 1} = \log(\sqrt{x^2 + 1} - x)$, then show that $(x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$.

[5 marks]

Solution:

Concept:

- In implicit differentiation, y is a function of x . The product rule is $\frac{d}{dx}(uv) = u'v + uv'$, where the prime denotes differentiation with respect to x .
- The chain rule gives $\frac{d}{dx} \log u = \frac{u'}{u}$ for $u > 0$, with $\log$ denoting the natural logarithm, and $\frac{d}{dx} \sqrt{1 + x^2} = \frac{x}{\sqrt{1 + x^2}}$.

Step 1: Identify the functions and their domains Put $s = \sqrt{x^2 + 1}$. Since $s > |x|$, both s and $s - x$ are positive for every real x . The given logarithm is therefore defined, and division by these expressions is permissible. In this notation, the given relation is

$$ys = \log(s - x), \quad s' = \frac{x}{s}.$$

Quick Tip to Solve: In the logarithmic derivative, rewrite $x/\sqrt{x^2 + 1} - 1$ as $-(\sqrt{x^2 + 1} - x)/\sqrt{x^2 + 1}$; the logarithm's argument then cancels immediately.

This notation keeps the square-root terms short while retaining their dependence on x ; neither y nor s is treated as a constant.

Step 2: Differentiate both sides using the appropriate rules Differentiate the left side by the product rule and the right side by the chain rule. The derivative of the inner expression $s - x$ is $x/s - 1$, so

$$s \frac{dy}{dx} + y \frac{x}{s} = \frac{x/s - 1}{s - x}.$$

Both terms on the left are necessary because both factors in ys vary with x . On the right, the derivative of the logarithm includes the derivative of its entire argument.

Step 3: Simplify the logarithmic derivative Express the numerator over a common denominator and use $x - s = -(s - x)$:

$$\begin{aligned} \frac{x/s - 1}{s - x} &= \frac{x - s}{s(s - x)} \\ &= -\frac{s - x}{s(s - x)} = -\frac{1}{s}. \end{aligned}$$

The negative sign is essential. The cancellation is valid because $s - x > 0$, as established above. The differentiated relation thus becomes

$$s \frac{dy}{dx} + \frac{xy}{s} = -\frac{1}{s}.$$

Step 4: Clear the square-root denominator Multiply every term by the nonzero quantity s , obtaining

$$s^2 \frac{dy}{dx} + xy = -1.$$

Finally substitute $s^2 = x^2 + 1$ and move the constant term to the left. This yields the required relation directly, without needing to solve explicitly for y .

Step 5: Conclusion Hence, $(x^2 + 1) \frac{dy}{dx} + xy + 1 = 0$, as required.

Quick Tip to Solve: In the logarithmic derivative, rewrite $x/\sqrt{x^2 + 1} - 1$ as $-(\sqrt{x^2 + 1} - x)/\sqrt{x^2 + 1}$; the logarithm's argument then cancels immediately.

OR

33(b). Find the differential of $x^{\cot x} + \frac{2x^2 - 3}{2x^2 - x + 2}$ with respect to x .

[5 marks]

Solution:

Concept:

- Logarithmic differentiation handles a variable base raised to a variable exponent: if $u = f(x)^{g(x)} > 0$, take $\log u = g(x) \log f(x)$, then use the product rule. Here $\log$ denotes the natural logarithm.
- The quotient rule is $(N/D)' = (DN' - ND')/D^2$, where $D \neq 0$, and the prime denotes differentiation with respect to x . Also, $(u + v)' = u' + v'$ and $dy = y' dx$.

Quick Tip to Solve: Take logarithms only for $x^{\cot x}$. In the rational term, expand both products before subtracting so that the final constant is -3 , not $+3$.

Step 1: Separate the two terms Write $y = u + v$, where $u = x^{\cot x}$ and $v = (2x^2 - 3)/(2x^2 - x + 2)$. Work on intervals with $x > 0$ and $\sin x \neq 0$, so logarithmic differentiation of u is valid. We differentiate the two terms separately.

Step 2: Differentiate the variable power Both base and exponent vary. Taking natural logarithms gives:

$$\log u = \cot x \log x.$$

The product rule, together with $(\cot x)' = -\csc^2 x$ and $(\log x)' = 1/x$, gives

$$\frac{u'}{u} = -\csc^2 x \log x + \frac{\cot x}{x}.$$

Multiply by $u = x^{\cot x}$ to obtain

$$u' = x^{\cot x} \left(\frac{\cot x}{x} - \csc^2 x \log x \right).$$

Step 3: Apply the quotient rule and simplify Set $N = 2x^2 - 3$, $D = 2x^2 - x + 2$; then $N' = 4x$ and $D' = 4x - 1$. Hence

$$v' = \frac{4x(2x^2 - x + 2) - (2x^2 - 3)(4x - 1)}{(2x^2 - x + 2)^2}.$$

Expand both products before subtracting:

$$\begin{aligned} 4x(2x^2 - x + 2) &= 8x^3 - 4x^2 + 8x, \\ (2x^2 - 3)(4x - 1) &= 8x^3 - 2x^2 - 12x + 3. \end{aligned}$$

The cubic terms cancel, leaving $-2x^2 + 20x - 3$. Thus $v' = (-2x^2 + 20x - 3)/(2x^2 - x + 2)^2$. Adding this to u' gives the required derivative. Multiplying the entire result by dx gives the differential.

Step 4: Conclusion The derivative with respect to x is

$$\frac{dy}{dx} = x^{\cot x} \left(\frac{\cot x}{x} - \csc^2 x \log x \right) + \frac{-2x^2 + 20x - 3}{(2x^2 - x + 2)^2}.$$

Equivalently, the differential is $dy = \left(\frac{dy}{dx} \right) dx$, using the derivative above.

Quick Tip to Solve: Take logarithms only for $x^{\cot x}$. In the rational term, expand both products before subtracting so that the final constant is -3 , not $+3$.

34. Sketch the curve $\{(x, y) : 100x^2 + 25y^2 = 2500\}$ and find the area of the region enclosed by it, using integration. [5 marks]

Solution:

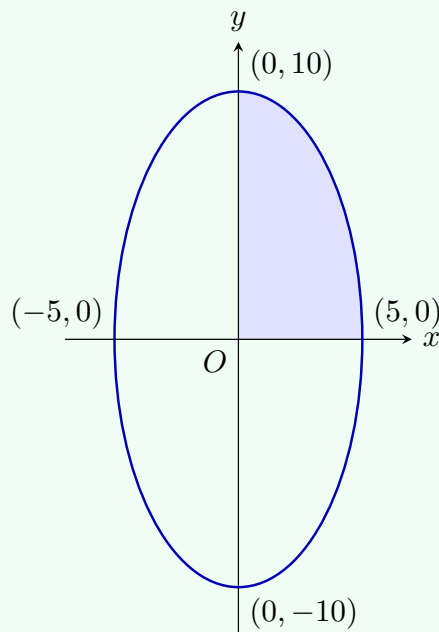
Concept:

- An ellipse centred at the origin has equation $x^2/a^2 + y^2/b^2 = 1$, where a and b are its intercept distances along the coordinate axes.
- The area below a nonnegative curve $y = f(x)$ from $x = u$ to $x = v$ is $\int_u^v f(x) dx$. Symmetry about both axes makes the enclosed area four times its first-quadrant area.
- For a square root of $a^2 - x^2$, the substitution $x = a \sin \theta$ simplifies the integrand by $1 - \sin^2 \theta = \cos^2 \theta$.

Step 1: Identify the ellipse and sketch its intercepts Divide the given equation by 2500:

$$\frac{x^2}{25} + \frac{y^2}{100} = 1.$$

The centre is $O(0, 0)$. Setting $y = 0$ gives $x = \pm 5$, while setting $x = 0$ gives $y = \pm 10$. Thus the major axis lies along the y -axis, with length 20, and the minor axis lies along the x -axis, with length 10. Both variables occur only as squares, so the curve is symmetric about each coordinate axis. Draw a smooth ellipse through the four intercepts as shown.



Ellipse with semiaxes 5 and 10; the shaded quadrant supplies one-fourth of the area.

Step 2: Set up the area integral using symmetry On the upper half of the ellipse, take the nonnegative

Quick Tip to Solve: Use the first-quadrant integral and multiply by four; the intercepts 5 and 10 also give the check $\pi(5)(10) = 50\pi$.

square root:

$$y = 10\sqrt{1 - \frac{x^2}{25}} = 2\sqrt{25 - x^2}.$$

The first-quadrant region extends from $x = 0$ to $x = 5$. A vertical strip has height y and width dx , so its area is $y dx$. If A denotes the entire enclosed area, all four quadrants contribute equally:

$$A = 4 \int_0^5 y dx = 8 \int_0^5 \sqrt{25 - x^2} dx.$$

Step 3: Evaluate the integral by trigonometric substitution Put $x = 5 \sin \theta$. Then $dx = 5 \cos \theta d\theta$. As x increases from 0 to 5, θ increases from 0 to $\pi/2$; hence $\cos \theta \geq 0$ and $\sqrt{25 - x^2} = 5 \cos \theta$. Therefore

$$\begin{aligned} A &= 200 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 100 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= 100 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}. \end{aligned}$$

The sine term vanishes at both limits. The remaining contribution is $100(\pi/2 - 0)$, in square units because an area is being calculated.

Step 4: Conclusion The required sketch is an ellipse with centre $(0, 0)$, horizontal intercepts $(\pm 5, 0)$ and vertical intercepts $(0, \pm 10)$. Its enclosed area, obtained by integration, is 50π square units.

Quick Tip to Solve: Use the first-quadrant integral and multiply by four; the intercepts 5 and 10 also give the check $\pi(5)(10) = 50\pi$.

35(a). Find the foot of the perpendicular from the point $(0, 2, 3)$ on the line $\frac{-x-3}{-5} = \frac{1-y}{-2} = \frac{3z+12}{9}$ and hence find the length of the perpendicular. [5 marks]

Solution:

Concept:

- A line through a point A , with direction vector $\vec{d}$, has points $H = A + t\vec{d}$, where t is a real parameter.
- At the foot H of the perpendicular from a point P , the joining vector $\overrightarrow{PH}$ is orthogonal to the line: $\overrightarrow{PH} \cdot \vec{d} = 0$.
- The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express the given line parametrically Let $P = (0, 2, 3)$. First simplify each ratio of the printed line without changing its value:

$$\frac{-x-3}{-5} = \frac{x+3}{5}, \quad \frac{1-y}{-2} = \frac{y-1}{2}, \quad \frac{3z+12}{9} = \frac{z+4}{3}.$$

Quick Tip to Solve: Simplify all three printed ratios first; then use one parameter and one dot-product equation instead of introducing an unknown perpendicular line.

Equating the common ratio to t gives

$$x = 5t - 3, \quad y = 2t + 1, \quad z = 3t - 4.$$

Thus $A = (-3, 1, -4)$ lies on the line, and a direction vector is $\vec{d} = (5, 2, 3)$. In particular, every possible foot on this line must have the coordinates $H = (5t - 3, 2t + 1, 3t - 4)$ for some t .

Step 2: Impose the perpendicularity condition The vector joining P to a general point H is

$$\overrightarrow{PH} = (5t - 3, 2t - 1, 3t - 7).$$

For H to be the required foot, this vector must have zero dot product with $\vec{d}$. Hence

$$\begin{aligned} 5(5t - 3) + 2(2t - 1) + 3(3t - 7) &= 0, \\ 25t - 15 + 4t - 2 + 9t - 21 &= 0, \\ 38t - 38 &= 0. \end{aligned}$$

This linear equation gives $t = 1$. Its coefficient $38 = 5^2 + 2^2 + 3^2$ is positive, so the perpendicular foot is uniquely determined.

Step 3: Substitute the parameter and verify the foot Substitute $t = 1$ in the coordinates of H :

$$H = (5 - 3, 2 + 1, 3 - 4) = (2, 3, -1).$$

Check that it belongs to the original line: its three ratios are $(-2 - 3)/(-5) = 1$, $(1 - 3)/(-2) = 1$, and $(3(-1) + 12)/9 = 1$. Moreover, $\overrightarrow{PH} = (2, 1, -4)$, and

$$(2, 1, -4) \cdot (5, 2, 3) = 10 + 2 - 12 = 0.$$

These two checks establish both necessary properties: the point lies on the line and the joining segment is perpendicular to it.

Step 4: Calculate the length of the perpendicular Apply the distance formula to $P = (0, 2, 3)$ and $H = (2, 3, -1)$:

$$PH = \sqrt{(2 - 0)^2 + (3 - 2)^2 + (-1 - 3)^2} = \sqrt{4 + 1 + 16}.$$

The nonnegative square root is used because this quantity is a length, even though one component of the displacement is negative.

Step 5: Conclusion The foot of the perpendicular is $(2, 3, -1)$, and its length is $\sqrt{21}$ units.

Quick Tip to Solve: Simplify all three printed ratios first; then use one parameter and one dot-product equation instead of introducing an unknown perpendicular line.

35(b). Find the value of p if the shortest distance between the lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (p\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$ is $\frac{3}{\sqrt{2}}$ units. [5 marks]

Solution:

Concept:

- For two non-parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, their shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$.
- The cross product $\vec{b}_1 \times \vec{b}_2$ is perpendicular to both direction vectors. The scalar projection of the joining vector on this common normal gives the distance; its absolute value is essential because a distance is non-negative.

Step 1: Identify the points and direction vectors From the given vector equations, choose the points whose position vectors are

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}, \quad \vec{a}_2 = p\hat{i} - \hat{j} - \hat{k}.$$

The corresponding direction vectors are

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}, \quad \vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}.$$

Thus the vector joining the first chosen point to the second is

$$\vec{a}_2 - \vec{a}_1 = (p - 1)\hat{i} - 3\hat{j} - 2\hat{k}.$$

Step 2: Obtain a common normal to the lines Evaluate the cross product by expanding the determinant:

$$\begin{aligned} \vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-2 - 1)\hat{i} - (2 - 2)\hat{j} + (1 + 2)\hat{k} \\ &= -3\hat{i} + 3\hat{k}. \end{aligned}$$

It is non-zero, so the lines are non-parallel and the stated shortest-distance formula applies. Its magnitude is

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + 0^2 + 3^2} = 3\sqrt{2}.$$

Step 3: Use the specified distance The scalar triple product in the numerator is

$$\begin{aligned} (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) &= -3(p - 1) + 0(-3) + 3(-2) \\ &= -3p - 3 = -3(p + 1). \end{aligned}$$

Quick Tip to Solve: Keep the modulus in the scalar-triple-product formula until the final equation; solving both signs of $p + 1$ prevents losing the negative solution.

Therefore the condition on the distance becomes

$$\frac{|-3(p+1)|}{3\sqrt{2}} = \frac{3}{\sqrt{2}} \implies |p+1| = 3.$$

An absolute-value equation has both a positive and a negative branch: $p+1 = 3$ or $p+1 = -3$. Neither branch may be discarded merely because one produces a negative value of p .

Step 4: Check both branches against the geometry Solving the two linear equations gives the candidates 2 and -4 . For each candidate, $|p+1| = 3$, so substituting in the distance expression gives exactly $3/\sqrt{2}$ units. The direction vectors remain non-parallel and the distance is positive in both cases, so both choices describe skew lines satisfying the question.

Step 5: Conclusion The required values are $p = 2$ or $p = -4$.

Quick Tip to Solve: Keep the modulus in the scalar-triple-product formula until the final equation; solving both signs of $p+1$ prevents losing the negative solution.

Section E (12 marks)

This section comprises 3 Case Study based questions of 4 marks each.

36. A school wants the students of class XII to do a project on 'Sustainability' keeping the world environment in mind. They select the student participants on the basis of an essay writing competition. 7 students out of 80 are selected for the project and are categorized into two sets such that:

Girl students belong to Set $A = \{G_1, G_2, G_3, G_4\}$,

Boy students belong to Set $B = \{B_1, B_2, B_3\}$.

Based on the above information, answer the following questions:

[4 marks]

36(i). How many relations are possible from Set $A \rightarrow$ Set B ?

[1 mark]

Solution:

Concept:

- A relation from A to B is any subset of the Cartesian product $A \times B$.
- If a finite set contains n elements, it has 2^n subsets: each element is either included or excluded independently.

Step 1: Count the available ordered pairs The given sets contain four girls and three boys, so $|A| = 4$ and $|B| = 3$. Each of the four elements of A can be paired with each of the three elements of B . Consequently,

$$|A \times B| = 4 \times 3 = 12.$$

Step 2: Count all subsets of the Cartesian product For each of these twelve distinct ordered pairs there are two choices: include the pair in the relation or leave it out. These choices are independent, giving 2^{12} subsets. This includes the empty relation and the full relation $A \times B$. No function condition is imposed, so we do not restrict each girl to exactly one image; the function-count formula 3^4 would answer a different question.

Quick Tip to Solve: For relations, first multiply the two set sizes and use that product as the exponent of 2; functions have a different counting rule.

Step 3: Conclusion The number of possible relations from A to B is $2^{12} = 4096$.

Quick Tip to Solve: For relations, first multiply the two set sizes and use that product as the exponent of 2; functions have a different counting rule.

36(ii). Let R be a relation from $A \rightarrow B$ such that $R = \{(G_1, B_1), (G_2, B_2), (G_3, B_2), (G_4, B_3), (G_1, B_2)\}$. Is R an injective function? Justify your answer. [1 mark]

Solution:

Concept:

- A function from A to B assigns exactly one image in B to every element of A .
- An injective function also assigns different images to distinct elements of its domain.

Step 1: Check whether the relation is a function The relation contains both (G_1, B_1) and (G_1, B_2) . Thus the same input G_1 has two distinct images, B_1 and B_2 , violating the uniqueness condition for a function.

Step 2: Examine the injectivity requirement Injectivity can hold only for a function. Moreover, the distinct inputs G_2 and G_3 are both paired with B_2 , so the relation also fails the distinct-images requirement.

Step 3: Conclusion No, R is not an injective function; it is not even a function because G_1 has two images.

Quick Tip to Solve: First scan repeated first coordinates: G_1 paired with two different boys immediately rules out a function.

36(iii)(a). Let the relation R from $A \rightarrow A$ be such that $R = \{(x, y), x, y \in A, x \text{ and } y \text{ are students from the same colony in the city}\}$. Verify if R is an equivalence relation. [2 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation if it is reflexive, symmetric and transitive.
- Here xRy means that the students x and y belong to the same colony; all three properties must be checked for arbitrary elements of A .

Step 1: Verify reflexivity Let $x \in A$. Every student belongs to the same colony as himself or herself, so $(x, x) \in R$. This holds for each student in A , and hence R is reflexive.

Step 2: Verify symmetry Let $x, y \in A$ and suppose $(x, y) \in R$. Then x and y are students from the same colony. Reversing their order does not change this fact: y and x also belong to the same colony. Therefore $(y, x) \in R$, proving symmetry.

Step 3: Verify transitivity Let $x, y, z \in A$, with $(x, y) \in R$ and $(y, z) \in R$. Students x and y belong to one colony, and students y and z belong to that same colony. Consequently x and z belong to the

Quick Tip to Solve: For a relation defined by the same colony, use an arbitrary student for reflexivity, reverse a pair for symmetry, and link two pairs through their common student for transitivity.

same colony, so $(x, z) \in R$. Thus R is transitive.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, it is an equivalence relation on A .

Quick Tip to Solve: For a relation defined by the same colony, use an arbitrary student for reflexivity, reverse a pair for symmetry, and link two pairs through their common student for transitivity.

OR

36(iii)(b). Verify if any function $f : B \rightarrow A$ is bijective. Give reason to support your answer. [2 marks]

Solution:

Concept:

- A bijective function is both one-one (injective) and onto (surjective).
- A function from a finite set with m elements can have at most m distinct images. An onto function must have every element of its codomain in its range.

Step 1: Compare the domain and codomain The given sets are $B = \{B_1, B_2, B_3\}$ and $A = \{G_1, G_2, G_3, G_4\}$. Hence the domain B has three elements whereas the codomain A has four. Each of the three boys must be assigned exactly one image by a function $f : B \rightarrow A$.

Step 2: Test whether the function can be onto Even if all three inputs have different images, the range can contain only three girls. If any images repeat, the range becomes smaller still. Thus

$$|f(B)| \leq 3 < 4 = |A|.$$

At least one element of A has no preimage under every possible function from B to A . Therefore no such function is onto. Although an injective function can be formed by choosing three different girls as images, this cannot make it bijective.

Step 3: Conclusion No function $f : B \rightarrow A$ is bijective, because a domain with three elements cannot map onto a codomain with four elements.

Quick Tip to Solve: For finite sets, compare their sizes first: a bijection requires equal cardinalities, and here $|B| = 3$ while $|A| = 4$.

37. There are three types of vaccines A_1, A_2, A_3 , available in the market to protect the population of the country from spread of certain infection. According to a survey conducted, it was found that 25% of the population was given Vaccine A_1 , 35% of the population was given Vaccine A_2 and 40% of the population was given Vaccine A_3 . The survey also stated that the probabilities that Vaccines A_1, A_2 and A_3 would protect against the infection were 60%, 55% and 50% respectively.

Based on the above information, answer the following questions:

Find the probability that:

[4 marks]

37(i). The person taking vaccine A_2 will get infected.

[1 mark]

Solution:

Concept:

- Let I denote infection and A_2 denote receiving vaccine A_2 . The requested probability is conditional: $P(I | A_2)$.
- For a fixed vaccine, infection and protection are complementary outcomes, so $P(I | A_2) = 1 - P(I^c | A_2)$.

Step 1: Identify the given protection probability The survey states that vaccine A_2 protects against infection with probability $55\% = 0.55$. Since the person is already known to have received this vaccine, use this conditional protection probability directly.

Step 2: Take the complement Subtract the protection probability from unity:

$$P(I | A_2) = 1 - 0.55 = 0.45.$$

Multiplying by 0.35 would instead find the joint probability of receiving this vaccine and becoming infected, which is a different event.

Step 3: Conclusion The required probability is $0.45 = \frac{9}{20}$, or 45%.

Quick Tip to Solve: When the vaccine is already specified, complement its protection rate; do not multiply by its population share.

37(ii). If a person is chosen randomly, he/she will be protected from the infection.

[1 mark]

Solution:

Concept:

- Let E denote protection from infection. The three vaccine groups A_1, A_2, A_3 form a mutually exclusive and exhaustive partition of the population.
- By total probability, $P(E) = \sum_{j=1}^3 P(A_j)P(E | A_j)$; each group's protection rate is weighted by its population share.

Step 1: Write the three weighted contributions The vaccine shares are 0.25, 0.35, 0.40, and their protection probabilities are 0.60, 0.55, 0.50, respectively. Thus the probabilities of belonging to each

Quick Tip to Solve: Pair each protection rate with its own population share before adding the three products.

group and being protected are

$$0.25(0.60) = 0.15, \quad 0.35(0.55) = 0.1925, \quad 0.40(0.50) = 0.20.$$

Step 2: Add the disjoint possibilities A protected person belongs to exactly one of these vaccine groups. Hence

$$P(E) = 0.15 + 0.1925 + 0.20 = 0.5425.$$

A simple unweighted average would be inappropriate because the vaccine groups have different sizes.

Step 3: Conclusion The probability that a randomly chosen person is protected is $0.5425 = \frac{217}{400}$.

Quick Tip to Solve: Pair each protection rate with its own population share before adding the three products.

37(iii)(a). The person was given Vaccine A_1 , given that the randomly chosen person is infected.

[2 marks]

Solution:

Concept:

- Let I denote infection and A_j denote receipt of vaccine A_j . Bayes' theorem gives $P(A_1 | I) = \frac{P(A_1)P(I | A_1)}{P(I)}$.
- Infection is complementary to protection within each vaccine group. Since the vaccine groups partition the population, $P(I) = \sum_{j=1}^3 P(A_j)P(I | A_j)$.

Step 1: Compute infection rates for each vaccine Subtract the stated protection probabilities from one. This gives

$$P(I | A_1) = 0.40, \quad P(I | A_2) = 0.45, \quad P(I | A_3) = 0.50.$$

The corresponding vaccine probabilities remain 0.25, 0.35, 0.40.

Step 2: Find the total probability of infection Include the infected people from all three vaccine groups in the denominator:

$$\begin{aligned} P(I) &= 0.25(0.40) + 0.35(0.45) + 0.40(0.50) \\ &= 0.10 + 0.1575 + 0.20 = 0.4575. \end{aligned}$$

Step 3: Apply Bayes' theorem The joint probability of receiving A_1 and being infected is $0.25(0.40) = 0.10$. Restricting the sample space to infected people gives

$$P(A_1 | I) = \frac{0.10}{0.4575} = \frac{1000}{4575} = \frac{40}{183}.$$

Step 4: Conclusion Given that the person is infected, the probability that they received vaccine A_1 is

$$\frac{40}{183}.$$

Quick Tip to Solve: For a vaccine given infection, put that vaccine's infected contribution on top and the total infected probability below.

37(iii)(b). The person was given Vaccine A_3 , given that the randomly chosen person is not infected.

[2 marks]

Solution:

Concept:

- Let E denote not being infected and A_j denote receiving vaccine A_j . Here E is the protection event stated in the survey.
- Bayes' theorem gives $P(A_3 | E) = P(A_3)P(E | A_3)/P(E)$, with $P(E) = \sum_{j=1}^3 P(A_j)P(E | A_j)$.

Step 1: Identify the joint probability for the third vaccine Vaccine A_3 is given to 40% of the population and protects 50% of its recipients. Therefore,

$$P(A_3 \cap E) = P(A_3)P(E | A_3) = 0.40(0.50) = 0.20.$$

This is the contribution of the third vaccine group to the uninfected population.

Step 2: Find the total uninfected probability A person can be protected after receiving any one of the three vaccines. Total probability gives

$$\begin{aligned} P(E) &= 0.25(0.60) + 0.35(0.55) + 0.40(0.50) \\ &= 0.15 + 0.1925 + 0.20 = 0.5425. \end{aligned}$$

Step 3: Condition on the person being uninfected Divide the third vaccine's uninfected contribution by the total uninfected probability:

$$P(A_3 | E) = \frac{P(A_3 \cap E)}{P(E)} = \frac{0.20}{0.5425} = \frac{2000}{5425} = \frac{80}{217}.$$

The given protection rate 0.50 is $P(E | A_3)$, so it is not itself the reverse conditional probability requested.

Step 4: Conclusion Given that the person is not infected, the probability that they received vaccine

A_3 is $\boxed{\frac{80}{217}}$.

Quick Tip to Solve: Use protection rates directly for the uninfected condition, and divide the relevant protected contribution by the total protected probability.

38. A company produces cylindrical tumblers, open from the top. Since they want uniformity in the product, they fix the surface area of the tumblers produced.

Based on the above information, answer the following questions:

If for a tumbler, V is its volume, h the height and r the radius of the circular base, then: [4 marks]

38(i). Differentiate its volume with respect to radius of the base, where the surface area is constant. [2 marks]

Solution:

Concept:

- For an open-top cylinder of radius r and height h , the fixed surface area is $S = \pi r^2 + 2\pi r h$; only one circular base is present.
- Its volume is $V = \pi r^2 h$. When S is fixed, the height changes with the radius, so eliminate h before differentiating.

Step 1: Express height using the fixed surface area Denote the constant surface area by S . The base contributes πr^2 , and the curved wall contributes $2\pi r h$. Therefore

$$S = \pi r^2 + 2\pi r h, \quad h = \frac{S - \pi r^2}{2\pi r}.$$

Here $r > 0$, and a positive height requires $r < \sqrt{S/\pi}$. Since S is constant, treating h as constant while varying r would not preserve the prescribed surface area.

Step 2: Differentiate the volume as a function of radius Substitute this expression for height into the cylinder's volume:

$$V = \pi r^2 \frac{S - \pi r^2}{2\pi r} = \frac{Sr}{2} - \frac{\pi r^3}{2}.$$

Differentiating with respect to r , with S constant, gives

$$\frac{dV}{dr} = \frac{S}{2} - \frac{3\pi r^2}{2}.$$

To write this in terms of the actual dimensions, substitute $S = \pi r^2 + 2\pi r h$ into the derivative.

Step 3: Conclusion At constant surface area, $\frac{dV}{dr} = \frac{S - 3\pi r^2}{2} = \pi r(h - r)$.

Quick Tip to Solve: For an open tumbler, count one base in the area and eliminate the changing height before differentiating the volume.

38(ii). If the company wants to maximize the volume of each tumbler, then establish a relation between its height and the radius of the base. [2 marks]

Solution:

Concept:

- A stationary value of a differentiable volume function occurs where $dV/dr = 0$; a negative second derivative there establishes a local maximum.
- For fixed surface area S , an open cylinder satisfies $S = \pi r^2 + 2\pi rh$ and $V = Sr/2 - \pi r^3/2$, with $r, h > 0$.

Step 1: Find the stationary radius Differentiate the volume while holding the surface area constant:

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2}.$$

The stationary condition is $S - 3\pi r^2 = 0$, giving $S = 3\pi r^2$ and the positive radius $r = \sqrt{S/(3\pi)}$. This radius lies in the physical interval $0 < r < \sqrt{S/\pi}$.

Step 2: Check the maximum and relate the dimensions The second derivative is

$$\frac{d^2V}{dr^2} = -3\pi r < 0 \quad (r > 0).$$

Thus the stationary point gives a maximum. It is the unique maximum on the physical interval: the first derivative is positive before $\sqrt{S/(3\pi)}$ and negative after it. At this radius, substitute $S = 3\pi r^2$ into the area constraint:

$$3\pi r^2 = \pi r^2 + 2\pi rh \implies 2\pi r^2 = 2\pi rh.$$

Since $r > 0$, division by $2\pi r$ relates the height directly to the radius.

Step 3: Conclusion The volume is maximum when $\boxed{h = r}$; the height of each tumbler must equal its base radius.

Quick Tip to Solve: After solving the stationary-volume condition, substitute it back into the open-cylinder area formula to obtain the height-to-radius ratio.