

# CBSE Class 12 Mathematics Question Paper 2026 with Solution PDF-Set 65-5-3

## Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

### General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.
- (iii) In **Section A**, Questions no. **1 to 18** are Multiple Choice Questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are Very Short Answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are Long Answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculator is not allowed.

## Section A (20 marks)

This section comprises Multiple Choice Questions (MCQs) of 1 mark each.

1. For any square matrix  $A$  with real entries, if  $A + A'$  is a symmetric matrix then:

[1 mark]

- (A)  $(A - A')$  cannot be a skew symmetric matrix
- (B)  $(A - A')$  is a skew symmetric matrix
- (C)  $A$  is always a symmetric matrix
- (D)  $A$  is always a skew symmetric matrix

**Correct Answer:** (B)  $(A - A')$  is a skew symmetric matrix

### Solution:

#### Concept:

- A matrix  $S$  is symmetric when  $S' = S$ .
- A matrix  $K$  is skew symmetric when  $K' = -K$ ; transpose reverses neither addition nor subtraction.

**Step 1: Transpose the difference** Put  $K = A - A'$ . Using  $(A')' = A$ , we obtain

$$K' = A' - A = -(A - A') = -K.$$

Thus the required transpose relation holds for every square matrix, regardless of the symmetric sum.

**Step 2: Check the alternatives** Option (A) contradicts this identity. Neither (C) nor (D) follows: for example  $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$  is neither symmetric nor skew symmetric, although  $A + A'$  is symmetric. The premise is an identity, not an additional restriction on  $A$ .

**Step 3: Conclusion**  $A - A'$  is skew symmetric. Hence the correct option is (B).

**Quick Tip to Solve:** Transpose the difference directly; the symmetric sum does not force  $A$  itself to be symmetric.

2. A matrix  $B = [b_{ij}]_{m \times m}$  is said to be a diagonal matrix, if:

[1 mark]

- (A)  $b_{ij} = 0$ , when  $i = j$
- (B)  $b_{ij} = 1$ , when  $i = j$
- (C)  $b_{ij} = 1$ , when  $i \neq j$
- (D)  $b_{ij} = 0$ , when  $i \neq j$

**Correct Answer:** (D)  $b_{ij} = 0$ , when  $i \neq j$

### Solution:

#### Concept:

- A diagonal matrix is square and every off-diagonal entry is zero.
- Diagonal entries  $b_{ii}$  can be arbitrary real numbers, including zero.

**Quick Tip to Solve:** Remember that diagonal refers to the positions where nonzero entries are allowed, not to a fixed diagonal value.

**Step 1: Identify the off-diagonal positions** The row and column indices are unequal off the main diagonal. Consequently the definition is

$$b_{ij} = 0 \quad \text{whenever } i \neq j.$$

No requirement that  $b_{ii} = 1$  is part of this definition.

**Step 2: Test the other statements** Option (A) instead makes the diagonal entries zero, allowing nonzero off-diagonal entries. Option (B) fixes only the diagonal entries to one and leaves other entries unrestricted; it does not ensure a diagonal matrix. Option (C) explicitly makes every off-diagonal entry one and therefore violates the definition for order greater than one.

**Step 3: Conclusion** The defining condition is  $b_{ij} = 0$  for  $i \neq j$ . Hence the correct option is (D).

**Quick Tip to Solve:** Remember that diagonal refers to the positions where nonzero entries are allowed, not to a fixed diagonal value.

**3. If  $A$  is a non-singular matrix, then which of the following is not true?**

[1 mark]

- (A)  $\text{adj } A$  is singular
- (B)  $(\text{adj } A)^{-1} = \text{adj}(A^{-1})$
- (C)  $|A| \neq 0$
- (D)  $A^{-1}$  exists

**Correct Answer:** (A)  $\text{adj } A$  is singular

**Solution:**

**Concept:**

- A non-singular matrix has  $|A| \neq 0$  and an inverse  $A^{-1}$ .
- The adjoint identity gives  $\text{adj } A = |A|A^{-1}$ .

**Step 1: Check whether the adjoint is singular** Since  $|A|$  is a nonzero scalar and  $A^{-1}$  is invertible, their product  $\text{adj } A$  is invertible. Indeed,

$$(\text{adj } A)^{-1} = \frac{1}{|A|} A.$$

Therefore the claim that the adjoint is singular is false.

**Step 2: Verify the remaining statements** Also

$$\text{adj}(A^{-1}) = |A^{-1}|(A^{-1})^{-1} = \frac{1}{|A|} A.$$

Hence option (B) is true. Options (C) and (D) are precisely the determinant and inverse characterisations of non-singularity, so they are true as well. The question asks for the statement that is not true.

**Step 3: Conclusion** The false statement is that  $\text{adj } A$  is singular. Hence the correct option is (A).

**Quick Tip to Solve:** For an invertible matrix, replace its adjoint by determinant times inverse before checking the statements.

4. If  $f(x) = \begin{cases} \frac{x^2 - 4x - 5}{x + 1}, & x \neq -1 \\ k, & x = -1 \end{cases}$  is continuous at  $x = -1$ , then the value of  $k$  is: [1 mark]

- (A) Any real value  
(B) 6  
(C) -1  
(D) -6

**Correct Answer:** (D) -6

**Solution:**

**Concept:**

- Continuity at  $a$  requires  $\lim_{x \rightarrow a} f(x) = f(a)$ .
- A removable common factor can be cancelled when computing a limit for  $x \neq a$ .

**Step 1: Simplify the expression near the point** The numerator factors as  $x^2 - 4x - 5 = (x - 5)(x + 1)$ . Thus for  $x \neq -1$ ,

$$f(x) = \frac{(x - 5)(x + 1)}{x + 1} = x - 5.$$

The cancellation is valid in the punctured neighbourhood used for a limit; it does not prescribe the value at the point.

**Step 2: Apply continuity** The two-sided limit is

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x - 5) = -1 - 5 = -6.$$

Since  $f(-1) = k$ , continuity fixes one unique value of  $k$ . Hence an arbitrary real value, 6, and -1 in the other choices all fail the equality between value and limit.

**Step 3: Conclusion**  $k = -6$ . Hence the correct option is (D).

**Quick Tip to Solve:** Factor before substituting at a removable discontinuity; substitute only after cancelling the common factor.

5. For the inverse trigonometric functions, which of the following Principal Value Branch is *not* correctly defined? [1 mark]

- (A)  $\tan^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$   
(B)  $\sec^{-1} : \mathbb{R} - (-1, 1) \rightarrow [0, \pi] - \{\pi/2\}$   
(C)  $\cot^{-1} : \mathbb{R} \rightarrow (0, \pi)$   
(D)  $\operatorname{cosec}^{-1} : \mathbb{R} - (-1, 1) \rightarrow [-\pi/2, \pi/2]$

**Correct Answer:** (D)  $\operatorname{cosec}^{-1} : \mathbb{R} - (-1, 1) \rightarrow [-\pi/2, \pi/2]$

**Solution:**

**Concept:**

- A principal branch must give one valid angle for each argument in the inverse function domain.

**Quick Tip to Solve:** Check excluded angles as carefully as endpoints: inverse secant excludes  $\pi/2$  and inverse cosecant excludes zero.

- Secant is undefined at  $\pi/2$ , and cosecant is undefined at 0.

**Step 1: Check tangent, secant and cotangent** The principal ranges  $(-\pi/2, \pi/2)$  for  $\tan^{-1}$  and  $(0, \pi)$  for  $\cot^{-1}$  are correct. For  $\sec^{-1}$ , the printed range  $[0, \pi] - \{\pi/2\}$  correctly excludes the angle where secant is undefined. Thus (A), (B) and (C) are correctly defined.

**Step 2: Check the cosecant branch** The correct principal range of inverse cosecant is

$$[-\pi/2, \pi/2] - \{0\}.$$

Both endpoints are allowed since their cosecants are  $-1$  and  $1$ , respectively. But zero must be excluded since  $\sin 0 = 0$ , so  $\operatorname{cosec} 0$  is undefined. The printed option (D) includes zero, which makes that branch incorrect.

**Step 3: Conclusion** The incorrectly defined principal value branch is (D).

**Quick Tip to Solve:** Check excluded angles as carefully as endpoints: inverse secant excludes  $\pi/2$  and inverse cosecant excludes zero.

6. Let  $A = \begin{pmatrix} 0 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} -3 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$ . If  $A + B + C = O$ , then matrix  $C$  is: [1 mark]

(A)  $\begin{pmatrix} -3 & -3 & 5 \\ 3 & 4 & 2 \end{pmatrix}$

(B)  $\begin{pmatrix} 3 & 3 & 5 \\ -3 & -4 & -2 \end{pmatrix}$

(C)  $\begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$

(D)  $\begin{pmatrix} -3 & -3 & -5 \\ 3 & 4 & 2 \end{pmatrix}$

**Correct Answer:** (C)  $\begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$

**Solution:**

**Concept:**

- Matrix addition is performed entry by entry for matrices of the same order.
- From  $A + B + C = O$ , the additive inverse gives  $C = -(A + B)$ .

**Step 1: Add the printed matrices** Using the negative entry in the first row of  $A$ ,

$$A + B = \begin{pmatrix} 0 - 3 & -3 + 0 & 4 + 1 \\ 1 + 2 & 0 + 4 & 2 + 0 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 5 \\ 3 & 4 & 2 \end{pmatrix}.$$

Both matrices have order  $2 \times 3$ , so this sum and its additive inverse are defined.

**Quick Tip to Solve:** First add entry by entry, then change every sign; check the upper-right entry to distinguish B from C.

**Step 2: Negate and compare each entry** Therefore

$$C = \begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}.$$

Option (A) has the wrong first and second rows, while (B) leaves the upper-right entry positive. Option (D) reverses only some of the required signs. Only option (C) cancels all six entries of  $A + B$ , so substituting it makes the sum the zero matrix.

**Step 3: Conclusion**  $C = \begin{pmatrix} 3 & 3 & -5 \\ -3 & -4 & -2 \end{pmatrix}$ . Hence the correct option is (C).

**Quick Tip to Solve:** First add entry by entry, then change every sign; check the upper-right entry to distinguish B from C.

**7. If vectors  $\vec{a} = 3\hat{i} + 2\hat{j} + \lambda\hat{k}$  and  $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ , represent the two strips of the Red Cross sign placed outside a doctor's clinic, then the value of  $\lambda$  is:** [1 mark]

- (A) 1
- (B)  $\frac{5}{2}$
- (C)  $\frac{2}{5}$
- (D) 0

**Correct Answer:** (C)  $\frac{2}{5}$

**Solution:**

**Concept:**

- The two strips of a Red Cross are perpendicular.
- Nonzero perpendicular vectors satisfy  $\vec{a} \cdot \vec{b} = 0$ , where the dot product sums products of corresponding components.

**Step 1: Use the geometry of the strips** The strips meet at a right angle, so

$$\vec{a} \cdot \vec{b} = 3(2) + 2(-4) + \lambda(5) = 0.$$

The components along  $\hat{i}, \hat{j}, \hat{k}$  are multiplied in corresponding pairs; mixed pairs do not contribute because the unit basis vectors are mutually perpendicular.

**Step 2: Solve and test the choices** The equation reduces to  $6 - 8 + 5\lambda = 0$ , or  $5\lambda = 2$ . Hence  $\lambda = 2/5$ . For the choices 1,  $5/2$ , and 0, the dot products are respectively 3,  $21/2$ , and  $-2$ , all nonzero. With  $2/5$ , the dot product is exactly zero.

**Step 3: Conclusion**  $\lambda = \frac{2}{5}$ . Hence the correct option is (C).

**Quick Tip to Solve:** Translate the right angle between the strips into a zero dot product before inserting the components.

8. If  $3P(A) = P(B) = \frac{3}{5}$  and  $P(A|B) = \frac{2}{3}$ , then  $P(A \cup B)$  is:

[1 mark]

- (A)  $\frac{3}{5}$
- (B)  $\frac{1}{5}$
- (C)  $\frac{2}{15}$
- (D)  $\frac{2}{5}$

**Correct Answer:** (D)  $\frac{2}{5}$

**Solution:**

**Concept:**

- Conditional probability satisfies  $P(A | B) = P(A \cap B)/P(B)$  when  $P(B) > 0$ .
- The union formula is  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .

**Step 1: Read the probabilities and compute the intersection** The printed equality gives  $P(B) = 3/5$  and  $P(A) = 1/5$ . Using the stated conditional probability,

$$P(A \cap B) = P(A | B)P(B) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}.$$

These values reveal an inconsistency: an intersection cannot have probability  $2/5$  when  $P(A) = 1/5$ .

**Step 2: Evaluate the intended union calculation** Applying the union identity algebraically to the printed data gives

$$P(A \cup B) = \frac{1}{5} + \frac{3}{5} - \frac{2}{5} = \frac{2}{5}.$$

Thus only option (D) matches the intended calculation; the other numerical options do not. However, this is a formal result from inconsistent data, since a union containing  $B$  would have to have probability at least  $3/5$ . No actual events satisfy all the printed conditions.

**Step 3: Conclusion** The intended algebraic answer is  $\frac{2}{5}$ , option (D); the printed probability conditions are inconsistent.

**Quick Tip to Solve:** Compute the intersection from the conditional probability, then also check that it does not exceed either event probability.

9. If the area of  $\triangle ABC$  with vertices  $A(3, 1)$ ,  $B(-2, 1)$  and  $C(0, k)$  is 5 sq. units, then values of  $k$  are: [1 mark]

- (A) 3, 1
- (B) -1, 3
- (C) -1, 2
- (D) 0, 2

**Correct Answer:** (B) -1, 3

**Solution:**

**Concept:**

- The area of a triangle is half its base multiplied by its perpendicular height.
- Area is nonnegative, so the height is an absolute value.

**Step 1: Find base and height** Both  $A(3, 1)$  and  $B(-2, 1)$  lie on  $y = 1$ . Thus  $AB = |3 - (-2)| = 5$ , and the perpendicular distance of  $C(0, k)$  from the base is  $|k - 1|$ .

**Step 2: Solve the area condition**

$$\frac{1}{2}(5)|k - 1| = 5 \implies |k - 1| = 2.$$

Hence  $k - 1 = 2$  or  $k - 1 = -2$ , giving  $k = 3$  or  $k = -1$ . Option (A) includes the degenerate value 1, while (C) and (D) include values with heights other than 2.

**Step 3: Conclusion** The values are  $\boxed{-1, 3}$ . Hence, the correct option is (B).

**Quick Tip to Solve:** For a horizontal base, use the absolute difference of the third vertex ordinate and the base ordinate.

10.  $\int \sqrt{\frac{1 + \cos x}{1 - \cos x}} dx$  is equal to: [1 mark]

- (A)  $2 \log \left| \sin \frac{x}{2} \right| + C$
- (B)  $\frac{1}{2} \log |\sin 2x| + C$
- (C)  $\log |1 - \cos 2x| + C$
- (D)  $\log |1 + \cos 2x| + C$

**Correct Answer:** (A)  $2 \log \left| \sin \frac{x}{2} \right| + C$

**Solution:**

**Concept:**

- Use  $1 + \cos x = 2 \cos^2(x/2)$  and  $1 - \cos x = 2 \sin^2(x/2)$ .
- The standard CBSE half-angle evaluation uses  $\int \cot u \, du = \log |\sin u| + C$ .

**Quick Tip to Solve:** Use half-angle identities first; remember that the principal square root of a square is an absolute value.

**Step 1: Apply the half-angle identity**

$$\sqrt{\frac{1 + \cos x}{1 - \cos x}} = \sqrt{\cot^2 \frac{x}{2}} = \left| \cot \frac{x}{2} \right|.$$

On an interval where  $\cot(x/2) \geq 0$ , this is  $\cot(x/2)$ , as used in the printed alternatives.

**Step 2: Integrate and check alternatives** Putting  $u = x/2$ , so  $dx = 2 du$ , gives

$$\int \cot \frac{x}{2} dx = 2 \log \left| \sin \frac{x}{2} \right| + C.$$

Its derivative is  $\cot(x/2)$ . Options (B), (C) and (D) differentiate to  $\cot 2x$ ,  $2 \cot x$ , and  $-2 \tan x$ , respectively, and therefore fail. On intervals where  $\cot(x/2) < 0$ , the principal-square-root integral has the negative of this antiderivative.

**Step 3: Conclusion** The intended printed answer is  $2 \log |\sin(x/2)| + C$ , option (A), on intervals where  $\cot(x/2) \geq 0$ .

**Quick Tip to Solve:** Use half-angle identities first; remember that the principal square root of a square is an absolute value.

**11. If  $\int_0^1 (6x^2 - 4x + k) dx = 0$ , then the value of  $k$  is:**

[1 mark]

- (A) 1
- (B) 0
- (C) 2
- (D) -1

**Correct Answer:** (B) 0

**Solution:**

**Concept:**

- Integrate each term independently using  $\int x^n dx = x^{n+1}/(n+1)$ .
- A definite integral is obtained by subtracting the lower-limit value from the upper-limit value.

**Step 1: Evaluate the integral** The printed integrand is  $6x^2 - 4x + k$ . Therefore

$$\int_0^1 (6x^2 - 4x + k) dx = [2x^3 - 2x^2 + kx]_0^1 = 2 - 2 + k = k.$$

The constant  $k$  integrates to  $kx$ , not to  $k^2/2$ .

**Step 2: Apply the given condition** The specified integral equals zero, so  $k = 0$ . Substitution confirms that  $\int_0^1 (6x^2 - 4x) dx = 0$ . The other printed values 1, 2, -1 produce integrals 1, 2, -1, respectively, and cannot satisfy the condition.

**Step 3: Conclusion**  $k = 0$ . Hence, the correct option is (B).

**Quick Tip to Solve:** Notice that the quadratic and linear terms have equal and opposite integrals over the interval.

12. If position vector  $\vec{p}$  of a point  $(24, n)$  is such that  $|\vec{p}| = 25$ , then the value of  $n$  is:

[1 mark]

- (A)  $\pm 49$
- (B)  $\pm 5$
- (C)  $\pm 1$
- (D)  $\pm 7$

**Correct Answer:** (D)  $\pm 7$

**Solution:**

**Concept:**

- The position vector of  $(a, b)$  is  $a\hat{i} + b\hat{j}$ .
- Its magnitude is  $\sqrt{a^2 + b^2}$ ; both signs of an unknown coordinate can give the same magnitude.

**Step 1: Use the magnitude formula** For the point  $(24, n)$ ,  $\vec{p} = 24\hat{i} + n\hat{j}$ . Thus

$$\sqrt{24^2 + n^2} = 25 \implies 576 + n^2 = 625.$$

Squaring is valid because both sides are nonnegative.

**Step 2: Solve for both possible ordinates**

$$n^2 = 625 - 576 = 49 \implies n = \pm 7.$$

Both points  $(24, 7)$  and  $(24, -7)$  have magnitude 25. Option (A) confuses  $n^2$  with  $n$ ; options (B) and (C) do not satisfy  $576 + n^2 = 625$ .

**Step 3: Conclusion**  $n = \pm 7$ . Hence, the correct option is (D).

**Quick Tip to Solve:** Recognise the 7-24-25 right-triangle triple, but retain both signs of the unknown coordinate.

13. The order and degree of the differential equation  $y = \left(\frac{d^2y}{dx^2}\right)^2 - \lambda \frac{dy}{dx}$  is:

[1 mark]

- (A) Order = 2, Degree = 2
- (B) Order = 2, Degree = 3
- (C) Order = 1, Degree = 2
- (D) Order = 2, Degree = 4

**Correct Answer:** (A) Order = 2, Degree = 2

**Solution:**

**Concept:**

- Order is the order of the highest derivative present.
- Degree is the exponent of the highest-order derivative when the equation is polynomial in derivatives.

**Step 1: Identify the order** The equation contains  $dy/dx$  and  $d^2y/dx^2$ . Its highest derivative is the

**Quick Tip to Solve:** First locate the highest derivative, then read its exponent; do not interchange order and degree.

second derivative, so its order is 2. The square on that derivative changes its degree, not its order.

**Step 2: Identify the degree and compare options** Writing

$$\left(\frac{d^2y}{dx^2}\right)^2 - \lambda \frac{dy}{dx} - y = 0$$

shows a polynomial in the derivatives. The highest-order derivative has exponent 2, so the degree is 2. Option (C) ignores the second derivative, while (B) and (D) assign exponents that do not occur in the equation.

**Step 3: Conclusion** Order = 2, Degree = 2. Hence, the correct option is (A).

**Quick Tip to Solve:** First locate the highest derivative, then read its exponent; do not interchange order and degree.

**14. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20) and (60, 0). If the objective function of an LPP is  $Z = 4x + 3y$ , then the maximum value is:** [1 mark]

- (A) 200
- (B) 300
- (C) 240
- (D) 120

**Correct Answer:** (B) 300

**Solution:**

**Concept:**

- A linear objective on a bounded polygonal feasible region attains its maximum at a corner point.
- Evaluate  $Z = 4x + 3y$  at every listed vertex before comparing values.

**Step 1: Evaluate the objective at the vertices**

$$\begin{aligned}Z(0, 0) &= 0, & Z(0, 40) &= 120, \\Z(20, 40) &= 80 + 120 = 200, \\Z(60, 20) &= 240 + 60 = 300, \\Z(60, 0) &= 240.\end{aligned}$$

Each substitution uses both coordinates of its particular corner point.

**Step 2: Compare all values** The largest of 0, 120, 200, 300, 240 is 300, achieved at (60, 20). The values 200, 240, 120 in the other options are objective values at other vertices, but none is the maximum. Choosing the largest individual coordinate alone would not establish the maximum.

**Step 3: Conclusion** The maximum is 300. Hence, the correct option is (B).

**Quick Tip to Solve:** Make a corner-point value list and compare every entry instead of selecting a vertex by appearance.

**15.** An ant is observed crawling on a sheet of paper along a straight line given by equation  $y = 2x - 4$ . Area of the surface covered by the ant bounded by  $y$ -axis,  $x$ -axis and  $x = 1$  is: [1 mark]

- (A) 1 sq. unit
- (B) 3 sq. units
- (C) 2 sq. units
- (D) 4 sq. units

**Correct Answer:** (B) 3 sq. units

**Solution:**

**Concept:**

- Geometric area between a curve and the  $x$ -axis is the integral of the absolute value of its ordinate.
- For  $0 \leq x \leq 1$ , the line  $y = 2x - 4$  lies below the  $x$ -axis.

**Step 1: Identify the height of the bounded strip** The boundaries  $y$ -axis and  $x = 1$  give limits 0 and 1. Throughout this interval,  $2x - 4 < 0$ . Therefore the vertical height between the  $x$ -axis and the line is  $0 - (2x - 4) = 4 - 2x$ , which is positive.

**Step 2: Integrate the positive height**

$$A = \int_0^1 (4 - 2x) dx = [4x - x^2]_0^1 = 4 - 1 = 3.$$

The signed integral of the line would be  $-3$ , whereas geometric area is 3. The endpoint heights are 4 and 2, so the trapezium check gives  $\frac{1}{2}(4 + 2)(1) = 3$ , excluding all other options.

**Step 3: Conclusion** The area is 3 square units. Hence, the correct option is (B).

**Quick Tip to Solve:** Check the sign of the line on the entire interval before integrating to find geometric area.

**16.** The general solution for the differential equation  $\frac{dy}{dx} = e^{3x-y}$  is: [1 mark]

- (A)  $3e^y = e^{3x} + C$
- (B)  $\log(3x - y) = C$
- (C)  $e^{3x-y} = C$
- (D)  $-e^y + 3e^{3x} = C$

**Correct Answer:** (A)  $3e^y = e^{3x} + C$

**Solution:**

**Concept:**

- Separate variables using  $e^{3x-y} = e^{3x}e^{-y}$ .
- The integrals are  $\int e^y dy = e^y$  and  $\int e^{3x} dx = e^{3x}/3$ .

**Step 1: Separate and integrate** Multiplying  $dy/dx = e^{3x-y}$  by  $e^y dx$  gives

$$e^y dy = e^{3x} dx.$$

**Quick Tip to Solve:** After separating exponential variables, check the factor from integrating the exponent  $3x$ .

Integration yields  $e^y = \frac{1}{3}e^{3x} + C_1$ . The factor  $1/3$  is necessary because the exponent on the right is  $3x$ .

**Step 2: Express the solution and check the options** Multiplying by 3 and renaming  $3C_1$  as  $C$  gives  $3e^y = e^{3x} + C$ . Differentiating this gives  $3e^y y' = 3e^{3x}$ , exactly the given equation. Options (B) and (C) force  $y' = 3$ ; option (D) gives  $y' = 9e^{3x-y}$ , so they fail.

**Step 3: Conclusion**  $3e^y = e^{3x} + C$ . Hence, the correct option is (A).

**Quick Tip to Solve:** After separating exponential variables, check the factor from integrating the exponent  $3x$ .

**17. Derivative of  $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$ ,  $-\frac{\pi}{4} < x < \frac{\pi}{4}$ , with respect to  $x$  is:**

[1 mark]

(A)  $-1$

(B)  $1$

(C)  $\frac{\pi}{4}$

(D)  $-\frac{\pi}{4}$

**Correct Answer:** (A)  $-1$

**Solution:**

**Concept:**

- The principal range of  $\cos^{-1} t$  is  $[0, \pi]$ . Thus  $\cos^{-1}(\cos u) = u$  only when  $u$  lies in this range.
- The addition identity gives  $(\sin x + \cos x)/\sqrt{2} = \cos(\pi/4 - x)$ .

**Step 1: Use the stated interval** For  $-\pi/4 < x < \pi/4$ , we have  $0 < \pi/4 - x < \pi/2$ . Therefore the angle  $\pi/4 - x$  lies within the principal range of inverse cosine, so the expression is exactly  $\pi/4 - x$ .

**Step 2: Differentiate and compare the options** Its derivative is  $d(\pi/4 - x)/dx = -1$ . Option (B) has the opposite sign; options (C) and (D) are angle values and are not the derivative of this linear expression.

**Step 3: Conclusion** The derivative is  $-1$ . Hence, the correct option is (A).

**Quick Tip to Solve:** Rewrite the argument as  $\cos(\pi/4 - x)$  and check the inverse-cosine principal range before differentiating.

**18. Absolute minimum value of  $f(x) = (x - 2)^2 + 5$  in the interval  $[-3, 2]$  is:**

[1 mark]

- (A)  $-3$
- (B)  $2$
- (C)  $5$
- (D)  $30$

**Correct Answer:** (C)  $5$

**Solution:**

**Concept:**

- For every real  $x$ ,  $(x - 2)^2 \geq 0$ , with equality only at  $x = 2$ .
- An absolute minimum on a closed interval must be compared with the endpoint values; an endpoint is allowed to attain it.

**Step 1: Establish the lower bound** We have  $f(x) = (x - 2)^2 + 5 \geq 5$  throughout  $[-3, 2]$ . Equality is possible at  $x = 2$ , which is included in the interval.

**Step 2: Check the endpoints and options** At the endpoints,  $f(-3) = (-5)^2 + 5 = 30$  and  $f(2) = 5$ . Thus  $30$  is the other endpoint value, while  $2$  is the location of the minimum, not its value. The value  $-3$  is merely the left endpoint of the domain.

**Step 3: Conclusion** The absolute minimum value is  $5$ , attained at  $x = 2$ . Hence, the correct option is (C).

**Quick Tip to Solve:** Distinguish the minimising input  $x = 2$  from the minimum output  $f(2) = 5$ .

**19. Assertion (A): A relation  $R$  on the set  $\{1, 2, 3\}$  defined as  $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$  is an equivalence relation.**

**Reason (R): A relation that is reflexive, symmetric and transitive is an equivalence relation.** [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

**Correct Answer:** (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

**Solution:**

**Concept:**

- An equivalence relation is reflexive, symmetric and transitive.
- Here  $R = (\{1, 2\} \times \{1, 2\}) \cup \{(3, 3)\}$ , so related elements lie in the same one of two disjoint classes.

**Step 1: Judge Assertion (A)** All three diagonal pairs  $(1, 1), (2, 2), (3, 3)$  occur, so the relation is reflexive. The only off-diagonal pair  $(1, 2)$  occurs together with  $(2, 1)$ , so it is symmetric. Within  $\{1, 2\}$

**Quick Tip to Solve:** For transitivity, recognise the complete block on  $\{1, 2\}$  and the singleton block on  $\{3\}$ .

every possible pair is present, and 3 relates only to itself. Consequently any chain  $aRb, bRc$  stays within one class and has  $aRc$ ; it is transitive. Assertion (A) is true.

**Step 2: Judge Reason (R)** Reason (R) states the defining criterion for an equivalence relation, so it is true.

**Step 3: Check the explanatory link** The three properties verified above are precisely those listed in Reason (R). Applying this criterion establishes Assertion (A), so the reason correctly explains it.

**Step 4: Conclusion** Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

**Quick Tip to Solve:** For transitivity, recognise the complete block on  $\{1, 2\}$  and the singleton block on  $\{3\}$ .

**20. Assertion (A):** Consider a Linear Programming Problem with minimise  $Z = x + 2y$  subject to constraints  $2x + y \geq 3$ ,  $x + 2y \geq 6$ ,  $x, y \geq 0$  which gives minimum  $Z$  at infinitely many points. The corner points of feasible region are  $(0, 3)$  and  $(6, 0)$ .

**Reason (R):** If two corner points produce the same minimum value of the objective function, then every point on the line segment joining the points will give the same minimum value. [1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
- (C) Assertion (A) is true, but Reason (R) is false.
- (D) Assertion (A) is false, but Reason (R) is true.

**Correct Answer:** (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

### Solution:

#### Concept:

- The constraint  $x + 2y \geq 6$  gives the lower bound  $Z \geq 6$ .
- A feasible region defined by linear inequalities is convex, and a linear objective has the same value throughout a segment whose endpoints have that value.

**Step 1: Judge Assertion (A)** On  $x + 2y = 6$  in the first quadrant, put  $y = (6 - x)/2$ , where  $0 \leq x \leq 6$ . Then  $2x + y = 3 + 3x/2 \geq 3$ , so the whole segment from  $(0, 3)$  to  $(6, 0)$  is feasible and gives  $Z = 6$ . Also  $2x + y \geq 3$  is redundant here, since  $2x + y = \frac{1}{2}(x + 2y) + \frac{3}{2}x \geq 3$ . The stated two points are therefore the corner points and the minimum occurs at infinitely many points. Assertion (A) is true.

**Step 2: Judge Reason (R)** For two feasible minimising corners  $P, Q$ , every point  $(1 - t)P + tQ$ ,  $0 \leq t \leq 1$ , is feasible. Linearity gives  $Z((1 - t)P + tQ) = (1 - t)Z(P) + tZ(Q)$ , the same minimum. Reason (R) is true.

**Step 3: Check the explanatory link** Both stated corners give  $Z(0, 3) = Z(6, 0) = 6$ , which is the lower bound. The reason explains why every point joining them also minimises the objective.

**Quick Tip to Solve:** When the objective repeats a constraint expression, use that constraint as a bound and test its equality segment.

**Step 4: Conclusion** Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

**Quick Tip to Solve:** When the objective repeats a constraint expression, use that constraint as a bound and test its equality segment.

### Section B (10 marks)

This section comprises 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Evaluate  $\tan \left[ \cos^{-1} \left( \tan \frac{3\pi}{4} \right) \right]$ .

[2 marks]

**Solution:**

**Concept:**

- The identity  $\tan(\pi - \theta) = -\tan \theta$  evaluates tangent in the second quadrant.
- The principal range of  $\cos^{-1} t$  is  $[0, \pi]$ , so  $\cos^{-1}(-1) = \pi$ .
- Tangent is  $\tan \theta = \sin \theta / \cos \theta$  whenever  $\cos \theta \neq 0$ .

**Step 1: Evaluate the innermost tangent** Since  $3\pi/4 = \pi - \pi/4$ , the angle lies in the second quadrant, where tangent is negative. Thus

$$\tan \frac{3\pi}{4} = -\tan \frac{\pi}{4} = -1.$$

The resulting input -1 is within the domain  $[-1, 1]$  of inverse cosine.

**Step 2: Apply the principal inverse cosine** Write  $\alpha = \cos^{-1}(-1)$ . It must satisfy  $\cos \alpha = -1$  and  $0 \leq \alpha \leq \pi$ . The unique allowed angle is  $\alpha = \pi$ ; values such as  $-\pi$  do not belong to this principal range.

**Step 3: Evaluate the outer tangent** Substituting the principal angle gives

$$\tan \left[ \cos^{-1} \left( \tan \frac{3\pi}{4} \right) \right] = \tan \pi = \frac{\sin \pi}{\cos \pi} = \frac{0}{-1} = 0.$$

This tangent is defined because its cosine denominator is nonzero.

**Step 4: Conclusion** The required value is  $\boxed{0}$ .

**Quick Tip to Solve:** Evaluate nested functions from the inside outward, enforcing the principal range at the inverse-cosine step.

**22. Find the angle between the following pair of lines:**

$$\frac{x-2}{3} = \frac{y+5}{2} = \frac{1-z}{-6} \text{ and } \frac{x-7}{1} = \frac{y}{2} = \frac{6-z}{-2}.$$

[2 marks]

**Solution:**

**Concept:**

- For direction vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , the acute angle between lines satisfies  $\cos \theta = |\mathbf{a} \cdot \mathbf{b}| / (|\mathbf{a}| |\mathbf{b}|)$ .
- Direction ratios are read after writing each numerator as coordinate minus its fixed value. Reversing both numerator and denominator preserves a ratio.

**Step 1: Read the direction ratios carefully** For the first line,  $(1-z)/(-6) = (z-1)/6$ . Hence its standard symmetric form is

$$(x-2)/3 = (y+5)/2 = (z-1)/6,$$

giving  $\mathbf{a} = (3, 2, 6)$ . For the second line,  $(6-z)/(-2) = (z-6)/2$ , so  $\mathbf{b} = (1, 2, 2)$ . The positions of the lines do not affect their angle.

**Step 2: Apply the dot-product formula** The dot product and lengths are

$$\mathbf{a} \cdot \mathbf{b} = 3(1) + 2(2) + 6(2) = 19,$$

$$|\mathbf{a}| = \sqrt{9 + 4 + 36} = 7,$$

$$|\mathbf{b}| = \sqrt{1 + 4 + 4} = 3.$$

Therefore  $\cos \theta = |19| / (7 \cdot 3) = 19/21$ . This lies between 0 and 1 and defines the smaller angle between the two lines.

**Step 3: Conclusion** The angle between the lines is  $\theta = \cos^{-1} \left( \frac{19}{21} \right)$ .

**Quick Tip to Solve:** First convert  $(c-z)/(-k)$  to  $(z-c)/k$ ; reading the raw denominator gives the wrong direction sign.

**23. Determine the interval(s) in which  $f(x) = \frac{x}{3} + \frac{3}{x}$ ,  $x \neq 0$  is increasing.**

[2 marks]

**Solution:**

**Concept:**

- A differentiable function increases where its derivative is positive; stationary boundary points can be included in a maximal interval of monotonicity.
- The domain of  $f(x) = x/3 + 3/x$  excludes  $x = 0$ , and  $x^2 > 0$  everywhere in that domain.

**Step 1: Differentiate on the domain** Using the power rule on  $3x^{-1}$ , we get

$$f'(x) = \frac{1}{3} - \frac{3}{x^2} = \frac{x^2 - 9}{3x^2}, \quad x \neq 0.$$

The derivative vanishes at  $x = -3$  and  $x = 3$ ; the function is undefined at 0.

**Step 2: Solve the derivative inequality** Since  $3x^2 > 0$ , the derivative has the sign of  $x^2 - 9 =$

**Quick Tip to Solve:** Because the derivative denominator  $3x^2$  is positive, determine its sign solely from  $x^2 - 9$ .

$(x - 3)(x + 3)$ . Thus

$$f'(x) > 0 \iff x^2 > 9 \iff x < -3 \text{ or } x > 3.$$

It is negative on  $(-3, 0)$  and  $(0, 3)$ , so neither middle interval is increasing.

**Step 3: State the increasing intervals** The positive-derivative intervals are  $(-\infty, -3)$  and  $(3, \infty)$ . At  $-3$  and  $3$  the function exists and is continuous; inclusion of these stationary endpoints preserves strict increase on each corresponding interval. The domain break at  $0$  prevents combining the two branches.

**Step 4: Conclusion** The function is increasing for  $x < -3$  or  $x > 3$ . Its maximal increasing intervals, including the stationary endpoints, are  $(-\infty, -3]$  and  $[3, \infty)$ .

**Quick Tip to Solve:** Because the derivative denominator  $3x^2$  is positive, determine its sign solely from  $x^2 - 9$ .

**24(a). Differentiate  $x^x$  with respect to  $x \log x$ .**

[2 marks]

**Solution:**

**Concept:**

- For a positive base,  $x^x = e^{x \log x}$ , where  $\log x$  denotes the natural logarithm.
- To differentiate  $y$  with respect to  $t = t(x)$ , use  $\frac{dy}{dt} = \frac{dy/dx}{dt/dx}$  wherever  $dt/dx \neq 0$ .

**Step 1: Introduce the required variable** Let  $t = x \log x$  and  $y = x^x$ , with  $x > 0$ . Taking logarithms gives  $\log y = x \log x = t$ . Hence  $y = e^t$ . This substitution uses exactly the expression with respect to which the derivative is requested; it is not differentiation with respect to  $x$  alone.

**Step 2: Differentiate and check by the chain rule** Differentiating the exponential representation gives  $\frac{dy}{dt} = e^t$ . For verification, logarithmic differentiation with respect to  $x$  gives  $\frac{dy}{dx} = x^x(1 + \log x)$ , while  $\frac{dt}{dx} = 1 + \log x$ . Their ratio is  $x^x$  wherever  $1 + \log x \neq 0$ . The direct representation  $y = e^t$  also defines the same derivative at the stationary value of  $t(x)$ .

**Step 3: Conclusion** Thus,  $\frac{d(x^x)}{d(x \log x)} = x^x$ .

**Quick Tip to Solve:** Write  $x^x = e^{x \log x}$  first: the requested variable is already the exponent.

OR

**24(b). If  $y = P \cos(ux) + Q \sin(ux)$ , then show that  $\frac{d^2y}{dx^2} + u^2y = 0$ .**

[2 marks]

**Solution:**

**Concept:**

- The coefficients  $P, Q$  and the frequency parameter  $u$  are constants with respect to  $x$ .
- The chain rule gives  $\frac{d}{dx} \cos(ux) = -u \sin(ux)$  and  $\frac{d}{dx} \sin(ux) = u \cos(ux)$ .

**Quick Tip to Solve:** After two derivatives, both sine and cosine return multiplied by  $-u^2$ ; factor the original function.

**Step 1: Find the first derivative** For the given function  $y = P \cos(ux) + Q \sin(ux)$ , differentiate each term separately. Since  $P$  and  $Q$  are constants and the derivative of the inner argument  $ux$  is  $u$ , we obtain

$$\frac{dy}{dx} = -Pu \sin(ux) + Qu \cos(ux).$$

The negative sign belongs to the derivative of the cosine term.

**Step 2: Differentiate once more and substitute** Apply the same chain rule to both terms of the first derivative:

$$\begin{aligned}\frac{d^2y}{dx^2} &= -Pu^2 \cos(ux) - Qu^2 \sin(ux) \\ &= -u^2 [P \cos(ux) + Q \sin(ux)] = -u^2 y.\end{aligned}$$

Both trigonometric terms acquire the same factor  $-u^2$ . Moving the resulting term to the left establishes the requested relation, irrespective of the values of the constants.

**Step 3: Conclusion** Hence  $\boxed{\frac{d^2y}{dx^2} + u^2y = 0}$ , as required.

**Quick Tip to Solve:** After two derivatives, both sine and cosine return multiplied by  $-u^2$ ; factor the original function.

**25(a).** Three honey bees were found flying along the vectors  $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ ,  $\vec{b} = 4\hat{j} - 2\hat{k}$  and  $\vec{c} = 3\hat{i} + 2\hat{k}$  respectively. Find the value of  $\lambda$  such that the path for  $\vec{a} + \lambda\vec{b}$  is perpendicular to  $\vec{c}$ . [2 marks]

**Solution:**

**Concept:**

- Two nonzero vectors are perpendicular if their scalar product is zero.
- In an orthonormal Cartesian basis,  $(a_1, a_2, a_3) \cdot (c_1, c_2, c_3) = a_1c_1 + a_2c_2 + a_3c_3$ .

**Step 1: Form the resultant direction** The given vectors are  $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ ,  $\vec{b} = 4\hat{j} - 2\hat{k}$ , and  $\vec{c} = 3\hat{i} + 2\hat{k}$ . Multiplying  $\vec{b}$  by  $\lambda$  and adding corresponding components gives

$$\vec{a} + \lambda\vec{b} = 2\hat{i} + (-3 + 4\lambda)\hat{j} + (1 - 2\lambda)\hat{k}.$$

The  $\hat{i}$  component remains 2 because  $\vec{b}$  has no  $\hat{i}$  component.

**Step 2: Apply the perpendicularity condition** The required path must have zero dot product with  $\vec{c}$ . Thus

$$\begin{aligned}(\vec{a} + \lambda\vec{b}) \cdot \vec{c} &= 2(3) + (-3 + 4\lambda)(0) + (1 - 2\lambda)(2) \\ &= 8 - 4\lambda = 0.\end{aligned}$$

The  $\hat{j}$  component contributes nothing because  $\vec{c}$  has zero  $\hat{j}$  component. Solving this linear equation yields the required parameter; substituting it back makes the scalar product vanish.

**Step 3: Conclusion** Therefore,  $\boxed{\lambda = 2}$ .

**Quick Tip to Solve:** Use  $\vec{a} \cdot \vec{c} + \lambda(\vec{b} \cdot \vec{c}) = 0$  to avoid writing the full resultant vector.

**25(b).** If  $A$ ,  $B$  and  $C$  be three non-collinear points such that  $\vec{AB} = \hat{i} + 2\hat{j} - \hat{k}$  and  $\vec{AC} = 2\hat{i} - 3\hat{j}$ , then find the area of  $\triangle ABC$ . [2 marks]

**Solution:**

**Concept:**

- The magnitude  $|\vec{AB} \times \vec{AC}|$  is the area of the parallelogram generated by these two vectors.
- The triangle uses half that parallelogram, so its area is  $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ .

**Step 1: Evaluate the cross product** The two sides from the common vertex  $A$  are  $\vec{AB} = (1, 2, -1)$  and  $\vec{AC} = (2, -3, 0)$ . Expanding the determinant along its first row gives

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -3 & 0 \end{vmatrix} \\ &= -3\hat{i} - 2\hat{j} - 7\hat{k}.\end{aligned}$$

In particular, the middle component has the negative sign from the cofactor expansion.

**Step 2: Take the magnitude and halve it** The length of this normal vector is

$$|\vec{AB} \times \vec{AC}| = \sqrt{(-3)^2 + (-2)^2 + (-7)^2} = \sqrt{62}.$$

This is the area of the parallelogram, not yet the triangle. The diagonal divides that parallelogram into two equal-area triangles, so the required triangle area is one half of this magnitude. The nonzero magnitude also agrees with the stated non-collinearity.

**Step 3: Conclusion** The area of  $\triangle ABC$  is  $\boxed{\frac{\sqrt{62}}{2} \text{ square units}}.$

**Quick Tip to Solve:** Use the two vectors from the same vertex and remember to halve the cross-product magnitude for a triangle.

### Section C (18 marks)

This section comprises 6 Short Answer (SA) type questions of 3 marks each.

**26.** The volume of a wooden block in the shape of a cube increases at a constant rate as the air becomes moist during the rainy season. Show that the rate of change of its surface area varies inversely as the length of edge of the cube. [3 marks]

**Solution:**

**Concept:**

- For a cube of edge length  $x > 0$ , volume is  $V = x^3$  and total surface area is  $S = 6x^2$ .
- If  $x = x(t)$  changes with time, the chain rule relates  $dV/dt$  and  $dS/dt$  through  $dx/dt$ .

**Quick Tip to Solve:** Eliminate  $dx/dt$  by dividing  $dS/dx = 12x$  by  $dV/dx = 3x^2$ , then multiply by the constant  $dV/dt$ .

- An inverse variation with  $x$  means the rate has the form  $K/x$ , where  $K$  is independent of time.

**Step 1: Translate the constant volume growth** Let  $t$  denote time and let  $x = x(t) > 0$  be the edge of the cubical block. Write the given constant rate of increase of its volume as  $\frac{dV}{dt} = k$ , where  $k > 0$  is constant. As the block remains a cube,  $V = x^3$ . Differentiating with respect to time gives

$$k = \frac{dV}{dt} = 3x^2 \frac{dx}{dt}, \quad \frac{dx}{dt} = \frac{k}{3x^2}.$$

The edge itself therefore does not grow at a constant rate: its rate depends on its current length.

**Step 2: Relate the surface-area rate to the edge** The cube has six square faces, each of area  $x^2$ , so its total surface area is  $S = 6x^2$ . Differentiating with respect to time, rather than merely with respect to  $x$ , gives

$$\begin{aligned} \frac{dS}{dt} &= 12x \frac{dx}{dt} \\ &= 12x \left( \frac{k}{3x^2} \right) = \frac{4k}{x}. \end{aligned}$$

Thus  $x \frac{dS}{dt} = 4k$ , a constant. Since  $k$  is fixed by the stated volume growth, the coefficient of  $1/x$  is independent of time. For example, if the edge doubles while the volume-growth rate stays fixed, the surface-area-growth rate becomes half its earlier value. This explains the inverse dependence directly.

**Step 3: Conclusion** Hence  $\boxed{\frac{dS}{dt} = \frac{4k}{x} \propto \frac{1}{x}}$ : the surface-area rate varies inversely as the edge length.

**Quick Tip to Solve:** Eliminate  $dx/dt$  by dividing  $dS/dx = 12x$  by  $dV/dx = 3x^2$ , then multiply by the constant  $dV/dt$ .

**27(a). A die is rolled. Consider events:**  $A = \{1, 2, 5\}$ ,  $B = \{3, 5\}$ ,  $C = \{2, 3, 4, 5\}$  **and hence find:**

(i)  $P(A|C)$  and  $P(C|A)$

(ii)  $P(A \cap B|C)$  and  $P(A \cup B|C)$

[3 marks]

**Solution:**

**Concept:**

- For equally likely outcomes,  $P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)}$ , provided  $P(F) > 0$ . Conditioning restricts the sample space to  $F$ .
- The intersection contains outcomes common to both events, while the union contains every outcome belonging to either event, with repeated outcomes counted only once.

**Step 1: List the relevant intersections** The die has sample space  $S = \{1, 2, 3, 4, 5, 6\}$ , with each outcome having probability  $1/6$ . Here  $n(A) = 3$ ,  $n(B) = 2$ , and  $n(C) = 4$ . Since  $A = \{1, 2, 5\}$  and  $C = \{2, 3, 4, 5\}$ , their common outcomes give  $A \cap C = \{2, 5\}$ . Thus  $P(A \cap C) = 2/6$ , whereas  $P(A) = 3/6$  and  $P(C) = 4/6$ .

**Step 2: Evaluate the two conditional probabilities in part (i)** Use the conditioning event in the

**Quick Tip to Solve:** For each probability conditioned on  $C$ , count only within  $\{2, 3, 4, 5\}$ ; the denominator is always 4.

denominator each time:

$$P(A | C) = \frac{2/6}{4/6} = \frac{1}{2},$$
$$P(C | A) = \frac{2/6}{3/6} = \frac{2}{3}.$$

Although the numerator is the same, the denominators differ, so these two conditional probabilities need not be equal.

**Step 3: Evaluate the probabilities in part (ii)** We have  $A \cap B = \{5\}$ , and this outcome also belongs to  $C$ . Hence  $(A \cap B) \cap C = \{5\}$ . Also  $A \cup B = \{1, 2, 3, 5\}$ , so  $(A \cup B) \cap C = \{2, 3, 5\}$ . Consequently,

$$P(A \cap B | C) = \frac{1/6}{4/6} = \frac{1}{4},$$
$$P(A \cup B | C) = \frac{3/6}{4/6} = \frac{3}{4}.$$

The outcome 1 is excluded from the conditional union because it is outside  $C$ .

**Step 4: Conclusion** (i)  $P(A | C) = \frac{1}{2}, \quad P(C | A) = \frac{2}{3}$ . (ii)  $P(A \cap B | C) = \frac{1}{4}, \quad P(A \cup B | C) = \frac{3}{4}$ .

**Quick Tip to Solve:** For each probability conditioned on  $C$ , count only within  $\{2, 3, 4, 5\}$ ; the denominator is always 4.

OR

**27(b).** A box contains 6 cards numbered 1 to 6. A student is asked to pick up two cards, one by one after replacement and note down the numbers on the cards. Let  $A$  be the event of getting sum of the numbers on two cards as 10, and  $B$ , the event of a number other than 4 on the first card selected. Find  $P(A \text{ and } B)$  and find whether the events  $A$  and  $B$  are independent events or not. [3 marks]

**Solution:**

**Concept:**

- Replacement makes the two draws independent and leaves six possible numbers on each draw. The ordered pairs  $(i, j)$ , with  $1 \leq i, j \leq 6$ , are the 36 equally likely elementary outcomes.
- Events  $A$  and  $B$  are independent precisely when  $P(A \cap B) = P(A)P(B)$ . Independent draws do not automatically make every event defined using those draws independent.

**Step 1: Find the outcomes giving a sum of 10** The first and second numbers can sum to 10 only through the ordered pairs

$$(4, 6), (5, 5), (6, 4).$$

Each pair has probability  $(1/6)(1/6) = 1/36$ , because the first card is replaced before the second selection. Therefore  $A = \{(4, 6), (5, 5), (6, 4)\}$ , and  $P(A) = 3/36 = 1/12$ . Order matters here:  $(4, 6)$  and  $(6, 4)$  are distinct outcomes.

**Step 2: Find the simultaneous event** Event  $B$  permits the first number to be any of 1, 2, 3, 5, 6, while

**Quick Tip to Solve:** List the ordered pairs with sum 10 first, then delete those whose first coordinate is 4 before testing independence.

the second number may be any of the six numbers. Thus  $B$  contains  $5 \times 6 = 30$  outcomes and  $P(B) = 30/36 = 5/6$ . Among the three outcomes in  $A$ , the pair  $(4, 6)$  violates  $B$ . The other two satisfy both events, so

$$A \cap B = \{(5, 5), (6, 4)\}, \quad P(A \text{ and } B) = \frac{2}{36} = \frac{1}{18}.$$

**Step 3: Test the independence condition** The product of the individual probabilities is

$$P(A)P(B) = \frac{1}{12} \cdot \frac{5}{6} = \frac{5}{72}.$$

But the probability of their intersection is  $1/18 = 4/72$ , which differs from  $5/72$ . Equivalently,  $P(B | A) = 2/3$ , whereas  $P(B) = 5/6$ : knowing that the sum is 10 changes the chance that the first number is other than 4.

**Step 4: Conclusion**  $P(A \text{ and } B) = \frac{1}{18}$ . Since  $P(A \cap B) \neq P(A)P(B)$ , the events  $A$  and  $B$  are not independent.

**Quick Tip to Solve:** List the ordered pairs with sum 10 first, then delete those whose first coordinate is 4 before testing independence.

## 28. Solve the differential equation $(x - \sin y) dy + \tan y dx = 0$ .

[3 marks]

**Solution:**

**Concept:**

- A linear differential equation in the dependent variable  $x$  has form  $\frac{dx}{dy} + P(y)x = Q(y)$ . Its integrating factor is  $I(y) = e^{\int P(y) dy}$ .
- After multiplication by  $I(y)$ , the left side is  $\frac{d}{dy}[xI(y)]$ . Treat  $y$  as the independent variable because the given coefficients make this choice linear.

**Step 1: Put the equation in linear form** The given equation is  $(x - \sin y) dy + \tan y dx = 0$ . On an interval where the required division is valid, divide by  $\tan y dy$  and rearrange:

$$\frac{dx}{dy} + x \cot y = \sin y \cot y = \cos y.$$

Thus  $P(y) = \cot y$  and  $Q(y) = \cos y$ . The original equation contains  $\tan y$ , so its domain already excludes values at which  $\cos y = 0$ .

**Step 2: Calculate the integrating factor** Integrating the coefficient of  $x$ , we obtain

$$I(y) = e^{\int \cot y dy} = e^{\log |\sin y|}.$$

On an interval with  $\sin y \neq 0$ , a constant sign can be absorbed into the integrating factor, and we may

**Quick Tip to Solve:** When an equation is nonlinear in  $y(x)$ , check whether it becomes linear in  $x(y)$ ; here the integrating factor is  $\sin y$ .

take  $I(y) = \sin y$ . Multiplying the linear equation by this factor gives

$$\sin y \frac{dx}{dy} + x \cos y = \sin y \cos y.$$

The left side is exactly the derivative of  $x \sin y$  by the product rule.

**Step 3: Integrate and verify the resulting family** Integration with respect to  $y$  yields

$$x \sin y = \int \sin y \cos y dy = \frac{1}{2} \sin^2 y + C.$$

For a check, differentiating this relation gives

$$\sin y dx + (x - \sin y) \cos y dy = 0,$$

which recovers the original equation after division by  $\cos y$  on its domain. The constant solutions  $y = n\pi$ , where  $n$  is an integer, also satisfy the original equation and are included in the implicit family when  $C = 0$ .

**Step 4: Conclusion** The solution is  $x \sin y - \frac{1}{2} \sin^2 y = C$ , where  $C$  is an arbitrary constant, on intervals where the original equation is defined.

**Quick Tip to Solve:** When an equation is nonlinear in  $y(x)$ , check whether it becomes linear in  $x(y)$ ; here the integrating factor is  $\sin y$ .

**29(a).** Let three toys  $A$ ,  $B$  and  $C$  be placed in the same straight line. If the position vectors of  $A$ ,  $B$  and  $C$  are  $55\hat{i} - 2\hat{j}$ ,  $5\hat{i} + 8\hat{j}$  and  $a\hat{i} - 52\hat{j}$  respectively, find the value of ' $a$ '. [3 marks]

**Solution:**

**Concept:**

- Three distinct points  $A, B, C$  are collinear when their displacement vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are scalar multiples:  $\overrightarrow{AC} = \lambda \overrightarrow{AB}$  for some real number  $\lambda$ .
- For points given by position vectors, subtract the initial position vector from the final position vector to obtain a displacement. Equality of vectors requires equality of each corresponding component.

**Step 1: Calculate displacements from the same point** The position vectors are  $\vec{r}_A = 55\hat{i} - 2\hat{j}$ ,  $\vec{r}_B = 5\hat{i} + 8\hat{j}$ , and  $\vec{r}_C = a\hat{i} - 52\hat{j}$ . Hence

$$\begin{aligned} \overrightarrow{AB} &= \vec{r}_B - \vec{r}_A \\ &= (5 - 55)\hat{i} + (8 + 2)\hat{j} \\ &= -50\hat{i} + 10\hat{j}, \\ \overrightarrow{AC} &= \vec{r}_C - \vec{r}_A \\ &= (a - 55)\hat{i} + (-52 + 2)\hat{j} \\ &= (a - 55)\hat{i} - 50\hat{j}. \end{aligned}$$

**Quick Tip to Solve:** Subtract the position vector of  $A$  from both others, then use the known vertical components to find the common scalar multiplier.

Using the same initial point makes the proportionality condition direct.

**Step 2: Use the vertical component to find the multiplier** Since the toys lie on one straight line, set  $\vec{AC} = \lambda \vec{AB}$ . Comparing the  $\hat{j}$  components gives

$$-50 = 10\lambda, \quad \lambda = -5.$$

The negative sign means that  $C$  lies on the opposite side of  $A$  from  $B$ . Comparing the  $\hat{i}$  components now gives  $a - 55 = (-5)(-50) = 250$ , and therefore  $a = 305$ .

**Step 3: Check both components for the calculated value** For  $a = 305$ , the second displacement becomes  $\vec{AC} = 250\hat{i} - 50\hat{j}$ . This equals  $-5(-50\hat{i} + 10\hat{j}) = -5\vec{AB}$ , so both components have exactly the same multiplier. Thus the calculated coordinate satisfies collinearity, rather than merely matching one component of the position vectors.

**Step 4: Conclusion** The required value is  $\boxed{a = 305}$ .

**Quick Tip to Solve:** Subtract the position vector of  $A$  from both others, then use the known vertical components to find the common scalar multiplier.

OR

**29(b).** If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are unit vectors, then prove that  $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \leq 9$ .

[3 marks]

**Solution:**

**Concept:**

- For any vector  $\vec{v}$ , its squared magnitude is  $|\vec{v}|^2 = \vec{v} \cdot \vec{v} \geq 0$ . A unit vector has squared magnitude equal to 1.
- The dot-product identity  $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}$  converts squared distances into dot products.

**Step 1: Expand the three squared magnitudes** Put  $T = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ . Since all three vectors are unit vectors, each difference contributes two unit squared magnitudes. Thus the sum  $S$  in the question becomes

$$\begin{aligned} S &= (2 - 2\vec{a} \cdot \vec{b}) + (2 - 2\vec{b} \cdot \vec{c}) + (2 - 2\vec{c} \cdot \vec{a}) \\ &= 6 - 2T. \end{aligned}$$

We therefore need a lower bound for the combined dot products, rather than separate bounds for each pair.

**Step 2: Use the nonnegative squared magnitude of their sum** Expand the magnitude of  $\vec{a} + \vec{b} + \vec{c}$ , using symmetry of the dot product:

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2T \\ &= 3 + 2T \geq 0. \end{aligned}$$

Consequently  $T \geq -\frac{3}{2}$ . Equivalently, substitution into the expression for  $S$  gives the useful exact

**Quick Tip to Solve:** Expand the squared magnitude of the sum of the three vectors; it bounds all three pairwise dot products together.

identity

$$S = 9 - |\vec{a} + \vec{b} + \vec{c}|^2.$$

The subtracted squared magnitude cannot be negative. Equality occurs precisely when  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ , so the upper bound is also attainable.

**Step 3: Conclusion**

$$|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 \leq 9.$$

Hence the required inequality is proved.

**Quick Tip to Solve:** Expand the squared magnitude of the sum of the three vectors; it bounds all three pairwise dot products together.

**30(a). Find:**  $\int \frac{x - \sin x}{1 - \cos x} dx$ .

[3 marks]

**Solution:**

**Concept:**

- The half-angle identities are  $1 - \cos x = 2 \sin^2(x/2)$  and  $\sin x / (1 - \cos x) = \cot(x/2)$ , wherever the denominator is nonzero.
- Integration by parts gives  $\int u dv = uv - \int v du$ . Also  $\frac{d}{dx} \cot(x/2) = -\frac{1}{2} \csc^2(x/2)$ .

**Step 1: Split the integrand and use half-angle identities** Let  $I$  denote the required indefinite integral. On any interval where  $1 - \cos x \neq 0$ , rewrite it as

$$\begin{aligned} I &= \int \frac{x}{1 - \cos x} dx - \int \frac{\sin x}{1 - \cos x} dx \\ &= \frac{1}{2} \int x \csc^2(x/2) dx - \int \cot(x/2) dx. \end{aligned}$$

The first term contains a linear factor multiplied by a derivative of a cotangent. This is the reason integration by parts is effective here.

**Step 2: Integrate by parts and cancel the remaining integrals** For the first integral take  $u = x$ ,  $du = dx$ , and  $dv = \frac{1}{2} \csc^2(x/2) dx$ . The chain rule gives  $v = -\cot(x/2)$ . Therefore,

$$\frac{1}{2} \int x \csc^2(x/2) dx = -x \cot(x/2) + \int \cot(x/2) dx.$$

Substituting this into  $I$ , the two cotangent integrals cancel exactly. To check the result independently, differentiate the remaining product:

$$\begin{aligned} \frac{d}{dx} [-x \cot(x/2)] &= -\cot(x/2) + \frac{x}{2} \csc^2(x/2) \\ &= \frac{x - \sin x}{1 - \cos x}. \end{aligned}$$

This verifies the antiderivative on each interval of definition; an arbitrary additive constant is included.

**Quick Tip to Solve:** Try differentiating a product of  $x$  and  $\cot(x/2)$ : the two terms reproduce the numerator after half-angle conversion.

**Step 3: Conclusion**

$$\int \frac{x - \sin x}{1 - \cos x} dx = -x \cot\left(\frac{x}{2}\right) + C.$$

**Quick Tip to Solve:** Try differentiating a product of  $x$  and  $\cot(x/2)$ : the two terms reproduce the numerator after half-angle conversion.

OR

**30(b). Evaluate:**  $\int_0^2 \frac{1}{\sqrt{x^2 + 2x + 3}} dx.$

[3 marks]

**Solution:****Concept:**

- Complete the square:  $x^2 + 2x + 3 = (x + 1)^2 + 2$ . This identifies the standard positive quadratic under the square root.
- For  $a > 0$ ,  $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln|u + \sqrt{u^2 + a^2}| + C$ . In a definite integral, the substitution must also change both limits.

**Step 1: Complete the square and transform the limits** Write the given integral as

$$I = \int_0^2 \frac{dx}{\sqrt{(x+1)^2 + 2}}.$$

Set  $u = x + 1$ , so  $du = dx$ . The lower limit  $x = 0$  becomes  $u = 1$ , and the upper limit  $x = 2$  becomes  $u = 3$ . Hence

$$I = \int_1^3 \frac{du}{\sqrt{u^2 + 2}}.$$

The radicand is strictly positive throughout this interval, and we use its positive square root.

**Step 2: Apply the standard antiderivative and evaluate** Using the logarithmic formula with  $a = \sqrt{2}$ ,

$$\begin{aligned} I &= \left[ \ln|u + \sqrt{u^2 + 2}| \right]_1^3 \\ &= \ln(3 + \sqrt{11}) - \ln(1 + \sqrt{3}). \end{aligned}$$

Both logarithm arguments are positive because  $u \geq 1$ , so absolute-value signs may be removed after inserting the limits. The formula can also be checked directly: differentiating  $\ln(u + \sqrt{u^2 + 2})$  gives

$$\frac{1 + u/\sqrt{u^2 + 2}}{u + \sqrt{u^2 + 2}} = \frac{1}{\sqrt{u^2 + 2}}.$$

Finally, use  $\ln A - \ln B = \ln(A/B)$  to express the answer as one logarithm. The ratio exceeds one, consistently giving a positive integral of a positive function.

**Quick Tip to Solve:** Complete the square first, then shift  $x$  to  $x+1$  and immediately change the limits from 0,2 to 1,3.

**Step 3: Conclusion**

$$I = \ln \left( \frac{3 + \sqrt{11}}{1 + \sqrt{3}} \right).$$

**Quick Tip to Solve:** Complete the square first, then shift  $x$  to  $x+1$  and immediately change the limits from 0,2 to 1,3.

**31. Solve the following Linear Programming Problem graphically:**

**Maximize**  $Z = \frac{2x}{5} + \frac{3y}{10}$  **subject to constraints**

$$2x + y \leq 1000$$

$$x + y \leq 800$$

$$x, y \geq 0.$$

[3 marks]

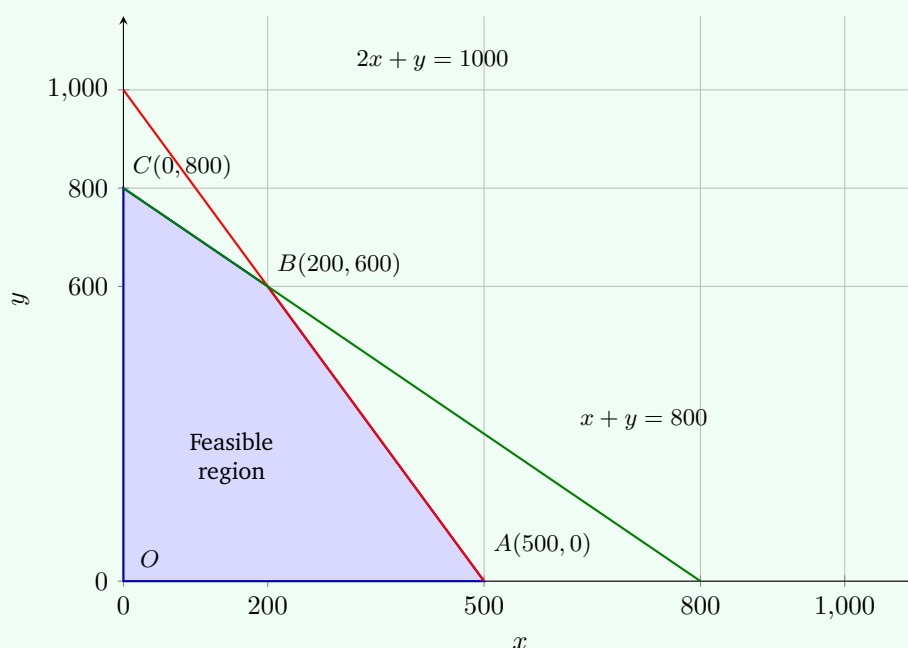
**Solution:**

**Concept:**

- The feasible region is the intersection of the half-planes satisfying all constraints, including the non-negative quadrant  $x \geq 0, y \geq 0$ .
- For a bounded feasible polygon, a linear objective  $Z = \frac{2}{5}x + \frac{3}{10}y$  attains its maximum at a corner point. Evaluate every vertex, rather than choosing the largest coordinate separately.

**Step 1: Plot the boundary lines and shade their common region** For  $2x + y = 1000$ , the intercepts are (500, 0) and (0, 1000). For  $x + y = 800$ , they are (800, 0) and (0, 800). Draw each line through its two intercepts. The origin satisfies both inequalities, so take the side of each line containing the origin. Restrict this intersection to the first quadrant. The shaded polygon in the graph is therefore the feasible region; both boundary lines are included because the inequalities are non-strict.

**Quick Tip to Solve:** Find the intersection of the two constraint lines first, but compare its objective value with both axis vertices before declaring a maximum.



Feasible region OABC for the two constraints

**Step 2: Find all corner points** The two lines meet where

$$2x + y = 1000,$$

$$x + y = 800.$$

Subtracting gives  $x = 200$ , and then  $y = 600$ . On the positive  $x$ -axis both inequalities require  $x \leq 500$ ; on the positive  $y$ -axis both require  $y \leq 800$ . Thus the vertices are  $O(0, 0)$ ,  $A(500, 0)$ ,  $B(200, 600)$  and  $C(0, 800)$ .

**Step 3: Evaluate the objective at the vertices** Substitute each vertex into the objective:

| Vertex $(x, y)$ | $Z = \frac{2}{5}x + \frac{3}{10}y$ |
|-----------------|------------------------------------|
| $(0, 0)$        | 0                                  |
| $(500, 0)$      | 200                                |
| $(200, 600)$    | $80 + 180 = 260$                   |
| $(0, 800)$      | 240                                |

These are all the vertices of the bounded feasible region. The value at  $B$  exceeds the value at each other vertex, so the corner-point criterion determines the optimum.

**Step 4: Conclusion** The maximum value is  $Z_{\max} = 260$ , attained at  $x = 200, y = 600$ .

**Quick Tip to Solve:** Find the intersection of the two constraint lines first, but compare its objective value with both axis vertices before declaring a maximum.

## Section D (20 marks)

This section comprises 4 Long Answer (LA) type questions of 5 marks each.

**32. Three students A, B and C go to a book-store to buy art books, story books and puzzle solving books. A buys one of each type of book for a total of Rs. 21. B buys 4 art books, 3 story books and 2 puzzle solving books for Rs. 60. C buys 6 art books, 2 story books and 3 puzzle solving books and pays Rs. 10 more than B. Use matrix method to find the cost of each type of book.** [5 marks]

### Solution:

#### Concept:

- Let  $x, y, z$  denote the costs, in rupees, of one art book, one story book and one puzzle solving book, respectively. Each purchase gives a linear equation in these same three unknowns.
- A system  $AX = B$  with  $\det A \neq 0$  has the unique solution  $X = A^{-1}B = \frac{1}{\det A} \text{adj}(A)B$ . The adjugate is the transpose of the cofactor matrix.

**Step 1: Translate the three purchases into a matrix equation** Student A buys one book of each type, so  $x + y + z = 21$ . Student B's purchase gives  $4x + 3y + 2z = 60$ . Student C pays Rs.  $60 + 10 = 70$ , so  $6x + 2y + 3z = 70$ . Hence

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 21 \\ 60 \\ 70 \end{pmatrix}}_B.$$

The column order is art, story, puzzle throughout; changing this order in only one matrix would change the meaning of the answer.

**Step 2: Calculate the determinant and the adjugate** Expanding along the first row,

$$\det A = 1(9 - 4) - 1(12 - 12) + 1(8 - 18) = 5 - 0 - 10 = -5.$$

Thus A is invertible. Using  $C_{ij} = (-1)^{i+j} M_{ij}$ , where  $M_{ij}$  is the minor obtained by deleting row  $i$  and column  $j$ , gives

$$C = \begin{pmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{pmatrix}, \quad \text{adj}(A) = C^T = \begin{pmatrix} 5 & -1 & -10 \\ 0 & -3 & 4 \\ -1 & 2 & -1 \end{pmatrix}.$$

For example,  $C_{23} = -(1 \cdot 2 - 1 \cdot 6) = 4$ ; retaining the alternating cofactor signs is essential.

**Step 3: Apply the inverse matrix to the payment column** Multiply the adjugate by B, then divide by

**Quick Tip to Solve:** Write the third payment as 60+10 before forming the matrix, keep a fixed book order, and verify the inverse result against all three totals.

the determinant:

$$\begin{aligned} X &= -\frac{1}{5} \begin{pmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{pmatrix} \begin{pmatrix} 21 \\ 60 \\ 70 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 105 - 60 - 70 \\ -180 + 140 \\ -210 + 240 - 70 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -25 \\ -40 \\ -40 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \end{pmatrix}. \end{aligned}$$

The first, second and third entries correspond to art, story and puzzle solving books respectively.

**Step 4: Check the costs against all three purchases** With these costs,  $A$ 's total is  $5+8+8 = 21$ , while  $B$ 's total is  $4(5)+3(8)+2(8) = 20+24+16 = 60$ . Student  $C$ 's total is  $6(5)+2(8)+3(8) = 30+16+24 = 70$ , exactly Rs. 10 more than  $B$ 's payment. All three original conditions are satisfied, and the nonzero determinant ensures uniqueness.

**Step 5: Conclusion** The cost of one art book is Rs. 5; one story book costs Rs. 8; and one puzzle solving book costs Rs. 8.

**Quick Tip to Solve:** Write the third payment as  $60+10$  before forming the matrix, keep a fixed book order, and verify the inverse result against all three totals.

**33(a).** If  $y\sqrt{x^2+1} = \log(\sqrt{x^2+1}-x)$ , then show that  $(x^2+1)\frac{dy}{dx} + xy + 1 = 0$ .

[5 marks]

**Solution:**

**Concept:**

- Implicit differentiation differentiates both sides of an equation with respect to  $x$ , treating  $y$  as a function of  $x$ . Thus a product containing  $y$  requires the product rule.
- The chain rule gives  $\frac{d}{dx} \log u = \frac{u'}{u}$  for  $u > 0$ , and  $\frac{d}{dx} \sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$ .
- For every real  $x$ ,  $\sqrt{x^2+1} > |x|$ ; consequently the logarithm's argument is positive and the square root never vanishes.

**Step 1: Differentiate the left-hand side using the product rule** Put  $s = \sqrt{x^2+1}$ , so that  $s' = x/s$ . The given relation is  $ys = \log(s-x)$ . Since both  $y$  and  $s$  depend on  $x$ , differentiating their product yields

$$\frac{d}{dx}(ys) = s \frac{dy}{dx} + y \frac{ds}{dx} = s \frac{dy}{dx} + \frac{xy}{s}.$$

Retaining the term containing  $dy/dx$  is essential:  $y$  is the dependent variable, not a constant.

**Quick Tip to Solve:** Set  $s = \sqrt{x^2+1}$ ; in the logarithmic derivative cancel  $x-s = -(s-x)$  before doing any further algebra.

**Step 2: Differentiate and simplify the logarithm** Applying the chain rule to the right-hand side gives

$$\begin{aligned}\frac{d}{dx} \log(s - x) &= \frac{s' - 1}{s - x} \\ &= \frac{x/s - 1}{s - x} \\ &= \frac{x - s}{s(s - x)} = -\frac{1}{s}.\end{aligned}$$

In the last step,  $x - s = -(s - x)$ . The cancelled factor is nonzero because  $s - x > 0$ . This cancellation is the reason the derivative of the apparently complicated logarithm has a simple form.

**Step 3: Combine the derivatives and remove the denominator** Equating the two differentiated expressions, we obtain

$$s \frac{dy}{dx} + \frac{xy}{s} = -\frac{1}{s}.$$

Multiplication by  $s$ , which is strictly positive for all real  $x$ , gives

$$s^2 \frac{dy}{dx} + xy = -1.$$

Substitute  $s^2 = x^2 + 1$  and transfer the constant to the left. No division by  $x$  was used, so the result also holds at  $x = 0$ .

**Step 4: Conclusion** Hence the required identity is proved:

$$(x^2 + 1) \frac{dy}{dx} + xy + 1 = 0.$$

**Quick Tip to Solve:** Set  $s = \sqrt{x^2 + 1}$ ; in the logarithmic derivative cancel  $x - s = -(s - x)$  before doing any further algebra.

OR

**33(b). Find the differential of  $x^{\cot x} + \frac{2x^2 - 3}{2x^2 - x + 2}$  with respect to  $x$ .**

[5 marks]

**Solution:**

**Concept:**

- Logarithmic differentiation handles a variable base raised to a variable exponent: if  $u = x^{\cot x}$ , then  $\log u = \cot x \log x$  on the real domain  $x > 0$ ,  $\sin x \neq 0$ .
- The derivatives used are  $\frac{d}{dx} \cot x = -\csc^2 x$  and  $\frac{d}{dx} \log x = 1/x$ . A product of functions is differentiated by the product rule.
- The quotient rule is  $\left(\frac{p}{q}\right)' = \frac{p'q - pq'}{q^2}$ , where  $q \neq 0$ . For a differentiable function  $f$ , its differential is  $df = f'(x) dx$ .

**Quick Tip to Solve:** Use logarithmic differentiation for  $x^{\cot x}$ , then expand the quotient-rule numerator separately before subtracting.

**Step 1: Differentiate the variable-power term** Let  $u = x^{\cot x}$ . Taking natural logarithms gives  $\log u = \cot x \log x$ . Differentiate both sides with respect to  $x$ :

$$\begin{aligned}\frac{u'}{u} &= (-\csc^2 x) \log x + \cot x \frac{1}{x}, \\ u' &= x^{\cot x} \left( \frac{\cot x}{x} - \csc^2 x \log x \right).\end{aligned}$$

Both the base and the exponent vary, so treating  $\cot x$  as a constant exponent would omit the logarithmic term.

**Step 2: Apply the quotient rule to the rational term** Put  $v = \frac{2x^2 - 3}{2x^2 - x + 2}$ . Its numerator has derivative  $4x$ , while its denominator has derivative  $4x - 1$ . Therefore

$$v' = \frac{4x(2x^2 - x + 2) - (2x^2 - 3)(4x - 1)}{(2x^2 - x + 2)^2}.$$

Expand the two products separately to preserve the minus sign:

$$\begin{aligned}4x(2x^2 - x + 2) &= 8x^3 - 4x^2 + 8x, \\ (2x^2 - 3)(4x - 1) &= 8x^3 - 2x^2 - 12x + 3.\end{aligned}$$

Subtracting gives  $-2x^2 + 20x - 3$ , and hence

$$v' = \frac{-2x^2 + 20x - 3}{(2x^2 - x + 2)^2}.$$

**Step 3: Add the derivatives and form the differential** Let  $f(x) = u + v$ . The derivative of a sum is the sum of the derivatives, so add the expressions found above. The denominator is never zero for real  $x$ , since

$$2x^2 - x + 2 = 2 \left( x - \frac{1}{4} \right)^2 + \frac{15}{8} > 0.$$

Thus the restrictions for the real expression come from the variable-power term:  $x > 0$  and  $\sin x \neq 0$ . Multiplying  $f'(x)$  by  $dx$  gives its differential.

**Step 4: Conclusion** The required differential is

$$df = \left[ x^{\cot x} \left( \frac{\cot x}{x} - \csc^2 x \log x \right) + \frac{-2x^2 + 20x - 3}{(2x^2 - x + 2)^2} \right] dx.$$

The expression in square brackets is the derivative with respect to  $x$ .

**Quick Tip to Solve:** Use logarithmic differentiation for  $x^{\cot x}$ , then expand the quotient-rule numerator separately before subtracting.

34. Sketch the graph defined by  $\left\{ (x, y) : \frac{x^2}{25} + \frac{y^2}{25} = 1 \right\}$ . Find the area of the region of minor segment cut off by the line  $x = \frac{5}{2}$ , using integration. [5 marks]

**Solution:**

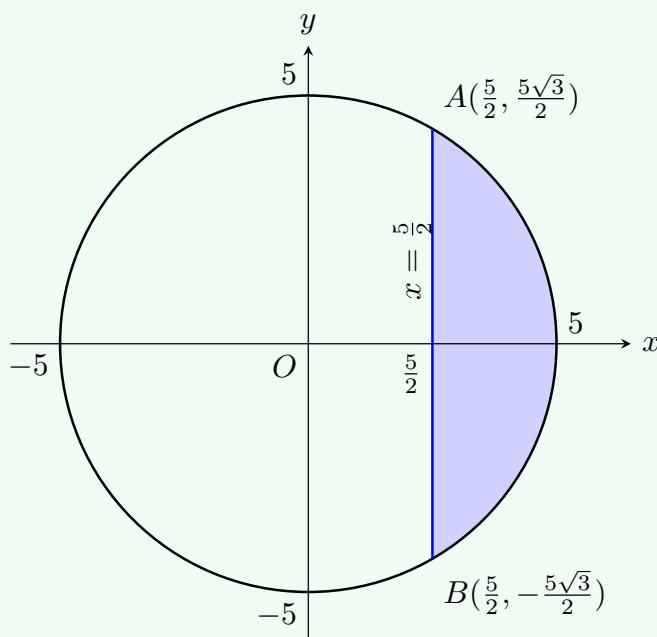
**Concept:**

- Multiplying the given equation by 25 gives  $x^2 + y^2 = 25$ , a circle with centre  $(0, 0)$  and radius 5. Its upper and lower semicircles are  $y = \sqrt{25 - x^2}$  and  $y = -\sqrt{25 - x^2}$ .
- The area between two curves over an interval is  $\int (y_{\text{upper}} - y_{\text{lower}}) dx$ . The minor segment lies to the right of the vertical chord, so its limits are  $x = 5/2$  and  $x = 5$ .

**Step 1: Locate the chord and sketch the required region** The circle meets the axes at  $(\pm 5, 0)$  and  $(0, \pm 5)$ . At  $x = 5/2$ , its equation gives

$$y^2 = 25 - \frac{25}{4} = \frac{75}{4}, \quad y = \pm \frac{5\sqrt{3}}{2}.$$

Thus the chord joins  $A(5/2, 5\sqrt{3}/2)$  and  $B(5/2, -5\sqrt{3}/2)$ . The shaded cap bounded by this chord and the shorter arc through  $(5, 0)$  is the minor segment. The larger region to the left of the chord is excluded.



Circle and shaded minor segment to the right of the chord  $x = 5/2$ .

**Step 2: Express the area as an integral** For each  $x$  between  $5/2$  and  $5$ , a vertical strip extends from the lower semicircle to the upper semicircle. Its height is  $2\sqrt{25 - x^2}$ , hence the required area  $S$  is

$$S = \int_{5/2}^5 \left( \sqrt{25 - x^2} - (-\sqrt{25 - x^2}) \right) dx = 2 \int_{5/2}^5 \sqrt{25 - x^2} dx.$$

This also uses symmetry about the  $x$ -axis: the upper half of the cap and its lower half have equal

**Quick Tip to Solve:** For a vertical chord on the right of the centre, integrate twice the upper semicircle from the chord to the rightmost point.

areas.

**Step 3: Evaluate the integral using a trigonometric substitution** Put  $x = 5 \sin \theta$ . On the required interval,  $\theta$  increases from  $\pi/6$  to  $\pi/2$ , so  $\cos \theta \geq 0$ ,  $dx = 5 \cos \theta d\theta$ , and  $\sqrt{25 - x^2} = 5 \cos \theta$ . Therefore

$$\begin{aligned} S &= 50 \int_{\pi/6}^{\pi/2} \cos^2 \theta d\theta \\ &= 25 \int_{\pi/6}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= 25 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/6}^{\pi/2} \\ &= 25 \left( \frac{\pi}{2} - \frac{\pi}{6} - \frac{\sin(\pi/3)}{2} \right) \\ &= \frac{25\pi}{3} - \frac{25\sqrt{3}}{4}. \end{aligned}$$

Both limits correspond to points on the right-hand semicircle; the substitution therefore measures exactly the shaded cap.

**Step 4: Conclusion** The graph is a circle of radius 5 centred at the origin. The area of its minor segment cut off by  $x = 5/2$  is  $\boxed{\frac{25\pi}{3} - \frac{25\sqrt{3}}{4}}$  square units.

**Quick Tip to Solve:** For a vertical chord on the right of the centre, integrate twice the upper semicircle from the chord to the rightmost point.

**35(a).** Find the foot of the perpendicular from the point  $(0, 2, 3)$  on the line  $\frac{-x-3}{-5} = \frac{1-y}{-2} = \frac{3z+12}{9}$  and hence find the length of the perpendicular. [5 marks]

**Solution:**

**Concept:**

- A point on a line can be written as  $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ , where  $\mathbf{a}$  is a fixed point on the line and  $\mathbf{d}$  is its direction vector. Equating the symmetric ratios gives its parametric coordinates.
- If  $H$  is the foot of the perpendicular from  $P$ , then  $H$  lies on the line and  $\overrightarrow{PH} \cdot \mathbf{d} = 0$ . The dot product is zero for perpendicular vectors.
- The perpendicular length is the distance  $PH = \sqrt{(x_H - x_P)^2 + (y_H - y_P)^2 + (z_H - z_P)^2}$ .

**Step 1: Put the printed line in parametric form** Set each printed ratio equal to  $t$ :

$$\frac{-x-3}{-5} = \frac{1-y}{-2} = \frac{3z+12}{9} = t.$$

The three equations give  $-x-3 = -5t$ ,  $1-y = -2t$ , and  $3z+12 = 9t$ , respectively. Therefore

$$x = 5t - 3, \quad y = 2t + 1, \quad z = 3t - 4.$$

**Quick Tip to Solve:** First isolate  $x$ ,  $y$  and  $z$  from the printed ratios; a coefficient in the numerator changes the direction component.

A fixed point on the line is  $A = (-3, 1, -4)$ , obtained at  $t = 0$ , and a direction vector is  $\mathbf{d} = (5, 2, 3)$ . In particular, the third direction component is 3, since the numerator there is  $3z + 12$ .

**Step 2: Apply the perpendicularity condition** Let the foot be  $H = (5t - 3, 2t + 1, 3t - 4)$ . From the given point  $P = (0, 2, 3)$ , the displacement to the foot is

$$\overrightarrow{PH} = (5t - 3, 2t - 1, 3t - 7).$$

Its dot product with the line direction must vanish:

$$\begin{aligned} 5(5t - 3) + 2(2t - 1) + 3(3t - 7) &= 0, \\ 25t - 15 + 4t - 2 + 9t - 21 &= 0, \\ 38t - 38 &= 0, \\ t &= 1. \end{aligned}$$

The nonzero coefficient  $38 = \mathbf{d} \cdot \mathbf{d}$  ensures a unique foot of the perpendicular.

**Step 3: Find the coordinates and verify the foot** Substituting  $t = 1$  gives

$$H = (5 - 3, 2 + 1, 3 - 4) = (2, 3, -1).$$

For these coordinates, the three original ratios are

$$\frac{-2 - 3}{-5} = 1, \quad \frac{1 - 3}{-2} = 1, \quad \frac{3(-1) + 12}{9} = 1.$$

Thus  $H$  is on the printed line. Also,  $\overrightarrow{PH} = (2, 1, -4)$ , and  $(2, 1, -4) \cdot (5, 2, 3) = 10 + 2 - 12 = 0$ . This verifies both conditions needed for a perpendicular foot.

**Step 4: Calculate the perpendicular distance** Taking the magnitude of the verified perpendicular displacement,

$$PH = \sqrt{(2 - 0)^2 + (3 - 2)^2 + (-1 - 3)^2} = \sqrt{4 + 1 + 16} = \sqrt{21}.$$

The distance is positive, as  $P$  is distinct from  $H$ , and it is measured along the perpendicular segment from the point to the line.

**Step 5: Conclusion** The foot of the perpendicular is  $(2, 3, -1)$ , and the length of the perpendicular is  $\sqrt{21}$  units.

**Quick Tip to Solve:** First isolate  $x$ ,  $y$  and  $z$  from the printed ratios; a coefficient in the numerator changes the direction component.

**35(b).** Find the value of  $p$  if the shortest distance between the lines  $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$  and  $\vec{r} = (p\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$  is  $\frac{3}{\sqrt{2}}$  units. [5 marks]

**Solution:**

**Concept:**

- For non-parallel lines with position vectors  $\vec{a}_1, \vec{a}_2$  and direction vectors  $\vec{b}_1, \vec{b}_2$ , their shortest distance is  $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$ .
- The vector  $\vec{b}_1 \times \vec{b}_2$  is perpendicular to both line directions. Projecting a vector joining points on the two lines onto this common normal gives their separation; the absolute value makes this distance non-negative.
- An equation  $|u| = c$ , with  $c > 0$ , has two cases:  $u = c$  and  $u = -c$ . Both cases must be retained when finding an unspecified real parameter.

**Step 1: Identify the points and directions** Comparing each line with  $\vec{r} = \vec{a} + t\vec{b}$ , we obtain

$$\begin{aligned}\vec{a}_1 &= (1, 2, 1), & \vec{b}_1 &= (1, -1, 1), \\ \vec{a}_2 &= (p, -1, -1), & \vec{b}_2 &= (2, 1, 2).\end{aligned}$$

Thus a vector from a point on the first line to a point on the second is  $\vec{a}_2 - \vec{a}_1 = (p-1, -3, -2)$ . The direction vectors are independent of  $p$ , so the common normal can be computed before determining the parameter.

**Step 2: Compute the common normal** Expanding the cross product gives

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} \\ &= -3\hat{i} + 3\hat{k}.\end{aligned}$$

Its magnitude is  $\sqrt{9+9} = 3\sqrt{2}$ . Since this cross product is nonzero, the lines are not parallel and the stated shortest-distance formula applies.

**Step 3: Substitute the given distance and resolve both signs** The scalar triple product is

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = -3(p-1) + 3(-2) = -3p - 3.$$

Therefore

$$\begin{aligned}d &= \frac{|-3p-3|}{3\sqrt{2}} = \frac{|p+1|}{\sqrt{2}}, \\ \frac{|p+1|}{\sqrt{2}} &= \frac{3}{\sqrt{2}}, \\ |p+1| &= 3.\end{aligned}$$

**Quick Tip to Solve:** Simplify the absolute scalar triple product before solving for  $p$ ; dropping the absolute value loses one valid value.

The two equations are  $p + 1 = 3$  and  $p + 1 = -3$ , giving  $p = 2$  and  $p = -4$ . Substituting either value in the distance expression produces  $3/\sqrt{2}$ , so both satisfy the required separation.

**Step 4: Conclusion** The required values are  $p = 2$  or  $p = -4$ .

**Quick Tip to Solve:** Simplify the absolute scalar triple product before solving for  $p$ ; dropping the absolute value loses one valid value.

### Section E (12 marks)

This section comprises 3 Case Study based questions of 4 marks each.

**36. A company produces cylindrical tumblers, open from the top. Since they want uniformity in the product, they fix the surface area of the tumblers produced.**

**Based on the above information, answer the following questions:**

**If for a tumbler,  $V$  is its volume,  $h$  the height and  $r$  the radius of the circular base, then:** [4 marks]

**36(i). Differentiate its volume with respect to radius of the base, where the surface area is constant.**

[2 marks]

**Solution:**

**Concept:**

- For a cylindrical tumbler open at the top, its fixed surface area is  $S = \pi r^2 + 2\pi r h$ : one circular base and the curved surface. Here  $r > 0$  is the base radius and  $h > 0$  is its height.
- The volume is  $V = \pi r^2 h$ . Since  $S$  remains constant as  $r$  varies, the height must vary with  $r$ ; eliminate  $h$  before differentiating.

**Step 1: Express height using the fixed area** Let the constant total surface area be  $S$ . The open top contributes no circular area. Thus

$$S = \pi r^2 + 2\pi r h \implies h = \frac{S - \pi r^2}{2\pi r}.$$

The physically meaningful radii satisfy  $0 < r < \sqrt{S/\pi}$ , so that the resulting height is positive.

**Step 2: Differentiate the volume** Substitute this height into the volume formula:

$$V = \pi r^2 \left( \frac{S - \pi r^2}{2\pi r} \right) = \frac{Sr}{2} - \frac{\pi r^3}{2}.$$

Taking the derivative with  $S$  constant gives

$$\frac{dV}{dr} = \frac{S}{2} - \frac{3\pi r^2}{2}.$$

Equivalently, substitute  $S = \pi r^2 + 2\pi r h$  into this derivative to obtain  $\pi r h - \pi r^2$ . Treating  $h$  as constant would omit the area constraint and give the wrong derivative.

**Quick Tip to Solve:** For an open cylinder, count only one base; eliminate the variable height using the fixed area before differentiating.

**Step 3: Conclusion**

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2} = \pi r(h - r)$$

where  $S$  is the fixed surface area of the open tumbler.

**Quick Tip to Solve:** For an open cylinder, count only one base; eliminate the variable height using the fixed area before differentiating.

**36(ii).** If the company wants to maximize the volume of each tumbler, then establish a relation between its height and the radius of the base. [2 marks]

**Solution:****Concept:**

- With fixed surface area  $S$ , an open cylindrical tumbler satisfies  $S = \pi r^2 + 2\pi rh$ , where  $r > 0$  and  $h > 0$ . Its volume is  $V = \pi r^2 h$ .
- A differentiable volume has an interior extremum where  $dV/dr = 0$ . A negative second derivative verifies that this stationary volume is a maximum.

**Step 1: Find the stationary radius** Eliminating the height with the fixed-area condition gives

$$h = \frac{S - \pi r^2}{2\pi r}, \quad V = \frac{Sr - \pi r^3}{2}.$$

Therefore

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2} = 0 \implies S = 3\pi r^2.$$

This radius lies in the allowed interval  $0 < r < \sqrt{S/\pi}$ .

**Step 2: Verify the maximum and obtain the height** The second derivative is

$$\frac{d^2V}{dr^2} = -3\pi r < 0 \quad (r > 0),$$

so the stationary value is a maximum. Also the volume tends to zero at either endpoint of the allowed radius interval, confirming the interior maximum is the largest attainable volume. Using the stationary condition in the surface-area equation,

$$3\pi r^2 = \pi r^2 + 2\pi rh \implies 2\pi r^2 = 2\pi rh.$$

Dividing by  $2\pi r$ , which is positive, gives the required relation.

**Step 3: Conclusion**

$$h = r$$

The volume is maximum when the tumbler height equals the radius of its base.

**Quick Tip to Solve:** For the fixed-area open cylinder, use one base; the maximum-volume condition gives height equal to radius.

37. There are three types of vaccines  $A_1, A_2, A_3$ , available in the market to protect the population of the country from spread of certain infection. According to a survey conducted, it was found that 25% of the population was given Vaccine  $A_1$ , 35% of the population was given Vaccine  $A_2$  and 40% of the population was given Vaccine  $A_3$ . The survey also stated that the probabilities that Vaccines  $A_1, A_2$  and  $A_3$  would protect against the infection were 60%, 55% and 50% respectively.

Based on the above information, answer the following questions:

Find the probability that:

[4 marks]

37(i). The person taking vaccine  $A_2$  will get infected.

[1 mark]

**Solution:**

**Concept:**

- Let  $E$  denote protection against infection and  $E^c$  denote getting infected. For a person already known to have received vaccine  $A_2$ , these complementary outcomes have conditional probabilities summing to one.
- The question asks for  $P(E^c | A_2)$ , with vaccine  $A_2$  already specified; the population share receiving this vaccine is not multiplied into a conditional probability.

**Step 1: Identify the given protection probability** The survey states that vaccine  $A_2$  protects against infection with probability 55%. Therefore, within the group of people receiving this vaccine,

$$P(E | A_2) = \frac{55}{100} = 0.55.$$

**Step 2: Apply the complement rule** Within the same vaccinated group, infection is the complement of protection. Consequently,

$$P(E^c | A_2) = 1 - P(E | A_2) = 1 - 0.55 = 0.45.$$

The factor 0.35, representing the population fraction receiving  $A_2$ , would be used for the joint event of receiving that vaccine and getting infected, which is a different event.

**Step 3: Conclusion**

$$P(E^c | A_2) = 0.45 = \frac{9}{20}$$

The required probability is 45%.

**Quick Tip to Solve:** When the vaccine is already specified, subtract its protection probability from one; do not multiply by its population share.

37(ii). If a person is chosen randomly, he/she will be protected from the infection.

[1 mark]

**Solution:**

**Concept:**

- Let  $E$  denote protection from infection, and  $A_i$  denote receipt of vaccine  $A_i$ . The vaccine groups form a partition of the population.

**Quick Tip to Solve:** Weight each protection rate by its vaccine group proportion before adding the three contributions.

- The total probability formula is  $P(E) = \sum_{i=1}^3 P(A_i)P(E | A_i)$ .

**Step 1: Identify the probabilities** The survey gives  $P(A_1) = 0.25$ ,  $P(A_2) = 0.35$ , and  $P(A_3) = 0.40$ . The corresponding probabilities of protection are 0.60, 0.55, and 0.50. These conditional protection probabilities must be weighted by the proportions receiving each vaccine; they cannot simply be averaged.

**Step 2: Apply total probability** For a randomly chosen person, the mutually exclusive ways of being protected are protection after receiving each of the three vaccines. Thus

$$P(E) = 0.25(0.60) + 0.35(0.55) + 0.40(0.50) = 0.15 + 0.1925 + 0.20 = 0.5425.$$

**Step 3: Conclusion** The required probability of protection is  $\boxed{0.5425}$ , or 54.25%.

**Quick Tip to Solve:** Weight each protection rate by its vaccine group proportion before adding the three contributions.

**37(iii)(a).** The person was given Vaccine  $A_1$ , given that the randomly chosen person is infected.

[2 marks]

**Solution:**

**Concept:**

- Let  $I$  denote infection. The vaccine groups  $A_1, A_2, A_3$  are mutually exclusive and exhaustive, with probabilities 0.25, 0.35, 0.40.
- Complementation gives  $P(I | A_i) = 1 - P(I^c | A_i)$ . Bayes theorem gives  $P(A_1 | I) = P(A_1)P(I | A_1)/P(I)$ .

**Step 1: Find the infection rates** The stated protection rates are 0.60, 0.55, 0.50 for the three vaccines. Their infection rates are therefore

$$P(I | A_1) = 0.40, \quad P(I | A_2) = 0.45, \quad P(I | A_3) = 0.50.$$

The joint probability of receiving vaccine  $A_1$  and becoming infected is  $P(A_1 \cap I) = 0.25(0.40) = 0.10$ .

**Step 2: Compute the probability of infection** Using total probability over all three vaccine groups,

$$P(I) = 0.25(0.40) + 0.35(0.45) + 0.40(0.50) = 0.10 + 0.1575 + 0.20 = 0.4575.$$

This is the denominator because the person is already known to belong to the infected population.

**Step 3: Apply Bayes theorem** The condition is infection, while the event sought is receipt of vaccine  $A_1$ . Hence

$$P(A_1 | I) = \frac{P(A_1 \cap I)}{P(I)} = \frac{0.10}{0.4575} = \frac{1000}{4575} = \frac{40}{183}.$$

This reverses the direction of conditioning from the vaccine-specific infection rate supplied in the question.

**Quick Tip to Solve:** For a posterior conditioned on infection, use failure rates in every weighted term of the denominator.

**Step 4: Conclusion** The required probability is  $\frac{40}{183}$ .

**Quick Tip to Solve:** For a posterior conditioned on infection, use failure rates in every weighted term of the denominator.

OR

**37(iii)(b).** The person was given Vaccine  $A_3$ , given that the randomly chosen person is not infected.

[2 marks]

**Solution:**

**Concept:**

- Let  $E$  denote protection, equivalently not being infected, and let  $A_i$  denote receipt of vaccine  $A_i$ .
- Use total probability  $P(E) = \sum_i P(A_i)P(E | A_i)$ , followed by Bayes theorem  $P(A_3 | E) = \frac{P(A_3)P(E | A_3)}{P(E)}$ .

**Step 1: Find the joint probability for vaccine  $A_3$**  The population proportion receiving vaccine  $A_3$  is  $P(A_3) = 0.40$ , and its protection probability is  $P(E | A_3) = 0.50$ . Therefore

$$P(A_3 \cap E) = P(A_3)P(E | A_3) = 0.40(0.50) = 0.20.$$

This numerator counts people who both received vaccine  $A_3$  and escaped infection.

**Step 2: Find the overall probability of protection** The protected person could have received any of the three vaccines. Their mutually exclusive contributions must all be included:

$$P(E) = 0.25(0.60) + 0.35(0.55) + 0.40(0.50) = 0.15 + 0.1925 + 0.20 = 0.5425.$$

The denominator is the probability of the given condition, namely not being infected.

**Step 3: Reverse the conditioning using Bayes theorem** Among the protected population, the fraction receiving vaccine  $A_3$  is

$$P(A_3 | E) = \frac{P(A_3 \cap E)}{P(E)} = \frac{0.20}{0.5425} = \frac{2000}{5425} = \frac{80}{217}.$$

Notice that 0.50 describes protection given vaccine  $A_3$ ; it is different from the reversed conditional probability requested here.

**Step 4: Conclusion** The required probability is  $\frac{80}{217}$ .

**Quick Tip to Solve:** When the person is known to be uninfected, retain protection rates in the Bayes numerator and denominator.

**38.** A school wants the students of class XII to do a project on ‘Sustainability’ keeping the world environment in mind. They select the student participants on the basis of an essay writing competition. 7 students out of 80 are selected for the project and are categorized into two sets such that:

Girl students belong to Set  $A = \{G_1, G_2, G_3, G_4\}$ ,

Boy students belong to Set  $B = \{B_1, B_2, B_3\}$ .

Based on the above information, answer the following questions:

[4 marks]

**38(i).** How many relations are possible from Set  $A \rightarrow$  Set  $B$ ?

[1 mark]

**Solution:**

**Concept:**

- A relation from  $A$  to  $B$  is any subset of the Cartesian product  $A \times B$ .
- If a finite set has  $n$  elements, it has  $2^n$  subsets; each ordered pair may independently be included or excluded.

**Step 1: Count the available ordered pairs** The given sets have  $|A| = 4$  and  $|B| = 3$ . For each of the four girls, any of the three boys can form an ordered pair. Hence

$$|A \times B| = 4 \times 3 = 12.$$

**Step 2: Count subsets of the Cartesian product** Each of these 12 ordered pairs has two choices: present in the relation or absent from it. Thus the number of relations is

$$2^{12} = 4096.$$

This count includes the empty relation and the entire Cartesian product. The expression  $3^4$  would count functions, where every girl must have exactly one image, rather than all relations.

**Step 3: Conclusion** The number of relations from  $A$  to  $B$  is 4096.

**Quick Tip to Solve:** For all relations, first multiply the set sizes and then use that product as the exponent of 2.

**38(ii).** Let  $R$  be a relation from  $A \rightarrow B$  such that  $R = \{(G_1, B_1), (G_2, B_2), (G_3, B_2), (G_4, B_3), (G_1, B_2)\}$ . Is  $R$  an injective function? Justify your answer.

[1 mark]

**Solution:**

**Concept:**

- A function from  $A$  to  $B$  must assign exactly one image in  $B$  to each element of  $A$ .
- An injective function additionally satisfies  $f(x) = f(y) \Rightarrow x = y$ ; distinct inputs cannot share an image.

**Step 1: Check whether the relation is a function** The relation contains both  $(G_1, B_1)$  and  $(G_1, B_2)$ . Since  $B_1 \neq B_2$ , the same input  $G_1$  has two distinct images. This violates the uniqueness condition for a function, although all four elements of  $A$  occur as first components.

**Step 2: Apply the requirement for injectivity** Injectivity is a property of a function, so a relation

**Quick Tip to Solve:** Check repeated first components before injectivity: two different images for one input already disprove that a relation is a function.

that fails to be a function cannot be an injective function. Moreover, the pairs  $(G_2, B_2)$  and  $(G_3, B_2)$  also assign the same second component to distinct first components, contradicting injectivity if one attempted to treat these pairs as function values.

**Step 3: Conclusion**  $R$  is not an injective function, because  $G_1$  is related to both  $B_1$  and  $B_2$ ; in fact,  $R$  is not a function.

**Quick Tip to Solve:** Check repeated first components before injectivity: two different images for one input already disprove that a relation is a function.

**38(iii)(a).** Let the relation  $R$  from  $A \rightarrow A$  be such that  $R = \{(x, y) : x, y \in A, x \text{ and } y \text{ are students from the same colony in the city}\}$ . Verify if  $R$  is an equivalence relation. [2 marks]

**Solution:**

**Concept:**

- A relation on  $A$  is an equivalence relation if it is reflexive, symmetric and transitive.
- Here  $xRy$  means that the students  $x$  and  $y$  belong to the same colony. Each property must hold for arbitrary students in  $A$ .

**Step 1: Check reflexivity** Let  $x \in A$  be any student. The student  $x$  belongs to the same colony as herself. Therefore  $(x, x) \in R$  for every  $x \in A$ , and  $R$  is reflexive. This does not require all the students in  $A$  to live in one colony.

**Step 2: Check symmetry** Let  $x, y \in A$  and suppose  $(x, y) \in R$ . Then  $x$  and  $y$  are students from the same colony. Reversing their order leaves this fact unchanged:  $y$  and  $x$  are also from the same colony. Hence  $(y, x) \in R$ , so  $R$  is symmetric.

**Step 3: Check transitivity** Let  $x, y, z \in A$  with  $(x, y) \in R$  and  $(y, z) \in R$ . The first pair says that  $x$  lives in the same colony as  $y$ , and the second says that  $z$  lives in the same colony as  $y$ . Consequently  $x$  and  $z$  are from that same colony, giving  $(x, z) \in R$ . Thus  $R$  is transitive.

**Step 4: Conclusion** Since  $R$  is reflexive, symmetric and transitive,  $R$  is an equivalence relation on  $A$ .

**Quick Tip to Solve:** Test same-colony membership with one student, with a reversed pair, and with a chain of two pairs to establish all three properties.

OR

**38(iii)(b).** Verify if any function  $f : B \rightarrow A$  is bijective. Give reason to support your answer. [2 marks]

**Solution:**

**Concept:**

- A bijective function must be both injective (distinct inputs have distinct images) and surjective (every element of the codomain is an image).

**Quick Tip to Solve:** For a possible bijection between finite sets, compare their sizes first; fewer inputs than outputs immediately rules out surjectivity.

- A bijection between finite sets requires equal cardinalities; the image of a function cannot contain more elements than its domain.

**Step 1: Compare the domain and codomain sizes** For a function  $f : B \rightarrow A$ , the domain is  $B = \{B_1, B_2, B_3\}$ , which has three elements. Its codomain is  $A = \{G_1, G_2, G_3, G_4\}$ , which has four elements. Therefore the domain and codomain have unequal cardinalities:

$$|B| = 3 < 4 = |A|.$$

**Step 2: Check the surjectivity requirement** Each of the three elements of  $B$  has exactly one image under  $f$ . Even if those three images are all distinct, they cover only three elements of  $A$ . Hence

$$|f(B)| \leq 3 < 4 = |A|.$$

At least one girl in  $A$  has no preimage, so no function  $f : B \rightarrow A$  is surjective. If images repeat, still fewer girls are covered. An injective function can exist, but cannot become bijective because this failure of surjectivity persists.

**Step 3: Conclusion** No function  $f : B \rightarrow A$  is bijective, since three domain elements cannot cover a codomain of four elements.

**Quick Tip to Solve:** For a possible bijection between finite sets, compare their sizes first; fewer inputs than outputs immediately rules out surjectivity.